

Isogenies and the Discrete Logarithm Problem in Jacobians of Genus 3 Hyperelliptic Curves^{*,†}

Benjamin Smith

INRIA Saclay–Île-de-France, Laboratoire d’Informatique de l’École polytechnique (LIX),
91128 Palaiseau Cedex, France
smith@lix.polytechnique.fr; url: <http://www.lix.polytechnique.fr/Labo/Benjamin.Smith/>

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Abstract. We describe the use of explicit isogenies to translate instances of the Discrete Logarithm Problem (DLP) from Jacobians of hyperelliptic genus 3 curves to Jacobians of non-hyperelliptic genus 3 curves, where they are vulnerable to faster index calculus attacks. We provide explicit formulae for isogenies with kernel isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$ (over an algebraic closure of the base field) for any hyperelliptic genus 3 curve over a field of characteristic not 2 or 3. These isogenies are rational for a positive fraction of all hyperelliptic genus 3 curves defined over a finite field of characteristic $p > 3$. Subject to reasonable assumptions, our constructions give an explicit and efficient reduction of instances of the DLP from hyperelliptic to non-hyperelliptic Jacobians for around 18.57% of all hyperelliptic genus 3 curves over a given finite field. We conclude with a discussion on extending these ideas to isogenies with more general kernels.

Key words. Hyperelliptic curve cryptography, Discrete logarithm problem, Isogeny, Genus 3

1. Introduction

After the great success of elliptic curves in public-key cryptography, researchers have naturally been drawn to their higher-dimensional generalisations: Jacobians of higher-genus curves. Curves of genus 1 (elliptic curves), 2, and 3 are widely believed to offer the best balance of security and efficiency. This article is concerned with the security of curves of genus 3.

There are two classes of curves of genus 3: hyperelliptic and non-hyperelliptic. Each class has a distinct geometry: The canonical morphism of a hyperelliptic curve is a double cover of a curve of genus 0, while the canonical morphism of a non-hyperelliptic

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curve of genus 3 is a birational map to a nonsingular plane quartic curve. A hyperelliptic curve cannot be isomorphic (or birational) to a non-hyperelliptic curve. From a cryptological point of view, the Discrete Logarithm Problem (DLP) in Jacobians of hyperelliptic curves of genus 3 over \mathbb{F}_q may be solved in $\tilde{O}(q^{4/3})$ group operations, using the index calculus algorithm of Gaudry, Thomé, Thériault, and Diem [8]. Jacobians of non-hyperelliptic curves of genus 3 over \mathbb{F}_q are amenable to Diem's index calculus algorithm [5], which requires only $\tilde{O}(q)$ group operations to solve the DLP (for comparison, Pollard/baby-step-giant-step methods require $\tilde{O}(q^{3/2})$ group operations to solve the DLP in Jacobians of genus 3 curves over \mathbb{F}_q). The security of non-hyperelliptic genus 3 curves is therefore widely held to be lower than that of their hyperelliptic cousins.

Our aim is to construct explicit homomorphisms to provide a means of efficiently translating instances of the DLP from Jacobians of hyperelliptic curves of genus 3 to Jacobians of non-hyperelliptic curves, where faster index calculus is available. In the context of DLP-based cryptography, we may assume that our Jacobians are absolutely simple. In this situation, every nontrivial homomorphism of Jacobians of curves of genus 3 is an *isogeny*, that is, a surjective homomorphism with finite kernel.

To be specific, suppose we are given a hyperelliptic curve H of genus 3 over a finite field \mathbb{F}_q , together with an instance $P = [n]Q$ of the DLP in $J_H(\mathbb{F}_q)$; our task is to recover n given P and Q . After applying the standard Pohlig–Hellman reduction [19], we may assume that P and Q have prime order. We want to solve this DLP instance by solving an equivalent DLP instance in a non-hyperelliptic Jacobian. Suppose we have an isogeny $\phi : J_H \rightarrow J_C$, where C is a non-hyperelliptic curve of genus 3. Further, suppose that ϕ is explicit (that is, we have equations for C and an efficient map on divisor classes representing ϕ) and defined over \mathbb{F}_q , so it maps $J_H(\mathbb{F}_q)$ into $J_C(\mathbb{F}_q)$. Provided $\phi(Q) \neq 0$, we can recover n by solving the DLP instance $\phi(P) = [n]\phi(Q)$ in $J_C(\mathbb{F}_q)$ with Diem's algorithm.

The approach outlined above is conceptually straightforward; the difficulty lies in computing explicit isogenies of Jacobians of genus 3 curves. Automorphisms, integer multiplications, and Frobenius maps aside, we know of no explicit and general formulae for isogenies from Jacobians of hyperelliptic curves of genus 3 apart from those presented below.

In Sects. 3 through 6, we derive explicit formulae for isogenies whose kernels are generated by differences of Weierstrass points, following the construction of Donagi and Livné [7]. The key step is making Recillas' trigonal construction [20] completely explicit. This gives us a curve X of genus 3 and an explicit isogeny $J_H \rightarrow J_X$. While X may be hyperelliptic, naïve moduli space dimension arguments suggest (and experience confirms) that X will be non-hyperelliptic with an overwhelming probability, and thus explicitly isomorphic to a nonsingular plane quartic curve C . We can therefore compute an explicit isogeny $\phi : J_H \rightarrow J_C$; if ϕ is defined over \mathbb{F}_q , then we can use it to reduce DLP instances. We note that the trigonal construction (and hence our formulae) does not apply in characteristics 2 and 3.

We show in Sect. 8 that, subject to some reasonable assumptions, given a uniformly randomly chosen hyperelliptic curve H of genus 3 over a sufficiently large finite field \mathbb{F}_q of characteristic at least 5, our algorithms succeed in constructing an explicit isogeny defined over \mathbb{F}_q from J_H to a non-hyperelliptic Jacobian with probability ≈ 0.1857 . In particular, instances of the DLP can be solved in $\tilde{O}(q)$ group operations for around 18.57%

of all Jacobians of hyperelliptic curves of genus 3 over finite fields of characteristic at least 5.

We discuss more general isogenies in Sect. 9. Given explicit formulae for these isogenies, we expect that most, if not all, instances of the DLP in Jacobians of hyperelliptic curves of genus 3 over any finite field could be reduced to instances of the DLP in non-hyperelliptic Jacobians.

Our results have a number of interesting implications for curve-based cryptography, at least for curves of genus 3. First, the difficulty of the DLP in a subgroup G of J_H depends not only on the size of the subgroup G , but upon the existence of other rational subgroups of J_H that can be used to form quotients. Second, the security of a given hyperelliptic genus 3 curve depends significantly upon the factorisation of its hyperelliptic polynomial. Neither of these results has any parallel in genus 1 or 2.

The constructions of Sects. 3 through 6 and 9 require some nontrivial algebraic geometry. We have included enough mathematical detail here to enable the reader to compute examples, to justify our claim that the construction is efficient, and to support our heuristics.

A Note on the Text

This article presents an extended version of work that appeared in the proceedings of the EUROCRYPT 2008 conference [23]. The chief results are the same; we have made some (minor) changes to our notation, expanded the derivation in Sect. 6, given further details and proofs throughout, and added an appendix with algorithms to compute sets of tractable subgroups.

2. Notation and Conventions for Hyperelliptic Curves

We will work over \mathbb{F}_q throughout this article,¹ where q is a power of a prime $p > 3$. We let \mathcal{G} denote the Galois group $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, which is (topologically) generated by the q th power Frobenius map.

Suppose we are given a hyperelliptic curve H of genus 3 over \mathbb{F}_q . We will use both an affine model

$$H : y^2 = F(x),$$

where F is a squarefree polynomial of degree 7 or 8, and a weighted projective plane model

$$H : w^2 = \tilde{F}(u, v)$$

for H (here u , v , and w have weights 1, 1, and 4, respectively). The coordinates of these models are related by $x = u/v$ and $y = w/v^4$. The polynomial \tilde{F} is squarefree of total degree 8, with $\tilde{F}(u, v) = v^8 F(u/v)$ and $F(x) = \tilde{F}(x, 1)$. We emphasise that F need not be monic. By a *randomly chosen hyperelliptic curve* we mean the hyperelliptic curve defined by $w^2 = \tilde{F}(u, v)$, where \tilde{F} is a uniformly randomly chosen squarefree homogeneous bivariate polynomial of degree 8 over \mathbb{F}_q .

¹ Some of the theory carries over to more general base fields. In particular, the results of Sects. 5 and 6 are valid over fields of characteristic not 2 or 3.

The canonical *hyperelliptic involution* ι of H is defined by $(x, y) \mapsto (x, -y)$ in the affine model, $(u : v : w) \mapsto (u : v : -w)$ in the projective model, and induces the negation map $[-1]$ on J_H . The quotient $\pi : H \rightarrow H/\langle \iota \rangle \cong \mathbb{P}^1$ sends $(u : v : w)$ to $(u : v)$ in the projective model, and (x, y) to x in the affine model (where it maps onto the affine patch of \mathbb{P}^1 where $v \neq 0$).

To compute in J_H , we fix an isomorphism from J_H to the group of degree-0 divisor classes on H , denoted $\text{Pic}^0(H)$. Recall that divisors are formal sums of points in $H(\overline{\mathbb{F}}_q)$, and if $D = \sum_{P \in H} n_P(P)$ is a divisor, then $\sum_{P \in H} n_P$ is the *degree* of D . We say D is *principal* if $D = \text{div}(f) := \sum_{P \in H} \text{ord}_P(f)(P)$ for some function f on H , where $\text{ord}_P(f)$ denotes the number of zeroes (or the negative of the number of poles) of f at P . Since H is complete, every principal divisor has degree 0. The group $\text{Pic}^0(H)$ is defined to be the group of divisors of degree 0 modulo principal divisors; the equivalence class of a divisor D is denoted by $[D]$. We let $J_H[l]$ denote the l -torsion subgroup of J_H : that is, the kernel of the multiplication-by- l map. If l is prime to q , then $J_H[l](\overline{\mathbb{F}}_q)$ is isomorphic to $(\mathbb{Z}/l\mathbb{Z})^6$.

3. The Kernel of the Isogeny

The eight points of $H(\overline{\mathbb{F}}_q)$ where $w = 0$ are called the *Weierstrass points* of H . Each Weierstrass point W corresponds to a linear factor

$$L_W := v(W)u - u(W)v$$

of \tilde{F} , which is defined up to scalar multiples. If W_1 and W_2 are Weierstrass points, then $2(W_1) - 2(W_2) = \text{div}(L_{W_1}/L_{W_2})$, so $2[(W_1) - (W_2)] = 0$; hence, $[(W_1) - (W_2)]$ represents an element of $J_H[2](\overline{\mathbb{F}}_q)$. In particular, $[(W_1) - (W_2)] = [(W_2) - (W_1)]$, so the divisor class $[(W_1) - (W_2)]$ corresponds to the pair $\{W_1, W_2\}$ of Weierstrass points, and hence to the quadratic factor $L_{W_1}L_{W_2}$ of \tilde{F} (up to scalar multiples).

