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Bernstein–Jackson Inequalities on Gaussian Hilbert Spaces

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Abstract

Estimates of best approximations by exponential type analytic functions in Gaussian random variables with respect to the Malliavin derivative in the form of Bernstein–Jackson inequalities with exact constants are established. Formulas for constants are expressed through basic parameters of approximation spaces. The relationship between approximation Gaussian Hilbert spaces and classic Besov spaces are shown.

Keywords Bernstein–Jackson inequalities · Approximation by Gaussian random variables · Best approximation constants

Mathematics Subject Classification 46N30 · 41A44 · 41A17

1 Introduction and Main Results

As is known (see [2, 7, 10, 26, 27]), the best approximations by differentiable functions in the classic analysis is based on a concept of the E-functional which characterizes the rapidity of approximations. This approach is constructive because it combines approximations with interpolation methods that provide explicit formulas for evaluating the approximations. In this area, many important inverse and direct theorems in the form of Bernstein–Jackson inequalities have been proven, in particular, in [3, 5, 11, 18, 19]. But approximation constants were not calculated that gives only asymptotic estimates of errors.

Our goal is to extend inverse and direct theorems in the form of Bernstein–Jackson inequalities on a more general case of best approximations by entire analytic in a Malliavin sense functions of random variables on Gaussian Hilbert spaces, and fur-

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thermore to calculate the explicit formulas for exact approximation constants in these inequalities.

The main results are presented in Sect. 3. Namely, in Theorem 2 it is established the bilateral version of Bernstein–Jackson inequalities

$$\begin{gathered} t^{-1+1/\theta} E(t, f) \leq C_{\theta,q} \|f\|_{E_{\theta,q}} \leq 2^{1/\theta} |f|_{\mathscr{E}^{p_0}}^{-1+1/\theta} \|f\|_{p_1}, \quad f \in \mathscr{E}^{p_0} \cap L^{p_1} \\ E(t, f) \leq t^{1-1/\theta} C_{\theta,q} \|f\|_{E_{\theta,q}}, \quad f \in E_{\theta,q}(\mathscr{E}^{p_0}, L^{p_1}), \end{gathered}$$

where the approximation Gaussian Hilbert space $E_{\theta,q}(\mathscr{E}^{p_0}, L^{p_1})$ is represented as a fractional power of the real interpolation space $K_{\theta,q}\,(\mathscr{E}^{p_0}, L^{p_1})$ in the form

$$E_{\theta,q}(\mathscr{E}^{p_0}, L^{p_1}) \simeq K_{\theta,q}\left(\mathscr{E}^{p_0}, L^{p_1}\right)^{1/\theta}, \quad p_0 \in (1, \infty), \quad p_1 \in (0, \infty)$$

which is a generalization of the known classic isomorphism (see e.g. [2, Theorem 7.1.7]) on the case of Gaussian Hilbert spaces.

One of main results in Theorem 2 is also the explicit formula (12) for the best approximation constants $C_{\theta,q}$ which for the case $q = 2$ receives the following simple form

$$C_{\theta,2} = \left(\frac{\sin \pi\theta}{\pi\theta}\right)^{1/2\theta}, \quad 0 < \theta < 1.$$

The above-mentioned Bernstein–Jackson inequalities two-sided characterize the rapidity of approximations in the space $L^{p_1}(\Omega, \mathcal{F}, P)$ by the dense quasi-normed subspace $(\mathscr{E}^{p_0}, |\cdot|_{\mathscr{E}^p})$ of convergent exponential types power series with respect to the Malliavin derivative ∇,

$$\sum_{k=0}^{\infty} \frac{\nabla^k f}{k!} z^k, \quad z \in \mathbb{C}.$$

More specific, we consider Gaussian Hilbert spaces of random variables ϕ_h defined on a complete probability space $(\Omega, \mathcal{F}, P)$ such that $\phi_h \sim \mathrm{N}(0, \|h\|_H^2)$, where h belongs to a separable real Hilbert space H and σ-field $\mathcal{F}$ is generated by a Gaussian field $H \ni h \mapsto \phi_h$. This means that ϕ_h is a family of Gaussian random variables with covariance structure $\mathsf{E}\,\phi_h\phi_g = \langle h \mid g\rangle$, where $\mathsf{E}\,\phi_h$ is the expectation of ϕ_h relative to $(\Omega, \mathcal{F}, P)$ (see e.g. [15, Theorem 1.23]).

One of the main tools that is used to characterize on Gaussian Hilbert spaces of entire analytic functions is the notion of an exponential type, introduced in Sect. 2. This notion is a generalization of analytic vectors in the Nelson sense [21] for an abstract linear unbounded operator on the case of Malliavin's derivative ∇.

Note that the case of non-stochastic entire analytic vectors in the Nelson sense were early analyzed in [8, 9].

The following properties of exponential-type random Gaussian variables are proved in Theorem 1. For this purpose, we introduce the quasi-normed space

$$\mathscr{E}^p(\Omega, \mathcal{F}, P) = \bigcup \mathscr{E}^{\nu,p}(\Omega, \mathcal{F}, P)$$

of entire analytic functions in random variables of all exponential types $\nu > 0$, which is dense in $L^p(\Omega, \mathcal{F}, P)$ with $p \in (0, \infty)$ and the interpolation couple

$$\left(\mathscr{E}^{p_0}(\Omega, \mathcal{F}, P), L^{p_1}(\Omega, \mathcal{F}, P)\right), \quad p_0 \in (1, \infty), \quad p_1 \in (0, \infty),$$

which is compatible in the interpolation theory sense. Using these spaces, we define the best approximation E-functional to be

$$E\left(t, f; \mathscr{E}^{p_0}, L^{p_1}\right) = \inf\left\{\|f - f_0\|_{p_1} : |f_0|_{\mathscr{E}^{p_0}} < t\right\}, \quad f \in L^{p_1}.$$

It is proved that each restriction $\nabla|_{\mathscr{E}^{\nu,p}}$ to the subspace $\mathscr{E}^{\nu,p}(\Omega, \mathcal{F}, P)$ with a fixed exponential type $\nu > 0$ has the finite norm $\leq \nu$ and that $\mathscr{E}^p(\Omega, \mathcal{F}, P)$ and $\mathscr{E}^{\nu,p}(\Omega, \mathcal{F}, P)$ are complete. The completeness is proved with the help of Bernstein compactness theorem for entire analytic functions of an exponential type [22, Theorem 3.3.6].

Notice additionally that in the considered case, the interpolation Gaussian Hilbert space $K_{\theta,q}\,(\mathscr{E}^{p_0}, L^{p_1})$ is determined through the quadratically modified (adapted to the case of Hilbert spaces) form of the K-functional, which was used, in particular, in [17].

Finally, the approach developed in this work naturally includes the case of functions with independent random variables, defined on infinite dimensional Banach spaces (see Example 1).

In Example 2 it is also shown that for the Gaussian space $L^p\left(\mathbb{R}^d, \mathcal{F}, \gamma_d\right)$ with $p \in (1, \infty)$, endowed with the gaussian measure γ_d on the Borel σ-field $\mathcal{F} = \mathcal{B}(\mathbb{R}^d)$, the previous approximation space has the form

$$E_{\theta,q}(\mathscr{E}^p, L^p) = \left\{f_g \in L^p\left(\mathbb{R}^d, \mathcal{F}, \gamma_d\right) : g \in B^s_{p,\tau}(\mathbb{R}^d)\right\},$$

where the space $B^s_{p,\tau}(\mathbb{R}^d)$ with $s = -1 + 1/\theta$ and $\tau = q\theta$ exactly coincides with the classic Besov space (see e.g. [33, p.197]). Above, the element $f_g = g(\phi_{h_1}, \ldots, \phi_{h_d}) \in \mathscr{E}^p\left(\mathbb{R}^d, \mathcal{F}, \gamma_d\right)$ means the cylindrical random function determined by an entire analytic function $g(z_1, \ldots, z_d)$ on $\mathbb{C}^d$ of an exponential type. In this case for $q = 2$ and $\tau = 2\theta$ the Bernstein–Jackson inequalities take the form

$$\|f_g\|_{E_{\theta,2}} \leq 2^{1/2\theta} \left(\frac{\pi\theta}{\sin\pi\theta}\right)^{1/2\theta} |f_g|^{-1+1/\theta}_{\mathscr{E}^p} \|g\|_{L^p(\mathbb{R}^d)}, \quad f_g \in \mathscr{E}^p,$$

$$E(t, f_g) \leq t^{1-1/\theta} \left(\frac{\sin\pi\theta}{\pi\theta}\right)^{1/2\theta} \|g\|_{B^s_{p,\tau}(\mathbb{R}^d)}, \quad g \in B^s_{p,\tau}(\mathbb{R}^d).$$

It is important to note that among other widely known universal approaches to approximating functions in Gaussian variables, the known Stein method [31, 32] and its subsequent modifications should be specially recorded. The following publications of recent years [12, 13, 23, 24, 30] are devoted to the development of these studies using the Malliavin calculus.

