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Sharp Hardy's Inequality for Orthogonal Expansions in H^p Spaces

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Abstract

Hardy's inequality on H^p spaces, $p \in (0, 1]$, in the context of orthogonal expansions is investigated for general bases on a wide class of domains in $\mathbb{R}^d$ with Lebesgue measure. The obtained result is applied to various Hermite, Laguerre, and Jacobi expansions. For that purpose some delicate estimates of the higher order derivatives for the underlying functions and of the associated heat or Poison kernels are proved. Moreover, sharpness of studied Hardy's inequalities is justified by a construction of an explicit counterexample, which is adjusted to all considered settings.

Keywords Hardy's inequality · Hardy space · Laguerre expansions · Hermite expansions · Jacobi expansions

Mathematics Subject Classification Primary 42C10; Secondary 42B30 · 42B05 · 33C45

1 Introduction

The classical Hardy inequality (see [15]) for Fourier coefficients states that

$$\sum_{k\in\mathbb{Z}} \frac{|\hat{f}(k)|}{|k|+1} \lesssim \|f\|_{\mathrm{Re}\,H^1}, \tag{1.1}$$

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where Re H^1 is the real Hardy space composed of the real parts of functions in the Hardy space $H^1(\mathbb{D})$. Here $\mathbb{D}$ denotes the unit disk in the complex plane. Analogues of (1.1) were considered by Kanjin [17], and $\hat{f}(k)$ were replaced by the expansion coefficients in two orthonormal bases: the Hermite and standard Laguerre function systems. In general, such inequalities are of the form

$$\sum_{k\in\mathbb{N}} \frac{|\langle f, \varphi_k\rangle|}{(k+1)^E} \lesssim \|f\|_{H^1}, \tag{1.2}$$

where φ_k is an orthonormal basis in a certain L^2 space, $\langle\cdot,\cdot\rangle$ denotes the inner product in this L^2, H^1 is an appropriate Hardy space, and E is a positive number which we refer to as the admissible exponent. The difficulty in establishing versions of (1.2) is twofold. Firstly, given an orthonormal basis one can ask if such an inequality is valid for a certain E. Secondly, there is a question of the sharpness of the admissible exponent. We say that E is sharp if it is the smallest positive number for which (1.2) holds. Moreover, some generalizations of (1.2) are possible, such as replacing H^1 by H^p, $p \in (0, 1]$, or considering the multi-dimensional situation.

In the last two decades many authors were interested in various Hardy's inequalities. As mentioned above, Kanjin initiated the studies for the Hermite functions (he obtained $E = 29/36$) and the standard Laguerre functions ($E = 1$). For the latter system Satake [37] generalized this result for $p \in (0, 1)$ with $E = 2 - p$, and for the former expansions Radha [34] investigated the multi-dimensional situation $d \geq 1$ with $E > (17d + 12)/(24 + 12d)$. A few years later Radha and Thangavelu [35] proved Hardy's inequality associated with Hermite expansions for $d \geq 2$ and $p \in (0, 1]$ with the admissible exponent $E = 3d(2 - p)/4$. The lacking case $d = 1$ was partially covered by Balasubramanian and Radha [3], but the exponent was strictly larger than the expected value $3(2 - p)/4$ (see also Kanjin [18]). The inequality with this admissible exponent was proved ten years later by Z. Li, Y. Yu, and Y. Shi [23]. On the other hand, the Jacobi trigonometric function expansions were studied by Kanjin and K. Sato [19, 20]. There are also some other papers concerning various Hardy's inequalities in the context of orthogonal expansions, see for instance [9, 22, 38, 39].

The author has already written a few articles on this topic. In [30] the system of Laguerre functions of Hermite type was studied. Secondly, in [33] a general multi-dimensional method of proving Hardy's inequalities on H^1 was introduced. It consists in estimating integral kernels of a certain family of operators closely related to the associated heat (or Poison) semigroup. The method was applied to two Laguerre systems: standard and of convolution type. We stress that in the latter case the underlying measure is not Lebesgue measure. Furthermore, in the same paper sharpness of the obtained admissible exponents was proved. Up to our knowledge, it was the first explicit construction of such counterexamples known in the topic. On the other hand, the long study of Hardy's inequality for Hermite expansions were concluded by the author in [31], where it was justified that the known exponent $E = 3d/4$ (for $p = 1$) is sharp. Finally, four Jacobi systems were also investigated, see [32].

In this paper we prove Hardy's inequalities in frameworks of various orthogonal function systems including generalized Hermite, standard Laguerre, Laguerre of Hermite type, and trigonometric Jacobi expansions in H^p spaces, $p \in (0, 1]$. We focus on systems associated with Lebesgue measure. The main reason behind this restriction is that the atomic H^p spaces are not well defined for all $p \in (0, 1)$ when the underlying measure is only assumed to be doubling. On the other hand, if $p = 1$, then there is no need for such restraint, see [33, Theorem 2.2].

Although we prove Hardy's inequality for certain orthogonal systems, we are interested in establishing a general method which works in the known settings. Therefore, we enhance the approach from [33] and adjust it for the case $p \in (0, 1]$. It requires estimating derivatives of an arbitrary order of the kernels $R_r(x, y)$ (see (2.3) for the definition). In most cases it turns out to be not as difficult as one could expect once we have analogous asymptotics for the functions composing the considered basis. However, for the Laguerre expansions of Hermite type it is much more involved, see the proof of Proposition 4.6. This result can be viewed in terms of the heat kernel, see Sect. 4.3. Moreover, by some minor modifications we were able to add the parameter $s \in [p, 2]$ in Theorem 2.4.

Another novelty of the paper is the unified approach to sharpness. Instead of finding separate counterexamples in each setting, we construct one sequence of piecewise constant atoms which, with an additional assumption, justifies that the admissible exponent is sharp. In order to verify the added condition in the specific settings we have to subtly bound the derivatives of the functions in orthonormal basis, see Lemmas 3.4, 4.3, and 5.2. These estimates can be interesting on their own.

The main result of the paper is Hardy's inequality for a general setting, see Theorem 2.4, and sharpness of the admissible exponent, see Propositions 2.5 and 2.6. This theorem is then applied in several settings, see Theorems 3.5, 4.9, 4.13, and 5.9, which generalize many results already known in the literature (see [3, 17, 18, 20, 23, 34, 35, 37]), but also answer some open questions (for instance sharpness of the multi-dimensional inequality on H^p for Laguerre expansions).

Organization of the paper is as follows. In Sect. 2 we prepare the necessary tools to prove Hardy's inequality, like Hardy, BMO, and Lipschitz spaces. Moreover, we enhance the method from [33] so that it works for H^p spaces with $p \in (0, 1]$. Furthermore, we construct a counterexample to justify that the obtained formula for the admissible exponent is sharp. In Sect. 3 we discuss the standard Laguerre functions and estimate their derivatives near zero. This allows us to apply the general theorem. Section 4 is devoted to Laguerre expansions of Hermite type. Similarly as before we estimate the derivatives of the functions from the basis. However, this time it is not immediate to obtain such bounds for the corresponding kernels $R_r(x, y)$. For that purpose we need to use the integral formula for the Bessel function, see (4.9) and Proposition 4.6. We also interpret this estimate in terms of the heat kernel. Moreover, we deduce Hardy's inequality in the generalized Hermite framework. Lastly, in Sect. 5 the Jacobi trigonometric function system is analysed. This time the analysis is focused on the corresponding Poison kernels.

Notation

Throughout this paper $d \geq 1$ denotes the dimension, u, v are real variables, and $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d)$ are vectors from $\mathbb{R}^d$. We use k, i, j for integers belonging to $\mathbb{N} = \mathbb{N}_+ \cup \{0\} = \{0, 1, \ldots\}$, and $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ for the multi-indices. Let $|n| = n_1 + \ldots + n_d$ stand for the length of n. We denote the type indices α and β with the same symbol in both situations $d = 1$ and $d \geq 1$. In the latter case we use the same convention for $|\alpha|$ and $|\beta|$ as for $|n|$. For $u \in \mathbb{R}$ we denote the largest integer not greater than u by $\lfloor u \rfloor$, and the smallest integer not smaller than u by $\lceil u \rceil$. We write $\lesssim$ for inequalities with non-negative entries which hold with a multiplicative constant. It may depend on the quantities stated beforehand, but not on the ones quantified afterwards. If $X \lesssim Y$ and $Y \lesssim X$ simultaneously, then we write $X \simeq Y$.

2 Hardy's Inequality

In this section we develop a method of proving Hardy's inequality on H^p spaces, $0 < p \leq 1$, associated with orthonormal expansions. This is a generalization of the technique described in [33, Sect. 2]. However, Hardy spaces, even in the sense of Coifman-Weiss [10], are not well defined for all p if the underlying measure is only assumed to be doubling. Hence, we will focus our attention on orthogonal expansions in $L^2(X)$, where X is a subset of $\mathbb{R}^d$ equipped with Lebesgue measure.

2.1 Hardy Spaces

Recall that given any Schwartz function Φ such that $\int \Phi \neq 0$, one can define the Hardy space $H^p(\mathbb{R}^d)$, $0 < p \leq 1$, as the space of all distributions satisfying

$$\sup_{t>0} |f * \Phi_t| \in L^p(\mathbb{R}^d),$$

where $\Phi_t(x) = t^{-d}\Phi(x/t)$. The L^p-norm of the quantity above can be taken as a (maximal) "norm" $\|\cdot\|_{\mathrm{m},H^p(\mathbb{R}^d)}$ in $H^p(\mathbb{R}^d)$. We remark that $\|\cdot\|_{\mathrm{m},H^p(\mathbb{R}^d)}$ is indeed a norm only for $p = 1$. In fact, $H^1(\mathbb{R}^d)$ is a Banach space. In general, if $p \leq 1$, then $\|\cdot\|^p_{\mathrm{m},H^p(\mathbb{R}^d)}$ is subadditive, hence $d(f, g) = \|f - g\|^p_{\mathrm{m},H^p(\mathbb{R}^d)}$ defines a complete metric on $H^p(\mathbb{R}^d)$.

A measurable function a supported in a Euclidean ball B is called a (p, q)-atom for $0 < p \leq 1$ and $q \in [1, \infty] \cap (p, \infty]$, if it satisfies

$$\int_B a(x)x^n \, dx = 0 \qquad \text{and} \qquad \|a\|_{L^q(\mathbb{R}^d)} \leq |B|^{\frac{1}{q}-\frac{1}{p}},$$

where $x^n = x_1^{n_1} \ldots x_d^{n_d}$, $|n| \leq \lfloor d(p^{-1} - 1) \rfloor$, and $|B|$ denotes the Lebesgue measure of B. In this paper we only consider $(p, 2)$-atoms, from now on simply called p-atoms.

Every $f \in H^p(\mathbb{R}^d)$ admits an atomic decomposition, namely there exist a sequence of p-atoms $\{a_j\}_{j\in\mathbb{N}}$ and a sequence of complex coefficients $\{\lambda_j\}_{j\in\mathbb{N}}$ such that

$$f(x) = \sum_{j\in\mathbb{N}} \lambda_j a_j(x), \qquad \sum_{j\in\mathbb{N}} |\lambda_j|^p < \infty,$$

where the former series is convergent in $H^p(\mathbb{R}^d)$.

There are several possibilities to define "norms" in $H^p(\mathbb{R}^d)$, equivalent to $\|\cdot\|_{\mathrm{m},H^p(\mathbb{R}^d)}$. For our purposes we choose the atomic one, which is given by

$$\|f\|_{H^p(\mathbb{R}^d)} = \inf \Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p \Big)^{\frac{1}{p}}, \tag{2.1}$$

where the infimum is taken over all atomic decompositions of f.

Let X be a convex open set equipped with Lebesgue measure and the Euclidean metric. There is a number of possible definitions of H^p spaces on subsets of $\mathbb{R}^d$, see for instance [6, 7, 25]. We select the following one (cf. Definition 3.1 in [7])

$$\Big\{ F \in H^p(\mathbb{R}^d) : \ F \equiv 0 \text{ on } (\bar{X})^{\mathrm{c}} \Big\} / \Big\{ F \in H^p(\mathbb{R}^d) : \ F \equiv 0 \text{ on } X \Big\}.$$

We define $H^p(X)$ (in some sources denoted as $H^p_z(X)$) to be composed of $f = F|_X$ with F as above.

In this paper we will work mostly with p-atoms. That is why our final assumption on X (apart from that it is open and convex) is that every $f \in H^p(X)$ admits an atomic decomposition with atoms supported in X (the supporting balls are not necessarily completely contained in X, but their centers are, cf. [16, Theorem 5.3]). We shall refer to such X as the admissible domains.

There are many examples of such domains. For instance, special Lipschitz domains (i.e. the set above a graph of a Lipschitz function on $\mathbb{R}^{d-1}$) or bounded Lipschitz domains, see [7]. In our paper we consider atoms supported on balls, not cubes, so we can also allow rotations of special Lipschitz domains. In our applications, the examples of X shall be $(0, \infty)^d$ and $(0, \pi)^d$.

We set $\|f\|_{H^p(X)}$ similarly as in (2.1). Observe that for f and F as above we have

$$\|F\|_{H^p(\mathbb{R}^d)} \le \|f\|_{H^p(X)}, \tag{2.2}$$

since for F the underlying infimum is taken over a possibly larger set.

2.2 Dual Type Spaces

We need to give some meaning to the paring $\langle f, \varphi_n \rangle$ for φ_n from a given orthonormal basis and $f \in H^p(\mathbb{R}^d)$ or, more generally, for $f \in H^p(X)$. For this purpose we shall make use of the duality relation between $H^p(\mathbb{R}^d)$ and $BMO(\mathbb{R}^d)$ and Lipschitz spaces.

Recall that a locally integrable function f is in $BMO(\mathbb{R}^d)$ (bounded mean oscillation space) if

$$\|f\|_{BMO(\mathbb{R}^d)} := \sup_B \frac{1}{|B|} \int_B |f(x) - f_B|\, dx < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^d$ and $f_B = |B|^{-1} \int_B f$ is the mean value of f over B. Observe that the expression above vanishes for constant functions. In fact, it is usual to define $BMO(\mathbb{R}^d)$ as the quotient of the above space by the space of constant functions. Then $BMO(\mathbb{R}^d)$ with the norm $\|\cdot\|_{BMO(\mathbb{R}^d)}$ becomes a Banach space. For more details we refer to the literature, see [14, 40].

Now let $\Lambda_\nu(\mathbb{R}^d)$, $\nu > 0$, denote the Lipschitz space. If $\nu \notin \mathbb{N}_+$, then $\Lambda_\nu(\mathbb{R}^d)$ is composed of all bounded functions $g \in \mathcal{C}^{(\lfloor \nu \rfloor)}(\mathbb{R}^d)$ satisfying the condition

$$\|g\|_{\Lambda_\nu(\mathbb{R}^d)} := \|g\|_{L^\infty(\mathbb{R}^d)} + \max_{|n|=\lfloor \nu \rfloor} \sup_{\substack{x,h\in\mathbb{R}^d \\ h\neq 0}} \frac{\left|\partial^n g(x+h) - \partial^n g(x)\right|}{|h|^{\nu-\lfloor \nu \rfloor}} < \infty,$$

where $|h|$ denotes the Euclidean norm of h. If $\nu \in \mathbb{N}_+$, then the above condition is replaced by

$$\|g\|_{\Lambda_\nu(\mathbb{R}^d)} := \|g\|_{L^\infty(\mathbb{R}^d)} + \max_{|n|=\nu-1} \sup_{\substack{x,h\in\mathbb{R}^d \\ h\neq 0}} \frac{\left|\partial^n g(x+h) - 2\partial^n g(x) + \partial^n g(x-h)\right|}{|h|} < \infty$$

for bounded $g \in \mathcal{C}^{(\nu-1)}(\mathbb{R}^d)$. Finally, for $\nu = 0$ we set $\Lambda_0(\mathbb{R}^d) := BMO(\mathbb{R}^d)$.

It is known that $BMO(\mathbb{R}^d)$ is the dual of $H^1(\mathbb{R}^d)$ (see [11, 12]), whereas for $H^p(\mathbb{R}^d)$, $p < 1$, the duals are the Campanato spaces (see [4] and for instance [24, p. 55]). Nonetheless, the Lipschitz spaces described above and $H^p(\mathbb{R}^d)$, $p \in (0, 1]$, have a duality property too (see [13, 14, 40, 43]). Moreover, they are easier to handle and completely sufficient for our purposes.

The aforementioned duality relation is the following: if $g \in \Lambda_{d(\frac{1}{p}-1)}(\mathbb{R}^d)$, then for the linear functional

$$T_g(f) = \int_{\mathbb{R}^d} g(x) f(x)\, dx,$$

there is

$$|T_g(f)| \lesssim \|g\|_{\Lambda_{d\left(\frac{1}{p}-1\right)}(\mathbb{R}^d)} \|f\|_{H^p(\mathbb{R}^d)},$$

uniformly in finite linear combinations of p-atoms f. Furthermore, since those linear combinations are dense in $H^p(\mathbb{R}^d)$ (see e.g. [24, p. 54]), the functional T_g has a unique bounded extension to the whole $H^p(\mathbb{R}^d)$ with the same bound.

Now we show two properties of one-dimensional functions which are in $\Lambda_\nu(\mathbb{R})$. The first result is known (see for instance [14, Corollary 1.4.11]), and for the second we provide a short justification.

Lemma 2.1 *Let $\nu > 0$ and $g \in \Lambda_\nu(\mathbb{R})$. Then, for $0 \le k \le \lceil \nu \rceil - 1$, $g^{(k)} \in \Lambda_{\nu-k}(\mathbb{R})$ and*

$$\|g^{(k)}\|_{\Lambda_{\nu-k}(\mathbb{R})} \le C_\nu \|g\|_{\Lambda_\nu(\mathbb{R})}, \qquad g \in \Lambda_\nu(\mathbb{R}),$$

for some positive constant C_ν independent of g.

Lemma 2.2 *Let $\nu > 0$ and $g_1, \dots, g_d \in \Lambda_\nu(\mathbb{R})$. If $\nu = 1$, then we additionally assume that g_i', $i = 1, \dots, d$, exist and are bounded. Then the function*

$$g(x) = g_1(x_1) \cdot \dots \cdot g_d(x_d), \qquad x \in \mathbb{R}^d,$$

belongs to $\Lambda_\nu(\mathbb{R}^d)$ and

$$\|g\|_{\Lambda_\nu(\mathbb{R}^d)} \lesssim \prod_{i=1}^{d} \|g_i\|_{\Lambda_\nu(\mathbb{R})},$$

where the underlying constant does not depend on g if $\nu \ne 1$, although it may depend on $\max_{1 \le i \le d} \|g_i'\|_{L^\infty(\mathbb{R})}$ if $\nu = 1$.