Proposition 1. *To every \mathcal{G} -stable partition of the eight Weierstrass points of H into four disjoint pairs, we may associate an \mathbb{F}_q -rational subgroup of $J_H[2](\overline{\mathbb{F}}_q)$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$.*

Proof. Let $\{\{W'_1, W''_1\}, \{W'_2, W''_2\}, \{W'_3, W''_3\}, \{W'_4, W''_4\}\}$ be a partition of the set of Weierstrass points of H into four disjoint pairs. Each pair $\{W'_i, W''_i\}$ corresponds to the 2-torsion divisor class $[(W'_i) - (W''_i)]$ in $J_H[2](\overline{\mathbb{F}}_q)$. We associate the subgroup $S := \langle [(W'_i) - (W''_i)] : 1 \leq i \leq 4 \rangle$ to the partition. Observe that

$$\sum_{i=1}^4 [(W'_i) - (W''_i)] = \left[\text{div} \left(w / \prod_{i=1}^4 L_{W''_i} \right) \right] = 0;$$

this is the only relation on the classes $[(W'_i) - (W''_i)]$, so $S \cong (\mathbb{Z}/2\mathbb{Z})^3$. The action of \mathcal{G} on $J_H[2](\overline{\mathbb{F}}_q)$ corresponds to its action on the Weierstrass points, so if the partition is \mathcal{G} -stable, then the subgroup S is \mathcal{G} -stable. □

Remark 1. By “an \mathbb{F}_q -rational subgroup of $J_H[2](\overline{\mathbb{F}}_q)$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$ ”, we mean a \mathcal{G} -stable subgroup that is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$ over $\overline{\mathbb{F}}_q$. We emphasise that the subgroup need *not* be contained in $J_H(\mathbb{F}_q)$.

Remark 2. Requiring the pairs of Weierstrass points in Proposition 1 to be disjoint ensures that the associated subgroup is isotropic with respect to the 2-Weil pairing. We will see in Sect. 9 that this is necessary for the quotient by the subgroup to be an isogeny of principally polarised abelian varieties, and hence for the quotient to be an isogeny of Jacobians.

Definition 1. We call the subgroups corresponding to partitions of the Weierstrass points of H as in Proposition 1 *tractable subgroups*. We let $\mathcal{S}(H)$ denote the set of all \mathbb{F}_q -rational tractable subgroups of $J_H[2](\overline{\mathbb{F}}_q)$.

Remark 3. Not every subgroup of $J_H[2](\overline{\mathbb{F}}_q)$ that is the kernel of an isogeny of Jacobians is a tractable subgroup. For example, if W_1, \dots, W_8 are the Weierstrass points of H , then the subgroup

$$\langle [(W_1) - (W_i) + (W_j) - (W_k)] : (i, j, k) \in \{(2, 3, 4), (2, 5, 6), (3, 5, 7)\} \rangle$$

is a maximal 2-Weil isotropic subgroup of $J_H(\overline{\mathbb{F}}_q)$, and hence is the kernel of an isogeny of Jacobians (see Sect. 9). However, this subgroup contains no nontrivial differences of Weierstrass points, and therefore cannot be a tractable subgroup.

Computing $\mathcal{S}(H)$ is straightforward if we identify each tractable subgroup with its corresponding partition of Weierstrass points. Recall that each pair of Weierstrass points $\{W'_i, W''_i\}$ corresponds to a quadratic factor of \tilde{F} (up to scalar multiples). Since the pairs are disjoint, the corresponding quadratic factors are pairwise coprime, so we may take them to form a factorisation of \tilde{F} . We therefore have a correspondence of tractable subgroups, partitions of Weierstrass points into pairs, and sets of quadratic polynomials (up to scalar multiples):

$$S \longleftrightarrow \{ \{W'_i, W''_i\} : 1 \leq i \leq 4 \} \longleftrightarrow \{F_1, F_2, F_3, F_4\}, \quad \text{where } \tilde{F} = F_1 F_2 F_3 F_4.$$

The action of \mathcal{G} on $J_H[2](\overline{\mathbb{F}}_q)$ corresponds to its action on the set of Weierstrass points, so the action of \mathcal{G} on a tractable subgroup S corresponds to the action of \mathcal{G} on the corresponding set $\{F_1, F_2, F_3, F_4\}$ (assuming the F_i have been scaled appropriately). In particular, S is \mathbb{F}_q -rational precisely when $\{F_1, F_2, F_3, F_4\}$ is fixed by \mathcal{G} . The factors F_i are themselves defined over \mathbb{F}_q precisely when the corresponding points of S are \mathbb{F}_q -rational.

We can use this information to compute $\mathcal{S}(H)$. The set of pairs of Weierstrass points contains a \mathcal{G} -orbit $(\{W'_1, W''_1\}, \dots, \{W'_n, W''_n\})$ if and only if (possibly after exchanging some of the W'_k with the W''_k) either both (W'_1, \dots, W'_n) and (W''_1, \dots, W''_n) are \mathcal{G} -orbits or $(W'_1, \dots, W'_n, W''_1, \dots, W''_n)$ is a \mathcal{G} -orbit. Every \mathcal{G} -orbit of Weierstrass points corresponds to an \mathbb{F}_q -irreducible factor of F , so the size of $\mathcal{S}(H)$ depends only on the factorisation of F . A table relating the size of $\mathcal{S}(H)$ to the factorisation of \tilde{F} appears

in Lemma 1 below; this will be useful for our analysis in Sect. 8. For completeness, we have included a naïve algorithm for enumerating $\mathcal{S}(H)$ in Appendix A.

Lemma 1. *Let $H : w^2 = \tilde{F}(u, v)$ be a hyperelliptic curve of genus 3 over \mathbb{F}_q . The cardinality of the set $\mathcal{S}(H)$ depends only on the degrees of the \mathbb{F}_q -irreducible factors of \tilde{F} , and is described by the following table:*

Degrees of \mathbb{F}_q -irreducible factors of \tilde{F}	$\#\mathcal{S}(H)$
(8), (6, 2), (6, 1, 1), (4, 2, 1, 1)	1
(4, 2, 2), (4, 1, 1, 1, 1), (3, 3, 2), (3, 3, 1, 1)	3
(4, 4)	5
(2, 2, 2, 1, 1)	7
(2, 2, 1, 1, 1, 1)	9
(2, 1, 1, 1, 1, 1, 1)	15
(2, 2, 2, 2)	25
(1, 1, 1, 1, 1, 1, 1, 1)	105
Other	0

Proof. This is a routine combinatorial exercise after noting that every \mathcal{G} -orbit of pairs of Weierstrass points corresponds to either an even-degree factor of F , or a pair of factors of F of the same degree. □

4. The Trigonal Construction

We will now briefly outline the theoretical aspects of constructing isogenies with tractable kernels. We will make the construction completely explicit in Sects. 5 and 6.

Definition 2. Suppose $S = \langle [(W'_i) - (W''_i)] : 1 \leq i \leq 4 \rangle$ is a tractable subgroup. We say that a morphism $g : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is a *trigonal map* for S if g has degree 3 and $g(\pi(W'_i)) = g(\pi(W''_i))$ for $1 \leq i \leq 4$.

Given a trigonal map g for some tractable subgroup S , Recillas’ trigonal construction [20] specifies a curve X of genus 3 and a map $f : X \rightarrow \mathbb{P}^1$ of degree 4.² The isomorphism class of X depends only on S , and is independent of the choice of g (see Recillas [20], Donagi [6, Theorem 2.11], and Remark 5 below). Theorem 1, due to Donagi and Livné, states that if g is a trigonal map for S , then S is the kernel of an isogeny from J_H to J_X .

Theorem 1 (Donagi and Livné [7, Sect. 5]). *Let S be a tractable subgroup in $\mathcal{S}(H)$, and let $g : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a trigonal map for S . If X is the curve formed from g by Recillas’ trigonal construction, then there is an isogeny $\phi : J_H \rightarrow J_X$ (defined over $\overline{\mathbb{F}}_q$) with kernel S .*

² Recillas’ original trigonal construction is defined where π is an étale double cover; the trigonal construction we apply here is, in fact, the flat limit of Recillas’ construction (see [7, Sect. 3] for details).

We will give only a brief description of the geometry of X here, concentrating instead on its explicit construction; we refer the reader to Recillas [20], Vakil [24], Donagi [6, Sect. 2], and Birkenhake and Lange [1, Sect. 12.7] for proofs and further detail. The isogeny of Theorem 1 is analogous to the well-known Richelot isogeny in genus 2 (see Bost and Mestre [3], and Donagi and Livné [7, Sect. 4] for details), and to the explicit isogeny described by Lehavi and Ritzenthaler in [14] for Jacobians of non-hyperelliptic genus 3 curves.

In abstract terms, if U is the subset of the codomain of g above which $g \circ \pi$ is unramified, then X is by definition the closure of the curve over U representing the pushforward to U of the sheaf of sections of $\pi : (g \circ \pi)^{-1}(U) \rightarrow g^{-1}(U)$ (in the étale topology). This means in particular that the $\overline{\mathbb{F}}_q$ -points of X over an $\overline{\mathbb{F}}_q$ -point P of U represent partitions of the six $\overline{\mathbb{F}}_q$ -points of $(g \circ \pi)^{-1}(P)$ into two sets of three exchanged by the hyperelliptic involution. The fibre product of H and X over \mathbb{P}^1 with respect to $g \circ \pi$ and f is the union of two isomorphic curves, R and R' , which are exchanged by the involution on $H \times_{\mathbb{P}^1} X$ induced by the hyperelliptic involution. The natural projections induce coverings $\pi_H : R \rightarrow H$ and $\pi_X : R \rightarrow X$ of degrees 2 and 3, respectively, so R is a $(3, 2)$ -correspondence between H and X .

The maps π_H and π_X induce homomorphisms $(\pi_H)^* : J_H \rightarrow J_R$ (the *pullback*) and $(\pi_X)_* : J_R \rightarrow J_X$ (the *pushforward*). In terms of divisor classes, the pullback is defined by

$$(\pi_H)^* \left(\left[\sum_{P \in H} n_P(P) \right] \right) = \left[\sum_{P \in H} n_P \sum_{Q \in \pi_H^{-1}(P)} (Q) \right],$$

with appropriate multiplicities where π_H ramifies; the pushforward is defined by

$$(\pi_X)_* \left(\left[\sum_{Q \in R} m_Q(Q) \right] \right) = \left[\sum_{Q \in R} m_Q(\pi_X(Q)) \right].$$

Composing $(\pi_X)_*$ with $(\pi_H)^*$, we obtain an isogeny $\phi : J_H \rightarrow J_X$ with kernel S . If we replace R with R' in the above, we obtain an isogeny isomorphic to $-\phi$. Thus, up to isomorphism, the construction of the isogeny depends only on the subgroup S . The curves and Jacobians described above form the commutative diagrams shown in Fig. 1.