2 Exponential Type with Respect to the Malliavin Derivative

Let a real separable Hilbert space H with scalar product and norm, denoted by $\langle \cdot \mid \cdot \rangle$ and $\|\cdot\|_H = \langle \cdot \mid \cdot \rangle^{1/2}$, has an orthonormal basis $\{\mathfrak{e}_i : i \in \mathbb{N}\}$.

There exists a linear isometry $H \ni h \to \phi_h$ into a Gaussian Hilbert space of real-valued functions defined on a complete probability space $(\Omega, \mathcal{F}, P)$, where the σ-field $\mathcal{F}$ is generated by H (see e.g. [15, Theorem 1.23]). We suppose that ϕ_h is centered and has the covariance $\mathsf{E}\,\phi_h \phi_g = \langle h \mid g \rangle_H$ (see [25, no 1.1]).

The L^p-norm of real-valued functions f on $(\Omega, \mathcal{F}, P)$ is defined by

$$\|f\|_p = \begin{cases} (\mathsf{E}\,|f|^p)^{1/p} & \text{if } p \in (0, \infty) \\ \operatorname{ess\,sup} |f| & \text{if } p = \infty. \end{cases}$$

All L^p-norms with $p \in (0, \infty)$ are proportional [15, Theorem 1.4], since $\|f\|_p = \kappa(p)\|f\|_2$, where $\kappa(p) = \sqrt{2}(\Gamma((p+1)/2)/\sqrt{\pi})^{1/p}$. By definition, the space $L^p = L^p(\Omega, \mathcal{F}, P)$ is endowed with the L^p-norm. The space of all measurable functions $L^0 = L^0(\Omega, \mathcal{F}, P)$ is equipped with the topology of convergence in probability, metrizable by $\|f\|_0 = \mathsf{E}\min(|f|, 1)$.

The L^p-norm of Y-valued functions $\xi = f \otimes y$ in $(\Omega, \mathcal{F}, P)$ is defined to be $\|\xi\|_p = \begin{cases} (\mathsf{E}\,\|\xi\|_Y^p)^{1/p} & \text{if } p \in (0, \infty) \\ \operatorname{ess\,sup} \|\xi\|_Y & \text{if } p = \infty \end{cases}$, where y belongs to a Banach space $(Y, \|\cdot\|_Y)$. Let $L^p(Y)$ be the completion of linear span of $\xi = \phi \otimes y$ with respect to this L^p-norm.

Consider the class of smooth functions of cylindrical forms

$$f = F(\phi_{h_1}, \ldots, \phi_{h_n}) \quad \text{with some} \quad n \in \mathbb{N},$$

where $\phi_{h_1}, \ldots, \phi_{h_n} \in L^0$ with $h_i \in H$ and $F \in C_b^\infty(\mathbb{R}^n)$ is a smooth function with bounded partial derivatives ∂_i. By definition the Malliavin derivative ∇ of f is the H-valued random variable

$$\nabla f = \sum_{i=1}^{n} \partial_i F(\phi_{h_1}, \ldots, \phi_{h_n}) h_i, \quad h_1, \ldots, h_n \in H$$

(see e.g. [25, no 1.2.1]). In particular, $\nabla \phi_h = h$ for every $h \in H$.

For $p \in [1, \infty)$ the domain $W^{1,p}$ of ∇ is the closure in L^p of all functions with respect to the graph-norm

$$\|f\|_{1,p} = (\mathsf{E}\,|f|^p)^{1/p} + (\mathsf{E}\,\|\nabla f\|_H^p)^{1/p}.$$

The completion of linear spans of tensor products $\psi_n = h_1 \otimes \ldots \otimes h_n$, $(h_\iota \in H)$ endowed with $\|\psi_n\|_{H^{\otimes n}} = \langle \psi_n \mid \psi_n \rangle^{1/2}$ is denoted by $H^{\otimes n}$, where $\langle \psi_n \mid \psi'_n \rangle = \langle h_1 \mid h'_1 \rangle \ldots \langle h_n \mid h'_n \rangle$. Let $h^{\otimes n} := h \otimes \ldots \otimes h$. The symmetric tensor power $H^{\odot n} \subset H^{\otimes n}$ is defined to be a range of the orthogonal projector

$$H^{\otimes n} \ni h_1 \otimes \ldots \otimes h_n \mapsto h_1 \odot \ldots \odot h_n := \frac{1}{n!} \sum_{\sigma \in S_n} h_{\sigma(1)} \otimes \ldots \otimes h_{\sigma(n)},$$

where S_n means n-elements permutations. The corresponding symmetric Fock space $\Gamma(H) = \bigoplus_0^\infty H^{\odot n}$ of elements $\psi = \bigoplus \psi_n$ with $\psi_n \in H^{\odot n}$ and $H^{\odot 0} = \mathbb{R}$ is endowed with the norm

$$\|\psi\|_\Gamma = \langle \psi \mid \psi \rangle_\Gamma^{1/2}, \quad \langle \psi \mid \psi' \rangle_\Gamma = \sum_{n=0}^{\infty} n! \langle \psi_n \mid \psi'_n \rangle. \tag{1}$$

The iterated derivative $\nabla^k f$ with $k > 1$ is a random variable with values in $H^{\odot k}$. Its domain $W^{k,p}$ coincides with the closure of Malliavin-smooth random variables with respect to the graph-norm

$$\|f\|_{k,p} = (\mathsf{E}\,|f|^p)^{1/p} + (\mathsf{E}\,\|\nabla^k f\|^p_{H^{\odot k}})^{1/p}, \quad p \in [1, \infty).$$

The operators $\nabla^k \colon W^{k,p} \to L^p(H^{\odot k})$ are closed and $\bigcap_{k=0}^\infty W^{k,p}$ is dense in $L^p(\Omega, \mathcal{F}, P)$ because one contains all Hermite polynomials (see e.g. [25, 1.5]).

Definition 1 A function $f \in \bigcap_{k=0}^\infty W^{k,p}$ with $p \in (1, \infty)$ of Gaussian random variables on $(\Omega, \mathcal{F}, P)$ we call the exponential type $\nu > 0$ with respect to the Malliavin derivative ∇ if the power series

$$\hat{f}(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!} \left(\mathsf{E}\,\|\nabla^k f\|^p_{H^{\odot k}}\right)^{2/p}, \quad \left(\mathsf{E}\,|\nabla^0 f|^p\right)^{2/p} = \|f\|_p^2 \tag{2}$$

is an entire analytic function in the complex variable $z \in \mathbb{C}$ of the exponential type ν, that is, for which the following condition is satisfied (see, e.g. [4, Theorem 1.1.1]),

$$\nu = \limsup_{r \to \infty} \frac{\ln \mu(r)}{r} \quad \text{with} \quad \mu(r) = \max_{|z|=r} |\hat{f}(z)|.$$

Definition 1 can be considered as a generalization of analytic vectors for a linear unbounded operator in the sense of Nelson (see [21]) on the case of derivative ∇. As we will see below, the series

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} \nabla^k f, \quad f \in \bigcap_{k=0}^{\infty} W^{k,p}$$

is pointwise absolutely convergent on a non-trivial dense subspace of L^p.