Proof Obviously $g \in L^\infty(\mathbb{R}^d)$. Firstly assume that ν is a non-integer positive number. Let n be a multi-index such that $|n| = \lfloor \nu \rfloor$. Then, for $h = (h_1, \dots, h_d) \in \mathbb{R}^d \setminus \{0\}$, we write the difference $\partial^n g(x+h) - \partial^n g(x)$ as

$$\begin{aligned}&\big(g_1^{(n_1)}(x_1+h_1) - g_1^{(n_1)}(x_1)\big) g_2^{(n_2)}(x_2+h_2) \cdot \dots \cdot g_d^{(n_d)}(x_d+h_d) \\ &\quad + g_1^{(n_1)}(x_1)\big(g_2^{(n_2)}(x_2+h_2) - g_2^{(n_2)}(x_2)\big) g_3^{(n_3)}(x_3+h_3) \cdot \dots \cdot g_d^{(n_d)}(x_d+h_d) \\ &\quad + \dots \\ &\quad + g_1^{(n_1)}(x_1) \cdot \dots \cdot g_{d-1}^{(n_{d-1})}(x_{d-1})\big(g_d^{(n_d)}(x_d+h_d) - g_d^{(n_d)}(x_d)\big).\end{aligned}$$

If $n_i < |n|$, then by the mean value theorem

$$\big|g_i^{(n_i)}(x_i+h_i) - g_i^{(n_i)}(x_i)\big| \le |h_i| \big\|g_i^{(n_i+1)}\big\|_{L^\infty(\mathbb{R})} \le |h_i| \big\|g_i^{(n_i+1)}\big\|_{\Lambda_{\nu-n_i-1}(\mathbb{R})}.$$

Obviously, if $|h_i|$ is large, then we could immediately estimate this by $2\|g_i^{(n_i)}\|_{\Lambda_{\nu-n_i}(\mathbb{R})}$. On the other hand, if $n_i = |n|$, then

$$\big|g_i^{(n_i)}(x_i+h_i) - g_i^{(n_i)}(x_i)\big| \le |h_i|^{\nu-|n|} \big\|g_i^{(|n|)}\big\|_{\Lambda_{\nu-|n|}(\mathbb{R})}.$$

In each case, by Lemma 2.1 we have

$$|h_i|^{\nu-\lfloor\nu\rfloor} \big|g_i^{(n_i)}(x_i+h_i) - g_i^{(n_i)}(x_i)\big| \le \big\|g_i\big\|_{\Lambda_\nu(\mathbb{R})}.$$

Hence,

$$\frac{\big|\partial^n g(x+h) - \partial^n g(x)\big|}{|h|^{\nu-\lfloor\nu\rfloor}} \lesssim \prod_{i=1}^{d} \|g_i\|_{\Lambda_\nu(\mathbb{R})}, \qquad x, h \in \mathbb{R}^d,\ h \ne 0.$$

Next, suppose that $\nu \in \mathbb{N}$ is such that $\nu \geq 2$. Then, for $|n| = \nu - 1$, we estimate an expression of the form

$$|h|^{-1}\big|g_1^{(n_1)}(x_1+h_1)\cdot\ldots\cdot g_d^{(n_d)}(x_d+h_d) - 2g_1^{(n_1)}(x_1)\cdot\ldots\cdot g_d^{(n_d)}(x_d) \\ + g_1^{(n_1)}(x_1-h_1)\cdot\ldots\cdot g_d^{(n_d)}(x_d-h_d)\big|.$$

Observe that if n is a multi-index which has at least two non-zero components, then by Lemma 2.1 the expression above is estimated by a constant times $\prod_{i=1}^d \|g_i\|_{\Lambda_\nu(\mathbb{R})}$. Indeed, it easily follows from the mean value theorem, or more precisely, from the estimate

$$\big|g_i^{(n_i)}(x_i+h_i) - g_i^{(n_i)}(x_i)\big| \leq \big\|g_i^{(n_i+1)}\big\|_{L^\infty(\mathbb{R})}|h_i| \leq \big\|g_i\big\|_{\Lambda_\nu(\mathbb{R})}|h_i|,$$

which holds since $n_i \leq \nu - 2$. Otherwise, we can assume that $n = (\nu-1, 0, \ldots, 0)$. Hence, denoting $\bar{x} = (x_2, \ldots, x_d)$, $\bar{g}(\bar{x}) = \prod_{i=2}^d g_i(x_i)$ and $\bar{h} = (h_2, \ldots, h_d)$, we write

$$\begin{aligned} &g_1^{(\nu-1)}(x_1+h_1)\bar{g}(\bar{x}+\bar{h}) - 2g_1^{(\nu-1)}(x_1)\bar{g}(\bar{x}) + g_1^{(\nu-1)}(x_1-h_1)\bar{g}(\bar{x}-\bar{h}) \\ &\quad = g_1^{(\nu-1)}(x_1+h_1)\big(\bar{g}(\bar{x}+\bar{h}) - \bar{g}(\bar{x})\big) + \big(g_1^{(\nu-1)}(x_1+h_1) - 2g_1^{(\nu-1)}(x_1) \\ &\qquad + g_1^{(\nu-1)}(x_1-h_1)\big)\bar{g}(\bar{x}) - g_1^{(\nu-1)}(x_1-h_1)\big(\bar{g}(\bar{x}) - \bar{g}(\bar{x}-\bar{h})\big). \end{aligned}$$

Again, it suffices to use the mean value theorem and Lemma 2.1 to get the required bound.

Notice that this argument is valid also for $\nu = 1$ provided that we assume that g_i' exist and are bounded. This finishes the proof. □

Now, we define the Lipschitz (and BMO) spaces on X and prove similar duality as on $\mathbb{R}^d$. We say that a function g defined on X belongs to $\Lambda_\nu(X)$, $\nu \geq 0$, if there exists $G \in \Lambda_\nu(\mathbb{R}^d)$ such that $G\big|_X = g$. Note that this type of definition differs from the one of $H^p(X)$, where we assume that the extension vanishes outside X. In this case this is not possible because of the smoothness requirement. This choice of $\Lambda_\nu(\mathbb{R}^d)$ agrees for $p = 1$ with the space BMO_{r}, dual to H^1_z, see [2].

Moreover, we set

$$\|g\|_{\Lambda_\nu(X)} = \inf \|G\|_{\Lambda_\nu(\mathbb{R}^d)},$$

where the infimum is taken over all G extending g to $\mathbb{R}^d$. With those definitions the following lemma holds.

Lemma 2.3 *If $p \in (0, 1]$ and $g \in \Lambda_{d(\frac{1}{p}-1)}(X)$, then for the linear functional*

$$T_g(f) := \int_X g(x) f(x)\, dx,$$

there holds

$$|T_g(f)| \lesssim \|g\|_{\Lambda_{d\left(\frac{1}{p}-1\right)}(X)} \|f\|_{H^p(X)},$$

uniformly in finite linear combinations of p-atoms f supported in X. Consequently, T_g *has a (unique) bounded extension to whole* $H^p(X)$ *such that* $|T_g(f)| \lesssim \|g\|_{\Lambda_{d(\frac{1}{p}-1)}(X)} \|f\|_{H^p(X)}$, $f \in H^p(X)$.

Proof We have already mentioned before that the claim is valid for $X = \mathbb{R}^d$. Fix $p \in (0, 1]$. Let f be a finite linear combination of p-atoms supported in X. We trivially extend f to $F \in H^p(\mathbb{R}^d)$ so that $F = 0$ on X^c. Hence, F is a finite linear combination of p-atoms as well. Similarly, for any $g \in \Lambda_{d(\frac{1}{p}-1)}(X)$ let $G \in \Lambda_{d(\frac{1}{p}-1)}(\mathbb{R}^d)$ be an extension of g to $\mathbb{R}^d$. Then we have

$$\left| \int_X f(x)g(x)\,dx \right| = \left| \int_{\mathbb{R}^d} F(x)G(x)\,dx \right| \lesssim \|F\|_{H^p(\mathbb{R}^d)} \|G\|_{\Lambda_{d\left(\frac{1}{p}-1\right)}(\mathbb{R}^d)}$$
$$\leq \|f\|_{H^p(X)} \|G\|_{\Lambda_{d\left(\frac{1}{p}-1\right)}(\mathbb{R}^d)},$$

where in the last inequality we used (2.2). By taking the infimum over G we obtain the required bound.

Now let f be an arbitrary element of $H^p(X)$ and F be the trivial extension to $H^p(\mathbb{R}^d)$. For $\tilde{G} \in \Lambda_{d(\frac{1}{p}-1)}(\mathbb{R}^d)$ let $\tilde{T}_{\tilde{G}}$ be the linear functional on $H^p(\mathbb{R}^d)$ corresponding to $\tilde{G}$ so that there holds

$$|\tilde{T}_{\tilde{G}}(\tilde{F})| \lesssim \|\tilde{F}\|_{H^p(\mathbb{R}^d)} \|\tilde{G}\|_{\Lambda_{d\left(\frac{1}{p}-1\right)}(\mathbb{R}^d)}, \qquad \tilde{F} \in H^p(\mathbb{R}^d).$$

We choose an extension G of g and define T_g on $H^p(X)$ by $T_g(f) = \tilde{T}_G(F)$ with the notation as above. Hence,

$$|T_g(f)| \lesssim \|F\|_{H^p(\mathbb{R}^d)} \|G\|_{\Lambda_{d\left(\frac{1}{p}-1\right)}(\mathbb{R}^d)} \leq \|f\|_{H^p(X)} \|G\|_{\Lambda_{d\left(\frac{1}{p}-1\right)}(\mathbb{R}^d)}.$$

It suffices to take the infimum over G to get the claim. □

One comment is in order here. Note that T_g defined as in the proof of Lemma 2.3 does not depend on the chosen extension G. Indeed, let G_1 and G_2 be some extensions of g to $\mathbb{R}^d$. Fix $f \in H^p(X)$ and let $F \in H^p(\mathbb{R}^d)$ be its trivial extension to $\mathbb{R}^d$. We chose an atomic decomposition of F with atoms supported in X, and set $\{F_k\}_{k\in\mathbb{N}}$ to be the partial sums of the decomposition. Now fix $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so that

$$\|F - F_N\|_{H^p(\mathbb{R}^d)} \leq \frac{\varepsilon}{\|G_1\|_{\Lambda_{d\left(\frac{1}{p}-1\right)}(\mathbb{R}^d)} + \|G_2\|_{\Lambda_{d\left(\frac{1}{p}-1\right)}(\mathbb{R}^d)}}.$$

Observe that

$$|\tilde{T}_{G_1}(F) - \tilde{T}_{G_2}(F)| \leq |\tilde{T}_{G_1}(F_N) - \tilde{T}_{G_2}(F_N)| + |\tilde{T}_{G_1}(F - F_N)| + |\tilde{T}_{G_2}(F - F_N)| \lesssim \varepsilon,$$

since $\tilde{T}_{G_1}(F_N) = \tilde{T}_{G_2}(F_N)$ as F_N is a finite linear combination of atoms. This justifies that $T_g(f)$ does not depend on the chosen extension G.

2.3 Main Theorem

Fix $p \in (0, 1]$ and let $\{\varphi_n\}_{n\in\mathbb{N}^d}$, where $\varphi_n \in \Lambda_{d(\frac{1}{p}-1)}(X)$, be an orthonormal basis in $L^2(X)$. We define the family of operators $\{R_r\}_{r\in(0,1)}$ via

$$R_r f = \sum_{n\in\mathbb{N}^d} r^{|n|}\langle f, \varphi_n\rangle\varphi_n, \tag{2.3}$$

where

$$\langle f, \varphi_n\rangle = \int_X f(x)\overline{\varphi_n(x)}\, dx.$$

Note that the integral makes sense for finite linear combinations of p-atoms. In order to apply R_r to all elements of $H^p(X)$ we need to give a more general meaning to $\langle f, \varphi_n\rangle$. This can be done by the means of Lemma 2.3, namely

$$\langle f, \varphi_n\rangle = T_{\varphi_n}(f).$$

Recall that T_{φ_n} is unique (see the comment below the lemma).

We assume that R_r, $r \in (0, 1)$, are integral operators

$$R_r f(x) = \int_X R(x, y) f(y)\, dy,$$

where the associated kernels $R_r(x, y)$ belong to $\mathcal{C}^P(X)$ (as functions of x, for any $y \in X$) for $P = \lfloor d(p^{-1} - 1)\rfloor$, which means that all of their partial derivatives ∂_x^n, $|n| \le P$, exist and are continuous. Then, at least formally,

$$R_r(x, y) = \sum_{n\in\mathbb{N}^d} r^{|n|}\varphi_n(x)\varphi_n(y).$$

Moreover, we impose the following condition on $R_r(x, y)$: there exist a constant $\gamma > 0$ and a finite set Δ composed of positive numbers δ strictly greater than $d(p^{-1}-1)-P$, such that

$$\begin{aligned}&\Big\| R_r(x,\cdot) - \sum_{|n|\le P} \frac{\partial_x^n R_r(x',\cdot)}{n_1! \cdot \ldots \cdot n_d!} \prod_{j=1}^d (x_j - x_j')^{n_j} \Big\|_{L^2(X)} \\ &\lesssim \sum_{\delta\in\Delta} (1-r)^{-(d+2k+2\delta)\gamma} |x - x'|^{P+\delta},\end{aligned} \tag{C}$$

uniformly in $r \in (0, 1)$ and $x, x' \in X$ such that $|x - x'| \le 1/2$. Since X is open and convex, we stress that if $R_r(\cdot, y)$ are in $\mathcal{C}^{P+1}(X)$, then (C) with $\Delta = \{1\}$ is implied by the easier estimate

$$\sup_{x\in X} \|\partial_x^n R_r(x,\cdot)\|_{L^2(X)} \lesssim (1-r)^{-(d+2|n|)\gamma},$$

uniformly in $r \in (0,1)$ and for $|n| = P+1$. Indeed, it suffices to use Taylor's theorem.

Theorem 2.4 *Let $p \in (0,1]$, $s \in [p,2]$, and X be an admissible domain in $\mathbb{R}^d$. Assume that the functions $\{\varphi_n\}_{n\in\mathbb{N}^d}$ belong to $\Lambda_{d(\frac{1}{p}-1)}(X)$, form an orthonormal basis in $L^2(X)$, and the associated kernels $R_r(x,y)$ satisfy condition (C) with $\gamma > 0$. Then the inequality*

$$\sum_{n\in\mathbb{N}^d} \frac{|\langle f, \varphi_n\rangle|^s}{(|n|+1)^E} \lesssim \|f\|^s_{H^p(X)}, \tag{2.4}$$

holds uniformly in $f \in H^p(X)$, where

$$E = \frac{(2-p)sd\gamma}{p} + \frac{(2-s)d}{2}. \tag{2.5}$$

We remark that the above parameter γ is not the same as γ in [33, Theorem 2.2]; in fact if in the cited theorem μ is Lebesgue measure (and hence $N = d$), then both γ's are equal up to the multiplicative constant $(d+2)$.

Proof Fix $p \in (0,1]$ and $s \in [p,2]$. Firstly, we prove the theorem for p-atoms, and then we justify that it holds for all $f \in H^p(X)$. Let a be a p-atom supported in a ball B with the center in x'. Similarly as in [33] and [23] in the first step we use an asymptotic estimate for the Beta function obtaining

$$\begin{aligned}
\sum_{n\in\mathbb{N}^d} \frac{|\langle a, \varphi_n\rangle|^s}{(|n|+1)^E} &\lesssim \sum_{n\in\mathbb{N}^d} \int_0^1 r^{2|n|}(1-r)^{E-1}|\langle a, \varphi_n\rangle|^s\, dr \\
&\le \int_0^1 (1-r)^{E-1}\Big(\sum_{n\in\mathbb{N}^d} r^{2|n|}\Big)^{\frac{2-s}{2}}\Big(\sum_{n\in\mathbb{N}^d}\big(r^{s|n|}|\langle a, \varphi_n\rangle|^s\big)^{\frac{2}{s}}\Big)^{\frac{s}{2}} dr \\
&\lesssim \int_0^1 (1-r)^{E-1}(1-r)^{-\frac{(2-s)d}{2}}\|R_r a\|^s_{L^2(X)}\, dr \\
&= \int_0^1 (1-r)^{\frac{(2-p)sd\gamma}{p}-1}\|R_r a\|^s_{L^2(X)}\, dr.
\end{aligned}$$

Observe that

$$\|R_r a\|^s_{L^2(X)} \le \|a\|^s_{L^2(X)} \le |B|^{\left(\frac{1}{2}-\frac{1}{p}\right)s}.$$

Thus, the claim holds if $|B| \ge 1$. On the other hand, by (C) we have

$$\begin{aligned}
\|R_r a\|^s_{L^2(X)} &= \Big(\int_X \Big|\int_{B\cap X} R_r(x,y)a(x)\, dx\Big|^2 dy\Big)^{\frac{s}{2}} \\
&= \Big(\int_X \Big|\int_{B\cap X} \Big(R_r(x,y) - \sum_{\substack{n\in\mathbb{N}^d\\ |n|\le P}} \frac{\partial_x^n R_r(x',y)}{n_1!\cdot\ldots\cdot n_d!}\prod_{i=1}^d (x_i - x_i')^{n_i}\Big)a(x)\, dx\Big|^2 dy\Big)^{\frac{s}{2}} \\
&\lesssim \Big(\sum_{\delta\in\Delta}\int_{B\cap X} |a(x)||x-x'|^{P+\delta}(1-r)^{-(d+2P+2\delta)\gamma}\, dx\Big)^s
\end{aligned}$$

$$\lesssim \sum_{\delta\in\Delta}(1-r)^{-s(d+2P+2\delta)\gamma}|B|^{s\left(\frac{P+\delta}{d}+1-\frac{1}{p}\right)}.$$

Notice that by the definition of P and Δ there is $\frac{P+\delta}{d}+1-\frac{1}{p}>0$ for every $\delta\in\Delta$. Hence,

$$\begin{aligned}&\int_0^1 \|R_r a\|^s_{L^2(X)}(1-r)^{\frac{(2-p)sd\gamma}{p}-1}dr\\ &\lesssim \sum_{\delta\in\Delta}\int_0^{1-|B|^{\frac{1}{2d\gamma}}}|B|^{s\left(\frac{P+\delta}{d}+1-\frac{1}{p}\right)}(1-r)^{\frac{(2-p)sd\gamma}{p}-1-s\gamma(d+2P+2\delta)}dr\\ &\quad+\int_{1-|B|^{\frac{1}{2d\gamma}}}^1 |B|^{s\left(\frac{1}{2}-\frac{1}{p}\right)}(1-r)^{\frac{(2-p)sd\gamma}{p}-1}dr\end{aligned}$$

and this quantity is bounded by a universal constant uniformly in B such that $|B|\le 1$. The obtained estimate is independent of a. This finishes the proof of the theorem for atoms.

