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AUTHOR CORRECTION

Correction: Maximal Operator in Variable Stummel Spaces

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We use all the definitions and notations from [1]. In particular, we say that a function $g : \mathbb{R}^n \to \mathbb{R}$ is *locally* log-*Hölder continuous* if there exists $c_{\log}(g) \geq 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{\log}(g)}{\log(\mathrm{e} + 1/|x - y|)}, \quad \text{for all } x,\ y \in \mathbb{R}^n. \tag{1.1}$$

The function g is said to satisfy the log-*Hölder continuity condition at infinity*, also known as the *decay condition*, if there exist $g_\infty \in \mathbb{R}$ and $c^*_{\log}(g) \geq 0$ such that

$$|g(x) - g_\infty| \leq \frac{c^*_{\log}(g)}{\log(\mathrm{e} + |x|)}, \quad \text{for all } x \in \mathbb{R}^n. \tag{1.2}$$

In what follows, we always take $c_{\log}(g)$ and $c^*_{\log}(g)$ as the smallest constant satisfying (1.1) and (1.2), respectively.

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We say that g is log-*Hölder continuous* when it satisfies conditions (1.1) and (1.2) simultaneously. The class $\mathcal{P}^{\log}(\mathbb{R}^n)$ collects all (measurable) bounded exponents $p : \mathbb{R}^n \to [1, \infty)$ which are log-Hölder continuous.

Given a set $\Omega \subseteq \mathbb{R}^n$, let $\mathfrak{S}_\Omega^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ be the Stummel space consisting of all measurable functions f on $\mathbb{R}^n$ such that

$$\left\| f \right\|_{\mathfrak{S}_\Omega^{p(\cdot),\lambda(\cdot)}} := \sup_{x\in\Omega} \left\| \frac{f}{|x-\cdot|^{\lambda(\cdot)}} \right\|_{p(\cdot)} < \infty.$$

When $\Omega = \mathbb{R}^n$ we write simply $\|\cdot\|_{\mathfrak{S}^{p(\cdot),\lambda(\cdot)}}$ for the norm and $\mathfrak{S}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ for the space (as considered in [1]).

The corrected version of [1, Theorem 5.2] runs as follows:

Theorem 1.1 Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$, λ satisfies (1.1), (1.2) with $\lambda^- \ge 0$, and $(\lambda p)^+ < n$. Then, for a bounded set Ω, there holds

$$M : \mathfrak{S}_\Omega^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n) \hookrightarrow \mathfrak{S}_\Omega^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n).$$

If p and λ are constant, then the result holds also with $\Omega = \mathbb{R}^n$.

In [1, Theorem 5.2] we just considered $\Omega = \mathbb{R}^n$. That result holds, for instance, for constant exponents $p \in (1, \infty)$ and $\lambda \in [0, n/p)$. However, for variable log-Hölder exponents in general we need to accommodate the influence of translations on the decay logarithmic constants. We give details below.

The small changes needed in the proof essentially rely on additional quantitative information on the dependency of the main constants involved in [1, Lemmas 4.5, 4.7, 4.9 and Propositions 4.1, 4.8] with respect to the logarithmic constants of the exponents. We briefly summarize the quantitative estimates for those constants, which may have independent interest.

If $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $1 < p^- \le p^+ < \infty$, and $w \in \mathcal{A}_{p(\cdot)}$, then we have

$$\|(Mf)\,w\|_{p(\cdot)} \le C_M\,\|f\,w\|_{p(\cdot)}, \tag{1.3}$$

for some constant $C_M > 0$ independent of $f \in L_w^{p(\cdot)}(\mathbb{R}^n)$ (cf. [4, Theorem 1.3]). In [1, Remark 3.2] we highlighted the dependency of the constant C_M appearing in the inequality above with respect to the weight w by writing

$$\begin{aligned} C_M &= c(n,\,p) \times \left[w^{1/s}\right]^s_{\mathcal{A}_{sp(\cdot)}} \times \left[w^{-1/s}\right]^s_{\mathcal{A}_{sp'(\cdot)}} \\ &= c(n,\,p) \times \left[w^{1/s}\right]^s_{\mathcal{A}_{sp(\cdot)}} \times \left[w^{1/s}\right]^s_{\mathcal{A}_{(sp')'(\cdot)}} \end{aligned} \tag{1.4}$$

for some $c(n,\,p) > 0$ depending on n and p only. A further inspection to the proof given in [4] shows that the factor $c(n,p)$ in (1.4) grows continuously and exponentially with the constants $c_{\log}(p)$ and $c^*_{\log}(p)$ of the exponent p.

From the proof of [1, Lemma 4.5] it is not hard to check that the constant C_0 given there can be written more precisely as

$$C_0 = c(n,\ p^-,\ p^+) \max \{\mathrm{e}^{nc_{\log}(p)},\ \mathrm{e}^{6nc^*_{\log}(p)}\} \left[w(\cdot)^{p(\cdot)}\right]_{A_1}^{p^+-p^-} (1 + \varrho_{p(\cdot)}(w\mathbf{1}_{B_0})^{-1})^{p^+-p^-} \tag{1.5}$$

with $c(n,\ p^-,\ p^+) \geq 1$ independent of the logarithmic constants of p. The quantity $6nc^*_{\log}(p)$ can be slightly improved taking into account the exact value of the measure of the n-dimensional unit ball, but this is unimportant for our goals.