Definition 2 (i) Let $\mathscr{E}^{\nu,p}$ with $p \in (1,\infty)$ be the subspace in L^p of functions f in random variables with the finite Hilbertian norm

$$\|f\|_{\mathscr{E}^{\nu,p}} = \left(\sum_{k=0}^{\infty} \frac{1}{\nu^{2k}} \left(\mathsf{E}\,\|\nabla^k f\|^p_{H^{\odot k}} \right)^{2/p} \right)^{1/2}, \quad \nu > 0. \tag{3}$$

(ii) Let the subspace of functions f in L^p,

$$\mathscr{E}^p = \bigcup_{\nu>0} \mathscr{E}^{\nu,p},$$

is endowed with the quasi-norm

$$|f|_{\mathscr{E}^p} = \|f\|_p + \inf\left\{\nu > 0\colon\, f \in \mathscr{E}^{\nu,p}\right\}. \tag{4}$$

Theorem 1 (a) *The spaces* $(\mathscr{E}^{\nu,p}, \|\cdot\|_{\mathscr{E}^{\nu,p}})$ *and* $(\mathscr{E}^p, |\cdot|_{\mathscr{E}^p})$ *are complete.*

(b) *Each restriction* $\nabla|_{\mathscr{E}^{\nu,p}}$ *is a linear operator with a finite norm* $\le \nu$ *on the space* $(\mathscr{E}^{\nu,p}, \|\cdot\|_{\mathscr{E}^{\nu,p}})$. *The following contractive inclusions hold,*

$$\mathscr{E}^{\nu,p} \hookrightarrow \mathscr{E}^{\mu,p} \hookrightarrow L^p, \quad \mu > \nu > 1. \tag{5}$$

(c) *The space* $\mathscr{E}^p$ *with* $p \in (1,\infty)$ *is dense in* L^q *for any* $q \in (0,\infty)$.

(d) *The interpolation couple* $(\mathscr{E}^{p_0}, L^{p_1})$ *for any* $p_0 \in (1,\infty)$ *and* $p_1 \in (0,\infty)$ *is compatible.*

Proof (a) Check that $|\cdot|_{\mathscr{E}^p}$ is a quasi-norm. For any $f \in \mathscr{E}^{t,p}$ and $g \in \mathscr{E}^{s,p}$,

$$\begin{aligned} |f+g|_{\mathscr{E}^p} &= \|f+g\|_p + \inf\left\{t+s > 0\colon\, f+g \in \mathscr{E}^{(t+s),p}\right\} \\ &\le \|f\|_p + \|g\|_p + \inf\left\{t+s > 0\colon\, f \in \mathscr{E}^{t,p},\ g \in \mathscr{E}^{s,p}\right\} \le |f|_{\mathscr{E}^p} + |g|_{\mathscr{E}^p}. \end{aligned}$$

This in particular ensures that $\mathscr{E}^p$ is a quasi-normed linear subspace.

Prove the completeness of the space $\mathscr{E}^{\nu,p}$. Let (f_n) be a fundamental sequence in $\mathscr{E}^{\nu,p}$, i.e.,

$$\forall\varepsilon > 0,\ \exists n_\varepsilon\colon \|f_n - f_m\|_{\mathscr{E}^{\nu,p}} < \varepsilon \quad \text{for all} \quad n, m > n_\varepsilon.$$

From the representation (3) for $\|\cdot\|^2_{\mathscr{E}^{\nu,p}}$ as a sum of positive addends, it follows that the sequences (f_n) and $\left(\nabla^k f_n/\nu^k\right)$ with $k \ge 1$ are fundamental in L^p and $L^p(H^{\odot k})$, respectively.

Hence, there are elements $f \in L^p$ and $g_k \in L^p(H^{\odot k})$ such that $f_n \to f$ in L^p and $\nabla^k f_n/\nu^k \to g_k$ in $L^p(H^{\odot k})$ for any $k \ge 1$. By closeness of ∇^k, the equality $g_k = \nabla^k f/\nu^k$ holds, i.e.,

$$\nabla^k f_n/\nu^k \underset{n\to\infty}{\longrightarrow} \nabla^k f/\nu^k \quad \text{for all} \quad k \ge 0.$$

Since $\|f_n\|_{\mathscr{E}^{\nu,p}} \le \|f_n - f_{n_\varepsilon}\|_{\mathscr{E}^{\nu,p}} + \|f_{n_\varepsilon}\|_{\mathscr{E}^{\nu,p}} \le \varepsilon + \|f_{n_\varepsilon}\|_{\mathscr{E}^{\nu,p}}$ for all $n \ge n_\varepsilon$, we find $\|f\|_{\mathscr{E}^{\nu,p}} \le \varepsilon + \|f_{n_\varepsilon}\|_{\mathscr{E}^{\nu,p}}$ by taking the limit in L^p as $n \to \infty$. As a result, $f \in \mathscr{E}^{\nu,p}$, since $f_{n_\varepsilon} \in \mathscr{E}^{\nu,p}$. Thus, $\mathscr{E}^{\nu,p}$ is complete.

Further, we note that the Laplace transform of a function (2) has the form

$$\mathsf{L}[\hat{f}](z) := \int_0^\infty \hat{f}(t)e^{-zt}\,dt = \sum_{k=0}^\infty \frac{1}{z^{k+1}}\left(\mathsf{E}\,\|\nabla^k f\|^p_{H^{\odot k}}\right)^{2/p}. \tag{6}$$

Hence, for the norm in $\mathscr{E}^{\nu,p}$, we get the integral representation

$$\|f\|^2_{\mathscr{E}^{\nu,p}} = \nu^2 \cdot \mathsf{L}[\hat{f}]\left(\nu^2\right), \quad \nu > 0. \tag{7}$$

Let now (f_n) be a fundamental sequence in the quasi-normed space $\mathscr{E}^p$. Hence, there exists $\nu > 0$ such that $|f_n|_{\mathscr{E}^p} < \nu$ for all $n \in \mathbb{N}$ thus

$$\inf\left\{\mu\colon (f_n) \subset \mathscr{E}^p\right\} < \nu. \tag{8}$$

It means that $(f_n) \subset \mathscr{E}^{\nu,p}$. Consider the restriction to $\mathbb{R}$ of the correspondent sequence of complex entire functions $(\hat{f}_n)$ of an exponential type ν, defined by (2). By (8), the following sequence is bounded by a constant $K_\nu > 0$,

$$\left\{[0,\infty) \ni t \longmapsto (\hat{f}_n - \hat{f}_m)(t)\exp\left(-t\nu^2\right) : n \in \mathbb{N}\right\}.$$

Hence, in accordance with Bernstein's compactness theorem [22, Theorem 3.3.6] there exists a convergent subsequence $\left\{(\hat{f}_{n_i} - \hat{f}_{m_i})(t)\exp(-t\nu^2)\colon i \in \mathbb{N}\right\}$ with respect to the uniform convergence in the variable $t \in [0, r]$ for any $r > 0$.

Thus, $\forall\varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$:

$$\sup_{t\in[0,r_\varepsilon]}\left(\hat{f}_{n_i} - \hat{f}_{m_i}\right)(t)\exp(-t\nu^2) < \varepsilon \quad \text{for all} \quad n_i, m_i \ge n_\varepsilon,$$

where $r = r_\varepsilon$ is chosen large enough that $K_\nu \exp\left(-r_\varepsilon\nu^2\right) < \varepsilon$. Using the integral representation (7), we obtain

$$\begin{aligned}\left\|f_{n_i} - f_{m_i}\right\|^2_{\mathscr{E}^{2\nu,p}} &\le 4\nu^2\left(\int_0^{r_\varepsilon} + \int_{r_\varepsilon}^\infty\right)\left(\hat{f}_{n_i} - \hat{f}_{m_i}\right)(t)\exp(-2t\nu^2)\,dt\\ &\le 4\nu^2\varepsilon\int_0^{r_\varepsilon}\exp\left(-t\nu^2\right)dt + 4\nu^2 K_\nu\int_{r_\varepsilon}^\infty \exp\left(-t\nu^2\right)dt < 8\varepsilon\end{aligned}$$

for all $n_i, m_i \ge n_\varepsilon$. As a result, (f_{n_i}) is fundamental in $\mathscr{E}^{2\nu,p}$. According to the completeness of $\mathscr{E}^{2\nu,p}$, there exists an element $f \in \mathscr{E}^{2\nu,p}$ such that $f_{n_i} \to f$ as $i \to \infty$. Thus $\mathscr{E}^p$ is complete.