In order to complete the proof let us now justify that the claim holds for any $f\in H^p(X)$. Fix $f\in H^p(X)$ and its atomic decomposition $f=\sum_{j\in\mathbb{N}}\lambda_j a_j$. Denote $f_J=\sum_{j=0}^J\lambda_j a_j$. Observe that for $s\in[p,1]$ and $J>I$ we have

$$\begin{aligned}\sum_{n\in\mathbb{N}^d}\frac{|\langle f_J-f_I,\varphi_n\rangle|^s}{(|n|+1)^E}&\le\sum_{j=I+1}^J|\lambda_j|^s\sum_{n\in\mathbb{N}^d}\frac{|\langle a_j,\varphi_n\rangle|^s}{(|n|+1)^E}\\ &\lesssim\sum_{j=I+1}^J|\lambda_j|^s\le\Big(\sum_{j=I+1}^J|\lambda_j|^p\Big)^{s/p}.\end{aligned}$$

On the other hand, if $s\in[1,2]$, then we use Minkowski's inequality and get

$$\begin{aligned}\Big(\sum_{n\in\mathbb{N}^d}\frac{|\langle f_J-f_I,\varphi_n\rangle|^s}{(|n|+1)^E}\Big)^{1/s}&\le\sum_{j=I+1}^J|\lambda_j|\Big(\sum_{n\in\mathbb{N}^d}\frac{|\langle a_j,\varphi_n\rangle|^s}{(|n|+1)^E}\Big)^{1/s}\lesssim\sum_{j=I+1}^J|\lambda_j|\\ &\le\Big(\sum_{j=I+1}^J|\lambda_j|^p\Big)^{1/p}.\end{aligned}$$

This proves that $\left\{\{\langle f_J,\varphi_n\rangle\}_{n\in\mathbb{N}^d}\right\}_{J\in\mathbb{N}}$ is a Cauchy sequence in $\ell^s\left(\mathbb{N}^d,(|n|+1)^{-E}\right)$, $s\in[p,2]$, so it is convergent there. Therefore, there exists $\{c_n\}_{n\in\mathbb{N}^d}\in\ell^s\left(\mathbb{N}^d,(|n|+1)^{-E}\right)$ such that

$$\lim_{J\to\infty}\sum_{n\in\mathbb{N}^d}\frac{|\langle f_J,\varphi_n\rangle-c_n|^s}{(|n|+1)^E}=0.$$

We justify that $c_n = \langle f, \varphi_n \rangle$. The above equality yields

$$\lim_{J\to\infty} \sum_{n\in\mathbb{N}^d} \frac{|\langle f_J, \varphi_n\rangle - c_n|^s}{(|n|+1)^{d+E}\left(1+\|\varphi_n\|^s_{\Lambda_{d\left(\frac{1}{p}-1\right)}(X)}\right)} = 0.$$

On the other hand, by Lemma 2.3 we see that

$$\begin{aligned}
&\lim_{J\to\infty} \sum_{n\in\mathbb{N}^d} \frac{|\langle f_J - f, \varphi_n\rangle|^s}{(|n|+1)^{d+E}\left(1+\|\varphi_n\|_{\Lambda_{d\left(\frac{1}{p}-1\right)}(X)}\right)} \\
&\lesssim \lim_{J\to\infty} \sum_{n\in\mathbb{N}^d} \frac{\|f_J - f\|^s_{H^p(X)}\|\varphi_n\|^s_{\Lambda_{d\left(\frac{1}{p}-1\right)}(X)}}{(|n|+1)^{d+E}\left(1+\|\varphi_n\|^s_{\Lambda_{d\left(\frac{1}{p}-1\right)}(X)}\right)},
\end{aligned}$$

and the latter limit is equal to zero. Hence, by the uniqueness of the limit we proved that $c_n = \langle f, \varphi_n \rangle$.

Finally, fix $\varepsilon > 0$ and $J \in \mathbb{N}$ such that $\|\langle f_J - f, \varphi_n\rangle\|^s_{\ell^s(\mathbb{N}^d,(|n|+1)^{-E})} < \varepsilon$. We estimate for $s \in [p, 1]$

$$\begin{aligned}
\sum_{n\in\mathbb{N}^d} \frac{|\langle f, \varphi_n\rangle|^s}{(|n|+1)^E} &\le \sum_{n\in\mathbb{N}^d} \frac{|\langle f - f_J, \varphi_n\rangle|^s}{(|n|+1)^E} + \sum_{n\in\mathbb{N}^d} \frac{|\langle f_J, \varphi_n\rangle|^s}{(|n|+1)^E} \\
&\le \varepsilon + \sum_{j=0}^{J} |\lambda_j|^s \sum_{n\in\mathbb{N}^d} \frac{|\langle a_j, \varphi_n\rangle|^s}{(|n|+1)^E} \\
&\lesssim \varepsilon + \left(\sum_{j=0}^{J} |\lambda_j|^p\right)^{s/p} \\
&\lesssim \varepsilon + \|f\|^s_{H^p(X)}.
\end{aligned}$$

If $s \in [1, 2]$, then we proceed as before using Minkowski's inequality. This finishes the proof of the theorem. □

2.4 Sharpness

In this subsection we prove that the admissible exponent in Theorem 2.4 cannot be lowered, provided that some additional assumptions on the basis $\{\varphi_n\}_{n\in\mathbb{N}^d}$ are satisfied. In fact, we focus only on the case $\varphi_n(x) = \prod_{i=1}^d \varphi_{n_i}(x_i)$. Therefore, we state our results in the one-dimensional situation and then make an appropriate comment on the general case $d \ge 1$.

We remark that although conditions (2.9) and (2.12) may seem hard to meet, they turn out to be very natural for the classical orthonormal bases, such as Laguerre, Hermite, or Jacobi function expansions.

Firstly, we construct a one-dimensional auxiliary atom a. Let $p \in (0,1]$, $P = \lfloor p^{-1} - 1 \rfloor$, $A \geq 1$ and $0 < \delta \leq \frac{1}{2(P+1)}$. Consider the following function

$$a(u) = 2^{-(P+2)} A^{1/p} \begin{cases} -1, & u \in (0, \delta A^{-1}), \\ C_j, & u \in (j\delta A^{-1}, (j+1)\delta A^{-1}), \ j = 1, \ldots, P, \\ C_{P+1}, & u \in ((P+1)\delta A^{-1}, A^{-1}), \\ 0, & \text{otherwise}, \end{cases} \tag{2.6}$$

where C_i, $i = 1, \ldots, P+1$, are some constants to be determined. Note that if $|C_i| \leq 2^{P+2}$, then we have the bound $\|a\|_{L^\infty} \leq |B|^{-1/p}$, where $B = (0, A^{-1})$. If additionally C_i are such that $\int u^k a(u)\, du = 0$, $k = 0, \ldots, P$, then a is a p-atom.

Observe that by the equality

$$\int_{i\delta A^{-1}}^{(i+1)\delta A^{-1}} u^k \, du = \frac{1}{k+1} A^{-k-1} \delta^{k+1} ((i+1)^{k+1} - i^{k+1}), \qquad k, i = 0, \ldots, P,$$

the cancellation properties come down to

$$\sum_{i=1}^{P} C_i \delta^{k+1} ((i+1)^{k+1} - i^{k+1}) + C_{P+1} (1 - ((P+1)\delta)^{k+1}) = \delta^{k+1}, \qquad k = 0, \ldots, P. \tag{2.7}$$

This is a system of linear equations on $C_1, \ldots, C_{P+1}$ and one can solve it using Cramer's rule. A calculation shows that

$$C_i = \sum_{\ell=0}^{i} \binom{P+1}{\ell} (-1)^{\ell-1} \frac{1}{1 - \ell\delta}, \qquad i = 1, \ldots, P+1.$$

Indeed, inserting this into left hand side of (2.7) we obtain

$$\begin{aligned} &\delta^{k+1} \sum_{i=1}^{P} \left((i+1)^{k+1} - i^{k+1}\right) \left(\sum_{\ell=0}^{i} \binom{P+1}{\ell} (-1)^{\ell-1} \frac{1}{1-\ell\delta} \right) \\ &\quad + (1 - ((P+1)\delta)^{k+1}) \left(\sum_{\ell=0}^{P+1} \binom{P+1}{\ell} (-1)^{\ell-1} \frac{1}{1-\ell\delta} \right) \\ &= \delta^{k+1} + \sum_{\ell=0}^{P+1} \binom{P+1}{\ell} (-1)^{\ell-1} \frac{1 - (\ell\delta)^{k+1}}{1 - \ell\delta} \\ &= \delta^{k+1} + \sum_{j=0}^{k} \delta^j \sum_{\ell=0}^{P+1} \binom{P+1}{\ell} (-1)^{\ell-1} \ell^j. \end{aligned}$$

Notice that for each j the inner sum vanishes since $k \le P$ and hence (2.7) holds.

We clearly see that $|C_i| \le 2^{P+2}$, $i \in \{1, \ldots, P+1\}$. Thus, the function a defined in (2.6) is a p-atom with this choice of C_i.

Additionally, we will require a more precise estimate for C_{P+1}. Observe that

$$C_{P+1} = \sum_{\ell=0}^{P+1} \binom{P+1}{\ell} (-1)^{\ell-1} \frac{1}{1-\ell\delta} = (-1)^P \int_0^1 (u^{-\delta} - 1)^{P+1}\, du.$$

Since

$$(-\log u) \le \frac{u^{-\delta} - 1}{\delta} \le (-\log u) u^{-1/(2P+2)}, \qquad u \in (0,1),\ \delta \in (0, (2P+2)^{-1}),$$

it is easily seen that

$$|C_{P+1}| \simeq \delta^{P+1}, \qquad \delta \in (0, (2P+2)^{-1}). \tag{2.8}$$

Proposition 2.5 *Let the one-dimensional version of the assumptions of Theorem 2.4 be satisfied. Moreover, we assume that $(0, c) \subset X$ for some $c > 0$ and that there exists $\tau > 2\gamma(p^{-1} - 1) + \gamma - \frac{1}{2}$ such that for some $0 < m \le M$*

$$m(k+1)^\tau u^{\frac{1+2\tau-2\gamma}{4\gamma}} \le |\varphi_k(u)| \le M(k+1)^\tau u^{\frac{1+2\tau-2\gamma}{4\gamma}}, \tag{2.9}$$

uniformly in $u \in (0, cK^{-2\gamma})$, $k \le K$ and $K \in \mathbb{N}_+$, and $\varphi_k(u)$ does not change the sign in this interval. Then the admissible exponent in (2.4) cannot be lowered.

Proof In order to prove this lemma we construct an explicit sequence of atoms a_K, such that for E defined in (2.5) and any $\varepsilon > 0$

$$\sum_{k \in \mathbb{N}} \frac{|\langle a_K, \varphi_k \rangle|^s}{(k+1)^{E-\varepsilon}} \gtrsim K^\varepsilon, \qquad K \in \mathbb{N}_+. \tag{2.10}$$

Let $K \in \mathbb{N}_+$ and a_K be an atom defined in (2.6) with $A = K^{2\gamma}/c$ and some sufficiently small δ. We will show that

$$|\langle a_K, \varphi_k \rangle| \gtrsim K^{\frac{2\gamma}{p} - \frac{1}{2} - \tau - \gamma} (k+1)^\tau, \qquad 0 \le k \le K. \tag{2.11}$$

This suffices to prove (2.10). Indeed, we get

$$\sum_{k \in \mathbb{N}} \frac{|\langle a_K, \varphi_k \rangle|^s}{(k+1)^{\frac{(2-p)s\gamma}{p} + \frac{2-s}{2} - \varepsilon}} \gtrsim K^{s\left(\frac{2\gamma}{p} - \frac{1}{2} - \tau - \gamma\right)} \sum_{k=1}^{K} k^{s\left(\tau - \frac{2\gamma(1-p)}{p} - \gamma + \frac{1}{2}\right) - 1 + \varepsilon} \simeq K^\varepsilon.$$

Let us now justify (2.11). We have

$$\int_0^{A^{-1}} a_K(u)\varphi_k(u)\,du = C_{P+1}\int_{(P+1)\delta A^{-1}}^{A^{-1}} \varphi_k(u)\,du + \sum_{j=0}^{P} C_j \int_{j\delta A^{-1}}^{(j+1)\delta A^{-1}} \varphi_k(u)\,du,$$

where $C_0 = -1$. Thus, by using the fact that $|C_j| \le 2^{P+2}$, $0 \le j \le P$, we see that the absolute value of the quantity above is bounded from below by

$$\begin{aligned} &A^{\frac{1}{p}-\frac{1+2\tau+2\gamma}{4\gamma}} \frac{4M\gamma}{1+2\gamma+2\tau}(k+1)^\tau \Big(\frac{m|C_{P+1}|}{M2^{P+2}}\big(1-((P+1)\delta)^{\frac{1+2\tau+2\gamma}{4\gamma}}\big) - \big((P+1)\delta\big)^{\frac{1+2\tau+2\gamma}{4\gamma}} \Big) \\ &\gtrsim K^{\frac{2\gamma}{p}-\frac{1}{2}-\tau-\gamma}(k+1)^\tau \delta^{P+1} \Big(\frac{m|C_{P+1}|}{M2^{P+2}\delta^{P+1}}\big((P+1)^{-\frac{1+2\tau+2\gamma}{4\gamma}} - \delta^{\frac{1+2\tau+2\gamma}{4\gamma}}\big) - \delta^{\frac{1+2\tau+2\gamma}{4\gamma}-(P+1)} \Big). \end{aligned}$$

Observe that by taking δ sufficiently small we obtain (2.11) because of (2.8) and the fact that τ is large enough. □

Sometimes the condition (2.9) does not hold and hence Proposition 2.5 cannot be applied in order to prove sharpness. However, estimate (2.9) can be replaced by its analogue for the derivatives of φ_k. We describe this situation in the following proposition.

Proposition 2.6 *Let the one-dimensional version of the assumptions of Theorem 2.4 be satisfied. Moreover, we assume that $(0, c) \subset X$ for some $c > 0$, φ_k are $(P+1)$-times differentiable, where $P = \lfloor p^{-1} - 1 \rfloor$, and that there exists $\tau > 2\gamma(p^{-1} - 1) + \gamma - \frac{1}{2}$ such that for some $0 < m \le M$ there holds*

$$m(k+1)^\tau u^{\frac{1+2\tau-2\gamma}{4\gamma}-(P+1)} \le |\varphi_k^{(P+1)}(u)| \le M(k+1)^\tau u^{\frac{1+2\tau-2\gamma}{4\gamma}-(P+1)}, \tag{2.12}$$

uniformly in $u \in (0, cK^{-2\gamma})$, $k \le K$ and $K \in \mathbb{N}_+$, and $\varphi_k^{(P+1)}(u)$ does not change the sign in this interval. Then the admissible exponent in (2.4) cannot be lowered.

Proof Fix $p \in (0, 1]$ and set $P = \lfloor p^{-1} - 1 \rfloor$. Let $K \in \mathbb{N}$ and a_K be the same $H^p(X)$ atom as in Proposition 2.5. We verify (2.11). Observe that we have for some ξ_u between u and $(P+1)\delta/A$ the following equality

$$\begin{aligned} \int_0^{A^{-1}} a_K(u)\varphi_k(u)\,du &= \int_0^{A^{-1}} a_K(u)\Big(\varphi_k(u) - \sum_{j=0}^{P} \frac{\varphi_k^{(j)}\big(\frac{(P+1)\delta}{A}\big)}{j!}\Big(u - \frac{(P+1)\delta}{A}\Big)^j\Big)\,du \\ &= \int_0^{A^{-1}} a_K(u)\frac{1}{(P+1)!}\varphi_k^{(P+1)}(\xi_u)\Big(u - \frac{(P+1)\delta}{A}\Big)^{P+1}\,du. \end{aligned}$$

The absolute value of the latter integral can be estimated from below by

$$\int_{\frac{(P+1)\delta}{A}}^{A^{-1}} \frac{|C_{P+1}|}{2^{P+2}} A^{1/p} \frac{m}{(P+1)!}(k+1)^\tau \xi_u^{\frac{1+2\tau-2\gamma}{4\gamma}-(P+1)}\Big(u - \frac{(P+1)\delta}{A}\Big)^{P+1}\,du$$

$$
\begin{aligned}
& - \int_0^{\frac{(P+1)\delta}{A}} A^{1/p} \frac{M}{(P+1)!} (k+1)^\tau \xi_u^{\frac{1+2\tau-2\gamma}{4\gamma}-(P+1)} \Big(\frac{(P+1)\delta}{A} - u \Big)^{P+1} du \\
& \geq \frac{M}{(P+1)!} A^{1/p} (k+1)^\tau \Big(\frac{(P+1)\delta}{A} \Big)^{\frac{1+2\tau-2\gamma}{4\gamma}-(P+1)} \Big(\frac{m|C_{P+1}|}{M 2^{P+2}} \\
& \quad \times \int_{\frac{(P+1)\delta}{A}}^{A^{-1}} \Big(u - \frac{(P+1)\delta}{A} \Big)^{P+1} du - \int_0^{\frac{(P+1)\delta}{A}} \Big(\frac{(P+1)\delta}{A} - u \Big)^{P+1} du \Big) \\
& = \frac{M}{(P+2)!} ((P+1)\delta)^{\frac{1+2\tau-2\gamma}{4\gamma}+P+1} (k+1)^\tau A^{\frac{1}{p}-\frac{1+2\tau+2\gamma}{4\gamma}} \\
& \quad \times \Big(\frac{m|C_{P+1}|}{M 2^{P+2} \delta^{P+1}} \Big((P+1)^{-(P+1)} - (P+1)\delta^{P+2} \Big) - (P+1)\delta \Big) \\
& \gtrsim (k+1)^\tau A^{\frac{1}{p}-\frac{1+2\tau+2\gamma}{4\gamma}},
\end{aligned}
$$

for δ sufficiently small, since we have (2.8).