The hyperelliptic Jacobians form a codimension-1 subspace \mathcal{H}_g of the moduli space of 3-dimensional principally polarised abelian varieties—which, by the theorem of Oort and Ueno [18], is also the moduli space \mathcal{M}_g of Jacobians of genus 3 curves. The Weil hypotheses imply that $\#\mathcal{H}_g(\mathbb{F}_q)/\#\mathcal{M}_g(\mathbb{F}_q) \sim 1/q$ for sufficiently large q (cf. [13, Theorem 1]). In particular, for cryptographically relevant sizes of q , the probability that a uniformly randomly chosen curve X of genus 3 over \mathbb{F}_q should be hyperelliptic is negligible. We will suppose that the same is true for the curve X constructed in Theorem 1 for a uniformly randomly chosen H and S in $\mathcal{S}(H)$. This is consistent with our experimental observations, so we postulate Hypothesis 1.

Hypothesis 1. The probability that the curve X constructed by the trigonal construction for a randomly chosen H/\mathbb{F}_q and S in $\mathcal{S}(H)$ is hyperelliptic is negligible for sufficiently large q .

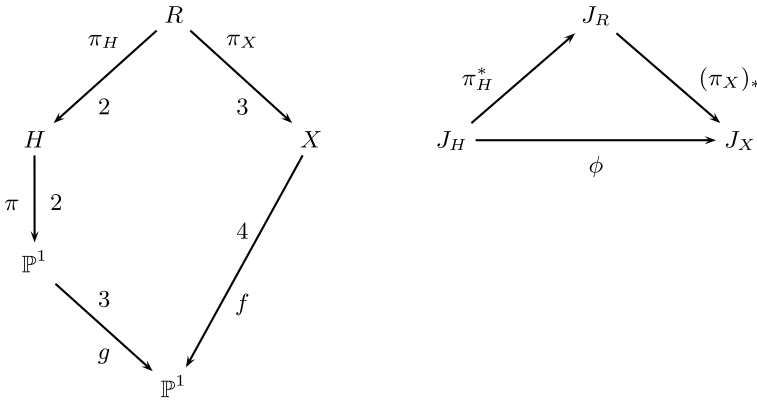


Fig. 1. The curves, Jacobians, and morphisms of Sect. 4.

5. Computing Trigonal Maps

Suppose we are given a tractable subgroup S of $J_H[2](\overline{\mathbb{F}}_q)$, corresponding to a partition $\{\{W'_i, W''_i\} : 1 \leq i \leq 4\}$ of the Weierstrass points of H into pairs. The first step in the explicit trigonal construction is to compute a trigonal map g for S . We will compute polynomials $N = x^3 + n_1x + n_0$ and $D = x^2 + d_1x + d_0$ such that the rational map

$$g : x \mapsto t = \frac{N(x)}{D(x)} = \frac{x^3 + n_1x + n_0}{x^2 + d_1x + d_0} \tag{1}$$

defines a trigonal map for S . The derivation is an exercise in classical geometry; we include it here to demonstrate its efficiency and to justify Hypothesis 2, which will be important in determining the expectation of success of our reduction in Sect. 8. The reader prepared to admit the existence of efficiently computable trigonal maps in the form of (1) may skip the remainder of this section on first reading.

By definition, $g : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is a degree-3 map with $g(\pi(W'_i)) = g(\pi(W''_i))$ for $1 \leq i \leq 4$. We will express g as a composition $g = p \circ e$, where $e : \mathbb{P}^1 \rightarrow \mathbb{P}^3$ is the rational normal embedding defined by

$$e : (u : v) \mapsto (u_0 : u_1 : u_2 : u_3) = (u^3 : u^2v : uv^2 : v^3),$$

and $p : \mathbb{P}^3 \rightarrow \mathbb{P}^1$ is the projection defined as follows. For each $1 \leq i \leq 4$, we let L_i denote the line in \mathbb{P}^3 passing through $e(\pi(W'_i))$ and $e(\pi(W''_i))$. There exists at least one line L intersecting all four of the L_i (in fact there are two, though they may coincide; we will compute them below). We take p to be the projection away from L ; then $p(e(\pi(W'_i))) = p(e(\pi(W''_i)))$ for $1 \leq i \leq 4$, so $g = p \circ e$ is a trigonal map for S . Given linear equations for L in the coordinates u_i , we can use Gaussian elimination to compute elements n_1, n_0, d_1 , and d_0 of \mathbb{F}_q such that

$$L = V(u_0 + n_1u_2 + n_0u_3, u_1 + d_1u_2 + d_0u_3).$$

The projection $p : \mathbb{P}^3 \rightarrow \mathbb{P}^1$ away from L is then defined by

$$p : (u_0 : u_1 : u_2 : u_3) \mapsto (u_0 + n_1u_2 + n_0u_3 : u_1 + d_1u_2 + d_0u_3),$$

so our trigonal map $g = p \circ e$ is defined by

$$g : (u : v) \mapsto (u^3 + n_1uv^2 + n_0v^3 : u^2v + d_1uv^2 + d_0v^3).$$

Therefore, if we set $N(x) := x^3 + n_1x + n_0$ and $D(x) := x^2 + d_1x + d_0$, then g will be defined by the rational map $x \mapsto t = N(x)/D(x)$.

To compute equations for L , we will use the classical theory of *Grassmannian varieties*. The elementary Lemmas 2 and 3 will be stated without proof; we refer the reader to Griffiths and Harris [9, Sect. 1.5] and Harris [10, Lecture 6] for details.

The set of lines in \mathbb{P}^3 has the structure of an algebraic variety $\text{Gr}(1, 3)$, called the Grassmannian. There is a convenient model for $\text{Gr}(1, 3)$ as a quadric hypersurface in \mathbb{P}^5 : if v_0, \dots, v_5 are coordinates on \mathbb{P}^5 , then we may take

$$\text{Gr}(1, 3) := V(v_0v_3 + v_1v_4 + v_2v_5) \subset \mathbb{P}^5.$$

Lemma 2. *There is a bijection between points of $\text{Gr}(1, 3)(\overline{\mathbb{F}}_q)$ and lines in \mathbb{P}^3 , defined as follows.*

1. *The point of $\text{Gr}(1, 3)(\overline{\mathbb{F}}_q)$ corresponding to the line through $(p_0 : p_1 : p_2 : p_3)$ and $(q_0 : q_1 : q_2 : q_3)$ in \mathbb{P}^3 has coordinates*

$$\left(\begin{array}{c|c} p_0 & p_1 \\ \hline q_0 & q_1 \end{array} \middle| : \begin{array}{c|c} p_0 & p_2 \\ \hline q_0 & q_2 \end{array} \middle| : \begin{array}{c|c} p_0 & p_3 \\ \hline q_0 & q_3 \end{array} \middle| : \begin{array}{c|c} p_2 & p_3 \\ \hline q_2 & q_3 \end{array} \middle| : \begin{array}{c|c} p_3 & p_1 \\ \hline q_3 & q_1 \end{array} \middle| : \begin{array}{c|c} p_1 & p_2 \\ \hline q_1 & q_2 \end{array} \right).$$

2. *The line in \mathbb{P}^3 corresponding to a point $(\gamma_0 : \dots : \gamma_5)$ of $\text{Gr}(1, 3)(\overline{\mathbb{F}}_q)$ is defined by*

$$V \left(\begin{array}{c} 0u_0 - \gamma_3u_1 - \gamma_4u_2 - \gamma_5u_3, \\ \gamma_3u_0 + 0u_1 - \gamma_2u_2 + \gamma_1u_3, \\ \gamma_4u_0 + \gamma_2u_1 + 0u_2 - \gamma_0u_3, \\ \gamma_5u_0 - \gamma_1u_1 + \gamma_0u_2 + 0u_3 \end{array} \right)$$

(two of the equations will be redundant linear combinations of the others).

Lemma 3. *Let L be the line in \mathbb{P}^3 corresponding to a point $(\gamma_0 : \dots : \gamma_5)$ of $\text{Gr}(1, 3)(\overline{\mathbb{F}}_q)$. The points in $\text{Gr}(1, 3)(\overline{\mathbb{F}}_q)$ corresponding to lines in \mathbb{P}^3 that intersect nontrivially with L are precisely the points lying in the hyperplane defined by $\sum_{i=0}^5 \gamma_i v_{i+3} = 0$ (where the subscripts are taken modulo 6).*

Suppose S is represented by a set $\{F_i = a_iu^2 + b_iuv + c_iv^2 : 1 \leq i \leq 4\}$ of quadratic factors of \tilde{F} (as in Sect. 3), with each factor F_i corresponding to a pair $\{W'_i, W''_i\}$ of Weierstrass points. Applying Lemma 2, we see that the line L_i through $e(\pi(W'_i))$ and $e(\pi(W''_i))$ corresponds to the point

$$(c_i^2 : -c_ib_i : b_i^2 - a_ic_i : a_i^2 : a_ib_i : a_ic_i)$$

on $\text{Gr}(1, 3)$. If $(\gamma_0 : \dots : \gamma_5)$ in $\text{Gr}(1, 3)(\overline{\mathbb{F}}_q)$ corresponds to a candidate for L , then by Lemma 3 we have $M(\gamma_0, \dots, \gamma_5)^T = 0$, where

$$M = \begin{pmatrix} a_1^2 & a_1b_1 & a_1c_1 & c_1^2 & -c_1b_1 & (b_1^2 - a_1c_1) \\ a_2^2 & a_2b_2 & a_2c_2 & c_2^2 & -c_2b_2 & (b_2^2 - a_2c_2) \\ a_3^2 & a_3b_3 & a_3c_3 & c_3^2 & -c_3b_3 & (b_3^2 - a_3c_3) \\ a_4^2 & a_4b_4 & a_4c_4 & c_4^2 & -c_4b_4 & (b_4^2 - a_4c_4) \end{pmatrix}. \tag{2}$$