(b) First note that according to the known classical formula (see e.g. [4, Theorem 1.1.1]), the function (2) has the exponential type ν^2 if and only if its Laplace transform (6) satisfies the following condition

$$\nu = \limsup_{k\to\infty} \left(\mathsf{E} \, \|\nabla^k f\|^p_{H^{\odot k}} \right)^{1/pk}. \tag{9}$$

It follows, in particular, that the formula (3) defines the norm on the space $\mathscr{E}^{\nu,p}$ correctly. Moreover, using that for every $f \in \mathscr{E}^{\nu,p}$ the inequality

$$\|\nabla f\|^2_{\mathscr{E}^{\nu,p}} = \nu^2 \sum_{k=0}^{\infty} \frac{1}{\nu^{2k}} \left(\mathsf{E} \, \|\nabla^k f\|^p_{H^{\odot k}} \right)^{2/p} \leq \nu^2 \|f\|^2_{\mathscr{E}^{\nu,p}}$$

holds, the restriction $\nabla|_{\mathscr{E}^{\nu,p}}$ is a bounded operator with a norm $\leq \nu$. The recursive reasoning gives $\|\nabla^k f\|_{\mathscr{E}^{\nu,p}} \leq \nu^k \|f\|_{\mathscr{E}^{\nu,p}}$ for all $k \geq 0$. It follows that

$$\limsup_{k\to\infty} \|\nabla^k f\|^{1/k}_{\mathscr{E}^{\nu,p}} \leq \nu \limsup_{k\to\infty} \|f\|^{1/k}_{\mathscr{E}^{\nu,p}} = \nu.$$

Thus, for $\mu > \nu$ the following convergent series satisfies the inequality

$$\begin{aligned} \|f\|^2_p \leq \|f\|^2_p + \sum_{k=1}^{\infty} \frac{1}{\mu^{2k}} \left(\mathsf{E} \, \|\nabla^k f\|^p_{H^{\odot k}} \right)^{2/p} \\ = \|f\|^2_{\mathscr{E}^{\mu,p}} \leq \|f\|^2_{\mathscr{E}^{\nu,p}} \end{aligned}$$

that give the inclusions (5) for $\mu > \nu$.

(c) Consider the Gaussian exponential defined for random variables ϕ_h,

$$\mathscr{G}_h = \exp\left(\phi_h - \mathsf{E} \, \phi_h^2/2 \right), \quad h \in H.$$

As is known (see [15, Theorem 3.33]), the corresponding exponential series is convergent in L^2 thus in L^p for $p \in (0, \infty)$. The equality

$$\nabla \mathscr{G}_h = \mathscr{G}_h \otimes h, \quad h \in H$$

follows from the property

$$\partial_g \exp(\phi_h) \exp\left(-\mathsf{E} \, \phi_h^2/2 \right) = \langle h \mid g \rangle \exp(\phi_h) \quad \text{for all} \quad \phi_h(g) = \langle h \mid g \rangle,$$

since the expression $\exp(-\mathsf{E} \, \phi_h^2/2)$ does not depend on all $g \in H$. Hence,

$$\mathscr{G}_h \otimes \exp(th) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \nabla^k \mathscr{G}_h, \quad \exp(h) := \bigoplus_{k=0}^{\infty} \frac{1}{k!} h^{\otimes k}$$

for any $t \in \mathbb{R}$, where the tensor exponential series $\exp(h)$ is convergent in the symmetric Fock space $\Gamma(H)$. Moreover, from the formula (1) for norm in $\Gamma(H)$ it follows $\|\exp(h)\|_\Gamma = \exp\|h\|$. So, for $f = \mathscr{G}_h$, we have the representation

$$\begin{aligned}\hat{f}(t) &:= \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\mathsf{E}\,\|\nabla^k \mathscr{G}_h\|^p_{H^{\odot k}}\right)^{2/p} \\ &= \left(\mathsf{E}\,|\mathscr{G}_h|^p\right)^{2/p} \exp(t\|h\|).\end{aligned}$$

Applying the formula (9) to this power series, we get that $f = \mathscr{G}_h$ has the following exponential type

$$\begin{aligned}\nu^2 &= \limsup_{k\to\infty} \left(\mathsf{E}\,\|\nabla^k \mathscr{G}_h\|^p_{H^{\odot k}}\right)^{2/pk} \\ &= \limsup_{k\to\infty} \left(\mathsf{E}\,|\mathscr{G}_h|^p\right)^{2/pk} \|h^{\otimes k}\|^{2p/pk}_{H^{\odot k}} \\ &= \limsup_{k\to\infty} \|h^{\otimes k}\|^{2/k}_{H^{\odot k}} = \|h\|^2_H.\end{aligned}$$

Hence, $\mathscr{G}_h \in \mathscr{E}^{\|h\|,p}$ for any $h \in H$.

On the other side, it is known (see [15, Theorem 2.12 & Corollary 3.40]) that

$$\left\{\mathscr{G}_h : h = \mathfrak{e}_i \in H,\ i \in \mathbb{N}\right\}$$

is total in L^q for any $q \in (0, \infty)$, where $\{\mathfrak{e}_i\}$ is an orthogonal basis in H. More specific, it follows from the fact that the family of all Hermite polynomials $\mathfrak{h}_n$ in random variables $\left\{\mathfrak{h}_n(\phi_h) : h \in H, n \in \mathbb{N} \cup \{0\}\right\}$ is total L^q for any $q \in (0, \infty)$, since L^q-norms are proportional to the L^2-norm (see [15, Theorem 1.4]). As a result, the subspace

$$\mathscr{E}^p = \bigcup_{h\in H} \mathscr{E}^{\|h\|,p}$$

with $p \in (1, \infty)$ is dense in L^q for any $q \in (0, \infty)$.

(d) This statement is a direct conclusion of (c). In fact, the couple quasi-normed spaces $(\mathscr{E}^{p_0}, L^{p_1})$ on the same $(\Omega, \mathcal{F}, P)$ can be consider as a dense subspace in the algebraic sum of spaces $L^{p_0} + L^{p_1}$ endowed with the quasi-norm

$$\|f\|_{L^{p_0}+L^{p_1}} = \inf_{f=f_0+f_1} \left(\|f_0\|_{L^{p_0}} + \|f_1\|_{L^{p_1}}\right)$$

which guarantees the compatibility (see e.g. [2, Lemma 3.10.3] or [16, no 1]). □

3 Exact Estimates of Best Approximations on Gaussian Hilbert Spaces

In what follows, our goal is to prove the inverse and direct approximation theorems on Gaussian Hilbert spaces by Malliavin-entire functions of random variables in the form of Bernstein–Jackson inequalities with exact constants.