Hence, we obtained (2.11), and this finishes the proof of this proposition. □

Remark 2.7 In the multi-dimensional situation, if the functions φ_n are of the form $\varphi_n(x) = \prod_{i=1}^d \varphi_{n_i}(x_i)$, then sharpness of E can be easily justified. We have to assume that each φ_{n_i}, $1 \leq i \leq d$ satisfies (2.9) or (2.12) with $\tau_i > 2\gamma d(p^{-1} - 1) + \gamma - \frac{1}{2}$. Indeed, denote $A_K(x) = \prod_{i=1}^d a_K(x_i)$, where a_K is the same as in Propositions 2.5 and 2.6, but this time with $P = \lfloor d(p^{-1} - 1) \rfloor$. Then A_K is a scaled p-atom in $\mathbb{R}^d$. By (2.11), for any $\varepsilon > 0$, we have the following lower bound

$$
\sum_{n \in \mathbb{N}} \frac{|\langle A_K, \varphi_n \rangle|^s}{(|n|+1)^{E-\varepsilon}} \gtrsim K^{sd\left(\frac{2\gamma}{p} - \frac{1}{2} - \gamma\right) - s|\tau|} \sum_{K/2 \leq n_i \leq K} \frac{\prod_{i=1}^d (n_i+1)^{s\tau_i}}{(|n|+1)^E} \gtrsim K^\varepsilon,
$$

uniformly in large K, where $|\tau| = \tau_1 + \ldots + \tau_d$.

Remark 2.8 Notice that (2.10), generalized to the multi-dimensional situation, and the uniform boundedness principle (in a stronger version than usual, see for instance [36, Theorem 2.5]) imply that there exists $f \in H^p(X)$ such that

$$
\sum_{n \in \mathbb{N}^d} \frac{|\langle f, \varphi_n \rangle|^s}{(|n|+1)^E} = \infty.
$$

This is consistent with what was proved in author's articles concerning Hardy's inequality on H^1, see [30–33].

3 Standard Laguerre Functions

The *standard Laguerre functions* $\{\mathcal{L}_k^\alpha\}_{k \in \mathbb{N}}$ of order $\alpha > -1$ are defined on $\mathbb{R}_+$ by

$$
\mathcal{L}_k^\alpha(u) = \Big(\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} \Big)^{1/2} L_k^\alpha(u) e^{-u/2} u^{\alpha/2}, \qquad u > 0, \tag{3.1}
$$

where $L_k^\alpha(u)$ are the Laguerre polynomials (see [42]). Moreover, in the multi-dimensional case $\mathcal{L}_n^\alpha(x)$ are defined as the tensor products of the one-dimensional functions, namely

$$\mathcal{L}_n^\alpha(x) = \prod_{i=1}^d \mathcal{L}_{n_i}^{\alpha_i}(x_i), \qquad x = (x_1, \dots, x_d) \in \mathbb{R}_+^d = (0, \infty)^d;$$

here $\alpha = (\alpha_1, \dots, \alpha_d) \in (-1, \infty)^d$ and $n = (n_1, \dots, n_d) \in \mathbb{N}^d$. The system $\{\mathcal{L}_n^\alpha\}_{n\in\mathbb{N}^d}$ forms an orthonormal basis in $L^2(\mathbb{R}_+^d, dx)$. The following estimates are known for the one-dimensional standard Laguerre functions (see [26, p. 435] and [1, p. 699])

$$|\mathcal{L}_k^\alpha(u)| \lesssim \begin{cases} (uk')^{\alpha/2}, & 0 < u \le 1/k', \\ (uk')^{-1/4}, & 1/k' < u \le k'/2, \\ (k'(k'^{1/3} + |u - k'|))^{-1/4}, & k'/2 < u \le (3k')/2, \\ \exp(-\gamma u), & 3k'/2 < u < \infty, \end{cases} \tag{3.2}$$

where $k' = \max(4k + 2\alpha + 2, 2)$ and $\gamma > 0$ depends only on α.

These estimates imply for all $\alpha \ge 0$ the bound (cf. [41, p. 94]),

$$\|\mathcal{L}_k^\alpha\|_{L^\infty(\mathbb{R}_+)} \lesssim 1, \qquad k \in \mathbb{N}.$$

Moreover, using the formula (see [41, p. 95])

$$(\mathcal{L}_k^\alpha)'(u) = -k^{1/2}u^{-1/2}\mathcal{L}_{k-1}^{\alpha+1}(u) + \frac{1}{2}\Big(\frac{\alpha}{u} - 1\Big)\mathcal{L}_k^\alpha(u), \tag{3.3}$$

where $\mathcal{L}_{-1}^{\alpha+1} \equiv 0$, for $\alpha \in \{0\} \cup [2, \infty)$ we obtain

$$\|(\mathcal{L}_k^\alpha)'\|_{L^\infty(\mathbb{R}_+)} \lesssim k + 1, \quad k \in \mathbb{N}.$$

More generally, for $j \in \mathbb{N}$ and $\alpha \in \{0, 2, \dots, 2j\} \cup (2j, \infty)$ there holds (see [37, Lemma 1])

$$\|(\mathcal{L}_k^\alpha)^{(j)}\|_{L^\infty(\mathbb{R}_+)} \lesssim (k + 1)^j, \qquad k \in \mathbb{N}. \tag{3.4}$$

Now we justify that $\mathcal{L}_n^\alpha$ belong to the Lipschitz spaces $\Lambda_\nu(\mathbb{R}_+^d)$. For that purpose we will indicate an extension $\tilde{\mathcal{L}}_n^\alpha \in \Lambda_\nu(\mathbb{R}^d)$ of $\mathcal{L}_n^\alpha$ to $\mathbb{R}^d$. Following the idea used in [39, p. 94] in the case $d = 1$ we define

$$\tilde{\mathcal{L}}_n^\alpha(x) = \prod_{i=1}^d \tilde{\mathcal{L}}_{n_i}^{\alpha_i}(x_i),$$

where, if α_i is not an even integer, then

$$\tilde{\mathcal{L}}_{n_i}^{\alpha_i}(x_i) = \begin{cases} \mathcal{L}_{n_i}^{\alpha_i}(x_i), & x_i > 0, \\ 0, & x_i \le 0, \end{cases}$$

and, if α_i is an even integer, then

$$\tilde{\mathcal{L}}_{n_i}^{\alpha_i}(x_i) = \psi(n_i x_i)\mathcal{L}_{n_i}^{\alpha_i}(x_i), \qquad x_i \in \mathbb{R}.$$

In the latter case the definition of $\mathcal{L}_{n_i}^{\alpha_i}$ is naturally extended to the whole real line by the initial formula (3.1), and ψ is a smooth function supported in $[-1, \infty)$ such that $\psi \equiv 1$ on $\mathbb{R}_+$ and $\|\psi^{(j)}\|_{L^\infty(\mathbb{R})} \lesssim 1$, $j \in \mathbb{N}$. For an example of such ψ see [39].

In view of [39, Corollary 2.4] we see that given $\nu > 0$ we have $\tilde{\mathcal{L}}_{n_i}^{\alpha_i} \in \Lambda_\nu(\mathbb{R})$ for $\alpha_i \in [2\nu, \infty)$. Secondly, if α_i is an even integer, then $\tilde{\mathcal{L}}_{n_i}^{\alpha_i} \in \Lambda_\nu(\mathbb{R})$ for all $\nu > 0$. Thus, by Lemma 2.2 if $p \in (0, 1)$ and $\alpha \in \big(\{0, 2, \ldots, 2P\} \cup [2d(p^{-1} - 1), \infty)\big)^d$, where $P = \lfloor d(\frac{1}{p} - 1)\rfloor$, then $\tilde{\mathcal{L}}_n^\alpha \in \Lambda_{d(\frac{1}{p}-1)}(\mathbb{R}^d)$, and therefore $\mathcal{L}_n^\alpha \in \Lambda_{d(\frac{1}{p}-1)}(\mathbb{R}_+^d)$. In order to verify the additional assumption in Lemma 2.2, we use the fact that for $\alpha_i \in \{0\} \cup [2, \infty)$ the functions $(\tilde{\mathcal{L}}_{n_i}^{\alpha_i})'$ exist and are bounded. Finally, for $\alpha \in [0, \infty)^d$ the functions $\tilde{\mathcal{L}}_n^\alpha$ lie in $L^\infty(\mathbb{R}^d)$, so they are also in $BMO(\mathbb{R}^d)$.

The family of operators $\{R_r^\alpha\}$ associated with $\{\mathcal{L}_n^\alpha\}_{n \in \mathbb{N}^d}$ and given by

$$R_r^\alpha f = \sum_{n \in \mathbb{N}^d} r^{|n|} \langle f, \mathcal{L}_n^\alpha \rangle \mathcal{L}_n^\alpha, \qquad r \in (0, 1),$$

is composed of integral operators, with the kernels of the form

$$R_r^\alpha(x, y) = \sum_{n \in \mathbb{N}^d} r^{|n|} \mathcal{L}_n^\alpha(x)\mathcal{L}_n^\alpha(y).$$

It can be explicitly written as the product of the kernels $R_r^{\alpha_i}(x_i, y_i)$ (cf. [33, 42])

$$R_r^{\alpha_i}(x_i, y_i) = (1-r)^{-1} r^{-\alpha_i/2} \exp\Big(-\frac{1}{2}\frac{1+r}{1-r}(x_i + y_i)\Big) I_{\alpha_i}\Big(\frac{2r^{1/2}}{1-r}\sqrt{x_i y_i}\Big),$$

where $I_s(u)$ denotes the modified Bessel function of the first kind and order s. For $s > -1$, it is a real, positive, and smooth function in $\mathbb{R}_+$.

In fact, we do not need this explicit formula for $R_r^\alpha(x, y)$ to prove Hardy's inequality. However, for the completeness of the presentation we gave it above. On the other hand, its analogue for Laguerre functions of Hermite type will be of paramount importance.

Now we are ready to verify condition (C) for the standard Laguerre functions. We begin with the two following observations.

Lemma 3.1 *For $j \in \mathbb{N}$ and $\alpha \in \{0, 2, \ldots, 2j\} \cup (2j, \infty)$ there holds*

$$\sup_{u>0} \Big\| \partial_u^j R_r^\alpha(u, \cdot) \Big\|_{L^2(\mathbb{R}_+)} \lesssim (1-r)^{-\frac{1+2j}{2}}, \qquad r \in (0, 1).$$

Proof We simply apply Parseval's identity and (3.4) obtaining

$$\sup_{u>0} \Big\| \partial_u^j R_r^\alpha(u, \cdot) \Big\|_{L^2(\mathbb{R}_+)} \le \Big(\sum_{k \in \mathbb{N}} r^{2k} \big\| (\mathcal{L}_k^\alpha)^{(j)} \big\|^2_{L^\infty(\mathbb{R}_+)} \Big)^{1/2} \lesssim (1-r)^{-\frac{1+2j}{2}},$$

uniformly in $r \in (0, 1)$. Notice that interchanging differentiation with summation is possible due to polynomial growth in k of $\|(\mathcal{L}_k^\alpha)^{(i)}\|_{L^\infty(\mathbb{R}_+)}$, $0 \le i \le j$ (see (3.4)), and the Lebesgue dominated convergence theorem. Analogous remarks apply to similar operations in this and the next sections. □

Lemma 3.2 *Let $j \in \mathbb{N}$ and $\alpha \in (2j, 2j+2)$. Then the estimate*

$$\left\|\partial_u^j R_r^\alpha(u,\cdot) - \partial_u^j R_r^\alpha(u',\cdot)\right\|_{L^2(\mathbb{R}_+)} \lesssim (1-r)^{-(1+\alpha)/2}|u-u'|^{\alpha/2-j},$$

holds uniformly in $r \in (0, 1)$ and $u, u' > 0$.

Proof Fix $j \in \mathbb{N}$. By [39, Lemma 2.2] we have for $\alpha \in (2j, 2j+2)$ the bound

$$\left|(\mathcal{L}_k^\alpha)^{(j)}(u) - (\mathcal{L}_k^\alpha)^{(j)}(u')\right| \lesssim (k+1)^{\alpha/2}|u-u'|^{\alpha/2-j}, \qquad u, u' > 0,\ k \in \mathbb{N}.$$

Hence, Parseval's identity implies

$$\begin{aligned}\left\|\partial_u^j R_r^\alpha(u,\cdot) - \partial_u^j R_r^\alpha(u',\cdot)\right\|_{L^2(\mathbb{R}_+)} &\le \Big(\sum_{k\in\mathbb{N}} r^{2k}(k+1)^\alpha\Big)^{1/2}|u-u'|^{\alpha/2-j}\\ &\lesssim (1-r)^{-(1+\alpha)/2}|u-u'|^{\alpha/2-j},\end{aligned}$$

uniformly in $u, u' \in \mathbb{R}_+$. □

Now we easily obtain the following proposition.

Proposition 3.3 *If $k \in \mathbb{N}$ and $\alpha \in \big(\{0, 2, \ldots, 2k\} \cup (2k, \infty)\big)^d$, then*

$$\left\|R_r^\alpha(x,\cdot) - \sum_{|n|\le k}\frac{\partial_x^n R_r^\alpha(x',\cdot)}{n_1!\cdot\ldots\cdot n_d!}\prod_{i=1}^d (x_i - x_i')^{n_i}\right\|_{L^2(\mathbb{R}_+^d)} \lesssim \sum_{\delta\in\Delta_k^\alpha}(1-r)^{-\frac{d+2k+2\delta}{2}}|x-x'|^{k+\delta},$$

uniformly in $r \in (0, 1)$ and $x, x' \in \mathbb{R}_+^d$, where

$$\Delta_k^\alpha = \{1\} \cup \{\alpha_i/2 - k :\ \alpha_i \in (2k, 2k+2),\ i = 1, \ldots, d\}.$$

Proof Fix $\alpha \in \big(\{0, 2, \ldots, 2k\} \cup (2k, \infty)\big)^d$. If for all $i = 1, \ldots, d$ there is $\alpha_i \notin (2k, 2k+2)$, then apply Taylor's theorem with the reminder of $(k+1)$-th order, and Lemma 3.1 with $j \le k+1$. On the other hand, if some $\alpha_i \in (2k, 2k+2)$, then proceed as before but with k-th order reminder, obtaining

$$\sum_{|n|=k}\frac{k!}{n_1!\cdot\ldots\cdot n_d!}\prod_{i=1}^d\Big(\big(\partial_{x_i}^{n_i}R_r^{\alpha_i}(\xi_i, y_i) - \partial_{x_i}^{n_i}R_r^{\alpha_i}(x_i', y_i)\big)(x_i - x_i')^{n_i}\Big),$$

where for every $i \in \{1, \ldots, d\}$ the number ξ_i lies between x_i and x_i'. Now for each difference above we apply Lemma 3.2 if $\alpha_i \in (2n_i, 2n_i+2)$, or the mean value theorem and Lemma 3.1 in the opposite situation. □

Although the following lemma will be applied strictly to prove sharpness of Hardy's inequality associated with the standard Laguerre expansions, we stress that this is an interesting result and possibly it could be widely used in other problems concerning the functions $\mathcal{L}_k^\alpha$.

Here and later on we use the following convention: $A \simeq -B$ for positive B means that A is negative and $(-A) \simeq B$.

Lemma 3.4 *Let $\alpha \geq 0$ and $j, \ell \in \mathbb{N}$ be given. There exists a constant $c > 0$ such that*

$$\frac{d^j}{du^j}\frac{\mathcal{L}_k^\alpha(u)}{u^{\alpha/2-\ell}} \simeq \begin{cases} (k+1)^{\alpha/2}u^{\ell-j}, & \textit{if } \ \ell \geq j, \\ (-1)^{j-\ell}(k+1)^{\alpha/2+j-\ell}, & \textit{if } \ \ell \leq j, \end{cases}$$

uniformly in $k \in \mathbb{N}$ and $u \in (0, c(k+1)^{-1})$.

Proof Fix $\ell \in \mathbb{N}$. We will apply the induction over j separately in both cases. Note that the claim holds for $j = 0$ (this is a known result, see [26, pp. 435, 453]). Suppose that it is valid for some $j \leq \ell$ and we will justify it for $j + 1$. Observe that by (3.3) we have

$$\frac{d^{j+1}}{du^{j+1}}\frac{\mathcal{L}_k^\alpha(u)}{u^{\alpha/2-\ell}} = \frac{d^j}{du^j}\ell\frac{\mathcal{L}_k^\alpha(u)}{u^{\alpha/2-\ell+1}} - \frac{1}{2}\frac{d^j}{du^j}\frac{\mathcal{L}_k^\alpha(u)}{u^{\alpha/2-\ell}} - \sqrt{k}\frac{d^j}{du^j}\frac{\mathcal{L}_{k-1}^{\alpha+1}(u)}{u^{(\alpha+1)/2-\ell}}. \tag{3.5}$$

Notice that if $\ell \geq j + 1$, then the components on the right hand side of (3.5) are of the sizes: $(k+1)^{\alpha/2}u^{\ell-j-1}$, $(k+1)^{\alpha/2}u^{\ell-j}$, and $(k+1)^{\alpha/2+1}u^{\ell-j}$, respectively, and the first one is the dominating.

It remains to justify the situation $j \geq \ell$. Note that the case $j = \ell$ is covered by the first part of the proof. Let us assume that for some $j \geq \ell$ the estimate holds. Then the second and the third summand on the right hand side of (3.5) are of the sizes (and the signs): $(-1)^{j-\ell+1}(k+1)^{\alpha/2+j-\ell}$ and $(-1)^{j-\ell+1}(k+1)^{\alpha/2+1+j-\ell}$, respectively. On the other hand, the first component we decompose and get

$$\frac{d^j}{du^j}\ell\frac{\mathcal{L}_k^\alpha(u)}{u^{\alpha/2-\ell+1}} = \ell\frac{d^{j-1}}{du^{j-1}}\Big((\ell-1)\frac{\mathcal{L}_k^\alpha(u)}{u^{\alpha/2-\ell+2}} - \frac{1}{2}\frac{\mathcal{L}_k^\alpha(u)}{u^{\alpha/2-\ell}} - \sqrt{k}\frac{\mathcal{L}_{k-1}^{\alpha+1}(u)}{u^{(\alpha+1)/2-\ell}}\Big).$$

Again, the first summand can be decomposed, and the two remaining are of the same size (and sign) as before. Moreover, note that the i-th decomposition of the first resulting component brings the multiplicative constant $\ell - i + 1$. But this proves that the component vanishes, since $j \geq \ell$. Hence, in this case (3.5) is of the size and the sign $(-1)^{j-\ell+1}(k+1)^{\alpha/2+1+j-\ell}$. This finishes the proof of the lemma. □

We are now ready to prove Hardy's inequality associated with the standard Laguerre functions.

