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Mathematical Logic

Preface

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The Workshop on Model Theory and Computable Model Theory took place February 5–10, 2007 at the University of Florida in Gainesville, as part of the National Science Foundation-sponsored Special Year in Logic. This special issue consists of selected papers from the conference. The workshop brought researchers in classical model theory and its applications together with researchers in computable model theory. Tutorials were given in these respective areas by Thomas Scanlon (Model Theory and Connections To Algebra) and Julia Knight (Computable Model Theory).

Model theory studies the relationship between structures and their first order properties, including the study of definable subsets of their universes. An introduction to the subject is given by Marker [21]. Several topics in model theory were presented, including three areas represented in this volume: abstract model theory with links to computable model theory (Laskowski), model theory of fields (Martin-Pizarro and Wagner), and definability in number fields (Shlapentokh). The remaining papers come from within computable model theory.

By effectivizing Henkin's construction, one shows that every consistent decidable theory has a decidable model. For a decidable uncountably categorical theory T, Harrington and Khisamiev (independently) showed that every countable model of T

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is isomorphic to a decidable model. For an undecidable uncountably categorical theory, it is possible that some of the countable models are isomorphic to computable ones, some not. Goncharov et al. [14] showed that trivial, strongly minimal theories are model complete after naming constants for a model, and hence are $\forall\exists$-axiomatizable. This implies that if a trivial, strongly minimal theory has a computable model, then all of its countable models are isomorphic to $\mathbf{0}''$-decidable models. Khoussainov et al. [19] later showed that this is the best possible bound for these theories.

Back in the context of abstract model theory, more specifically stability theory, Dolich et al. [7] established model completeness for every trivial, uncountably categorical theory of Morley rank 1 once one has named constants for a model. Previously, Marker [20] constructed a trivial, totally categorical theory of Morley rank 2 which cannot be made model complete by naming constants.

Michael C. Laskowski (The elementary diagram of a trivial, weakly minimal structure is near model complete) continues this line of research on bounded quantifier depth of the elementary diagram of a model. He proves that if $\mathcal{M}$ is any model of a trivial, weakly minimal theory, then the elementary diagram $T(\mathcal{M})$ eliminates quantifiers down to Boolean combinations of certain existential formulas. A trivial, weakly minimal theory has a well behaved forking notion defined by algebraic closure, for which the finite cover property fails. The existential formulas used in the quantifier elimination are obtained from a class of quantifier-free, mutually algebraic formulas $\psi(\overrightarrow{z})$ by partitioning $\overrightarrow{z}$ into $\overrightarrow{z} = \overrightarrow{x}\,\hat{}\,\overrightarrow{y}$ and existentially quantifying over $\overrightarrow{x}$.

A long-standing open question in applied model theory is whether there are stable theories of fields beyond those of finite and separably closed fields. Related questions arise in the context of simple and supersimple theories. In 1995, Pillay conjectured that all supersimple fields are perfect, pseudo algebraically closed (PAC), and with bounded absolute Galois group, that is, have finitely many open subgroups of index n for every n. In [23], Pillay and Poizat showed that supersimple fields are perfect and have bounded absolute Galois group. A perfect fields K is PAC if every absolutely irreducible plane curve over K has a K-rational point.

Amador Martin-Pizarro and Frank O. Wagner (Supersimplicity and quadratic extensions) prove that if K is a supersimple field with exactly one extension of degree 2 (up to isomorphism), then any elliptic curve E defined over K has an s-generic K-rational point, that is, a point $P \in E(K)$ such that $SU(P/F) = SU(K)$, where F is some small set of parameters over which E is defined. The importance of this theorem is that it holds for all elliptic curves. It uses the group law in the elliptic curves, and thus it is not clear how to handle curves of larger genus. They also ask whether it is possible to generalize the result to fields with more than one extension of degree 2.

A strong link between the computable and classical arises in the context of definability theory. Structural information sought by applied model theorists can arise from the study of what can, and cannot, be defined in a given structure. Similarly, the presence of certain definable sets can enable transfer of undecidability phenomena to the underlying theory.