Given the compatible interpolation couple of quasi-normed Gaussian spaces

$$(\mathscr{E}^{p_0}, L^{p_1}) \quad \text{with} \quad p_0 \in (1, \infty) \quad \text{and} \quad p_1 \in (0, \infty),$$

we define the best approximation E-functional

$$\begin{aligned} E(t, f) &:= E\left(t, f; \mathscr{E}^{p_0}, L^{p_1}\right) \\ &= \inf\left\{\|f - f_0\|_{p_1} : |f_0|_{\mathscr{E}^{p_0}} < t\right\}, \quad f \in L^{p_1}, \end{aligned} \tag{10}$$

where $f = f_0 + f_1$ belongs to the algebraic sum $\mathscr{E}^{p_0} + L^{p_1}$ such that $f_0 \in \mathscr{E}^{p_0}$ and $f_1 \in L^{p_1}$. For any pairs indexes

$$\{0 < \theta < 1,\ 0 < q < \infty\} \quad \text{or} \quad \{0 < \theta \leq 1,\ q = \infty\}$$

the corresponding best approximation scale is defined to be the following scale of quasi-normed Gaussian spaces

$$\begin{aligned} E_{\theta,q}(\mathscr{E}^{p_0}, L^{p_1}) &= \left\{f \in \mathscr{E}^{p_0} + L^{p_1} : \|f\|_{E_{\theta,q}} < \infty\right\}, \\ \|f\|_{E_{\theta,q}} &= \begin{cases} \left(\displaystyle\int_0^\infty \left[t^{-1+1/\theta} E(t, f)\right]^{q\theta} \frac{dt}{t}\right)^{1/q\theta} & \text{if } q < \infty, \\ \displaystyle\sup_{0<t<\infty} t^{-1+1/\theta} E(t, f) & \text{if } q = \infty. \end{cases} \end{aligned} \tag{11}$$

It is natural to call the space $E_{\theta,q}(\mathscr{E}^{p_0}, L^{p_1})$ approximation for the compatible interpolation couple $(\mathscr{E}^{p_0}, L^{p_1})$ on the same probability space $(\Omega, \mathcal{F}, P)$.

In what follows, we will prove that the approximation constants

$$C_{\theta,q} = \begin{cases} \dfrac{2^{1/2\theta}}{(q^2\theta)^{1/q\theta}} N_{\theta,q}^{1/\theta} & \text{if } q < \infty, \\ \quad 2^{1/2\theta} & \text{if } q = \infty, \end{cases} \tag{12}$$

determined by the normalization factor of Lions–Peetre's interpolation method,

$$N_{\theta,q} = \left(\int_0^\infty t^{-\theta} |g(t)|^q \frac{dt}{t}\right)^{-1/q}, \quad g(t) = \frac{t}{\sqrt{1 + t^2}}, \tag{13}$$

are exact for both Bernstein–Jackson inequalities. Note that $N_{\theta,q} = N_{1-\theta,q}$ (see e.g. [17, p. 99]). The approximation constant for $q = 2$ receives the form

$$C_{\theta,2} = \left(\frac{\sin \pi\theta}{\pi\theta}\right)^{1/2\theta}, \quad 0 < \theta < 1. \tag{14}$$

In fact, by integrating the above functions (see e.g. [20, Example B.5, Theorem B.7] or [17, p. 99]) it follows that for $q = 2$ the normalization factor (14) in the interpolation K-method employed here is equal to $N_{\theta,2} = (2 \sin \pi\theta/\pi)^{1/2}$.

The following approximation theorem is based on analytical properties of an exponential type of Gaussian random variables with respect to the Malliavin derivative which were established in Theorem 1.

Theorem 2 (a) *The Bernstein–Jackson bilateral inequalities*

$$t^{-1+1/\theta} E(t, f) \leq C_{\theta,q} \|f\|_{E_{\theta,q}} \leq 2^{1/2\theta} |f|_{\mathscr{E}^{p_0}}^{-1+1/\theta} \|f\|_{p_1} \tag{15}$$

with the approximation constant (12) for all $f \in \mathscr{E}^{p_0} \cap L^{p_1}$ hold.

(b) *The following isomorphism is valid up to norm equivalence,*

$$E_{\theta,q}(\mathscr{E}^{p_0}, L^{p_1}) \simeq K_{\theta,q} \left(\mathscr{E}^{p_0}, L^{p_1}\right)^{1/\theta}. \tag{16}$$

(c) *There is a unique extension of the left inequality in (15) to the following Jackson-type inequality on the whole Gaussian approximative space*

$$E(t, f) \leq t^{1-1/\theta} C_{\theta,q} \|f\|_{E_{\theta,q}} \quad \textit{for all} \quad f \in E_{\theta,q}(\mathscr{E}^{p_0}, L^{p_1}). \tag{17}$$

Proof (a) We will use the classical integral of the Lions–Peetre real interpolation method

$$\|f\|_{\theta,q} = \left(\int_0^\infty t^{-q\theta} |f(t)|^q \frac{dt}{t}\right)^{1/q}, \quad 0 < \theta < 1,\ 1 \leq q < \infty. \tag{18}$$

Consider the quadratic K-functional (see e.g. [17] or [20, App. B]) for the interpolation couple of quasi-normed spaces $(\mathscr{E}^{p_0}, L^{p_1})$ with $p_0 \in [1, \infty)$ and $p_1 \in (0, \infty)$,

$$\begin{aligned} K(t, f) &:= K(t, f; \mathscr{E}^{p_0}, L^{p_1}) \\ &= \inf_{f = f_0 + f_1} \left\{ \left(|f_0|^2_{\mathscr{E}^{p_0}} + t^2 \|f_1\|^2_{p_1}\right)^{1/2} : f_0 \in \mathscr{E}^{p_0}, f_1 \in L^{p_1} \right\}, \end{aligned}$$

determining the real interpolation space (both alternative notations are used),

$$\begin{aligned} \left(\mathscr{E}^{p_0}, L^{p_1}\right)_{\theta,q} &:= K_{\theta,q} \left(\mathscr{E}^{p_0}, L^{p_1}\right) \\ &= \left\{ f = f_0 + f_1 : \|K(\cdot, f)\|_{\theta,q} < \infty \right\} \end{aligned}$$

which is endowed with the norm

$$\|f\|_{K_{\theta,q}} = \begin{cases} N_{\theta,q}\|K(\cdot, f)\|_{\theta,q} & \text{if } q < \infty, \\ \sup\limits_{t\in(0,\infty)} t^{-\vartheta}K(t, f) & \text{if } q = \infty. \end{cases}$$

First, let $0 < q < \infty$. By integration both sides of the following inequality

$$g\left(\frac{v}{t}\right)^q K(t, f)^q \le K(v, f)^q,$$

we successively find

$$\int_0^\infty v^{-q\theta} g\left(\frac{v}{t}\right)^q \frac{dv}{v} K(t, f)^q \le \int_0^\infty v^{-q\theta} K(v, f)^q \frac{dv}{v} = N_{\theta,q}^{-q}\|f\|_{K_{\theta,q}}^q,$$
$$\int_0^\infty v^{-q\theta} g\left(\frac{v}{t}\right)^q \frac{dv}{v} = (t^\theta N_{\theta,q})^{-q},$$
$$\int_0^\infty v^{-q\theta} g\left(\frac{v}{t}\right)^q \frac{dv}{v} K(t, f)^q = \frac{K(t, f)^q}{(t^\theta N_{\theta,q})^q} \le \|f\|_{K_{\theta,q}}^q.$$

After summing up, it follows the inequality

$$K(t, f) \le t^\theta N_{\theta,q}^{-1}\|f\|_{K_{\theta,q}}, \quad f \in \left(\mathscr{E}^{p_0}, L^{p_1}\right)_{\theta,q}, \ t > 0. \tag{19}$$

Let $K_\infty(t, f) := \inf_{f=f_0+f_1} \max\left\{|f_0|_{\mathscr{E}^{p_0}}, t\|f_1\|_{p_1}\right\}$. It is easy to see that

$$K_\infty(t, f) \le K(t, f) \le 2^{1/2}K_\infty(t, f), \quad f \in \left(\mathscr{E}^{p_0}, L^{p_1}\right)_{\theta,q}. \tag{20}$$

By [2, Lemma 7.1.2] for every $t > 0$ there exists $v > 0$ such that

$$v^{-1+1/\theta}E(v, f)^\theta \le t^{-\theta}K_\infty(t, f). \tag{21}$$

Using (19) and (21), we get

$$v^{1-\theta}E(v, f)^\theta \le t^{-\theta}K_\infty(t, f) \le t^{-\theta}K(t, f) \le N_{\theta,q}^{-1}\|f\|_{K_{\theta,q}}.$$

It follows that

$$v^{-1+1/\theta}E(v, f) \le N_{\theta,q}^{-1/\theta}\|f\|_{K_{\theta,q}}^{1/\theta}, \quad f \in \left(\mathscr{E}^{p_0}, L^{p_1}\right)_{\theta,q}, \quad v > 0. \tag{22}$$