Theorem 3.5 *Let $p \in (0, 1)$, $s \in [p, 2]$, and denote $P := \lfloor d(p^{-1} - 1)\rfloor$. For*

$$\alpha \in \big(\{0, 2, \dots, 2P\} \cup (2d(p^{-1} - 1), \infty)\big)^d$$

there holds

$$\sum_{n\in\mathbb{N}^d} \frac{|\langle f, \mathcal{L}_n^\alpha\rangle|^s}{(|n|+1)^E} \lesssim \|f\|^s_{H^p(\mathbb{R}^d_+)}, \qquad f \in H^p(\mathbb{R}^d_+),$$

where $E = d + sd\big(p^{-1}-1\big)$*, and the exponent is sharp.*

Proof Proposition 3.3 ensures that the appropriate version of (C) holds for the standard Laguerre functions with $\gamma = 1/2$, and hence by Theorem 2.4 we obtain the associated Hardy's inequality.

On the other hand, by Lemma 3.4 (with $j = \ell = 0$) we have

$$m((k+1)u)^{\alpha/2} \le \mathcal{L}_k^\alpha(u) \le M((k+1)u)^{\alpha/2}, \qquad 0 < u \le \frac{c}{k+1}, \tag{3.6}$$

where $m, M, c > 0$. Observe that, since $\gamma = 1/2$, condition (2.9) holds for $\{\mathcal{L}_k^\alpha\}_{k\in\mathbb{N}}$ with $\tau = \alpha/2$. Hence, by Proposition 2.5 sharpness of the exponent E in the one-dimensional case follows for $\alpha > 2(p^{-1}-1)$. Moreover, if α is an even integer smaller that $2(p^{-1}-1)$, then we apply Proposition 2.6 with $\tau = P+1$ and Lemma 3.4 (with $\ell = \alpha/2$ and $j = P+1$).

This reasoning can be adapted to the multi-dimensional situation, see Remark 2.7.

□

4 Laguerre Functions of Hermite Type

The *Laguerre functions of Hermite type* φ_k^α, $k \in \mathbb{N}$, are defined by the following relation with the standard Laguerre functions

$$\varphi_k^\alpha(u) = \sqrt{2u}\,\mathcal{L}_k^\alpha(u^2) = \left(\frac{2\Gamma(k+1)}{\Gamma(k+\alpha+1)}\right)^{1/2} L_k^\alpha(u^2)e^{-u^2/2}u^{\alpha+1/2}, \tag{4.1}$$

where $u > 0$ and $\alpha > -1$. In the multi-dimensional situation $\varphi_n^\alpha(x)$ are defined as the tensor products of $\varphi_{n_i}^{\alpha_i}(x_i)$. The system $\{\varphi_n^\alpha\}_{n\in\mathbb{N}^d}$ is an orthonormal basis in $L^2(\mathbb{R}^d_+)$.

The functions φ_k are bounded on $\mathbb{R}_+$ for $\alpha \ge -1/2$. Moreover, by (4.1) and (3.2) we have

$$\|\varphi_k^\alpha\|_{L^\infty(\mathbb{R}_+)} \lesssim (k+1)^{-1/12}, \qquad k \in \mathbb{N}. \tag{4.2}$$

The following recurrence formula for the derivatives of φ_k^α holds (see [41, p. 100])

$$(\varphi_k^\alpha)'(u) = -2\sqrt{k}\varphi_{k-1}^{\alpha+1}(u) + \left(\frac{2\alpha+1}{2u} - u\right)\varphi_k^\alpha(u), \tag{4.3}$$

where $\varphi_{-1}^{\alpha+1} \equiv 0$. Hence, for $\alpha \in \{-1/2\} \cup [1/2, \infty)$, by using (4.1) and (3.2) one obtains

$$\big\|(\varphi_k^\alpha)'\big\|_{L^\infty(\mathbb{R}_+)} \lesssim (k+1)^{5/12}, \qquad k \in \mathbb{N}.$$

We discuss the boundedness of higher order derivatives in Lemma 4.4.

4.1 Lipschitz and *BMO* Properties

Obviously, $\varphi_n^\alpha \in BMO(\mathbb{R}_+^d)$ for $\alpha \in [-1/2, \infty)^d$, since φ_n^α are bounded. In order to justify that $\varphi_n^\alpha \in \Lambda_\nu(\mathbb{R}_+^d)$ for $\nu > 0$ and certain α's, we shall consider the one-dimensional situation, and then apply Lemma 2.2. To prove that $\varphi_k^\alpha \in \Lambda_\nu(\mathbb{R}_+)$ we will construct an extension $\tilde{\varphi}_k^\alpha$ of φ_k^α to $\mathbb{R}$, such that $\tilde{\varphi}_k^\alpha \in \Lambda_\nu(\mathbb{R})$.

For $\alpha + 1/2 \notin \mathbb{N}$ we simply put

$$\tilde{\varphi}_k^\alpha(u) = \begin{cases} \varphi_k^\alpha(u), & u > 0, \\ 0, & u \leq 0. \end{cases}$$

Mind that $\tilde{\varphi}_k^\alpha \in \mathcal{C}^{\lfloor \alpha+1/2 \rfloor}(\mathbb{R})$. On the other hand, if $\alpha + 1/2$ is an integer, then we can naturally extend the definition of φ_k (4.1) to the whole $\mathbb{R}$, and put

$$\tilde{\varphi}_k^\alpha(u) = \varphi_k^\alpha(u), \qquad u \in \mathbb{R}.$$

In this case $\tilde{\varphi}_k^\alpha \in \mathcal{C}^\infty(\mathbb{R})$.

Our first aim in this section is to prove the following lemma.

Lemma 4.1 *Let $\alpha \geq -1/2$. If $\alpha + 1/2 \notin \mathbb{N}$, then $\tilde{\varphi}_k^\alpha \in \Lambda_\nu(\mathbb{R})$ for $0 \leq \nu \leq \alpha + 1/2$, whereas if $\alpha + 1/2 \in \mathbb{N}$, then $\tilde{\varphi}_k^\alpha \in \Lambda_\nu(\mathbb{R})$ for all $\nu \geq 0$.*

Notice that for $\alpha \in \{-1/2\} \cup [1/2, \infty)$ the functions $(\varphi_k^\alpha)'$ exist and are bounded, and observe that Lemmas 4.1 and 2.2 yield that for a given $p \in (0, 1]$ and

$$\alpha \in \Big(\Big\{ -\frac{1}{2}, \frac{1}{2}, \dots, P - \frac{1}{2}\Big\} \cup \Big[d\big(\frac{1}{p} - 1\big) - \frac{1}{2}, \infty\Big)\Big)^d,$$

where $P = \lfloor d(p^{-1} - 1) \rfloor$, we have $\varphi_n^\alpha \in \Lambda_{d(\frac{1}{p}-1)}(\mathbb{R}_+^d)$.

For the proof of Lemma 4.1 we need some auxiliary results.

Lemma 4.2 *Let $\alpha \geq -1/2$ and $j \in \mathbb{N}$. Then, for any $c \in (0, 1]$, we have*

$$\big|(\varphi_k^\alpha)^{(j)}(u)\big| \lesssim \begin{cases} u^{\alpha+1/2-j}(k+1)^{\alpha/2}, & u \in \big(0, c(k+1)^{-1/2}\big), \\ (k+1)^{j/2-1/4}, & u \in \big(c(k+1)^{-1/2}, 1\big), \end{cases}$$

uniformly in u and $k \in \mathbb{N}$.

Proof Fix $c \in (0, 1]$. We apply the induction over j. For $j = 0$ the estimates are known (see [30, (1)], and for the original result [1, p. 699] and [26, p. 435]). We assume that the claim holds for $j \in \mathbb{N}$ and prove it for $j + 1$. By (4.3) we have

$$(\varphi_k^\alpha)^{(j+1)}(u) = \frac{d^j}{du^j}\Big(-2\sqrt{k}\varphi_{k-1}^{\alpha+1}(u) + \Big(\frac{2\alpha+1}{2u} - u\Big)\varphi_k^\alpha(u)\Big).$$

Thus, $\left|(\varphi_k^\alpha)^{(j+1)}(u)\right|$ can be estimated from above by a constant multiple of

$$\sqrt{k}\left|(\varphi_{k-1}^{\alpha+1})^{(j)}(u)\right| + \left|(\varphi_k^\alpha)^{(j-1)}(u)\right| + u\left|(\varphi_k^\alpha)^{(j)}(u)\right| + \sum_{\ell=0}^{j} u^{-\ell-1}\left|(\varphi_k^\alpha)^{(j-\ell)}(u)\right|,$$

where we set $(\varphi_k^\alpha)^{(-1)} \equiv 0$. Finally, by the inductive hypothesis we obtain

$$\left|(\varphi_k^\alpha)^{(j+1)}(u)\right| \lesssim u^{\alpha+1/2-j}(k+1)^{\alpha/2}\Big(u(k+1)+u^{-1}+u\Big) \lesssim u^{\alpha-1/2-j}(k+1)^{\alpha/2},$$

uniformly in $u \in (0, c(k+1)^{-1/2})$, and

$$\left|(\varphi_k^\alpha)^{(j+1)}(u)\right| \lesssim (k+1)^{(j+1)/2-1/4},$$

uniformly in $u \in (c(k+1)^{-1/2}, 1)$. This finishes the proof. □

The following result is an analogue of Lemma 3.4.

Lemma 4.3 *Let $\alpha \geq -1/2$ and $j, \ell \in \mathbb{N}$ be given. There exists a small constant $c > 0$ such that there holds*

$$\frac{d^j}{du^j}\frac{\varphi_k^\alpha(u)}{u^{\alpha+1/2-\ell}} \simeq \begin{cases} (k+1)^{\alpha/2}u^{\ell-j}, & \text{if } \ell \geq j, \\ (-1)^{\lceil\frac{j-\ell}{2}\rceil}(k+1)^{\alpha/2+\lceil\frac{j-\ell}{2}\rceil}u^{\frac{1-(-1)^{j-\ell}}{2}}, & \text{if } \ell \leq j, \end{cases}$$

uniformly in $k \in \mathbb{N}$ and $u \in (0, c(k+1)^{-1/2})$.

Proof The proof is similar to that of Lemma 3.4, therefore we will only sketch it. Fix $\ell \in \mathbb{N}$. If $j = 0$, then the estimate is well known (cf. (3.6)). For $j \geq 1$ we use the induction over j. By (4.3) we have

$$\frac{d^{j+1}}{du^{j+1}}\frac{\varphi_k^\alpha(u)}{u^{\alpha+1/2-\ell}} = \ell\frac{d^j}{du^j}\frac{\varphi_k^\alpha(u)}{u^{\alpha+1/2-\ell+1}} - \frac{d^j}{du^j}\frac{\varphi_k^\alpha(u)}{u^{\alpha-\ell-1/2}} - 2\sqrt{k}\frac{d^j}{du^j}\frac{\varphi_{k-1}^{\alpha+1}(u)}{u^{(\alpha+1)+1/2-\ell-1}}. \tag{4.4}$$

Note that if $j+1 \leq \ell$, then the first component on the right hand side of the above identity is of the greatest size, $(k+1)^{\alpha/2}u^{\ell-j-1}$, and the others are strictly smaller. This proves the first bound in the lemma.

On the other hand, if $j \geq \ell$, then we consider the $(j+1)$-th derivative as in (4.4). Notice that the second summand on the right hand side of (4.4) is of the size (and the sign)

$$(-1)^{\left\lceil\frac{j-\ell-1}{2}\right\rceil+1}(k+1)^{\alpha/2+\left\lceil\frac{j-\ell-1}{2}\right\rceil}u^{\frac{1-(-1)^{j-\ell-1}}{2}},$$

and the third

$$(-1)^{\left\lceil\frac{j-\ell-1}{2}\right\rceil+1}(k+1)^{\alpha/2+1+\left\lceil\frac{j-\ell-1}{2}\right\rceil}u^{\frac{1-(-1)^{j-\ell-1}}{2}}.$$

We see that the latter is the leading one. Moreover, by the simple identity $\lceil \frac{i-1}{2} \rceil + 1 = \lceil \frac{i+1}{2} \rceil$, $i \in \mathbb{N}$, it can be written in the following form:

$$(-1)^{\left\lceil \frac{j+1-\ell}{2} \right\rceil} (k+1)^{\alpha/2 + \left\lceil \frac{j+1-\ell}{2} \right\rceil} u^{\frac{1-(-1)^{j+1-\ell}}{2}}.$$

Furthermore, the first component in (4.4) can be decomposed similarly as in the proof of Lemma 4.3, and it gives the same growth and the size as the remaining summands.

This finishes the proof of the lemma. □

Lemma 4.4 *Let $j \in \mathbb{N}$. For $\alpha \geq -1/2$ there holds*

$$\left\| (\varphi_k^\alpha)^{(j)} \right\|_{L^\infty(1/2,\infty)} \lesssim (k+1)^{(6j-1)/12}, \qquad k \in \mathbb{N}, \tag{4.5}$$

whereas for $\alpha \in \{-1/2, 1/2, \ldots, j-1/2\} \cup (j-1/2, \infty)$ there is

$$\left\| (\varphi_k^\alpha)^{(j)} \right\|_{L^\infty(\mathbb{R}_+)} \lesssim (k+1)^{(6j-1)/12}, \qquad k \in \mathbb{N}. \tag{4.6}$$

Proof In order to prove (4.5) we justify an auxiliary result: for every $\ell \in \mathbb{N}$ there is

$$\sup_{u \geq 1/2} \left| u^\ell (\varphi_k^\alpha)^{(j)}(u) \right| \lesssim (k+1)^{(6(j+\ell)-1)/12}.$$

We use the induction over j. For $j = 0$ we simply apply (4.1) and (3.2). Now assume that the claim holds for some $j \in \mathbb{N}$. Observe that by (4.3) we have for any $\ell \in \mathbb{N}$

$$\begin{aligned} \left| u^\ell (\varphi_k^\alpha)^{(j+1)}(u) \right| &= u^\ell \left| \frac{d^j}{du^j} \Big(-2\sqrt{k} \varphi_{k-1}^{\alpha+1}(u) + \Big(\frac{2\alpha+1}{2u} - u \Big) \varphi_k^\alpha(u) \Big) \right| \\ &\lesssim (k+1)^{(6(j+\ell)+5)/12} + \sum_{i=0}^{j} u^{\ell-1-j+i} \left| (\varphi_k^\alpha)^{(i)}(u) \right| + u^{\ell+1} \left| (\varphi_k^\alpha)^{(j)}(u) \right| \\ &\lesssim (k+1)^{(6(j+\ell)+5)/12}, \end{aligned}$$

uniformly in $k \in \mathbb{N}$ and $u \geq 1/2$. This proves the auxiliary claim. Notice that for $\ell = 0$ we obtain (4.5).

To justify (4.6), it suffices to verify that for the considered α the required bound holds on the interval $(0, 1/2)$. In fact, this is true even with the smaller exponent $(2j-1)/4$. Indeed, if $\alpha \geq j - 1/2$, then we invoke Lemma 4.2, whereas in the case $j > \alpha + 1/2 \in \mathbb{N}$ we additionally apply Lemma 4.3 with $\ell = \alpha + 1/2$. This finishes the proof of the lemma. □

Lemma 4.5 *For $j \in \mathbb{N}$ and $\alpha \in (j-1/2, j+1/2]$ there holds*

$$\left| (\varphi_k^\alpha)^{(j)}(u) - (\varphi_k^\alpha)^{(j)}(u') \right| \lesssim (k+1)^{(2j+1)/4} |u-u'| + (k+1)^{\alpha/2} |u-u'|^{\alpha+1/2-j},$$

uniformly in $k \in \mathbb{N}$ and $u, u' \in (0, 1)$.

Proof Fix $1 > u > u' > 0$. Observe that (4.3) and Lemma 4.2 permit to estimate

$$\begin{aligned}
\big|(\varphi_k^\alpha)^{(j)}(u) - (\varphi_k^\alpha)^{(j)}(u')\big| &= \Big| \int_{u'}^{u} \frac{d^j}{ds^j}\Big(-2\sqrt{k}\varphi_{k-1}^{\alpha+1}(s) + \Big(\frac{2\alpha+1}{2s} - s\Big)\varphi_k^\alpha(s)\Big)\, ds \Big| \\
&\lesssim \int_{u'}^{u} \Big(\sqrt{k}\big|(\varphi_{k-1}^{\alpha+1})^{(j)}(s)\big| + \sum_{\ell=0}^{j} s^{-\ell-1}\big|(\varphi_k^\alpha)^{(j-\ell)}(s)\big| \\
&\quad + \big|(\varphi_k^\alpha)^{(j-1)}(s)\big| + s\big|(\varphi_k^\alpha)^{(j)}(s)\big| \Big)\, ds \\
&\lesssim |u-u'|(k+1)^{(2j+1)/4} + \sum_{\ell=0}^{j} \int_{u'}^{u} s^{-\ell-1}\big|(\varphi_k^\alpha)^{(j-\ell)}(s)\big|\, ds,
\end{aligned}$$

where we set $(\varphi_k^\alpha)^{(-1)} \equiv 0$. Now notice that Lemma 4.2 implies

$$\begin{aligned}
&\int_{u'}^{u} s^{-\ell-1}\big|(\varphi_k^\alpha)^{(j-\ell)}(s)\big|\, ds \\
&= \int_{[u',u]\cap[(k+1)^{-1/2},1)} s^{-\ell-1}\big|(\varphi_k^\alpha)^{(j-\ell)}(s)\big|\, ds \\
&\quad + \int_{[u',u]\cap(0,(k+1)^{-1/2})} s^{-\ell-1}\big|(\varphi_k^\alpha)^{(j-\ell)}(s)\big|\, ds \\
&\lesssim |u-u'|(k+1)^{(2j+1)/4} + (k+1)^{\alpha/2}\int_{u'}^{u} s^{\alpha-1/2-j}\, ds.
\end{aligned}$$

Finally, since $\alpha \in (j-1/2, j+1/2]$, we see that

$$\int_{u'}^{u} s^{\alpha-1/2-j}\, ds \lesssim |u-u'|^{\alpha+1/2-j}.$$

Combining the above gives the claim. □

Proof of Lemma 4.1 We verify that the functions $\tilde{\varphi}_k^\alpha$ satisfy the condition in definition of $\Lambda_\nu(\mathbb{R})$. If $\alpha + 1/2$ is an integer then the claim follows from (4.6). On the other hand, if $\alpha + 1/2 \notin \mathbb{N}$, then we apply (4.6), (4.5), and Lemma 4.5. □

4.2 Hardy's Inequality

The kernels of the operators R_r^α (cf. (2.3)) associated with the Laguerre functions of Hermite type, are defined by

$$R_r^\alpha(x,y) = \sum_{n\in\mathbb{N}^d} r^{|n|}\varphi_n^\alpha(x)\varphi_n^\alpha(y), \tag{4.7}$$

and, in the one-dimensional case, admit the explicit form (cf. [42])

$$R_r^\alpha(u,v) = \frac{2(uv)^{1/2}}{(1-r)r^{\alpha/2}} \exp\left(-\frac{1}{2}\frac{1+r}{1-r}(u^2+v^2)\right) I_\alpha\left(\frac{2r^{1/2}}{1-r}uv\right). \tag{4.8}$$

Unfortunately, it is highly complicated to proceed as in [30] while estimating derivatives of R_r^α of order higher than 2. The cancellations between the underlying Bessel functions are not well understood yet. Therefore, we choose an approach similar to the one applied in the case of the Jacobi expansions [32]. This method relies on the following formula

$$I_\alpha(z) = z^\alpha \int_{-1}^{1} e^{-zs}\,\Pi_\alpha(ds), \qquad |\arg z| < \pi,\ \alpha \geq -1/2, \tag{4.9}$$

where Π_α in the case $\alpha > -1/2$ is a measure with the density given by

$$\Pi_\alpha(ds) = \frac{(1-s^2)^{\alpha-1/2}ds}{\sqrt{\pi}\,\Gamma(\alpha+1/2)},$$

whereas for $\alpha = -1/2$ it is an atomic measure of the form $\Pi_{-1/2} = \frac{\delta_{-1}+\delta_1}{\sqrt{2\pi}}$.