The kernel of M is two-dimensional, corresponding to a line Λ in \mathbb{P}^5 . The kernel is independent of the ordering of the F_i , and does not change if we replace the F_i by scalar multiples; hence, Λ depends only on the subgroup S . Let $\{\underline{\alpha}, \underline{\beta}\}$ be a basis for $\ker M$, writing $\underline{\alpha} = (\alpha_0, \dots, \alpha_5)$ and $\underline{\beta} = (\beta_0, \dots, \beta_5)$. If S is \mathbb{F}_q -rational, then so is $\ker M$, so we may take the α_i and β_i to be in \mathbb{F}_q (see Cartier [4, Sect. I]). We want to find a point $P_L = (\alpha_0 + \lambda\beta_0 : \dots : \alpha_5 + \lambda\beta_5)$ where Λ intersects with $\text{Gr}(1, 3)$. The points $(u_0 : \dots : u_3)$ on the line L in \mathbb{P}^3 corresponding to P_L satisfy $(M_{\underline{\alpha}} + \lambda M_{\underline{\beta}})(u_0, \dots, u_3)^T = 0$, where

$$M_{\underline{\alpha}} := \begin{pmatrix} 0 & -\alpha_3 & -\alpha_4 & -\alpha_5 \\ \alpha_3 & 0 & -\alpha_2 & \alpha_1 \\ \alpha_4 & \alpha_2 & 0 & -\alpha_0 \\ \alpha_5 & -\alpha_1 & \alpha_0 & 0 \end{pmatrix} \quad \text{and} \quad M_{\underline{\beta}} := \begin{pmatrix} 0 & -\beta_3 & -\beta_4 & -\beta_5 \\ \beta_3 & 0 & -\beta_2 & \beta_1 \\ \beta_4 & \beta_2 & 0 & -\beta_0 \\ \beta_5 & -\beta_1 & \beta_0 & 0 \end{pmatrix}.$$

By part (2) of Lemma 2, the rank of $M_{\underline{\alpha}} + \lambda M_{\underline{\beta}}$ is 2. Using the expression

$$\det(M_{\underline{\alpha}} + \lambda M_{\underline{\beta}}) = \left(\frac{1}{2} \left(\sum_{i=0}^6 \beta_i \beta_{i+3} \right) \lambda^2 + \left(\sum_{i=0}^6 \alpha_i \beta_{i+3} \right) \lambda + \frac{1}{2} \sum_{i=0}^6 \alpha_i \alpha_{i+3} \right)^2 \tag{3}$$

(where the subscripts are taken modulo 6), we see that $M_{\underline{\alpha}} + \lambda M_{\underline{\beta}}$ has rank 2 precisely when $\det(M_{\underline{\alpha}} + \lambda M_{\underline{\beta}}) = 0$; we can therefore solve $\det(M_{\underline{\alpha}} + \lambda M_{\underline{\beta}}) = 0$ to determine a value for λ . Finally, we use Gaussian elimination to compute n_1, n_0, d_1 , and d_0 in $\mathbb{F}_q(\lambda)$ such that $(1, 0, n_1, n_0)$ and $(0, 1, d_1, d_0)$ generate the rowspace of $M_{\underline{\alpha}} + \lambda M_{\underline{\beta}}$. We then take $L = V(u_0 + n_1u_2 + n_0u_3, u_1 + d_1u_2 + d_0u_3)$, and compute p, e , and the trigonal map $g = p \circ e$ as above.

Since L is defined over $\mathbb{F}_q(\lambda)$, so is the projection p and the trigonal map g . But λ satisfies a quadratic equation with coefficients in \mathbb{F}_q , so $\mathbb{F}_q(\lambda)$ is at most a quadratic extension of \mathbb{F}_q . Computing the discriminant of $\det(M_{\underline{\alpha}} + \lambda M_{\underline{\beta}})$, we obtain a criterion for existence of trigonal maps over \mathbb{F}_q for a given tractable subgroup.

Proposition 2. *Suppose S is a tractable subgroup, and let $\{\underline{\alpha} = (\alpha_i), \underline{\beta} = (\beta_i)\}$ be any \mathbb{F}_q -rational basis of the nullspace of the matrix M defined in (2). There exists an \mathbb{F}_q -rational trigonal map for S if and only if*

$$\left(\sum_{i=0}^6 \alpha_i \beta_{i+3} \right)^2 - \left(\sum_{i=0}^6 \alpha_i \alpha_{i+3} \right) \left(\sum_{i=0}^6 \beta_i \beta_{i+3} \right) \tag{4}$$

is a square in \mathbb{F}_q , where the subscripts are taken modulo 6.

Proof. From the derivation above, we see that there exists an \mathbb{F}_q -rational trigonal map for S if and only if we can find a λ in \mathbb{F}_q such that $\det(M_{\underline{\alpha}} + \lambda M_{\underline{\beta}}) = 0$. By (3), we can find such a λ if and only if the quadratic polynomial

$$\frac{1}{2} \left(\sum_{i=0}^6 \beta_i \beta_{i+3} \right) T^2 + \left(\sum_{i=0}^6 \alpha_i \beta_{i+3} \right) T + \frac{1}{2} \sum_{i=0}^6 \alpha_i \alpha_{i+3}$$

has two roots in \mathbb{F}_q . This occurs precisely when the discriminant of this polynomial—the expression in (4) above—is a square in \mathbb{F}_q . □

Proposition 2 shows that the rationality of a trigonal map for a tractable subgroup S depends only upon whether an element of \mathbb{F}_q depending only on S is a square. It seems reasonable to assume that these field elements are uniformly distributed for uniformly random choices of H and S , and indeed this is consistent with our experimental observations. Since a uniformly randomly chosen element of \mathbb{F}_q is a square with probability $\sim 1/2$, we propose Hypothesis 2.

Hypothesis 2. The probability that there exists an \mathbb{F}_q -rational trigonal map for a subgroup S uniformly randomly chosen from $\mathcal{S}(H)$, where H is a randomly chosen hyperelliptic curve over \mathbb{F}_q , is $1/2$.

6. Equations for the Isogeny

Suppose we have a hyperelliptic curve H of genus 3, a tractable subgroup S in $\mathcal{S}(H)$, and a trigonal map g for S . We will now perform an explicit trigonal construction on g to compute a curve X and an isogeny $\phi : J_H \rightarrow J_X$ with kernel S .

We assume that g has been derived as in Sect. 5 and, in particular, that $g : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is defined by a rational map in the form

$$g : x \mapsto t = \frac{N(x)}{D(x)} = \frac{x^3 + n_1x + n_0}{x^2 + d_1x + d_0}.$$

Observe that g maps the point at infinity to the point at infinity (that is, $(1 : 0)$). For notational convenience, we define

$$G(t, x) = x^3 + g_2(t)x^2 + g_1(t)x + g_0(t) := N(x) - tD(x);$$

unless otherwise noted, we will view $G(t, x)$ as an element of $\mathbb{F}_q[t][x]$. We have

$$g_2(t) = -t, \quad g_1(t) = n_1 - d_1t, \quad \text{and} \quad g_0(t) = n_0 - d_0t.$$

We also define f_0, f_1 , and f_2 to be the elements of $\mathbb{F}_q[t]$ such that

$$f_0(t) + f_1(t)x + f_2(t)x^2 \equiv F(x) \pmod{G(t, x)}.$$

Let U be the subset of $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{(1 : 0)\}$ above which $g \circ \pi$ is unramified. With the notation above,

$$U = \text{Spec}(k[t]) \setminus V((f_1^2 - 4f_2f_0)(4g_2^3g_0 - g_2^2g_1^2 - 18g_2g_1g_0 + 4g_1^3 + 27g_0^2)).$$

We will derive equations for an affine model $X|_U$ of $f^{-1}(U)$ —that is, the open subset of X over U . We will not prove here that the normalisation of $X|_U$ is isomorphic to the curve X specified by Recillas, but we will exhibit a bijection on geometric points. If X is not hyperelliptic, then taking the canonical map of $X|_U$ into \mathbb{P}^2 will give us a nonsingular plane quartic curve C isomorphic to X .

By definition, every point P in $X|_U(\overline{\mathbb{F}}_q)$ corresponds to a pair of unordered triples of points in $H(\overline{\mathbb{F}}_q)$, exchanged by the hyperelliptic involution, with each triple supported on the fibre of $g \circ \pi$ over $f(P)$. To be more explicit, suppose Q is a generic point of U . Since $g \circ \pi$ is unramified above Q , we may choose three preimages P_1, P_2 , and P_3 of Q such that

$$(g \circ \pi)^{-1}(Q) = \{P_1, P_2, P_3, \iota(P_1), \iota(P_2), \iota(P_3)\}.$$

Viewing unordered triples of points as effective divisors of degree 3 (that is, as formal sums of three points), we have

$$f^{-1}(Q) = \left\{ \begin{array}{l} Q_1 \leftrightarrow \{P_1 + P_2 + P_3, \iota(P_1) + \iota(P_2) + \iota(P_3)\}, \\ Q_2 \leftrightarrow \{P_1 + \iota(P_2) + \iota(P_3), \iota(P_1) + P_2 + P_3\}, \\ Q_3 \leftrightarrow \{\iota(P_1) + P_2 + \iota(P_3), P_1 + \iota(P_2) + P_3\}, \\ Q_4 \leftrightarrow \{\iota(P_1) + \iota(P_2) + P_3, P_1 + P_2 + \iota(P_3)\} \end{array} \right\}. \tag{5}$$

Note that P_i and $\iota(P_i)$ never appear in the same divisor for any $1 \leq i \leq 3$. There is a one-to-one correspondence between effective divisors of degree 3 on H satisfying this condition, and ideals $(a(x), y - b(x))$ where a is a monic cubic polynomial and b is a quadratic polynomial satisfying $b^2 \equiv F \pmod{a}$ (this is the well-known *Mumford representation* [17, Sect. IIIa]). For example, $P_1 + P_2 + P_3$ corresponds to the ideal $(a(x), y - b(x))$ where $a(x) = \prod_i (x - x(P_i))$ and b satisfies $y(P_i) = b(x(P_i))$ for $1 \leq i \leq 3$ (with appropriate multiplicities); we may compute b using the Lagrange interpolation formula. A divisor is defined over \mathbb{F}_q if and only if a and b are defined over \mathbb{F}_q . The ideal $(a(x), y - b(x))$ corresponds to $P_1 + P_2 + P_3$ if and only if $(a(x), y + b(x))$ corresponds to $\iota(P_1) + \iota(P_2) + \iota(P_3)$; so each point of X over U corresponds to a pair $\{(a(x), y \pm b(x))\}$ of ideals. We will construct a curve parameterising these pairs of ideals, and take this as a model for $X|_U$.