Integrating by parts with the change of variables $v = t/E(t, f)$ and using the known properties of functionals that $v^{-\theta}K_\infty(v, f) \to 0$ as $v \to 0$ or $v \to \infty$ and $t^{-1+1/\theta}E(t, f) \to 0$ as $t \to 0$ or $t \to \infty$ (see [2, Theorem 7.1.7]), we get

$$\int_0^\infty \left(v^{-\theta}K_\infty(v, f)\right)^q \frac{dv}{v} = -\frac{1}{q\theta}\int_0^\infty K_\infty(v, f)^q dv^{-q\theta}$$

$$= \frac{1}{q\theta}\int_0^\infty v^{-q\theta}\, dK_\infty(v,x)^q = \frac{1}{q\theta}\int_0^\infty \left(\frac{t}{E(t,f)}\right)^{-q\theta} dt^q$$
$$= \frac{1}{q^2\theta}\int_0^\infty \left(t^{-1+1/\theta}E(t,f)\right)^{q\theta}\frac{dt}{t}.$$

Therefore, according to the first inequality (20) and the notation (11),

$$\begin{aligned}\frac{1}{q^2\theta}\|f\|_{E_{\theta,q}}^{q\theta} &= \frac{1}{q^2\theta}\int_0^\infty \left(t^{-1+1/\theta}E(t,f)\right)^{q\theta}\frac{dt}{t}\\ &= \int_0^\infty \left(v^{-\theta}K_\infty(v,f)\right)^q \frac{dv}{v}\\ &\le \int_0^\infty \left(v^{-\theta}K(v,f)\right)^q \frac{dv}{v} = N_{\theta,q}^{-q}\|f\|_{K_{\theta,q}}^q.\end{aligned}$$

On the other hand, from the second inequality (20) it follows

$$\begin{aligned}N_{\theta,q}^{-q}\|f\|_{K_{\theta,q}}^q &= \int_0^\infty \left(v^{-\theta}K(v,f)\right)^q \frac{dv}{v}\\ &\le 2^{q/2}\int_0^\infty \left(v^{-\theta}K_\infty(v,f)\right)^q \frac{dv}{v}\\ &= 2^{q/2}\frac{1}{q^2\theta}\int_0^\infty \left(t^{-1+1/\theta}E(t,f)\right)^{q\theta}\frac{dt}{t} = 2^{q/2}\frac{1}{q^2\theta}\|f\|_{E_{\theta,q}}^{q\theta}.\end{aligned}$$

Taking the root and combining the previous inequalities, we get

$$N_{\theta,q}^{-1}\|f\|_{K_{\theta,q}} \le 2^{1/2}(q^2\theta)^{-1/q}\|f\|_{E_{\theta,q}}^{\theta} \le 2^{1/2}N_{\theta,q}^{-1}\|f\|_{K_{\theta,q}}. \tag{23}$$

As a result, we obtain the isomorphism (16), which proves the claim (b), i.e., that

$$E_{\theta,q}(\mathscr{E}^{p_0}, L^{p_1}) \simeq \left(\mathscr{E}^{p_0}, L^{p_1}\right)_{\theta,q}^{1/\theta} := K_{\theta,q}\left(\mathscr{E}^{p_0}, L^{p_1}\right)^{1/\theta}.$$

Let $\alpha = |f|_{\mathscr{E}^{p_0}}/\|f\|_{p_1}$. Since $K(t,f) \le \min\left\{|f|_{\mathscr{E}^{p_0}}, t\|f\|_{p_1}\right\}$, we find

$$\begin{aligned}N_{\theta,q}^{-q}\|f\|_{K_{\theta,q}}^q &\le \|f\|_{p_1}^q\int_0^\alpha t^{-1+q(1-\theta)}dt + |f|_{\mathscr{E}^{p_0}}^q\int_\alpha^\infty t^{-1-\theta q}\,dt\\ &= \frac{1}{q(1-\theta)}\alpha^{q(1-\theta)}\|f\|_{p_1}^q + \frac{1}{\theta q}\alpha^{-\theta q}|f|_{\mathscr{E}^{p_0}}^q\\ &= \frac{\|f\|_{p_1}^q}{q(1-\theta)}\left(\frac{|f|_{\mathscr{E}^{p_0}}}{\|f\|_{p_1}}\right)^{q(1-\theta)} + \frac{|f|_{\mathscr{E}^{p_0}}^q}{\theta q}\left(\frac{|f|_{\mathscr{E}^{p_0}}}{\|f\|_{p_1}}\right)^{-\theta q}\\ &= \frac{1}{q\theta(1-\theta)}\left(|f|_{\mathscr{E}^{p_0}}^{1-\theta}\|f\|_{p_1}^{\theta}\right)^q.\end{aligned}$$

Taking the root above, this can be rewritten as

$$N_{\theta,q}^{-1}\|f\|_{K_{\theta,q}} \le [q\theta(1-\theta)]^{-1/q}|f|_{\mathscr{E}^{p_0}}^{1-\theta}\|f\|_{p_1}^{\theta}, \tag{24}$$

where $[q\theta(1-\theta)]^{-1/q} = \|\min\{1,\cdot\}\|_{\theta,q}$. Applying the integral (18) to the inequality $2^{-1/2}\min\{1,\cdot\} \le g(\cdot)$, we find $N_{\theta,q} \le 2^{1/2}\|\min\{1,\cdot\}\|_{\theta,q}^{-1}$. Hence,

$$\begin{aligned}[q\theta(1-\theta)]^{-1/q} &= \|\min\{1,\cdot\}\|_{\theta,q} \\ &\le 2^{1/2}\|g\|_{K_{\theta,q}} = 2^{1/2}N_{\theta,q}^{-1}.\end{aligned}$$

Now, by the second inequality in (23) and (12),(24), we find that

$$\begin{aligned}(q^2\theta)^{-1/q}\|f\|_{E_{\theta,q}}^{\theta} &\le N_{\theta,q}^{-1}\|f\|_{K_{\theta,q}} \le 2^{1/2}N_{\theta,q}^{-1}|f|_{\mathscr{E}^{p_0}}^{1-\theta}\|f\|_{p_1}^{\theta}, \\ C_{\theta,q}\|f\|_{E_{\theta,q}} &\le 2^{1/2\theta}|f|_{\mathscr{E}^{p_0}}^{-1+1/\theta}\|f\|_{p_1}.\end{aligned}$$

On the other hand, by (22), we have

$$v^{-1+1/\theta}E(v,f) \le N_{\theta,q}^{-1+1/\theta}\|f\|_{K_{\theta,q}}^{1/\theta} \le C_{\theta,q}\|f\|_{E_{\theta,q}}.$$

Combining the last inequalities, we get the desired inequality (15).

Let us consider the case $q=\infty$. Denote $\alpha = \|f\|_{p_1}/|f|_{\mathscr{E}^{p_0}}$ with a nonzero element $f \in \mathscr{E}^{p_0} \cap L^{p_1}$. Since

$$\begin{aligned}K(t,f)^2 &= \inf_{f=f_0+f_1}\left(|f_0|_{\mathscr{E}^{p_0}}^2 + t^2\|f_1\|_{p_1}^2\right) \\ &\le |f|_{\mathscr{E}^{p_0}}^2\min(1,\alpha^2t^2) = \min\left(|f|_{\mathscr{E}^{p_0}}^2, t^2\|f\|_{p_1}^2\right)\end{aligned}$$

or otherwise $K(t,f) \le |f|_{\mathscr{E}^{p_0}}\min(1,\alpha t) = \min\left(|f|_{\mathscr{E}^{p_0}}, t\|f\|_{p_1}\right)$, we get

$$t^{-\vartheta}K(t,f) \le \min\left(t^{-\vartheta}|f|_{\mathscr{E}^{p_0}}, t^{1-\vartheta}\|f\|_{p_1}\right).$$

Taking $t = |f|_{\mathscr{E}^{p_0}}/\|f\|_{p_1}$, we obtain

$$t^{-\vartheta}K(t,f) \le |f|_{\mathscr{E}^{p_0}}^{1-\vartheta}\|f\|_{p_1}^{\vartheta}.$$

So, the right side inequality in (15) holds. On the other hand,

$$t^{-1+1/\theta}E(t,f) \le \sup_{t>0} t^{-1+1/\theta}E(t,f) = \|f\|_{E_{\theta,\infty}}, \quad f \in E_{\theta,\infty}.$$

Thus, the inequality (17) also is valid for this case.