Hence, by (4.8) we have for $\alpha > -1/2$

$$R_r^\alpha(u,v) = \frac{2^{\alpha+1}(uv)^{\alpha+1/2}}{(1-r)^{\alpha+1}} E_r^\alpha(u,v),$$

where by $E_r^\alpha(u,v)$ we denote

$$\exp\Big(-\frac{1}{2}\frac{1+r}{1-r}(v-u)^2 - \frac{1-r}{(1+\sqrt{r})^2}uv\Big)\int_{-1}^{1}\exp\Big(-\frac{2\sqrt{r}}{1-r}uv(s+1)\Big)\frac{(1-s^2)^{\alpha-1/2}ds}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}. \tag{4.10}$$

Note that if $\alpha = -1/2$, then

$$R_r^{-1/2}(u,v) = \frac{2}{\sqrt{\pi}\sqrt{1-r}}\exp\Big(-\frac{1}{2}\frac{1+r}{1-r}(u^2+v^2)\Big)\cosh\Big(\frac{2\sqrt{r}uv}{1-r}\Big). \tag{4.11}$$

Now we have the following proposition.

Proposition 4.6 *For $j \in \mathbb{N}$ and $\alpha \in \{-1/2, 1/2, \dots, j-1/2\} \cup (j-1/2, \infty)$ there holds*

$$\sup_{u>0}\Big\|\partial_u^j R_r^\alpha(u,\cdot)\Big\|_{L^2(\mathbb{R}_+)} \lesssim (1-r)^{-\frac{1+2j}{4}}, \qquad r \in (0,1).$$

Proof Observe that Parseval's identity and (4.6) yield

$$\sup_{u>0}\Big\|\partial_u^j R_r^\alpha(u,\cdot)\Big\|_{L^2(\mathbb{R}_+)} \leq \Big(\sum_{k=0}^{\infty} 2^{-2k}\|(\varphi_n^\alpha)^{(j)}\|^2_{L^\infty(\mathbb{R}_+)}\Big)^{1/2} \lesssim 1,$$

uniformly in $r \in (0, 1/2]$. Hence, we can focus only on the case $r \in (1/2, 1)$.

We shall firstly consider the situation when $\alpha \geq 1/2$ and $j \in \mathbb{N}_+$ (for $j = 0$ see [30, Lemma 3.1]). Note that for $\ell \in \mathbb{N}$ such that $\ell \leq j$ we can write $\partial_u^\ell E_r^\alpha(u, v)$, where $E_r^\alpha(u, v)$ is defined in (4.10), as

$$\int_{-1}^{1} \exp\Big(-\frac{1}{2}\frac{1+r}{1-r}(v-u)^2 - \frac{1-r}{(1+\sqrt{r})^2}uv - \frac{2\sqrt{r}}{1-r}uv(s+1)\Big) \\ \times \sum_{\substack{k,i\geq 0\\ k+2i=\ell}} c_{k,i}^{\ell}(1-r)^{-k}\Big((1+r)(u-v) + \frac{(1-r)^2}{(1+\sqrt{r})^2}v + 2\sqrt{r}v(s+1)\Big)^k \Big(\frac{1+r}{1-r}\Big)^i \Pi_\alpha(ds),$$

where $c_{k,i}^{l}$ are certain constants (cf. [29, p. 812]). Consequently,

$$\Big|\partial_u^\ell E_r^\alpha(u, v)\Big| \\ \lesssim (1-r)^{-\ell/2} \exp\Big(-\frac{1}{2}\frac{1+r}{1-r}(v-u)^2\Big)\Big(\frac{|u-v|}{\sqrt{1-r}} + \min\Big(\frac{v}{\sqrt{1-r}}, \frac{\sqrt{1-r}}{u}\Big)\Big)^\ell \\ \times \int_{-1}^{1} \exp\Big(-\frac{\sqrt{r}}{1-r}uv(s+1)\Big)(1-s^2)^{\alpha-1/2}\, ds \\ \lesssim (1-r)^{-\ell/2} \exp\Big(-\frac{1}{2}\frac{(v-u)^2}{1-r}\Big) \int_{-1}^{1} \exp\Big(-\frac{\sqrt{r}}{1-r}uv(s+1)\Big)(1+s)^{\alpha-1/2}\, ds,$$

uniformly in $u, v > 0$ and $r \in (1/2, 1)$, where in the last inequality we used the simple estimate

$$\min(a+b, b^{-1}) \leq a+1, \qquad a, b > 0. \tag{4.12}$$

The latter integral is bounded by a constant. On the other hand, again uniformly in $u, v > 0$ and $r \in (1/2, 1)$,

$$\int_{-1}^{1} \exp\Big(-\frac{\sqrt{r}}{1-r}uv(s+1)\Big)(1+s)^{\alpha-1/2}\, ds \lesssim \Big(\frac{1-r}{uv}\Big)^{\alpha-1/2} \int_0^\infty \exp\Big(-\frac{\sqrt{r}}{1-r}uvs\Big)\, ds \\ \simeq \Big(\frac{1-r}{uv}\Big)^{\alpha+1/2}.$$

Now we are ready to establish the bound for $\partial_u^j R_r^\alpha(u, v)$. Combining the above we obtain

$$\big|\partial_u^j R_r^\alpha(u, v)\big| \leq \frac{2^{\alpha+1}v^{\alpha+1/2}}{(1-r)^{\alpha+1}} \sum_\ell \binom{j}{\ell} \big|\partial_u^\ell E_r^\alpha(u, v)\big| \big|\partial_u^{j-\ell} u^{\alpha+1/2}\big| \\ \lesssim (1-r)^{-(j+1)/2}\Big(\frac{uv}{1-r}\Big)^{\alpha+1/2} \min\Big(1, \Big(\frac{1-r}{uv}\Big)^{\alpha+1/2}\Big) \exp\Big(-\frac{1}{2}\frac{(v-u)^2}{1-r}\Big) \\ \times \sum_\ell \Big(\frac{\sqrt{1-r}}{u}\Big)^{j-\ell} \\ \lesssim (1-r)^{-(j+1)/2} \min\Big(\frac{uv}{1-r}, 1\Big)^{\alpha+1/2} \exp\Big(-\frac{1}{2}\frac{(v-u)^2}{1-r}\Big) \max_\ell \Big(\frac{\sqrt{1-r}}{u}\Big)^{j-\ell},$$

where $\ell \in \{0, \ldots, j\}$ if $j \leq \alpha + 1/2$, and $\ell \in \{j - (\alpha + 1/2), \ldots, j\}$ if $\alpha + 1/2$ is an integer and $j > \alpha + 1/2$. Observe that if $u \geq \sqrt{1-r}$, then

$$\left|\partial_u^j R_r^\alpha(u, v)\right| \lesssim (1-r)^{-(j+1)/2} \exp\Big(-\frac{1}{2}\frac{(v-u)^2}{1-r}\Big).$$

In the other case, $u \leq \sqrt{1-r}$ we estimate firstly assuming that $j \leq \alpha + 1/2$

$$\begin{aligned}\left|\partial_u^j R_r^\alpha(u, v)\right| &\lesssim (1-r)^{-(j+1)/2}\Big(\frac{uv}{1-r}\Big)^j \exp\Big(-\frac{1}{2}\frac{(v-u)^2}{1-r}\Big)\Big(\frac{\sqrt{1-r}}{u}\Big)^j \\ &\lesssim (1-r)^{-(j+1)/2}\Big(\frac{|v-u|+u}{\sqrt{1-r}}\Big)^j \exp\Big(-\frac{1}{2}\frac{(v-u)^2}{1-r}\Big) \\ &\lesssim (1-r)^{-(j+1)/2} \exp\Big(-\frac{1}{4}\frac{(v-u)^2}{1-r}\Big).\end{aligned}$$

If $\alpha + 1/2 \in \mathbb{N}_+$ is smaller than j, then analogously

$$\begin{aligned}\left|\partial_u^j R_r^\alpha(u, v)\right| &\lesssim (1-r)^{-(j+1)/2}\Big(\frac{uv}{1-r}\Big)^{\alpha+1/2} \exp\Big(-\frac{1}{2}\frac{(v-u)^2}{1-r}\Big)\Big(\frac{\sqrt{1-r}}{u}\Big)^{\alpha+1/2} \\ &\lesssim (1-r)^{-(j+1)/2}\Big(\frac{|v-u|+u}{\sqrt{1-r}}\Big)^{\alpha+1/2} \exp\Big(-\frac{1}{2}\frac{(v-u)^2}{1-r}\Big) \\ &\lesssim (1-r)^{-(j+1)/2} \exp\Big(-\frac{1}{4}\frac{(v-u)^2}{1-r}\Big).\end{aligned}$$

Combining the above we arrive at

$$\left|\partial_u^j R_r^\alpha(u, v)\right| \lesssim (1-r)^{-(j+1)/2} \exp\Big(-\frac{1}{4}\frac{(v-u)^2}{1-r}\Big), \qquad u, v > 0,\ r \in [1/2, 1). \tag{4.13}$$

Hence,

$$\begin{aligned}\sup_{u>0} \left\|\partial_u^j R_r^\alpha(u, \cdot)\right\|_{L^2(\mathbb{R}_+)} &\lesssim (1-r)^{-(j+1)/2} \sup_{u>0}\Big(\int_{\mathbb{R}_+} \exp\Big(-\frac{1}{2}\frac{(v-u)^2}{1-r}\Big)\,dv\Big)^{1/2} \\ &\simeq (1-r)^{-(2j+1)/4},\end{aligned} \tag{4.14}$$

and this completes the proof of the proposition for $\alpha \geq 1/2$.

Now we move on to the case $\alpha < 1/2$. In fact, we need to consider only $\alpha = -1/2$ and $j \in \mathbb{N}$, since for $\alpha \in (-1/2, 1/2)$ only $j = 0$ is allowed, and this was already done in author's previous paper (see [30, Lemma 3.1]). By (4.11) we obtain, for some constants $c_{k,i}^j$ and $\tilde{c}_{k,i}^j$, the following equality

$$\begin{aligned}\partial_u^j R_r^{-1/2}(u,v) = \frac{1}{\sqrt{\pi}\sqrt{1-r}}\Big(&\exp\Big(-\frac{1}{2}\frac{1+r}{1-r}(v-u)^2 - \frac{1-r}{(1+\sqrt{r})^2}uv - \frac{4\sqrt{r}}{1-r}uv\Big)\\ &\times \sum c_{k,i}^j\Big(\frac{1+r}{1-r}(u-v)+\frac{1-r}{(1+\sqrt{r})^2}v+\frac{4\sqrt{r}}{1-r}v\Big)^k\Big(\frac{1+r}{1-r}\Big)^i\\ &+\exp\Big(-\frac{1}{2}\frac{1+r}{1-r}(v-u)^2-\frac{1-r}{(1+\sqrt{r})^2}uv\Big)\\ &\times\sum \tilde{c}_{k,i}^j\Big(\frac{1+r}{1-r}(u-v)+\frac{1-r}{(1+\sqrt{r})^2}v\Big)^k\Big(\frac{1+r}{1-r}\Big)^i\Big),\end{aligned}$$

where in both sums the summation goes over all $k, i \geq 0$ such that $k+2i = j$. Hence,

$$\begin{aligned}\big|\partial_u^j R_r^{-1/2}(u,v)\big| &\lesssim (1-r)^{-(j+1)/2}\Big(1+\min\big(\frac{v}{\sqrt{1-r}},\frac{\sqrt{1-r}}{u}\big)\Big)^j\exp\Big(-\frac{1}{2}\frac{(v-u)^2}{1-r}\Big)\\ &\lesssim (1-r)^{-(j+1)/2}\Big(\frac{|v-u|}{\sqrt{1-r}}+1\Big)^j\exp\Big(-\frac{1}{2}\frac{(v-u)^2}{1-r}\Big)\\ &\lesssim (1-r)^{-(j+1)/2}\exp\Big(-\frac{1}{4}\frac{(v-u)^2}{1-r}\Big),\end{aligned}$$

where in the second inequality we used (4.12).

The last step is the same as in (4.14). This conclude the proof of the proposition. □

Before we state Hardy's inequality associated with the Laguerre functions of Hermite type we prove some auxiliary results. The next one complements the estimate from Proposition 4.6. Essentially, it says that the mentioned bound holds also for $\alpha \in (j-3/2, j-1/2)$, $j \in \mathbb{N}_+$, but only away from the origin.

Lemma 4.7 *If $j \in \mathbb{N}$ and $\alpha \in (j-1/2, j+1/2)$, then*

$$\sup_{u\geq 1/2}\Big\|\partial_u^{j+1}R_r^\alpha(u,\cdot)\Big\|_{L^2(\mathbb{R}_+)} \lesssim (1-r)^{-\frac{3+2j}{4}}, \qquad r\in(0,1).$$

Proof It suffices to proceed as in the proof of Proposition 4.6 with some minor changes. For $r \in (0, 1/2]$ use (4.5) instead of (4.6). If $r \in (1/2, 1)$, then we arrive at

$$\begin{aligned}\big|\partial_u^{j+1}R_r^\alpha(u,v)\big| &\lesssim \sum_{\ell=0}^{j+1}\Big(\frac{\sqrt{1-r}}{u}\Big)^{j+1-\ell}(1-r)^{-(j+2)/2}\exp\Big(-\frac{1}{2}\frac{(v-u)^2}{1-r}\Big)\Big(1+\frac{\sqrt{1-r}}{u}\Big)^\ell\\ &\quad\times\min\Big(\frac{uv}{1-r},1\Big)^{\alpha+1/2}\\ &\lesssim (1-r)^{-(j+2)/2}\sum_{\ell=0}^{j+1}\Big(\frac{\sqrt{1-r}}{u}\Big)^{j+1-\ell}\Big(1+\frac{\sqrt{1-r}}{u}\Big)^\ell\exp\Big(-\frac{1}{2}\frac{(v-u)^2}{1-r}\Big)\\ &\lesssim (1-r)^{-(j+2)/2}\exp\Big(-\frac{1}{2}\frac{(v-u)^2}{1-r}\Big),\end{aligned}$$

since $\sqrt{1-r} \lesssim u$. Then we estimate like in (4.14). This finishes the proof of the lemma. □

Notice that for $j \in \mathbb{N}$ and $\alpha \in (j-1/2, j+1/2)$ Lemmas 4.7 and 4.5 yield

$$\begin{aligned} &\left\| \partial_u^j R_r^\alpha(u,\cdot) - \partial_u^j R_r^\alpha(u',\cdot) \right\|_{L^2(\mathbb{R}_+)} \\ &\lesssim (1-r)^{-\frac{2j+3}{4}} |u-u'| + (1-r)^{-\frac{\alpha+1}{2}} |u-u'|^{\alpha+1/2-j}, \end{aligned} \tag{4.15}$$

uniformly in $r \in (0,1)$ and $u, u' > 0$ such that $|u-u'| \le 1/2$. Indeed, if $u, u' \in (0,1)$ then we apply Lemma 4.5 and Parseval's identity. On the other hand, if $u, u' \ge 1/2$, invoke the mean value theorem and Lemma 4.7.

Proposition 4.8 *If $k \in \mathbb{N}$ and $\alpha \in \big(\{-1/2, 1/2, \ldots, k-1/2\} \cup (k-1/2, \infty)\big)^d$, then*

$$\Big\| R_r^\alpha(x,\cdot) - \sum_{|n| \le k} \frac{\partial_x^n R_r^\alpha(x',\cdot)}{n_1! \cdot \ldots \cdot n_d!} \prod_{i=1}^d (x_i - x_i')^{n_i} \Big\|_{L^2(\mathbb{R}_+^d)} \lesssim \sum_{\delta \in \Delta_k^\alpha} (1-r)^{-\frac{d+2k+2\delta}{4}} |x-x'|^{k+\delta},$$

uniformly in $r \in (0,1)$ and $x, x' \in \mathbb{R}_+^d$ such that $|x-x'| \le 1/2$, where

$$\Delta_k^\alpha = \{1\} \cup \{\alpha_i + 1/2 - k : \ \alpha_i \in (k-1/2, k+1/2), \ i = 1, \ldots, d\}. \tag{4.16}$$

Proof The proof is analogous to the one of Proposition 3.3, thus we only sketch it.

Observe that if $\alpha_i \notin (k-1/2, k+1/2)$ for all $i = 1, \ldots, d$, then the claim, with $\Delta_k^\alpha = \{1\}$, follows from Taylor's theorem and Proposition 4.6 applied for $j = k+1$. On the other hand, if $\alpha_i \in (k-1/2, k+1/2)$ for some i then we apply Taylor's theorem, Proposition 4.6, and (4.15). Then the set Δ_k^α is as in (4.16). We omit the details. □

Now we are ready to state Hardy's inequality associated with the system of Laguerre functions of Hermite type.

Theorem 4.9 *Let $p \in (0,1)$, $s \in [p, 2]$, and denote $P := \lfloor d(\frac{1}{p} - 1) \rfloor$. For*

$$\alpha \in \big(\{-1/2, 1/2, \ldots, P-1/2\} \cup (d(p^{-1}-1) - 1/2, \infty)\big)^d,$$

there holds

$$\sum_{n \in \mathbb{N}^d} \frac{|\langle f, \varphi_n^\alpha \rangle|^s}{(|n|+1)^E} \lesssim \|f\|_{H^p(\mathbb{R}_+^d)}^s, \qquad f \in H^p(\mathbb{R}_+^d),$$

where $E = d + \frac{ds}{4p}(2-3p)$, and the exponent is sharp.