Suppose $\{(a(x), y \pm b(x))\}$ is a pair of ideals corresponding to one of the preimages of Q on $X|_U$. The product of the two ideals is equal to the principal ideal $(a(x))$; but products of ideals correspond to sums of divisors, so $(a(x))$ must cut out the divisor $P_1 + P_2 + P_3 + \iota(P_1) + \iota(P_2) + \iota(P_3)$ on H . This divisor is just $(g \circ \pi)^*(Q)$, which we know is cut out by $(G(t(Q), x))$; so we conclude that $a(x) = G(t(Q), x)$ for every pair of ideals $\{(a(x), y \pm b(x))\}$ corresponding to a point in $f^{-1}(Q)$. In particular, the generic point of $X|_U$ corresponds to a pair of ideals of the form $\{(G(t, x), y \pm (b_0 + b_1x + b_2x^2))\}$, where b_0, b_1 , and b_2 are algebraic functions of t such that

$$(b_0 + b_1x + b_2x^2)^2 \equiv F(x) \pmod{G(t, x)}. \tag{6}$$

Viewing b_0, b_1 , and b_2 as coordinates on \mathbb{A}^3 (over \mathbb{F}_q), we expand both sides of (6) modulo $G(t, x)$ and equate coefficients to obtain a variety \tilde{X} in $U \times \mathbb{A}^3$ parameterising ideals:

$$\tilde{X} = V(\tilde{c}_0(t, b_0, b_1, b_2), \tilde{c}_1(t, b_0, b_1, b_2), \tilde{c}_2(t, b_0, b_1, b_2)),$$

where

$$\begin{aligned}\tilde{c}_0(t, b_0, b_1, b_2) &= g_2(t)g_0(t)b_2^2 - 2g_0(t)b_2b_1 + b_0^2 - f_0(t), \\ \tilde{c}_1(t, b_0, b_1, b_2) &= (g_2(t)g_1(t) - g_0(t))b_2^2 - 2g_1(t)b_2b_1 + 2b_1b_0 - f_1(t), \quad \text{and} \\ \tilde{c}_2(t, b_0, b_1, b_2) &= (g_2(t)^2 - g_1(t))b_2^2 - 2g_2(t)b_2b_1 + 2b_2b_0 + b_1^2 - f_2(t).\end{aligned}$$

The ideals in each pair $\{(G(t, x), y \pm (b_2x^2 + b_1x + b_0))\}$ are exchanged by the involution $\iota_* : \tilde{X} \rightarrow \tilde{X}$ defined by

$$\iota_* : (t, b_0, b_1, b_2) \mapsto (t, -b_0, -b_1, -b_2);$$

the curve $X|_U$ is therefore the quotient of \tilde{X} by $\langle \iota_* \rangle$. To make this quotient explicit, let $m : U \times \mathbb{A}^3 \rightarrow U \times \mathbb{A}^6$ be the map defined by

$$m : (t, b_0, b_1, b_2) \mapsto (t, b_{00}, b_{01}, b_{02}, b_{11}, b_{12}, b_{22}) = (t, b_0^2, b_0b_1, b_0b_2, b_1^2, b_1b_2, b_2^2);$$

observe that

$$m(U \times \mathbb{A}^3) = V \left(\begin{array}{l} b_{01}^2 - b_{00}b_{11}, b_{01}b_{02} - b_{00}b_{12}, b_{02}^2 - b_{00}b_{22}, \\ b_{02}b_{11} - b_{01}b_{12}, b_{02}b_{12} - b_{01}b_{22}, b_{12}^2 - b_{11}b_{22} \end{array} \right) \subset U \times \mathbb{A}^6.$$

We have $X|_U = m(\tilde{X})$, so

$$X|_U = V \left(\begin{array}{l} c_0(t, b_{00}, \dots, b_{22}), c_1(t, b_{00}, \dots, b_{22}), c_2(t, b_{00}, \dots, b_{22}), \\ b_{01}^2 - b_{00}b_{11}, b_{01}b_{02} - b_{00}b_{12}, b_{02}^2 - b_{00}b_{22}, \\ b_{02}b_{11} - b_{01}b_{12}, b_{02}b_{12} - b_{01}b_{22}, b_{12}^2 - b_{11}b_{22} \end{array} \right) \subset U \times \mathbb{A}^6,$$

where c_0, c_1 , and c_2 are the polynomials defined by

$$\begin{aligned}c_0(t, b_{00}, b_{01}, b_{02}, b_{11}, b_{12}, b_{22}) &:= g_2g_0b_{22} - 2g_0b_{12} + b_{00} - f_0, \\ c_1(t, b_{00}, b_{01}, b_{02}, b_{11}, b_{12}, b_{22}) &:= (g_2g_1 - g_0)b_{22} - 2g_1b_{12} + 2b_{01} - f_1, \quad \text{and} \\ c_2(t, b_{00}, b_{01}, b_{02}, b_{11}, b_{12}, b_{22}) &:= (g_2^2 - g_1)b_{22} - 2g_2b_{12} + 2b_{02} + b_{11} - f_2.\end{aligned}$$

Observe that $X|_U$ is defined over the field of definition of g .

It remains to derive a correspondence R between H and $X|_U$ inducing the isogeny ϕ . We know that R is a component of the fibre product $H \times_{\mathbb{P}^1} X$ (with respect to $g \circ \pi$ and f). We may realise the open affine subset $H|_U \times_U X|_U$ as the subvariety $V(G(t, x))$ of $H|_U \times X|_U$; decomposing the ideal $(G(t, x))$ will therefore give us a model for R .

Lemma 4. *Let s be the polynomial in $\mathbb{F}_q[t]$ defined by*

$$\begin{aligned}s &:= f_0^3 - f_0^2 f_1 g_2 - 2f_0^2 f_2 g_1 + f_0^2 f_2 g_2^2 + f_0 f_1^2 g_1 + 3f_0 f_1 f_2 g_0 - f_0 f_1 f_2 g_1 g_2 \\ &\quad - 2f_0 f_2^2 g_0 g_2 + f_0 f_2^2 g_1^2 - f_1^3 g_0 + f_1^2 f_2 g_0 g_2 - f_1 f_2^2 g_0 g_1 + f_2^3 g_0^2,\end{aligned} \quad (7)$$

and let α be its leading coefficient. Then s has a square root in $\mathbb{F}_q(\sqrt{\alpha})[t]$.

Proof. The polynomial s is a square in $\mathbb{F}_q(\sqrt{\alpha})[t]$ if and only if each of its roots in $\overline{\mathbb{F}}_q$ occur with multiplicity 2. In the notation of (5), we have

$$s(t(Q)) = F(x(P_1))F(x(P_2))F(x(P_3)),$$

so $s(t(Q)) = 0$ if and only if $F(x(P_i)) = 0$ for some $1 \leq i \leq 3$ —that is, if and only if at least one of the P_i is a Weierstrass point of H . But the trigonal map g was constructed precisely so that the Weierstrass points of H appear in pairs in the fibres of g ; hence, exactly two of the P_i must be Weierstrass points, and so $F(x(P_1))F(x(P_2))F(x(P_3)) = 0$ and $s(t(Q)) = 0$ with multiplicity 2. \square

Proposition 3. *Let s be the polynomial of Lemma 4, and let $\delta_0, \delta_1, \delta_2,$ and δ_4 be the polynomials in $\mathbb{F}_{q^2}[t]$ defined by*

$$\begin{aligned} \delta_4 &:= -27g_0^2 + 18g_0g_1g_2 - 4g_0g_2^3 - 4g_1^3 + g_1^2g_2^2, \\ \delta_2 &:= 12f_0g_1 - 4f_0g_2^2 - 18f_1g_0 + 2f_1g_1g_2 + 12f_2g_0g_2 - 4f_2g_1^2, \\ \delta_1 &:= 8\sqrt{s}, \quad \text{and} \\ \delta_0 &:= -4f_0f_2 + f_1^2. \end{aligned}$$

On the curve $X|_U$, we have

$$(\delta_4(t)b_{22}^2 + \delta_2(t)b_{22} + \delta_0(t))^2 - \delta_1(t)^2b_{22} = 0. \tag{8}$$

Proof. Consider again the fibre of $f : X \rightarrow \mathbb{P}^1$ over the generic point $Q = (t)$ of U (as in (5)). If $\{P_1 + P_2 + P_3, \iota(P_1) + \iota(P_2) + \iota(P_3)\}$ is a pair of divisors corresponding to one of the points in the fibre, then by the Lagrange interpolation formula the value of b_{22} at the corresponding point of \tilde{X} is

$$b_{22} = \left(\sum y(P_i) / ((x(P_i) - x(P_j))(x(P_i) - x(P_k))) \right)^2, \tag{9}$$

where the sum is taken over the cyclic permutations (i, j, k) of $(1, 2, 3)$. After interpolating for each pair of divisors in the fibre, an elementary but involved symbolic calculation shows that b_{22} satisfies

$$\left(\Delta b_{22}^2 - 2 \left(\sum_i \Gamma_i \right) b_{22} + \frac{1}{\Delta} \left(2 \left(\sum_i \Gamma_i^2 \right) - \left(\sum_i \Gamma_i \right)^2 \right) \right)^2 - 64 \left(\prod_i \Gamma_i \right) b_{22} = 0, \tag{10}$$

where

$$\Gamma_i := (f_2(t)x(P_i)^2 + f_1(t)x(P_i) + f_0(t))\Delta_i = F(x(P_i))\Delta_i$$

with

$$\Delta_i := (x(P_j) - x(P_k))^2$$

for each cyclic permutation (i, j, k) of $(1, 2, 3)$, and where $\Delta := \Delta_1\Delta_2\Delta_3$.

Now Δ , $\sum_i \Gamma_i$, $\sum_i \Gamma_i^2$, and $\prod_i \Gamma_i$ are symmetric functions with respect to permutations of the points in the fibre $g^{-1}(Q) = g^{-1}(t)$. They are therefore polynomials in the homogeneous elementary symmetric functions

$$e_1 = \sum_i x(P_i), \quad e_2 = \sum_{i < j} x(P_i)x(P_j), \quad \text{and} \quad e_3 = \prod_i x(P_i),$$

which are polynomials in t . Indeed, the e_i are given by the coefficients of $G(t, x)$:

$$e_1 = -g_2(t), \quad e_2 = g_1(t), \quad \text{and} \quad e_3 = -g_0(t).$$

Expressing Δ , $\sum_i \Gamma_i$, $\sum_i \Gamma_i^2$, and $\prod_i \Gamma_i$ in terms of f_0, f_1, f_2, g_0, g_1 , and g_2 , and substituting the resulting expressions into (10), we obtain (8). \square

Equation (8) gives us a (singular) affine plane model for X . We can also use (8) to compute a square root for b_{22} on $X|_U$: we have

$$b_{22} = \rho^2, \quad \text{where } \rho := \frac{\delta_4(t)b_{22}^2 + \delta_2(t)b_{22} + \delta_0(t)}{\delta_1(t)}.$$

Returning to (9), we observe that b_{22} is a unit on $X|_U$, since its zeroes and poles occur only at points Q where $g \circ \pi$ is ramified over $f(Q)$, and these points were excluded from U . Since ρ is the square root of b_{22} , it must also be a unit on $X|_U$.