Interest in the questions of decidability and existential definability goes back to Hilbert's Tenth Problem (HTP) which asks for an algorithm to determine whether a given polynomial in several variables over $\mathbb{Z}$ has solutions in $\mathbb{Z}$. HTP was answered negatively by M. Davis, H. Putnam, J. Robinson and Yu. Matijasevich [12]. Similar questions arise for other fields and rings: given a computable ring R, is there an algorithm to determine whether an arbitrary polynomial in several variables over R has solutions in R? A survey of the area is given by Shlapentokh [26]. One route to a negative solution of this question over a ring R is to construct a Diophantine definition of $\mathbb{Z}$ over R. Using norm equations, Diophantine definitions have been obtained for $\mathbb{Z}$ over the rings of algebraic integers of various number fields [16] and also over certain "large" subrings of totally real number fields [27]. Another method of constructing Diophantine definitions [6] uses elliptic curves. If K is a totally real algebraic extension of $\mathbb{Q}$ and there exists an elliptic curve E over $\mathbb{Q}$ such that $[E(K) : E(\mathbb{Q})] < \infty$, then $\mathbb{Z}$ has a Diophantine definition over $\mathcal{O}_K$.

Alexandra Shlapentokh (Rings of algebraic numbers in infinite extensions of $\mathbb{Q}$ and elliptic curves retaining their rank) shows that elliptic curves whose Mordell–Weil groups are finitely generated over some infinite extensions of $\mathbb{Q}$ can be used to show the Diophantine undecidability of the rings of integers and larger rings contained in some infinite extensions of rational numbers. In particular, let K be a totally real possibly infinite extension of $\mathbb{Q}$ and let U be a finite extension of K such that there is an elliptic curve E defined over U with $E(U)$ finitely generated and of positive rank. Then $\mathbb{Z}$ is existentially definable and HTP is unsolvable over the ring of integers of K.

Computable model theory investigates the relationship between computability theoretic properties of countable structures and their theories and definable sets. Thus, computable model theory and computable algebra include the study of the computability of structures, substructures, isomorphisms and theories. We say that computable structures $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same computable isomorphism type if there is a computable isomorphism between them. The number of computable isomorphism types of $\mathcal{A}$, denoted by $\dim(\mathcal{A})$, is called the *computable dimension* of $\mathcal{A}$. It is obvious that $\dim(\mathcal{A}) = 1$ if and only if any two computable presentations of $\mathcal{A}$ are computably isomorphic. In case $\dim(\mathcal{A}) = 1$, then we say that $\mathcal{A}$ is *computably categorical*.

One of the central topics in computable model theory is the study of computable dimensions of structures and characterizations of computable categoricity. Goncharov proved that for any $n \in \omega \cup \{\omega\}$ there exists a structure of computable dimension n [13]. In [5] Cholak, Goncharov, Khoussainov and Shore gave an example of a computably categorical structure $\mathcal{A}$ such that for each $a \in A$ the structure $(\mathcal{A}, a)$ has computable dimension n, where $n \in \omega$. Goncharov and Remmel proved that a linearly ordered set is computably categorical if and only if the set of successive pairs in the order is finite [11,25]. Calvert, Cenzer, Harizanov and Morozov [1] showed that an equivalence structure is computably categorical if and only if there is a bound b on the sizes of finite equivalence classes, and there is at most one $t \in \{1, \ldots, b\} \cup \{\omega\}$ with infinitely many classes of size t.

Wesley Calvert, Sergey Goncharov, Jessica Millar and Julia Knight (Categoricity of computable infinitary theories) answer a question posed by J. Millar and Sacks, on the categoricity of the computable infinitary theories of structures with Scott rank ω_1^{CK}. In previous work, various subsets of the authors had produced computable structures of various kinds (trees [3], undirected graphs, fields, linear orderings [2]) with Scott rank ω_1^{CK}. J. Millar and Sacks asked whether it was possible that a computable structure with Scott rank ω_1^{CK} could have a computable infinitary theory that was $\aleph_0$-categorical [22]. It is natural to ask whether for known examples of computable structures of Scott rank ω_1^{CK} the theories are $\aleph_0$-categorical. The present paper gives an affirmative answer for several of the known examples; in particular, trees, undirected graphs, fields, and linear orderings.