Proof Similarly as in the proof of Theorem 3.5: the inequality follows from Theorem 2.4 and Proposition 4.8 (here $\gamma = 1/4$), whereas sharpness is a consequence of Propositions 2.5, 2.6, Lemma 4.3, and Remark 2.7. □

4.3 Heat Kernel Estimates

In this article we estimate the kernels $R_r(x, y)$ in various contexts. In the case of the standard Laguerre functions it was very easy and, as the reader shall see, the same is true for the Jacobi expansions. On the other hand, here the situation was more involved. In Proposition 4.6 we have obtained a result which can be interesting on its own, especially when expressed in terms of the associated heat kernel.

Recall that the heat semigroup $\{T_t^\alpha\}_{t\geq 0}$ is spectrally defined by

$$T_t^\alpha f = \sum_{n\in\mathbb{N}^d} e^{-t(4|n|+2|\alpha|+2d)}\langle f, \varphi_n^\alpha\rangle\varphi_n^\alpha, \qquad f \in L^2(\mathbb{R}_+^d).$$

It is known (cf. [28, p. 403]) that T_t are integral operators:

$$T_t^\alpha f(x) = \int_{\mathbb{R}_+^d} G_t^\alpha(x, y) f(y)\, dy, \qquad f \in L^2(\mathbb{R}_+^d),\ x \in \mathbb{R}_+^d,$$

where

$$G_t^\alpha(x, y) = \sum_{n\in\mathbb{N}^d} e^{-t(4|n|+2|\alpha|+2d)}\varphi_n^\alpha(x)\varphi_n^\alpha(y),$$

and explicitly (cf. [21, (4.17.6)])

$$G_t^\alpha(x, y) = (\sinh 2t)^{-d} \exp\Big(-\frac{1}{2}\coth(2t)(|x|^2 + |y|^2)\Big)\prod_{i=1}^d \sqrt{x_i y_i}\, I_{\alpha_i}\Big(\frac{x_i y_i}{\sinh 2t}\Big).$$

Observe that by the definition of G_t^α and (4.7) we have the following relation

$$G_t^\alpha(x, y) = e^{-2t(|\alpha|+d)} R^\alpha_{e^{-4t}}(x, y).$$

Hence, the results obtained for $R_r^\alpha(x, y)$ can be easily transferred to $G_t^\alpha(x, y)$. Therefore, by (4.13) we have the following one-dimensional estimate. By an obvious modification, this lemma can be generalized to $d \geq 1$.

Proposition 4.10 *If $j \in \mathbb{N}$ and $\alpha \in \{-1/2, 1/2, \dots, j - 1/2\} \cup (j - 1/2, \infty)$, then*

$$\big|\partial_u^j G_t^\alpha(u, v)\big| \lesssim \begin{cases} t^{-\frac{j+1}{2}} \exp\Big(-c\frac{(u-v)^2}{t}\Big), & t \leq 1, \\ e^{-2t(\alpha+1)} e^{-c(u-v)^2}, & t \geq 1, \end{cases}$$

uniformly in $u, v, t > 0$ and for some positive constant c. Moreover,

$$\sup_{u>0} \big\|\partial_u^j G_t^\alpha(u, \cdot)\big\|_{L^2(\mathbb{R}_+)} \lesssim \begin{cases} t^{-\frac{2j+1}{4}}, & t \leq 1, \\ e^{-2t(\alpha+1)}, & t \geq 1. \end{cases}$$

4.4 Generalized Hermite Functions

In this subsection we focus on the generalized Hermite function system. This basis was already studied in the context of Hardy's inequality on $H^1(\mathbb{R}^d)$ in [23, 31]. Due to its relation with the Laguerre expansions of Hermite type, we essentially deduce the desired results from that obtained above.

The *generalized Hermite functions* h_k^λ, $k \in \mathbb{N}$, of order $\lambda \geq 0$ on $\mathbb{R}$ are defined via

$$h_{2k}^\lambda(u) = (-1)^k 2^{-1/2} \varphi_k^{\lambda-1/2}(|u|),$$
$$h_{2k+1}^\lambda(u) = (-1)^k 2^{-1/2} \operatorname{sgn}(u) \varphi_k^{\lambda+1/2}(|u|), \quad u \in \mathbb{R},$$

where for $u = 0$ we naturally extend the definition of φ_k^α from (4.1). In higher dimensions these functions are defined as tensor products, similarly as in the previous sections. The system $\{h_n^\lambda\}_{n\in\mathbb{N}^d}$ forms an orthonormal basis in $L^2(\mathbb{R}^d)$. We remark that $\{h_n^{(0,\ldots,0)}\}_{n\in\mathbb{N}^d}$ is the Hermite function basis.

The generalized Hermite functions $\{h_k^\lambda\}_{k\in\mathbb{N}}$ are bounded (cf. (4.2)), and therefore they are in $BMO(\mathbb{R})$. Moreover, for $\lambda \in \{0, 2, \ldots\} \cup [p^{-1} - 1, \infty)$ they belong to the Lipschitz spaces $\Lambda_{\frac{1}{p}-1}(\mathbb{R})$, see [22, Proposition 1.2]. Hence, by Lemma 2.2 in the multi-dimensional situation we see that $h_n^\lambda \in \Lambda_{d(\frac{1}{p}-1)}(\mathbb{R}^d)$ for $\lambda \in \big(\{0, 2, \ldots\} \cup [d(p^{-1} - 1), \infty)\big)^d$ and $p \in (0, 1]$ (note that the additional assumption in the lemma is satisfied).

The family of kernels $R_r(u, v)$ associated with the generalized Hermite functions, in the case $d = 1$, is given by

$$\tilde{R}_r^\lambda(u, v) = \sum_{k\in\mathbb{N}} r^k h_k^\lambda(u) h_k^\lambda(v).$$

We use the symbol $\tilde{R}$ instead of R to distinguish this kernel from the one associated with the functions $\{\varphi_k^\alpha\}_{k\in\mathbb{N}}$, which will be of use in this subsection. Notice that

$$\tilde{R}_r^\lambda(u, v) = \frac{1}{2}\Big(R_{r^2}^{\lambda-1/2}(|u|, |v|) + \operatorname{sgn}(uv) r R_{r^2}^{\lambda+1/2}(|u|, |v|)\Big), \tag{4.17}$$

where $r \in (0, 1)$ and $u, v \in \mathbb{R}$. We naturally extended the definition of $R_r^{\lambda\pm1/2}$ for $u = 0$ and $v = 0$. Observe that if λ is an even integer, then $\tilde{R}_r^\lambda \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R})$. Moreover, given $j \in \mathbb{N}$ we see that $\tilde{R}_r^\lambda \in \mathcal{C}^j(\mathbb{R} \times \mathbb{R})$ for $\lambda > j$.

Fix $j \in \mathbb{N}$ and $\lambda \in \{0, 2, \ldots\} \cup (j, \infty)$. For $u \neq 0$ we have

$$\partial_u^j \tilde{R}_r^\lambda(u, v) = \frac{(\operatorname{sgn} u)^j}{2}\Big(\partial_u^j R_{r^2}^{\lambda-1/2}(|u|, |v|) + \operatorname{sgn}(vu) r \partial_u^j R_{r^2}^{\lambda+1/2}(|u|, |v|)\Big),$$

whereas for $u = 0$ we see that

$$\partial_u^j \tilde{R}_r^\lambda(0, v) = \frac{1}{2}\partial_u^j R_{r^2}^{\lambda-1/2}(0, |v|),$$

where in both cases $r \in (0,1)$ and $v \in \mathbb{R}$. In the latter equality we naturally extended the formula from (4.8) to $u = 0$.

Lemma 4.11 *Let $j \in \mathbb{N}$. For $\lambda \in \{0, 2, \ldots\} \cup (j, \infty)$ we have*

$$\sup_{u\in\mathbb{R}} \left\| \partial_u^j \tilde{R}_r^\lambda(u,\cdot) \right\|_{L^2(\mathbb{R})} \lesssim (1-r)^{(1+2j)/4},$$

uniformly in $r \in (0,1)$. Moreover, for $\lambda \in (j, j+1]$ we have

$$\left\| \partial_u^j \tilde{R}_r^\lambda(u,\cdot) - \partial_u^j \tilde{R}_r^\lambda(u',\cdot) \right\|_{L^2(\mathbb{R})} \lesssim (1-r)^{(1+2j)/4} |u-u'| + (1-r)^{(2\lambda+1)/4} |u-u'|^{\lambda-j}$$

uniformly in $r \in (0,1)$ and $u, u' \in \mathbb{R}$ such that $|u-u'| \le 1$.

Proof The first part is implied by (4.17) and Proposition 4.6. For the second one see (4.15). □

Now the version of (C) corresponding to the generalized Hermite setting follows easily. Then we immediately obtain the associated Hardy's inequality.

Proposition 4.12 *If $k \in \mathbb{N}$ and $\lambda \in \big(\{0, 2, \ldots\} \cup (k, \infty)\big)^d$, then*

$$\left\| \tilde{R}_r^\lambda(x,\cdot) - \sum_{|n|\le k} \frac{\partial_x^n \tilde{R}_r^\lambda(x',\cdot)}{n_1! \cdot \ldots \cdot n_d!} \prod_{i=1}^d (x_i - x_i')^{n_i} \right\|_{L^2(\mathbb{R}^d)} \lesssim \sum_{\delta\in\Delta_k^\alpha} (1-r)^{-\frac{d+2k+2\delta}{4}} |x-x'|^{k+\delta},$$

uniformly in $r \in (0,1)$ and $x, x' \in \mathbb{R}^d$ such that $|x-x'| \le 1/2$, where

$$\Delta_k^\lambda = \{1\} \cup \{\lambda_i - k : \lambda_i \in (k, k+1)\}.$$

Theorem 4.13 *Let $p \in (0,1)$, $s \in [p,2]$, and $P = \lfloor d(p^{-1}-1) \rfloor$. For*

$$\lambda \in \big(\{0, 2, \ldots, 2\lfloor P/2 \rfloor\} \cup (d(p^{-1}-1), \infty)\big)^d$$

there holds

$$\sum_{n\in\mathbb{N}^d} \frac{|\langle f, h_n^\lambda\rangle|^s}{(|n|+1)^E} \lesssim \|f\|_{H^p(\mathbb{R}^d)}^s, \qquad f \in H^p(\mathbb{R}^d),$$

where $E = d + \frac{ds}{4p}(2-3p)$, and the exponent is sharp.

Proof The inequality is a consequence of Proposition 4.12 and Theorem 2.4 and sharpness follows immediately from sharpness of the exponent in Theorem 4.9. □

We remark that for $\lambda = (0, \ldots, 0)$, that is in the case of the Hermite functions, the result agrees with the ones already known in the literature ([35] for $d \ge 2$, [23] for $d = 1$).

5 Jacobi Trigonometric Functions

The *Jacobi functions* $\phi_k^{\alpha,\beta}$, $k \in \mathbb{N}$, $\alpha, \beta > -1$, are defined by

$$\phi_k^{\alpha,\beta}(\theta) = \Big(\sin\frac{\theta}{2}\Big)^{\alpha+1/2}\Big(\cos\frac{\theta}{2}\Big)^{\beta+1/2}\mathcal{P}_k^{\alpha,\beta}(\theta), \qquad \theta \in (0,\pi), \tag{5.1}$$

where

$$\mathcal{P}_k^{\alpha,\beta}(\theta) = c_k^{\alpha,\beta} P_k^{\alpha,\beta}(\cos\theta),$$

and $P_k^{\alpha,\beta}$ denotes the Jacobi polynomial of type α, β and degree k. Here $c_k^{\alpha,\beta}$ is the normalizing constant,

$$c_k^{\alpha,\beta} = \left(\frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)\Gamma(k+1)}{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}\right)^{1/2},$$

where for $k = 0$ and $\alpha+\beta = -1$ we write 1 in place of $(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)$ in the numerator. Note that $c_k^{\alpha,\beta} \simeq (k+1)^{1/2}$, $k \in \mathbb{N}$. The system $\{\phi_k^{\alpha,\beta}\}_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2((0,\pi))$. In higher dimensions $\phi_n^{\alpha,\beta}(\theta)$ are defined as tensor products of $\phi_{n_i}^{\alpha_i,\beta_i}(\theta_i)$.

We are now interested in the L^∞ norms of the derivatives of $\phi_k^{\alpha,\beta}$ in various ranges of the parameters α and β and on different subintervals of $(0,\pi)$. Firstly, recall that for $\alpha, \beta \geq -1/2$ there is (see [27, (2.8)])

$$\big|\phi_k^{\alpha,\beta}(\theta)\big| \lesssim \begin{cases} \big((k+1)\theta\big)^{\alpha+1/2}, & 0 < \theta \leq (k+1)^{-1}, \\ 1, & (k+1)^{-1} \leq \theta \leq \pi - (k+1)^{-1}, \\ \big((k+1)\theta\big)^{\beta+1/2}, & \pi - (k+1)^{-1} \leq \theta < \pi. \end{cases} \tag{5.2}$$

Hence, for $\alpha, \beta \geq -1/2$

$$\|\phi_k^{\alpha,\beta}\|_{L^\infty(0,\pi)} \lesssim 1, \qquad k \in \mathbb{N}.$$

Secondly, we make use of the formula (cf. [42, (4.21.7)] or [8, p. 364] after an obvious simplification)

$$\frac{d}{d\theta}\phi_k^{\alpha,\beta}(\theta) = -k_{\alpha,\beta}\phi_{k-1}^{\alpha+1,\beta+1}(\theta) + \Big(\frac{2\alpha+1}{4}\cot\frac{\theta}{2} - \frac{2\beta+1}{4}\tan\frac{\theta}{2}\Big)\phi_k^{\alpha,\beta}(\theta), \tag{5.3}$$

where we put $\phi_{-1}^{\alpha+1,\beta+1} \equiv 0$ and $k_{\alpha,\beta} = \sqrt{k(k+\alpha+\beta+1)}$. Observe that (5.3) and (5.2) give for $\alpha \in \{-1/2\} \cup [1/2,\infty)$ and $\beta \geq -1/2$ the bound

$$\big\|(\phi_k^{\alpha,\beta})'\big\|_{L^\infty\left(0,\frac{2\pi}{3}\right)} \lesssim (k+1), \qquad k \in \mathbb{N},$$

and symmetrically for $\alpha \geq -1/2$ and $\beta \in \{-1/2\} \cup [1/2, \infty)$,

$$\big\|(\phi_k^{\alpha,\beta})'\big\|_{L^\infty(\frac{\pi}{3},\pi)} \lesssim (k+1), \qquad k \in \mathbb{N}.$$

For similar estimates for higher order derivatives see Lemma 5.3.

We will frequently make use of the formula

$$\phi_k^{\alpha,\beta}(\theta) = \phi_k^{\beta,\alpha}(\pi - \theta), \qquad \theta \in (0,\pi). \tag{5.4}$$

5.1 Lipschitz and *BMO* Properties

Let us firstly give some auxiliary lemmas and justify that the Jacobi functions belong to the Lipschitz spaces $\Lambda_\nu((0,\pi))$ for certain ν.

Lemma 5.1 *Let $j \in \mathbb{N}$ and $\alpha, \beta \geq -1/2$. Then, for any $c \in (0,1]$, we have*

$$\big|(\phi_k^{\alpha,\beta})^{(j)}(\theta)\big| \lesssim \begin{cases} \theta^{\alpha+1/2-j}(k+1)^{\alpha+1/2}, & \theta \in \big(0, c(k+1)^{-1}\big), \\ (k+1)^j, & \theta \in \big[c(k+1)^{-1}, \frac{2\pi}{3}\big), \end{cases}$$

and

$$\big|(\phi_k^{\alpha,\beta})^{(j)}(\theta)\big| \lesssim \begin{cases} \theta^{\beta+1/2-j}(k+1)^{\beta+1/2}, & \theta \in \big(\pi - c(k+1)^{-1}, \pi\big), \\ (k+1)^j, & \theta \in \big(\frac{\pi}{3}, \pi - c(k+1)^{-1}\big], \end{cases}$$

uniformly in θ and $k \in \mathbb{N}$.

Proof Notice that by (5.4) it suffices to verify the first estimate. We use the induction. For $j = 0$ see (5.2). Assume that the claim holds for $j \in \mathbb{N}$. By (5.3) we have

$$\begin{aligned}(\phi_k^{\alpha,\beta})^{(j+1)}(\theta) = &-k_{\alpha,\beta}(\phi_{k-1}^{\alpha+1,\beta+1})^{(j)}(\theta) \\ &+ \frac{d^j}{d\theta^j}\Big(\Big(\frac{2\alpha+1}{4}\cot\frac{\theta}{2} - \frac{2\beta+1}{4}\tan\frac{\theta}{2}\Big)\phi_k^{\alpha,\beta}(\theta)\Big),\end{aligned}$$

where we used (5.3). Mind that for any given $i \in \mathbb{N}$ there is

$$\Big|\big(\tan(\theta/2)\big)^{(i)}\Big| \lesssim 1 \qquad \text{and} \qquad \Big|\big(\cot(\theta/2)\big)^{(i)}\Big| \lesssim \theta^{-(i+1)}, \tag{5.5}$$

uniformly in $\theta \in \big(0, \frac{2\pi}{3}\big)$. Hence,

$$\begin{aligned}\Big|(\phi_k^{\alpha,\beta})^{(j+1)}(\theta)\Big| \lesssim &(k+1)\Big|(\phi_{k-1}^{\alpha+1,\beta+1})^{(j)}(\theta)\Big| \\ &+ \sum_{i=0}^{j}\theta^{-(i+1)}\Big|(\phi_k^{\alpha,\beta})^{(j-i)}(\theta)\Big| + \sum_{i=0}^{j}\Big|(\phi_k^{\alpha,\beta})^{(i)}(\theta)\Big|,\end{aligned}$$

uniformly in $k \in \mathbb{N}$ and $\theta \in \big(0, \frac{2\pi}{3}\big)$. Thus,

$$\Big|(\phi_k^{\alpha,\beta})^{(j+1)}(\theta)\Big| \lesssim (k+1)^{\alpha+1/2}\theta^{\alpha+1/2-j}\Big((k+1)^2\theta+\theta^{-1}+1\Big) \\ \lesssim (k+1)^{\alpha+1/2}\theta^{\alpha+1/2-j-1},$$

uniformly in $k \in \mathbb{N}$ and $\theta \in \big(0, c(k+1)^{-1}\big)$. Similarly,

$$\Big|(\phi_k^{\alpha,\beta})^{(j+1)}(\theta)\Big| \lesssim (k+1)^{j+1},$$

uniformly in $k \in \mathbb{N}$ and $\theta \in \big(c(k+1)^{-1}, \frac{2\pi}{3}\big)$. This finishes the proof. □

The following result is an analogue of Lemmas 3.4 and 4.3.