Given a point $(t, b_{00}, \dots, b_{22})$ of $X|_U$, the corresponding pair of divisors of degree 3 on H is cut out by the pair of ideals

$$\left\{ \left(G(t, x), y \pm \left(\frac{b_{02}}{\rho} + \frac{b_{12}}{\rho}x + \frac{b_{22}}{\rho}x^2 \right) \right) \right\}.$$

This is precisely the decomposition of $(G(t, x))$ that we need to compute the correspondence from $H|_U$ to $X|_U$: we have $V(G(t, x)) = R \cup R'$, where

$$R = V \left(G(t, x), y - \frac{1}{\rho}(b_{02} + b_{12}x + b_{22}x^2) \right) \tag{11}$$

and

$$R' = V \left(G(t, x), y + \frac{1}{\rho}(b_{02} + b_{12}x + b_{22}x^2) \right).$$

On the level of divisor classes, the isogeny $\phi : J_H \rightarrow J_X$ is made explicit by the map

$$\phi = (\pi_X)_* \circ (\pi_H)^*,$$

where $\pi_H : R \rightarrow H$ and $\pi_X : R \rightarrow X|_U$ are the natural projections defined by $(x, y, t, b_{00}, \dots, b_{22}) \mapsto (x, y)$ and $(x, y, t, b_{00}, \dots, b_{22}) \mapsto (t, b_{00}, \dots, b_{22})$, respectively. In terms of ideals cutting out effective divisors, ϕ is realised by the map

$$I_D \mapsto \left(I_D + \left(G(t, x), y - \frac{1}{\rho}(b_{02} + b_{12}x + b_{22}x^2) \right) \right) \cap \mathbb{F}_q[s, t, b_{00}, \dots, b_{22}].$$

Taking R' in place of R in the above gives an isogeny equal to $-\phi$. It remains to determine the field of definition of ϕ .

Proposition 4. *If S is a subgroup in $\mathcal{S}(H)$ with an \mathbb{F}_q -rational trigonal map g defined over \mathbb{F}_q , and $s(t)$ is the polynomial defined in Lemma 4, then the explicit trigonal construction on g described above yields an isogeny defined over \mathbb{F}_q if and only if the leading coefficient of $s(t)$ is a square in \mathbb{F}_q .*

Proof. We noted earlier that $X|_U$ is defined over the field of definition of g . The correspondence R , and hence the induced isogeny ϕ , are both defined over the field of definition of ρ , which is the field of definition of $\delta_4\delta_1$, δ_2/δ_1 , and δ_0/δ_1 . But δ_4 , δ_4 , and δ_0 are all defined over \mathbb{F}_q (cf. Proposition 3), while δ_1 is defined over $\mathbb{F}_q(\sqrt{\alpha})$ where α is the leading coefficient of s by Lemma 4. □

Remark 4. If ϕ is not defined over \mathbb{F}_q , then the Jacobian J_X is, in fact, a quadratic twist of the quotient J_H/S (see Sect. 9). In fact, when ϕ is not \mathbb{F}_q -rational, Frobenius exchanges ρ and $-\rho$, hence R and R' , and therefore ϕ and $-\phi$. This is a concrete realisation of the Galois cohomology referred to in the proof of Proposition 5 below. The obstruction to the existence of an isomorphism from J_H/S to J_X over \mathbb{F}_q is, in fact, the interaction of \mathcal{G} with $[\pm 1]$ on J_X .

If we assume that the leading coefficients of the polynomials $s(t)$ are uniformly distributed for randomly chosen H , S , and g , then the probability that s is a square in $\mathbb{F}_q[t]$ is $1/2$. Indeed, it is easily seen that $s(t)$ is a square for H if and only if it is not a square for the quadratic twist of H . Suppose $H : w^2 = \tilde{F}(u, v)$ is a hyperelliptic curve. Let c be a non-square in \mathbb{F}_q , and let $H' : w^2 = c\tilde{F}(u, v)$ be the quadratic twist of H . Suppose S in $\mathcal{S}(H)$ is a tractable subgroup, represented by a set $\{F_1, F_2, F_3, F_4\}$ of quadratic factors of \tilde{F} . The set $\{cF_1, F_2, F_3, F_4\}$ is a factorisation of $c\tilde{F}$, so it represents a tractable subgroup S' in $\mathcal{S}(H')$. We noted in Sect. 5 that scalar multiples of quadratic polynomials do not affect the construction of trigonal maps; so if S has a trigonal map g defined over \mathbb{F}_q , then g is also a trigonal map for S' . Let s be the polynomial computed from g and S in Lemma 4, and let s' be the corresponding polynomial computed for g and S' . Looking at the form of (7), we see that $s'(t) = c^3s(t)$. Therefore, the leading coefficient of s' is a square if and only if the leading coefficient of s is *not* a square. In particular, if S has a trigonal map defined over \mathbb{F}_q , then so does S' , and we can construct an isogeny of Jacobians with kernel S if and only if we cannot construct an isogeny of Jacobians with kernel S' .

This suggests that the probability that we can compute an isogeny defined over \mathbb{F}_q given a randomly chosen H and S in $\mathcal{S}(H)$ with a trigonal map defined over \mathbb{F}_q is $1/2$ —since we have a 50% chance of being on the “right” quadratic twist of H . This hypothesis is consistent with our experimental observations.

Hypothesis 3. For a randomly chosen hyperelliptic curve H and a uniformly randomly chosen subgroup S in $\mathcal{S}(H)$ with a trigonal map g defined over \mathbb{F}_q , the probability that we can compute an \mathbb{F}_q -rational isogeny ϕ with kernel S is $1/2$.

7. Computing Isogenies

Now we will put the ideas above into practice. Suppose we are given a hyperelliptic curve H of genus 3 over \mathbb{F}_q , and a DLP instance in $J_H(\mathbb{F}_q)$ to solve. Our goal is to compute a nonsingular plane quartic curve C and an explicit isogeny $J_H \rightarrow J_C$ defined over \mathbb{F}_q , so that we can solve our DLP instance in $J_C(\mathbb{F}_q)$.

We begin by computing the set $\mathcal{S}(H)$ of \mathbb{F}_q -rational tractable subgroups of the 2-torsion subgroup $J_H[2](\overline{\mathbb{F}}_q)$ (see Appendix A below). For each S in $\mathcal{S}(H)$, we apply Proposition 2 to determine whether there exists an \mathbb{F}_q -rational trigonal map g for S . If so, we use the formulae of Sect. 5 to compute g ; if not, we move on to the next S . Having computed g , we apply Proposition 4 to determine whether we can compute an isogeny over \mathbb{F}_q . If so, we use the formulae of Sect. 6 to compute equations for X and the isogeny $\phi : J_H \rightarrow J_X$; if not, we move on to the next S .

The formulae of Sect. 6 give an affine model of X in $\mathbb{A}^1 \times \mathbb{A}^6$. In order to apply Diem’s algorithm to the DLP in J_X , we need a nonsingular plane quartic model of X , that is, a nonsingular curve $C \subset \mathbb{P}^2$ isomorphic to X , cut out by a quartic form. Such a model exists if and only if X is not hyperelliptic. To find C , we compute a basis $\mathcal{B} = \{\psi_1, \psi_2, \psi_3\}$ of the Riemann–Roch space of a canonical divisor of X . This is a routine geometrical calculation; Hess [11] describes an efficient approach. In practice, the algorithms implemented in Magma [2,15] compute \mathcal{B} very quickly. The three functions in \mathcal{B} define a map $\psi : X \rightarrow \mathbb{P}^2$, mapping P to $(\psi_1(P) : \psi_2(P) : \psi_3(P))$. Up to automorphisms of \mathbb{P}^2 , the map ψ is independent of the choice of basis \mathcal{B} , and depends only on X . If the image of ψ is a conic (that is, if the ψ_i satisfy a quadratic relation), then X is hyperelliptic; in this situation we move on to the next S , since we will gain no advantage from index calculus on X . Otherwise, the image of ψ is a nonsingular plane quartic C , and ψ restricts to an isomorphism $\psi : X \rightarrow C$.

If the procedure outlined above succeeds for some S in $\mathcal{S}(H)$, then we have computed an explicit \mathbb{F}_q -rational isogeny $\psi_* \circ \phi : J_H \rightarrow J_C$. We can then map our DLP from $J_H(\mathbb{F}_q)$ into $J_C(\mathbb{F}_q)$, and solve it using Diem’s algorithm.

We emphasise that the entire procedure is very fast: The curve X and the isogeny can be constructed using just a few low-degree polynomial operations and some low-dimensional linear algebra (and hence the procedure is polynomial-time in $\log q$, the size of the base field). For a rough idea of the computational effort involved, given a random H over a 160-bit prime field with a tractable subgroup S in $\mathcal{S}(H)$, a naïve implementation of our algorithms in Magma computes the trigonal map g , the curve X , the nonsingular plane quartic C , and the isogeny $\phi : J_H \rightarrow J_C$ in a few seconds on a 1.2 GHz laptop. Since the difficulty of the construction depends only upon the difficulty of arithmetic in \mathbb{F}_q (and *not* upon the size of the DLP subgroup of $J_H(\mathbb{F}_q)$), we may conclude that instances of the DLP in 160-bit Jacobians chosen for cryptography may also be reduced to instances of the DLP in non-hyperelliptic Jacobians in very little time.

Example 1. We will give an example over a small field. Let H be the hyperelliptic curve over \mathbb{F}_{37} defined by

$$H : y^2 = x^7 + 28x^6 + 15x^5 + 20x^4 + 33x^3 + 12x^2 + 29x + 2.$$

Using the ideas in Sect. 3, or the algorithms in Appendix A, we find that J_H has one \mathbb{F}_{37} -rational tractable subgroup:

$$\mathcal{S}(H) = \{S\}, \quad \text{where } S = \left\{ \begin{array}{l} u^2 + \xi_1 uv + \xi_2 v^2, \quad u^2 + \xi_1^{37} uv + \xi_2^{37} v^2, \\ u^2 + \xi_1^{37^2} uv + \xi_2^{37^2} v^2, \quad uv + 20v^2 \end{array} \right\},$$

where ξ_1 is an element of \mathbb{F}_{37^3} satisfying $\xi_1^3 + 29\xi_1^2 + 9\xi_1 + 13 = 0$, and $\xi_2 = \xi_1^{50100}$. Applying the methods of Sect. 5, we compute a trigonal map $g : x \mapsto N(x)/D(x)$ for S , taking

$$N(x) = x^3 + 16x + 22 \quad \text{and} \quad D(x) = x^2 + 32x + 18;$$

clearly g is defined over \mathbb{F}_{37} . The formulae of Sect. 6 give us a curve $X \subset \mathbb{A}^1 \times \mathbb{A}^6$ of genus 3, defined by

$$X = V \left(\begin{array}{l} (18t^2+15t)b_{22}+(36t+30)b_{12}+b_{00}+19t^5+10t^4+12t^3+7t^2+t+30, \\ (32t^2+2t+15)b_{22}+(27t+5)b_{12}+2b_{01}+5t^5+26t^4+15t^3+23t^2+19t+17, \\ (t^2+32t+21)b_{22}+2tb_{12}+2b_{02}+b_{11}+36t^5+29t^4+7t^3+13t^2+21t+18, \\ b_{00}b_{11}-b_{01}^2, b_{00}b_{12}-b_{01}b_{02}, b_{00}b_{22}-b_{02}^2, b_{02}b_{11}-b_{01}b_{12}, b_{02}b_{12}-b_{01}b_{22}, b_{12}^2-b_{11}b_{22} \end{array} \right).$$