(c) By Theorem 1, $\mathscr{E}^{p_0}$ is complete, so $K_{\theta,q}(\mathscr{E}^{p_0}, L^{p_1})$ is complete, as interpolation of complete spaces. □

Remark 1 The relationship between the weight function $g(t) = t^2/(1 + t^2)$ and the square K-functional, and therefore also the E-functional, is explained by the formula

$$N_{\theta,q} = \|g\|_{\theta,q}^{-1} = \|K(\cdot, 1)\|_{\theta,q}^{-1}$$

[20, Example B.4]. It follows from $\min_{z=z_0+z_1}\left(\alpha_0|z_0|^2 + \alpha_1|z_1|^2\right) = \alpha_0\alpha_1|z|^2\alpha_0 + \alpha_1$ for a fixed $\alpha_0, \alpha_1 > 0$ and a complex z. This minimum is achieved when $\alpha_0 z_0 = \alpha_1 z_1 = \alpha_0\alpha_1 z/(\alpha_0 + \alpha_1)$. Thus, $K(t, 1)$ is minimized when f_0, f_1 are such that

$$f_0 = t^2 f_1 = \frac{t^2}{1+t^2}.$$

For the space $L^2(\Omega, \mathcal{F}, P)$ previous results can be made more specific.

Corollary 3 *On the space $E_{\theta,2}(\mathscr{E}^2, L^2)$ endowed with the quasi-norm*

$$\|f\|_{E_{\theta,2}} = \left(\int_0^\infty \left[t^{-1+1/\theta} E(t, f; \mathscr{E}^2, L^2)\right]^{2\theta} \frac{dt}{t}\right)^{1/2\theta}, \quad 0 < \theta < 1,$$

defined by the best approximation E-functional

$$E(t, f; \mathscr{E}^2, L^2) = \inf\left\{\|f - f_0\|_2 \colon |f_0|_{\mathscr{E}^2} < t\right\}, \quad f \in L^2,$$

the following Bernstein–Jackson type inequalities are satisfied,

$$\|f\|_{E_{\theta,2}} \le 2^{1/2\theta}\left(\frac{\pi\theta}{\sin\pi\theta}\right)^{1/2\theta} |f|_{\mathscr{E}^2}^{-1+1/\theta}\|f\|_2, \quad f \in \mathscr{E}^2 \tag{25}$$

$$E(\nu, f) \le \nu^{1-1/\theta}\left(\frac{\sin\pi\theta}{\pi\theta}\right)^{1/2\theta} \|f\|_{E_{\theta,2}}, \quad f \in E_{\theta,2}(\mathscr{E}^2, L^2). \tag{26}$$

Proof The inequalities (25) - (26) directly follow from Theorem 2 (a,c). □

Corollary 4 *The norm on the Hilbert space $\mathscr{E}^{\nu,2}$ satisfies the equality*

$$\begin{aligned} \|f\|_{\mathscr{E}^{\nu,2}} &:= \left(\sum_{k=0}^{\infty} \frac{1}{\nu^{2k}} \mathsf{E}\,\|\nabla^k f\|_{H^{\odot k}}^2\right)^{1/2} \\ &= \|\hat{F}\|_{\mathcal{H}^2(D_\nu)}, \quad D_\nu = \{z \in \mathbb{C} \colon |z| < \nu\}, \end{aligned} \tag{27}$$

where $\|\cdot\|_{\mathcal{H}^2(D_\nu)}$ in (27) is the Hilbertian norm for analytic functions

$$\hat{F} \colon z \longmapsto \frac{1}{z} \cdot \mathsf{L}[\hat{f}]\left(\frac{1}{z}\right), \quad |z| < \nu$$

belonging to the Hardy space $\mathcal{H}^2(D_\nu)$. Herewith, the isometric isomorphism

$$\mathscr{E}^{\nu,2} \simeq \mathcal{H}^2(D_\nu), \tag{28}$$

determined by the linear mapping $f \longmapsto \hat{F}$, holds.

Proof The isometry (27) follows from the properties (2) and (7) of the Laplace transform L for entire analytic functions, as well as, from the elementary fact that for every power series $\hat{F}(z) = \sum c_k z^k$ from $\mathcal{H}^2(D_\nu)$ its norm satisfies the equality

$$\|\hat{F}\|^2_{\mathcal{H}^2(D_\nu)} = \sum |c_k|^2 \nu^{2k}.$$

The isometric equation (28) is a consequence of the equality (7) for the norm $\|\cdot\|_{\mathscr{E}^{\nu,2}}$. □

Corollary 5 *The quasi-norm on $(\mathscr{E}^2, |\cdot|_{\mathscr{E}^2})$ admits the representation*

$$\begin{aligned} |f|_{\mathscr{E}^2} &:= \inf\left\{\nu > 0 \colon f \in \mathscr{E}^{\nu,2}\right\} \\ &= \limsup_{k\to\infty} \left(\mathsf{E}\,\|\nabla^k f\|^2_{H^{\odot k}}\right)^{1/2k}. \end{aligned} \tag{29}$$

Moreover, $\mathscr{E}^2$ has also a stronger nuclear topology of the inductive limits

$$\varinjlim \mathscr{E}^{\nu,2} \simeq \varinjlim \mathcal{H}^2(D_{1/\nu}) \quad as \quad \nu \to \infty$$

with compact inclusions.

Proof The proof of (29) follows from the known formula (9) [4, Theorem 1.1.1] for Taylor coefficients of complex entire analytic functions of an exponential type, expressed through its Laplace-image (see, the proof of Theorem 1(b)).

The compactness of inclusions (5) is proved Theorem 1(c) based on Bernstein's compactness theorem [22, Theorem 3.3.6]. Nuclearity of inductive limits with compact inclusions are a well-known fact (see e.g. [29, no 7.4]). □

Corollary 6 *The 1-parameter family of linear operators $T_s \colon L^2 \to L^2(\Gamma(H))$, uniquely defined by the mapping*

$$T_s\left(\mathscr{G}_h\right) = \mathscr{G}_h \otimes e^{-s\mathcal{N}} \exp(h), \quad s > 0, \quad h \in H, \tag{30}$$

satisfies the following invariant property

$$T_s\left(\mathscr{E}^2\right) = \mathscr{E}^2 \otimes e^{-s\mathcal{N}} \exp(h), \tag{31}$$

where the number operator $\mathcal{N}$ is determined on the Fock space $\Gamma(H)$. Moreover, the derivative $\nabla \colon W^{1,2} \to L^2(H)$ coincides with a universal annihilator of T_s, i.e.,

$$\left.\frac{dT_s f}{ds}\right|_{s=0} = \nabla f, \quad f \in W^{1,2}. \tag{32}$$

Proof From the proof of Theorem 1(c), we directly get

$$T_s \mathscr{G}_h = e^{-s\nabla} \mathscr{G}_h = \mathscr{G}_h \otimes \exp(-sh)$$

and, as a consequence, (31). On the other hand, $\mathcal{N}$ is the infinitesimal generator of the 1-parameter second-quantization semigroup $\Gamma(e^{-sI_H}) = e^{-s\nabla}$ with the identical operator I_H on H (see [25, no 1.4]), thus

$$e^{-s\nabla} \mathscr{G}_h = \mathscr{G}_h \otimes e^{-s\mathcal{N}} \exp(h), \quad h \in H.$$

It follows the equality (30). The uniqueness of extension T_s onto L^2 is due to totality $\mathscr{G}_h$ in L^2 and $\mathscr{G}_h \otimes \exp(h)$ in $L^2(\Gamma(H))$.