Lemma 5.2 *Let $j, \ell \in \mathbb{N}$ and $\alpha, \beta \geq -1/2$. There exists $c > 0$ such that*

$$\frac{d^j}{d\theta^j}\frac{\phi_k^{\alpha,\beta}(\theta)}{\big(\sin\frac{\theta}{2}\big)^{\alpha+1/2-\ell}} \simeq \begin{cases} (k+1)^{\alpha+1/2}\,\theta^{\ell-j}, & \ell \geq j, \\ (-1)^{\left\lceil\frac{j-\ell}{2}\right\rceil}(k+1)^{\alpha+1/2+2\left\lceil\frac{j-\ell}{2}\right\rceil}\theta^{\frac{1-(-1)^{j-\ell}}{2}}, & \ell \leq j, \end{cases}$$

uniformly in $k \in \mathbb{N}$ and $\theta \in \big(0, c(k+1)^{-1}\big)$, and

$$\frac{d^j}{d\theta^j}\frac{\phi_k^{\alpha,\beta}(\theta)}{\big(\cos\frac{\theta}{2}\big)^{\beta+1/2-\ell}} \simeq \begin{cases} (k+1)^{\beta+1/2}\,(\pi-\theta)^{\ell-j}, & \ell \geq j, \\ (-1)^{\left\lceil\frac{j-\ell}{2}\right\rceil}(k+1)^{\beta+1/2+2\left\lceil\frac{j-\ell}{2}\right\rceil}(\pi-\theta)^{\frac{1-(-1)^{j-\ell}}{2}}, & \ell \leq j, \end{cases}$$

uniformly in $k \in \mathbb{N}$ and $\theta \in \big(\pi - c(k+1)^{-1}, \pi\big)$.

Proof It is sufficient to prove the first estimate. The reasoning is similar to the ones used in the proofs of Lemmas 3.4 and 4.3, therefore we only sketch it.

Fix ℓ, α, β as in the hypothesis. We use the induction over j. For $j = 0$ see [32, (A.1) and (A.2)]. For the inductive step observe that

$$\frac{d^{j+1}}{d\theta^{j+1}}\frac{\phi_k^{\alpha,\beta}(\theta)}{\big(\sin\frac{\theta}{2}\big)^{\alpha+1/2-\ell}} = \frac{d^j}{d\theta^j}\bigg(\frac{\ell}{2}\frac{\cos\frac{\theta}{2}\,\phi_k^{\alpha,\beta}(\theta)}{\big(\sin\frac{\theta}{2}\big)^{\alpha+3/2-\ell}} - \frac{2\beta+1}{4}\frac{\phi_k^{\alpha,\beta}(\theta)}{\cos\frac{\theta}{2}\big(\sin\frac{\theta}{2}\big)^{\alpha-1/2-\ell}} \\ - \frac{k_{\alpha,\beta}\phi_{k-1}^{\alpha+1,\beta+1}(\theta)}{\big(\sin\frac{\theta}{2}\big)^{\alpha+1/2-\ell}}\bigg).$$

If $j \leq \ell - 1$, then the first implied component is the largest on the right hand side of the above equality. It is positive and of the desired size $(k+1)^{\alpha+1/2}\,\theta^{\ell-j-1}$. Secondly, the case $j = \ell$ can be checked directly. On the other hand, if $j \geq \ell + 1$, then the first term, after ℓ iterations, vanishes as did its counterparts from Lemma 3.4 and 4.3. Furthermore, for sufficiently small $c > 0$ the second summand is of the sign and the size

$$(-1)^{\left\lceil\frac{j-\ell-1}{2}\right\rceil+1}(k+1)^{\alpha+1/2+2\left\lceil\frac{j-\ell-1}{2}\right\rceil}\theta^{\frac{1-(-1)^{j-\ell-1}}{2}},$$

and the third

$$(-1)^{\left\lceil \frac{j-\ell-1}{2}\right\rceil+1}(k+1)^{\alpha+5/2+2\left\lceil \frac{j-\ell-1}{2}\right\rceil}\theta^{\frac{1-(-1)^{j-\ell-1}}{2}}.$$

The latter is dominant and it can be rewritten as

$$(-1)^{\left\lceil \frac{j+1-\ell}{2}\right\rceil}(k+1)^{\alpha+1/2+2\left\lceil \frac{j+1-\ell}{2}\right\rceil}\theta^{\frac{1-(-1)^{j+1-\ell}}{2}},$$

which finishes the inductive step. □

Lemma 5.3 *Let* $j \in \mathbb{N}$. *For* $\alpha \in \{-1/2, 1/2, \dots, j-1/2\} \cup (j-1/2, \infty)$, *and* $\beta \geq -1/2$, *there is*

$$\left\| \left(\phi_k^{\alpha,\beta}\right)^{(j)} \right\|_{L^\infty\left(0, \frac{2\pi}{3}\right)} \lesssim (k+1)^j, \qquad k \in \mathbb{N},$$

whereas for $\alpha \geq -1/2$ *and* $\beta \in \{-1/2, 1/2, \dots, j-1/2\} \cup (j-1/2, \infty)$, *we have*

$$\left\| \left(\phi_k^{\alpha,\beta}\right)^{(j)} \right\|_{L^\infty\left(\frac{\pi}{3}, \pi\right)} \lesssim (k+1)^j, \qquad k \in \mathbb{N}.$$

Proof Observe that the latter estimate follows from the former by (5.4). Thus we fix $j \in \mathbb{N}$, α, and β as in the first hypothesis. We justify the bound on $(0, \frac{2\pi}{3})$. Note that for $\alpha \geq j - 1/2$ it suffices to use Lemma 5.1. On the other hand, if $j > \alpha + 1/2 \in \mathbb{N}$, then use Lemma 5.2 with $\ell = \alpha + 1/2$. This concludes the proof. □

Lemma 5.4 *Let* $j \in \mathbb{N}$. *If* $\alpha \in (j-1/2, j+1/2)$ *and* $\beta \geq -1/2$, *then*

$$\left|(\phi_k^{\alpha,\beta})^{(j)}(\theta) - (\phi_k^{\alpha,\beta})^{(j)}(\theta')\right| \lesssim (k+1)^{j+1}|\theta-\theta'| + (k+1)^{\alpha+1/2}|\theta-\theta'|^{\alpha+1/2-j},$$

uniformly in $k \in \mathbb{N}$ *and* $\theta, \theta' \in (0, \frac{2\pi}{3})$. *Similarly, for* $\alpha \geq -1/2$ *and* $\beta \in (j-1/2, j+1/2)$,

$$\left|(\phi_k^{\alpha,\beta})^{(j)}(\theta) - (\phi_k^{\alpha,\beta})^{(j)}(\theta')\right| \lesssim (k+1)^{j+1}|\theta-\theta'| + (k+1)^{\beta+1/2}|\theta-\theta'|^{\beta+1/2-j},$$

uniformly in $k \in \mathbb{N}$ *and* $\theta, \theta' \in (\frac{\pi}{3}, \pi)$.

Proof Again, by (5.4) we verify only the first estimate. Fix j, α, β as in the hypothesis. For $0 < \theta' < \theta < \frac{2\pi}{3}$, by using (5.3), we write the difference $(\phi_k^{\alpha,\beta})^{(j)}(\theta) - (\phi_k^{\alpha,\beta})^{(j)}(\theta')$ as

$$\int_{\theta'}^{\theta} \frac{d^j}{d\omega^j}\Big(-k_{\alpha,\beta}\phi_{k-1}^{\alpha+1,\beta+1}(\omega) + \Big(\frac{2\alpha+1}{4}\cot\frac{\omega}{2} - \frac{2\beta+1}{4}\tan\frac{\omega}{2}\Big)\phi_k^{\alpha,\beta}(\omega)\Big)\, d\omega.$$

Thus, by the first bound in Lemma 5.1 and (5.5) we obtain

$$\left|(\phi_k^{\alpha,\beta})^{(j)}(\theta) - (\phi_k^{\alpha,\beta})^{(j)}(\theta')\right| \lesssim (k+1)^{j+1}|\theta-\theta'| + \int_{\theta'}^{\theta} \left|\frac{d^j}{d\omega^j}\Big(\cot\frac{\omega}{2}\phi_k^{\alpha,\beta}(\omega)\Big)\right| d\omega,$$

uniformly in $k \in \mathbb{N}$ and $\theta, \theta' \in (0, \frac{2\pi}{3})$. Using Lemma 5.1 we estimate the last integral uniformly in the indicated ranges, up to a multiplicative constant, by

$$(k+1)^{j+1}|\theta - \theta'| + (k+1)^{\alpha+1/2} \int_{\theta'}^{\theta} \omega^{\alpha-1/2-j}\, d\omega.$$

The conclusion follows since the last integral is bounded by a constant times $|\theta - \theta'|^{\alpha+1/2-j}$. □

Now we pass to the verification of Lipschitz and BMO properties of the Jacobi functions. Observe that for $\alpha, \beta \in [-1/2, \infty)^d$, $\phi_n^{\alpha,\beta} \in L^\infty((0,\pi)^d) \subset BMO((0,\pi)^d)$. We justify that $\phi_n^{\alpha,\beta} \in \Lambda_\nu\big((0,\pi)^d\big)$, $\nu > 0$, for appropriate parameters α and β. For this purpose we define an extension $\tilde{\phi}_k^{\alpha,\beta}$ of $\phi_k^{\alpha,\beta}$ to the whole $\mathbb{R}$ such that $\tilde{\phi}_k^{\alpha,\beta} \in \Lambda_\nu(\mathbb{R})$, and then apply Lemma 2.2 for the multi-dimensional situation.

Fix α and β such that $\alpha + 1/2, \beta + 1/2 \in \mathbb{N}$. We extend the initial definition of $\phi_k^{\alpha,\beta}(\theta)$, see (5.1), to the whole $\mathbb{R}$. Note that for $j \in \mathbb{N}$ and $\theta \in (j\pi, (j+1)\pi)$ there holds

$$\phi_k^{\alpha,\beta}(\theta) = \begin{cases} \phi_k^{\alpha,\beta}(\theta - j\pi), & j \equiv 0 \mod 4, \\ (-1)^{\beta+1/2}\phi_k^{\alpha,\beta}((j+1)\pi - \theta), & j \equiv 1 \mod 4, \\ (-1)^{\alpha+\beta+1}\phi_k^{\alpha,\beta}(\theta - j\pi), & j \equiv 2 \mod 4, \\ (-1)^{\alpha+1/2}\phi_k^{\alpha,\beta}((j+1)\pi - \theta), & j \equiv 3 \mod 4. \end{cases}$$

We remark that the second (fourth, resp.) line on the right hand side of the formula above makes sense also when $\alpha + 1/2$ ($\beta + 1/2$, resp.) is not an integer. Moreover, if $\alpha + 1/2 \in \mathbb{N}$ ($\beta + 1/2 \in \mathbb{N}$, resp.), then $\phi_k^{\alpha,\beta}(2j\pi)$ ($\phi_k^{\alpha,\beta}((2j+1)\pi)$, resp.) is naturally defined for $j \in \mathbb{N}$.

Now we define the extension $\tilde{\phi}_k^{\alpha,\beta}$ of $\phi_k^{\alpha,\beta}$. If both $\alpha + 1/2, \beta + 1/2 \in \mathbb{N}$, then

$$\tilde{\phi}_k^{\alpha,\beta}(\theta) = \phi_k^{\alpha,\beta}(\theta), \qquad \theta \in \mathbb{R}.$$

Secondly, if $\alpha + 1/2 \in \mathbb{N}$ and $\beta + 1/2 \notin \mathbb{N}$, then

$$\tilde{\phi}_k^{\alpha,\beta}(\theta) = \begin{cases} \phi_k^{\alpha,\beta}(\theta), & \theta \in (-\pi, \pi), \\ 0, & \theta \notin (-\pi, \pi). \end{cases}$$

Similarly, if $\alpha + 1/2 \notin \mathbb{N}$ and $\beta + 1/2 \in \mathbb{N}$, then

$$\tilde{\phi}_k^{\alpha,\beta}(\theta) = \begin{cases} \phi_k^{\alpha,\beta}(\theta), & \theta \in (0, 2\pi), \\ 0, & \theta \notin (0, 2\pi). \end{cases}$$

Finally, if both $\alpha + 1/2, \beta + 1/2 \notin \mathbb{N}$, then we put

$$\tilde{\phi}_k^{\alpha,\beta}(\theta) = \begin{cases} \phi_k^{\alpha,\beta}(\theta), & \theta \in (0, \pi), \\ 0, & \theta \notin (0, \pi). \end{cases}$$

Notice that $\tilde{\phi}_k^{\alpha,\beta} \in \mathcal{C}^{\min(\bar{\alpha},\bar{\beta})}(\mathbb{R})$, where we used the one-off notation $\bar{\alpha} = \lfloor \alpha + 1/2 \rfloor$ if $\alpha + 1/2 \notin \mathbb{N}$ and $\bar{\alpha} = \infty$ otherwise, and the same for $\bar{\beta}$.

Now, by Lemmas 5.3, 5.4, and 2.2, we have the following result.

Lemma 5.5 *If* $p \in (0, 1]$ *and* $\alpha, \beta \in \big(\{-1/2, 1/2, \dots, P - 1/2\} \cup [d(p^{-1} - 1) - 1/2, \infty)\big)^d$, *where* $P = \lfloor d(p^{-1} - 1) \rfloor$, *then* $\phi_n^{\alpha,\beta} \in \Lambda_{d(\frac{1}{p}-1)}((0, \pi)^d)$.

5.2 Hardy's Inequality

The one-dimensional kernels $R_r^{\alpha,\beta}(\theta, \varphi)$, $r \in (0, 1)$, $\theta, \varphi \in (0, \pi)$, associated with the Jacobi functions are defined via (cf. (2.3))

$$R_r^{\alpha,\beta}(\theta, \varphi) = \sum_{k \in \mathbb{N}} r^k \phi_k^{\alpha,\beta}(\theta) \phi_k^{\alpha,\beta}(\varphi).$$

For an explicit formula see [32].

Notice that by Parseval's identity and interchanging the differentiation with the summation, which is allowed due to Lemma 5.3 and the Lebesgue dominated convergence theorem, we obtain the following lemma.

Lemma 5.6 *If* $j \in \mathbb{N}$, *and* $\alpha, \beta \in \{-1/2, 1/2, \dots, j - 1/2\} \cup (j - 1/2, \infty)$, *then*

$$\sup_{\theta \in (0,\pi)} \big\| \partial_\theta^j R_r^{\alpha,\beta}(\theta, \cdot) \big\|_{L^2((0,\pi))} \lesssim (1 - r)^{-(j+1/2)}, \qquad r \in (0, 1).$$

In order to verify the appropriate version of (C) we firstly estimate differences of the derivatives of $R_r^{\alpha,\beta}(\theta, \varphi)$. We remark that in order to prove the below-stated proposition, one could use [5, Lemma 3.4] and the explicit form of the investigated kernels. However, Lemma 5.4 yields this result much quicker.

Proposition 5.7 *If* $j \in \mathbb{N}$ *and* $\alpha, \beta \in \{-1/2, 1/2, \dots, j - 1/2\} \cup (j - 1/2, \infty)$, *then*

$$\begin{aligned} &\big\| \partial_\theta^j R_r^{\alpha,\beta}(\theta, \cdot) - \partial_\theta^j R_r^{\alpha,\beta}(\theta', \cdot) \big\|_{L^2((0,\pi))} \\ &\quad \lesssim (1 - r)^{-(j+3/2)} |\theta - \theta'| + (1 - r)^{-(\alpha+1)} |\theta - \theta'|^{\alpha+1/2-j} \\ &\qquad + (1 - r)^{-(\beta+1)} |\theta - \theta'|^{\beta+1/2-j}, \end{aligned}$$

uniformly in $r \in (0, 1)$ *and* $\theta, \theta' \in (0, \pi)$, *where the second (third, resp.) summand on the right hand side of the estimate appears only if* α *(*β*, resp.) belongs to* $(j - 1/2, j + 1/2)$.

Proof In the case $\alpha, \beta \notin (j - 1/2, j + 1/2)$ we simply apply the mean value theorem and Lemma 5.6. On the other hand, if one or both of the parameters α and β is in $(j - 1/2, j + 1/2)$, then we apply Parseval's identity and Lemma 5.4. □

Now the following proposition follows easily (compare with Propositions 3.3 and 4.8).

Proposition 5.8 *If $k \in \mathbb{N}$ and $\alpha, \beta \in \big(\{-1/2, 1/2, \ldots, k-1/2\} \cup (k-1/2, \infty)\big)^d$, then*

$$\Big\| R_r^{\alpha,\beta}(\theta,\cdot) - \sum_{|n|\le k} \frac{\partial_\theta^n R_r^{\alpha,\beta}(\theta',\cdot)}{n_1!\cdot\ldots\cdot n_d!} \prod_{i=1}^d (\theta_i-\theta_i')^{n_i} \Big\|_{L^2((0,\pi)^d)} \lesssim \sum_{\delta\in\Delta_k^{\alpha,\beta}} (1-r)^{-\frac{d+2k+2\delta}{2}} |\theta-\theta'|^{k+\delta},$$

uniformly in $r \in (0,1)$ and $\theta, \theta' \in (0,\pi)^d$, where

$$\Delta_k^{\alpha,\beta} = \{1\} \cup \{\alpha_i + 1/2 - k : \alpha_i \in (k-1/2, k+1/2)\} \cup \{\beta_i + 1/2 - k : \beta_i \in (k-1/2, k+1/2)\}.$$

We are ready to state Hardy's inequality associated with the Jacobi trigonometric functions.

Theorem 5.9 *Let $p \in (0,1)$, $s \in [p,2]$, and $P = \lfloor d(p^{-1}-1)\rfloor$. For*

$$\alpha, \beta \in \big(\{-1/2, 1/2, \ldots, P-1/2\} \cup (d(p^{-1}-1)-1/2, \infty)\big)^d,$$

there holds

$$\sum_{n\in\mathbb{N}^d} \frac{|\langle f, \phi_n^{\alpha,\beta}\rangle|^s}{(|n|+1)^E} \lesssim \|f\|_{H^p((0,\pi)^d)}^s, \qquad f \in H^p((0,\pi)^d),$$

where $E = d + sd\big(p^{-1}-1\big)$, and the exponent is sharp.

Proof Similarly as in the proofs of Theorems 3.5 and 4.9 the inequality follows from Theorem 2.4 and Proposition 5.8, whereas sharpness is a consequence of Propositions 2.5, 2.6, Lemma 5.2 and Remark 2.7. □

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