The map on divisors inducing an isogeny from J_H to J_X with kernel S is induced by the correspondence R defined as in (11) with

$$\begin{aligned} G(t, x) &= x^3 - tx^2 - (32t - 16)x - 18t + 22, \\ \delta_0 &= 27t^{10} + 20t^9 + 33t^8 + 6t^7 + 16t^6 + 8t^5 + 9t^4 + 2t^3 + 31t^2 + 15t + 16, \\ \delta_1 &= 35t^3 + 8t^2 + 33t + 3, \\ \delta_2 &= 20t^7 + 18t^6 + 29t^5 + 14t^4 + 6t^3 + 20t^2 + 12t + 16, \quad \text{and} \\ \delta_4 &= 27t^4 + 36t^3 + 13t^2 + 21t. \end{aligned}$$

Computing the canonical morphism of X , we find that X is non-hyperelliptic, and isomorphic to the nonsingular plane quartic curve

$$C = V \left(\begin{array}{l} u^4 + 26u^3v + 2u^3w + 17u^2v^2 + 9u^2vw + 20u^2w^2 + 34uv^3 + 24uv^2w \\ + 5uvw^2 + 36uw^3 + 19v^4 + 13v^3w + v^2w^2 + 23vw^3 + 5w^4 \end{array} \right).$$

Composing the isomorphism with the isogeny $J_H \rightarrow J_X$, we obtain an explicit isogeny $\phi : J_H \rightarrow J_C$. We can verify that J_H and J_C are isogenous by checking that the zeta functions of H and C are identical. Indeed, direct calculation with Magma shows that

$$Z(H; T) = Z(C; T) = \frac{37^3 T^6 + 4 \cdot 37^2 T^5 - 6 \cdot 37 T^4 - 240 T^3 - 6 T^2 + 4 T + 1}{(37 T - 1)(T - 1)}.$$

Let $D = [(10 : 28 : 1) - (14 : 6 : 1)]$ and $D' = [(19 : 28 : 1) - (36 : 13 : 1)]$ be divisor classes on H ; we have $D' = [22359]D$. Applying ϕ , we find that

$$\begin{aligned} \phi(D) &= [(7 : 18 : 1) + (34 : 34 : 1) - (18 : 22 : 1) - (15 : 33 : 1)] \quad \text{and} \\ \phi(D') &= [(7 : 23 : 1) + (6 : 13 : 1) - (13 : 15 : 1) - (7 : 18 : 1)]; \end{aligned}$$

direct calculation verifies that $\phi(D') = [22359]\phi(D)$, as expected.

8. Expectation of Existence of Computable Isogenies

Our aim in this section is to estimate the proportion of genus 3 hyperelliptic Jacobians over \mathbb{F}_q for which the methods of this article produce an \mathbb{F}_q -rational isogeny—and thus for which the DLP may be solved using Diem’s algorithm—as q tends to infinity. We will assume that if we are given a selection of \mathbb{F}_q -rational tractable subgroups of a given Jacobian, then the probabilities that each will yield a rational isogeny are mutually independent. This hypothesis appears to be consistent with our experimental observations.

Hypothesis 4. For a randomly chosen hyperelliptic curve H , the probabilities that we can compute an \mathbb{F}_q -rational isogeny with kernel S for each S in $\mathcal{S}(H)$ are mutually independent.

Theorem 2. Assume Hypotheses 1, 2, 3, and 4. As q tends to infinity, the expectation that the algorithms in this article will give a reduction of the DLP in a subgroup of $J_H(\mathbb{F}_q)$ for a randomly chosen hyperelliptic curve H of genus 3 over \mathbb{F}_q to a subgroup of $J_C(\mathbb{F}_q)$ for some nonsingular plane quartic curve C is

$$\sum_{T \in \mathcal{T}} \left((1 - (1 - 1/4)^{s(T)}) / \prod_{n \in T} (v_T(n)! \cdot n^{v_T(n)}) \right) \approx 0.1857, \tag{12}$$

where \mathcal{T} denotes the set of integer partitions of 8 and $v_T(n)$ denotes the multiplicity of an integer n in a partition T , and $s(T) = \#\mathcal{S}(H)$, where H is any hyperelliptic curve over \mathbb{F}_q such that the multiset of degrees of the \mathbb{F}_q -irreducible factors of its hyperelliptic polynomial coincides with T .

Proof. Suppose H is a randomly chosen hyperelliptic curve of genus 3 over \mathbb{F}_q . Hypotheses 1, 2, and 3 together imply that for each S in $\mathcal{S}(H)$, the probability that we can compute an isogeny with kernel S defined over \mathbb{F}_q is $1/2 \cdot 1/2 \cdot 1 = 1/4$. Hypothesis 4 implies that we have an equal chance of constructing an isogeny from each S in $\mathcal{S}(H)$, so the probability that we can compute an isogeny over \mathbb{F}_q from J_H is $1 - (1 - 1/4)^{\#\mathcal{S}(H)}$. The expectation that we can compute an isogeny over \mathbb{F}_q given a curve over \mathbb{F}_q is therefore

$$E_q := \frac{\sum_{\tilde{F}} (1 - (3/4)^{\#\mathcal{S}(H)})}{\sum_{\tilde{F}} 1}, \tag{13}$$

where H is the curve defined by $w^2 = \tilde{F}(u, v)$, and \tilde{F} ranges over the set of all homogeneous squarefree polynomials of degree 8 over \mathbb{F}_q . Lemma 1 implies that $\#\mathcal{S}(H)$ depends only on the degrees of the \mathbb{F}_q -irreducible factors of \tilde{F} , so the map $T \mapsto s(T)$ is well-defined. For each T in \mathcal{T} , let $N_q(T)$ denote the number of homogeneous squarefree polynomials over \mathbb{F}_q whose multiset of degrees of \mathbb{F}_q -irreducible factors coincides with T . We can now rewrite (13) as

$$E_q = \frac{\sum_{T \in \mathcal{T}} (1 - (3/4)^{s(T)}) N_q(T)}{\sum_{T \in \mathcal{T}} N_q(T)}.$$

There are $N_q(n) = \frac{1}{n} \sum_{d|n} \mu(d)q^{n/d}$ monic irreducible polynomials of degree n over \mathbb{F}_q (here μ is the Möbius function). Clearly $N_q(T) = (q - 1) \prod_{n \in T} \binom{N_q(n)}{v_T(n)}$, so

$$N_q(T) = \left(\prod_{n \in T} (v_T(n)! \cdot n^{v_T(n)}) \right)^{-1} q^9 + O(q^8),$$

and $\sum_{T \in \mathcal{T}} N_q(T) = q^9 + O(q^8)$. Therefore, as q tends to infinity, we have

$$\lim_{q \rightarrow \infty} E_q = \sum_{T \in \mathcal{T}} \left((1 - (3/4)^{s(T)}) / \prod_{n \in T} (v_T(n)! \cdot n^{v_T(n)}) \right).$$

The result follows upon explicitly computing this sum, using the values for $s(T)$ listed in Lemma 1. □

Theorem 2 gives the expectation of our ability to construct an explicit isogeny for a randomly selected hyperelliptic curve. However, looking at the table in Lemma 1, we see that we can be sure that a particular curve has no isogenies with tractable kernels defined over \mathbb{F}_q if we use only curves whose hyperelliptic polynomials have an irreducible factor of degree 5 or 7 (or a single irreducible factor of degree 3). It may be difficult to efficiently construct a curve in this form if we are using a CM construction, for example, to ensure that the Jacobian has a large prime-order subgroup. In any case, it is interesting to note that the security of genus 3 hyperelliptic Jacobians depends significantly upon the factorisation of their hyperelliptic polynomials. This observation has no analogue for elliptic curves or Jacobians of curves of genus 2. Of course, if $E : y^2 = F(x)$ is an elliptic curve and F is completely reducible, then $\#E(\mathbb{F}_q)$ is divisible by 4 and, in particular, $\#E(\mathbb{F}_q)$ cannot be prime; but this does not reduce the security of $E(\mathbb{F}_q)$ to the extent that a completely reducible hyperelliptic polynomial does for a curve of genus 3.

Remark 5. We noted in Sect. 4 that the $\overline{\mathbb{F}}_q$ -isomorphism class of the curve X in the trigonal construction is independent of the choice of trigonal map. If there is no trigonal map defined over \mathbb{F}_q for a given subgroup S in $\mathcal{S}(H)$, then the methods of Sect. 5 construct a pair of Galois-conjugate trigonal maps g_1 and g_2 (corresponding to the roots of (3)) instead. Applying the trigonal construction to g_1 and g_2 , we obtain curves X_1 and X_2 over \mathbb{F}_{q^2} . If the isomorphism between X_1 and X_2 were made explicit, then we could descend it to compute a curve X over \mathbb{F}_q in the $\overline{\mathbb{F}}_q$ -isomorphism class of X_1 and X_2 , and hence a nonsingular plane quartic C over \mathbb{F}_q and an isogeny $J_H \rightarrow J_C$. We note that the isogeny may not be defined over \mathbb{F}_q , but this approach could still allow us to replace the 1/4 in (13) and (12) with 1/2, raising the expectation of success in Theorem 2 to 31.13%.

Example 2. Let $p = 1008945029102471339$. Note that p is a 60-bit prime; if H is a hyperelliptic curve of genus 3 over \mathbb{F}_p such that $J_H(\mathbb{F}_p)$ has a large prime-order subgroup and if Gaudry–Thomé–Thériault–Diem index calculus is the fastest algorithm for solving DLP instances in $J_H(\mathbb{F}_p)$, then J_H has roughly the same security level as an elliptic curve over a 160-bit field.

We generated one million random hyperelliptic curves of genus 3 over \mathbb{F}_p using Magma. For each curve H , we computed the set $\mathcal{S}(H)$ of tractable subgroups; then, for each S in $\mathcal{S}(H)$ we determined whether there was an \mathbb{F}_p -rational trigonal map for S , and if so whether there was an \mathbb{F}_p -rational isogeny with kernel S . Of these curves, 502005 (that is, 50.02%) had at least one rational tractable subgroup. Between them, the 10^6 curves had 1002244 rational tractable subgroups, of which 501629 had a rational trigonal map (that is, 50.05%, which is close to the 50% predicted by Hypothesis 2). Of these subgroups, 250560 led to a rational isogeny (that is, 49.95%, which is close to the 50% predicted by Hypothesis 3). We found that 185814 of the curves had at least one \mathbb{F}_p -rational isogeny, none of which had a hyperelliptic codomain (this is compatible with Hypothesis 1). In particular, we could move a discrete logarithm problem for 18.58% of these curves (recall that Theorem 2 predicts a success rate of about 18.57%).