Moreover, according to [1, no 3], the derivative ∇ is a universal annihilator and the equality (32) is valid. □

4 Application Examples

Example 1 Let us consider the space $L^p(X, \mathcal{F}, \gamma)$ with $1 \le p \le \infty$ of functions f in Gaussian random variables $X \ni x \mapsto \phi_h(x)$ for all $h \in H$, defined on the probability space $(X, \mathcal{F}, \gamma)$ over an abstract Wiener space (X, H) in the sense of Gross's theory [14]. Here, let X be a separable real Banach space, $H \subset X$ is a Cameron-Martin type reproducing kernel subspace, $\mathcal{F} = \mathcal{B}(X)$ is the Borel σ-field on X and, in addition, the probability measure γ on $\mathcal{F}$ is characterized by the property

$$\int_X \exp\left(\mathbb{i}\phi_h(x)\right) d\gamma(x) = \exp\left(-\frac{\|h\|_H^2}{2}\right), \quad x \in X.$$

The measure γ is Gaussian in the sense that each continuous linear functional $x^* \in X^*$, regarded as a random variable $x \mapsto x^*(x)$ on $(X, \mathcal{F}, \gamma)$, is Gaussian. The expectation for this case is defined to be

$$\mathsf{E} f = \int_X f \, d\gamma, \quad f \in L^p(X, \mathcal{F}, \gamma)$$

for all $p \ge 1$, where $L^p \subset L^1$ because $\gamma(X) = 1$ (see e.g. [34, Theorem 2]).

In this case, the interpolation structure of the approximative Gaussian space is described by the isomorphism

$$E_{\theta,q}(\mathscr{E}^{p_0}, L^{p_1}) \simeq K_{\theta,q}\left(\mathscr{E}^{p_0}, L^{p_1}\right)^{1/\theta}, \quad p_0 \in (1, \infty), \quad p_1 \in [1, \infty],$$

where $\{0 < \theta < 1, \ 0 < q < \infty\}$ or $\{0 < \theta \le 1, \ q = \infty\}$. According to Theorem 2 and Corrolary 3 the Bernstein–Jackson inequalities take the form

$$\|f\|_{E_{\theta,q}} \le 2^{1/2\theta} C_{\theta,q}^{-1} |f|_{\mathscr{E}^{p_0}}^{-1+1/\theta} \|f\|_{p_1}, \quad f \in \mathscr{E}^{p_0} \cap L^{p_1},$$

$$E(t,f) \le t^{1-1/\theta} C_{\theta,q} \|f\|_{E_{\theta,q}}, \quad f \in E_{\theta,q}(\mathscr{E}^{p_0}, L^{p_1})$$

with the exact approximation constant $C_{\theta,q}$ of the form (12), or (14) for the case $p_0 = p_1 = 2$.

Example 2 A special case of Example 1 is obtained for $X = (\mathbb{R}^d, |\cdot|)$. Consider the Banach space $L^p = L^p(\mathbb{R}^d, \mathcal{F}, \gamma_d)$ with $p \in (1,\infty)$ and $\mathcal{F} = \mathcal{B}(\mathbb{R}^d)$, which is just the space of measurable functions in random variables relative to the gaussian measure

$$\gamma_d(x) = (2\pi)^{-d/2} e^{-|x|^2/2}, \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

Each function $G \in L^p$ can be approximated by entire analytic functions g of an exponential type $t > 0$ with restrictions to $\mathbb{R}^d$ belonging to L^p (see e.g. [22]). The best approximations can be characterized by the functional

$$E\left(t, G; \mathscr{E}^p, L^p\right) = \inf\left\{\|G-g\|_p : |g|_{\mathscr{E}^p} < t\right\},$$

where the subspace $\mathscr{E}^p = \bigcup_{t>0} \mathscr{E}^{p,t}$ of L^p is endowed with the quasi-norm

$$|g|_{\mathscr{E}^p} = \|g\|_p + \sup\left\{|\zeta| : \zeta \in \operatorname{supp} \hat{g}\right\}, \quad g \in \mathscr{E}^p, \tag{33}$$

defined using the support of the Fourier-image $\hat{g}$ (see [2, no 7.2]). By Paley–Wiener theorem for entire analytic functions of an exponential type, this quasi-norm can be rewritten in the form (4) (see e.g. [9]).

Now, taking any cylindrical random function $f_g = g(\phi_{h_1}, \dots, \phi_{h_d})$ determined by functions g of an exponential type t and applying the formula (9), we get

$$\begin{aligned}
&\left(\mathsf{E}\,\|\nabla^k f_g\|^p_{H^{\odot k}}\right)^{1/pk} \\
&= \limsup_{k\to\infty} \left(\int_{\mathbb{R}^d} \left|\partial^{k_1+\ldots+k_d}_{h_1,\ldots,h_d} g\right|^p d\gamma_d\right)^{1/pk} \left\|h_1^{\otimes k_1} \odot \ldots \odot h_d^{\otimes k_d}\right\|^{1/k}_{H^{\odot k}} \\
&= t \cdot \limsup_{k\to\infty} \left(\frac{1}{d!}\|h_1\|^{k_1}_H \ldots \|h_d\|^{k_d}_H\right)^{1/k} = t, \quad k = k_1 + \ldots + k_d,
\end{aligned}$$

i.e., $f_g \in \mathscr{E}^{p,t}$. The subspace of all functions f_g with such g and any t is dense in $\mathscr{E}^p$, since it contains all polynomials of the random variables $\phi_{h_1}, \dots, \phi_{h_d}$.

On the other hand, it is known that if the space $\mathscr{E}^p$, consisting of all entire analytic functions g of an exponential type on $\mathbb{C}^d$, is endowed with the quasi-norm (33) then the suitable approximation space $E_{\theta,q}(\mathscr{E}^p, L^p)$ exactly coincides with the classic Besov space denoted by

$$B^s_{p,\tau}(\mathbb{R}^d) \quad \text{with} \quad s = -1 + 1/\theta, \quad \tau = q\theta$$

(see [33, p. 197]). Hence, the equality (16) from Theorem 2 may be rewritten in the form

$$E_{\theta,q}(\mathscr{E}^p, L^p) = \left\{ f_g \in L^p \left(\mathbb{R}^d, \mathcal{B}, \gamma_d \right) : g \in B^s_{p,\tau}(\mathbb{R}^d) \right\}.$$

Then the corresponding Bernstein–Jackson inequalities take the form

$$\begin{aligned} \|f_g\|_{E_{\theta,q}} &\le 2^{1/2\theta} C^{-1}_{\theta,q} |f_g|^{-1+1/\theta}_{\mathscr{E}^p} \|g\|_{L^p(\mathbb{R}^d)}, \quad f_g \in \mathscr{E}^p, \\ E(t, f_g) &\le t^{1-1/\theta} C_{\theta,q} \|g\|_{B^s_{p,\tau}(\mathbb{R}^d)}, \quad g \in B^s_{p,\tau}(\mathbb{R}^d), \end{aligned}$$

where the constant $C_{\theta,q}$ has the form (12), or (14) for the case $p = 2$.

Remark 2 The last example also shows that the Gaussian space

$$E_{\theta,q}(\mathscr{E}^{p_0}, L^{p_1}) \simeq K_{\theta,q} \left(\mathscr{E}^{p_0}, L^{p_1} \right)^{1/\theta}, \quad p_0 \in (1, \infty), \quad p_1 \in (0, \infty)$$

with $\{0 < \theta < 1,\ 0 < q < \infty\}$ or $\{0 < \theta \le 1,\ q = \infty\}$, which characterizes the best approximations in $L^{p_1} = L^{p_1}(\Omega, \mathcal{F}, P)$ with two-sided precision by entire analytic functions relative to the Malliavin derivative, are the closest generalization of Besov spaces on the case of functions in Gaussian random variables.

Significant new generalizations and connections between the approximation and Besov-type spaces in a wider context are presented in [13] (see also references therein).

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Declarations

Conflict of interest There are no conflicts and potential competing of interest to disclose.

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