Valentina Harizanov, Carl Jockusch and Julia Knight (Chains and antichains in partial orderings) study the complexity of infinite chains and antichains in computable partial orderings. It follows from a result of Jockusch [17] that a computable partial ordering has either an infinite Δ_2^0 chain or an infinite Δ_2^0 antichain, or else both an infinite Π_2^0 chain and an infinite Π_2^0 antichain. Hermann [15] constructed a computable partial ordering with no infinite Σ_2^0 chain or antichain. The present paper shows that there is a computable partial ordering which has an infinite chain but none that is Σ_1^1 or Π_1^1, and also obtains an analogous result for antichains. On the other hand, every computable partial ordering which has an infinite chain must have an infinite chain that is the difference of two Π_1^1 sets. The main result is that there is a computably axiomatizable theory of partial orderings which has a computable model with arbitarily long finite chains but no computable model with an infinite chain, and similarly for antichains. It is shown that if a computable partial ordering $\mathcal{A}$ has the feature that for every $\mathcal{B} \cong \mathcal{A}$, there is an infinite chain or antichain which is Δ_2^0 relative to $\mathcal{B}$, then there is a uniform dichotomy: either every copy $\mathcal{B}$ of $\mathcal{A}$ has an infinite chain which is Δ_2^0 relative to $\mathcal{B}$, or every copy $\mathcal{B}$ of $\mathcal{A}$ has an infinite antichain which is Δ_2^0 relative to $\mathcal{B}$.

Jennifer Chubb, Valentina Harizanov and Andrey Frolov (Degree spectra of the successor relation of computable linear orderings) determine a condition ensuring the Turing degree spectrum of the successor relation of a linear ordering will be closed upward in the c.e. Turing degrees. The condition applies to a broad class of linear orderings, and those to which it does not apply are characterized. The *Turing degree spectrum* of a relation on a linear ordering is the class of Turing degrees of that relation in computable copies of the linear ordering. Surprisingly little is known about the degree spectrum of the successor relation in computable linear orderings. Of course, the successor relation is always intrinsically co-c.e., and it is intrinsically computable when it is finite. Downey and Moses [9] provide an example where it is intrinsically complete. Downey, Goncharov and Hirschfeldt [10] ask whether the degree spectrum of the successor relation can consist of a single degree different from $\mathbf{0}$ and $\mathbf{0}'$, and a similar question for the degree spectrum of the atom relation of computable Boolean algebras with infinitely many atoms was resolved by Downey and Remmel. Remmel [24] established that such a spectrum is closed upward in the c.e. degrees, and Downey [8] showed that such a spectrum must contain an incomplete degree. The result in the present article provides that every upper cone of c.e. degrees is realized as the Turing degree spectrum of some computable linear ordering.

Douglas Cenzer, Barbara Csima and Bakhadyr Khoussainov (Linear orders with distinguished function symbol) study certain linear orders with a function on them, and discuss for which types of functions the resulting structure is or is not computably categorical. In [18] Khoussainov provided examples of structures of type $(\mathcal{A}, h)$ where h is a function from A to A, of computable dimension n with $n \in \omega$. In [28] Ventsov studied computable dimensions of $(L; \leq, P)$ where $(L; \leq)$ is a linearly ordered set and P is a unary predicate. This paper is a continuation of the above work with an emphasis on computable dimensions of linearly ordered sets with distinguished endomorphisms. Particular structures include computable copies of the rationals with a fixed-point free automorphism, and also ω with a non-decreasing function.

Douglas Cenzer, Rod Downey, Jeffrey Remmel and Zia Uddin (Space complexity of torsion-free Abelian groups) continue the study of complexity theoretic model theory and algebra developed by Nerode, Remmel and Cenzer; see the handbook article [4] for details. Much of the work of those authors focused on polynomial time models. The present paper develops the theory of $LOGSPACE$ structures and applies it to the study of $LOGSPACE$ Abelian groups. It is shown that all computable torsion Abelian groups have $LOGSPACE$ presentations and the authors show that the groups $\mathbb{Z}$, $\mathbb{Z}(p^{\infty})$, and the additive group of the rationals have $LOGSPACE$ presentations over a standard universe such as the tally representation and the binary representation of the natural numbers. The effective categoricity of such groups is also studied. For example, conditions are given under which two isomorphic $LOGSPACE$ structures will have a linear space isomorphism.

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