9. Other Isogenies

So far, we have concentrated on using isogenies with kernels generated by differences of Weierstrass points to move instances of the DLP from hyperelliptic to non-hyperelliptic Jacobians. More generally, we could use isogenies with other kernels. There are two important issues to consider here: the first is a theoretical restriction on the types of subgroups that can be kernels of isogenies of Jacobians, and the second is a practical restriction on the isogenies that we can currently compute.

Let H be a hyperelliptic curve of genus 3. We want to characterise the subgroups S of J_H that are kernels of isogenies of Jacobians, combining standard results from the theory of abelian varieties with some special results on curves of genus 3. For our purposes, it is enough to know that the l -Weil pairing is a nondegenerate, bilinear pairing on the l -torsion of an abelian variety, which can be efficiently evaluated in the case where the abelian variety is the Jacobian of a hyperelliptic curve; for further detail, we refer the reader to [12, Ex. A.7.8].

Definition 3. Let A be an abelian variety over \mathbb{F}_q , and let l be a positive integer coprime with q . We say a subgroup S of $A[l]$ is *maximal l -isotropic* if

1. the l -Weil pairing on $A[l]$ restricts trivially to S , and
2. S is not properly contained in any other subgroup of $A[l]$ satisfying (1).

If l is a prime not dividing q , then every maximal l -isotropic subgroup of $J_H(\overline{\mathbb{F}}_q)[l]$ is isomorphic to $(\mathbb{Z}/l\mathbb{Z})^3$. The situation is more complicated when l is not prime; for example, $J_H[2]$ is a maximal 4-isotropic subgroup of $J_H[4]$, but it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^6$ and not $(\mathbb{Z}/4\mathbb{Z})^3$.

Proposition 5. Let H be a hyperelliptic curve of genus 3 over \mathbb{F}_q such that J_H is absolutely simple. Let S be a finite, nontrivial, \mathbb{F}_q -rational subgroup of $J_H(\overline{\mathbb{F}}_q)$. There exists a curve X of genus 3 over \mathbb{F}_q , and an isogeny $\phi : J_H \rightarrow J_X$ with kernel S , if and only if S is a maximal l -isotropic subgroup of $J_H[l]$ for some positive integer l . The isogeny ϕ is defined over \mathbb{F}_{q^2} .

Proof. The quotient $J_H \rightarrow J_H/S$ always exists as an isogeny of abelian varieties, and is defined over \mathbb{F}_q (see Serre [21, Sect. III.3.12]). For the quotient to be an isogeny of Jacobians, there must be an integer l such that S is a maximal l -isotropic subgroup (see Proposition 16.8 of Milne [16]). This ensures that the canonical polarisation on J_H induces a principal polarisation on the quotient J_H/S . The theorem of Oort and Ueno [18] therefore guarantees that there will be an isomorphism of principally polarised abelian varieties over $\overline{\mathbb{F}}_q$ from J_H/S to the Jacobian J_X of some irreducible curve X (irreducibility of X follows from the fact that J_H , and hence J_H/S , is absolutely simple). Composing this isomorphism with the quotient map gives an isogeny of Jacobians from J_H to J_X with kernel S . Standard arguments from Galois cohomology (see Serre [22, Sect. III.1], for example) show that the isomorphism is defined over either \mathbb{F}_q or \mathbb{F}_{q^2} , and it follows that the isogeny $J_H \rightarrow J_X$ must be defined over \mathbb{F}_q or \mathbb{F}_{q^2} . \square

Remark 6. Proposition 5 does *not* hold in higher genus. For every $g \geq 4$, there are g -dimensional abelian varieties that are not isomorphic to Jacobians. Indeed, this is the generic situation. For $g \geq 2$ the moduli space of g -dimensional abelian varieties is $g(g+1)/2$ -dimensional, with the Jacobians occupying a subspace of dimension $(3g-3)$ —which is strictly less than $g(g+1)/2$ for $g \geq 4$. We should not therefore expect an arbitrary quotient of a Jacobian to be isomorphic to a Jacobian in genus $g \geq 4$. Proposition 5 does hold in genus 1 and 2, and in these cases the isogenies are always defined over \mathbb{F}_q .

We can expect the curve X of Proposition 5 to be non-hyperelliptic. To compute an \mathbb{F}_q -rational isogeny from J_H to a non-hyperelliptic Jacobian; therefore, the minimum requirement is an \mathbb{F}_q -rational l -isotropic subgroup of $J_H(\overline{\mathbb{F}}_q)$ isomorphic to $(\mathbb{Z}/l\mathbb{Z})^3$ for some prime l . We emphasise that this subgroup need *not* be contained in $J_H(\mathbb{F}_q)$. Indeed, there may be isogenies from J_H to non-hyperelliptic Jacobians over \mathbb{F}_q even when $J_H(\mathbb{F}_q)$ has prime order (which would be the desirable situation in cryptological applications).

The major obstruction to using more general isogenies to move DLP instances is the lack of general constructions for explicit isogenies in genus 3. Apart from integer multiplications, automorphisms, Frobenius isogenies, and the construction for isogenies with tractable kernels exhibited above, we know of no constructions for explicit isogenies of general Jacobians of genus 3 hyperelliptic curves. In particular, while we know that the curve X of Proposition 5 exists, we generally have no means of computing a defining equation for it, let alone equations for a correspondence between H and X that would allow us to move DLP instances from J_H to J_X . This situation stands in marked contrast to the case of isogenies of elliptic curves, which have been made completely explicit by Vélú [25]. Deriving general formulae for explicit isogenies in genus 3 (and 2) remains a significant problem in computational number theory.

Appendix A. Computing $\mathcal{S}(H)$

Given a hyperelliptic curve H of genus 3 over \mathbb{F}_q , we want to compute the set $\mathcal{S}(H)$ of \mathbb{F}_q -rational tractable subgroups of J_H . Algorithm 4 splits the hyperelliptic poly-

mial of H into Galois orbits of factors, before calling the recursive subroutine Algorithm 5 to enumerate $\mathcal{S}(H)$. This algorithm is included only for completeness, and is not particularly efficient (we suggest some optimisations in Remark 7 below.)

Algorithm 4. Given a hyperelliptic curve H of genus 3 over \mathbb{F}_q , enumerates the set $\mathcal{S}(H)$ of \mathbb{F}_q -rational tractable subgroups of $J_H[2](\overline{\mathbb{F}_q})$. Each subgroup in $\mathcal{S}(H)$ is represented by a set of four coprime quadratic factors of \tilde{F} .

Input The hyperelliptic polynomial $\tilde{F}(u, v)$ of H .

Output The set $\mathcal{S}(H)$.

Step 1 Let \mathcal{F} be the set of irreducible factors of \tilde{F} over its splitting field, scaled so that $\tilde{F} = \prod_{L \in \mathcal{F}} L$, and set $\mathcal{O} := \{\}$.

Step 2 Choose a polynomial L from \mathcal{F} . Set $\mathcal{O} := (\mathcal{O}, L)$, set $\mathcal{F} := \mathcal{F} \setminus \{L\}$, and set $L_1 := L$.

Step 3 Set $L := \sigma(L)$, where σ denotes the q th power Frobenius map.

If $L \neq L_1$, then append L to \mathcal{O} , set $\mathcal{F} := \mathcal{F} \setminus \{L\}$, and go to **Step 3**.

If $L = L_1$, then set $\mathcal{O} := \mathcal{O} \cup \{O\}$; if $\mathcal{F} \neq \emptyset$, then go to **Step 2**.

Step 4 Return the result of Algorithm 5 applied to \mathcal{O} .

Algorithm 5. Given a set of \mathcal{G} -orbits of coprime linear polynomials over $\overline{\mathbb{F}_q}$, returns the \mathcal{G} -invariant sets of coprime quadratic products of the polynomials.

Input A set \mathcal{O} of disjoint sequences of distinct linear polynomials. Each sequence $O = (O_1, \dots, O_m)$ in \mathcal{O} must satisfy $O_1 = \sigma(O_m)$ and $O_{i+1} = \sigma(O_i)$ for $1 \leq i < m$, where σ denotes the q th-power Frobenius map.

Output The set \mathcal{S} of \mathcal{G} -stable sets of coprime quadratic polynomials such that $\prod_{S \in \mathcal{S}} \prod_{Q \in S} Q = \prod_{O \in \mathcal{O}} \prod_{L \in O} L$.

Step 1 If \mathcal{O} is empty, then return $\mathcal{S} := \{\emptyset\}$.

Step 2 Choose a sequence O from \mathcal{O} , and set $m := \#O$.

If m is even, then let \mathcal{T} be the result of Algorithm 5 applied to $\mathcal{O} \setminus \{O\}$, and set $\mathcal{S} := \{O_i \cdot O_{(m/2)+i} : 1 \leq i \leq m/2\} \cup \mathcal{T} : \mathcal{T} \in \mathcal{T}\}$.

If m is odd, then set $\mathcal{S} := \{\}$.

Step 3 For each P in $\mathcal{O} \setminus \{O\}$ such that $\#P = \#O = m$,

Step 3i Set $\mathcal{U} := \{O_{1+i} \cdot P_{1+((i+j) \bmod m)} : 0 \leq i < m : 0 \leq j < m\}$.

Step 3ii Let \mathcal{V} be the result of Algorithm 5 applied to $\mathcal{O} \setminus \{O, P\}$.

Step 3iii Set $\mathcal{S} := \mathcal{S} \cup \{U \cup V : U \in \mathcal{U}, V \in \mathcal{V}\}$.

Step 4 Return \mathcal{S} .

Remark 7. As we noted above, Algorithms 4 and 5 are not particularly efficient. For conceptual simplicity, we worked over the splitting field of the hyperelliptic polynomial, and this can be extremely slow in practice. A number of simple optimisations will significantly improve the performance of this algorithm. The key is to avoid field extensions where possible, and to minimise their degree in any case. Before factoring \tilde{F} over its splitting field, we should factor it over \mathbb{F}_q , and then work on a case-by-case basis depending on the degrees of the \mathbb{F}_q -irreducible factors. For example, if \tilde{F} has an odd number of odd-degree factors, then $\mathcal{S}(H)$ is empty by Lemma 1, and we can simply

return the empty set. If \tilde{F} is \mathbb{F}_q -irreducible, then it is not necessary to factor \tilde{F} over its splitting field (which is \mathbb{F}_{q^8}); there is one tractable subgroup, and it corresponds to the four quadratic factors of \tilde{F} that we obtain by factoring \tilde{F} over \mathbb{F}_{q^4} . Making similar modifications for the cases where \tilde{F} has factors of degree 6, we can avoid working over any extensions of degree greater than 4. If desired, we can further avoid some field extensions in the case where \tilde{F} has only low-degree factors. These modifications resulted in a factor-of-50 speedup for our experiments with 60-bit prime fields; the unmodified Algorithms 4 and 5 should *not* be used in practice.

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