In [1, Lemma 4.7] the constant C_1 indicated there should be written as

$$C_1 = c(n,\ p^-,\ p^+)^{c^*_{\log}(p)} \left[w(\cdot)^{p(\cdot)}\right]_{A_1}^{\frac{2p^+}{p_\infty \cdot p^-} c^*_{\log}(p)} (1 + \varrho_{p(\cdot)}(w\,\mathbf{1}_{B_0}))^{\frac{2p^+}{p_\infty \cdot p^-} c^*_{\log}(p)}, \tag{1.6}$$

also for some constant $c(n,\ p^-,\ p^+) \geq 1$ not depending on the logarithmic constants of p.

Taking into account the explicit form of the constants in (1.5) and (1.6), the estimate given in [1, Proposition 4.8] should be written as

$$[w]_{\mathcal{A}_{p(\cdot)}} \leq c(n,\ p^-,\ p^+)^{\frac{1+c^*_{\log}(p)}{p^-}} \left[w(\cdot)^{p(\cdot)}\right]_{A_1}^{\frac{p^+}{p^-} + \frac{4p^+}{(p^-)^2} c^*_{\log}(p)} \times \times (1 + \varrho_{p(\cdot)}(w\mathbf{1}_{B_0}) + \varrho_{p(\cdot)}(w\mathbf{1}_{B_0})^{-1})^{\frac{p^+}{p^-} - 1 + \frac{4p^+}{(p^-)^2} c^*_{\log}(p)} \tag{1.7}$$

where $c(n,\ p^-,\ p^+) \geq 1$ depends only on n and the infimum and supremum of p.

Finally we point out a quantitative result on A_1 weights which generalizes [1, Lemma 4.9]:

Lemma 1.2 ([2]) Let $\gamma : \mathbb{R}^n \to [0, n)$ be a function satisfying (1.1) and (1.2), and $0 \leq \gamma^- \leq \gamma^+ < n$. Then, for each $x \in \mathbb{R}^n$, we have $|x - \cdot|^{-\gamma(\cdot)} \in A_1$ with

$$\left[|x - \cdot|^{-\gamma(\cdot)}\right]_{A_1} \leq \frac{c(n)}{n - \gamma^+} \max \{\mathrm{e}^{c_{\log}(\gamma)},\ \mathrm{e}^{c^*_{\log}(\gamma)}(\mathrm{e} + |x|)^{2c^*_{\log}(\gamma)}\}, \tag{1.8}$$

where $c(n) > 0$ depends only on the dimension $n \in \mathbb{N}$.

By the translation invariance of A_1 weights, under the same conditions of Lemma 1.2 we also have $|\cdot|^{-\gamma(x-\cdot)} \in A_1$ with the same A_1 constant.

Proof of Theorem 1.1 Following the proof given in [1, pp. 19–20], for each $x \in \Omega$, we have

$$\left\| \frac{Mf}{|x - \cdot|^{\lambda(\cdot)}} \right\|_{p(\cdot)} = \left\| \frac{M(\tau_x f)}{|\cdot|^{\lambda(x-\cdot)}} \right\|_{(\tau_x p)(\cdot)} \leq C_M \left\| \frac{\tau_x f}{|\cdot|^{\lambda(x-\cdot)}} \right\|_{(\tau_x p)(\cdot)} = C_M \left\| \frac{f}{|x - \cdot|^{\lambda(\cdot)}} \right\|_{p(\cdot)}$$

with

$$C_M \leq c(n,\ \tau_x p) \times \left[w^{1/s}\right]^s_{\mathcal{A}_{s(\tau_x p)(\cdot)}} \times \left[w^{1/s}\right]^s_{\mathcal{A}_{(s(\tau_x p)')'(\cdot)}}, \tag{1.9}$$

where $w(y) := |y|^{-\tau_x \lambda(y)}$ (and s is taken independent of x). Since, for $c^*_{\log}(p) > 0$,

$$\frac{1}{\log(\mathrm{e} + |x|)} \leq \frac{c^*_{\log}(\tau_x p)}{c^*_{\log}(p)} \leq \log(\mathrm{e} + |x|), \tag{1.10}$$

the log decay constant of $\tau_x p$ may depend on x (note that the local log-Hölder constant is translation invariant, i.e. $c_{\log}(\tau_x p) = c_{\log}(p)$). As observed before, the factor $c(n, \tau_x p)$ grows continuously and exponentially with respect to $c^*_{\log}(\tau_x p)$, so it may grow with respect to x. Taking into account the bounds in (1.7), (1.8) and (1.10), we see that the remaining factors on the right-hand side in (1.9) may depend also on x. Overall we can find uniform bounds with respect to x if it runs on bounded sets Ω. In the simpler case of constant exponents $p(x) \equiv p$ the translation as no influence on the bounds (with $c^*_{\log}(p) = 0$ in that case), so that one can admit $\Omega = \mathbb{R}^n$. □

Remark 1.3 The formulation of [1, Theorem 5.3] remains almost the same. We only have, with the additional assumption $p_\infty \lambda^+ < n$, to add Ω in the notation of the space and write

$$M : V_0 \mathfrak{S}^{p(\cdot), \lambda(\cdot)}_{\Omega}(\mathbb{R}^n) \hookrightarrow V_0 \mathfrak{S}^{p(\cdot), \lambda(\cdot)}_{\Omega}(\mathbb{R}^n)$$

instead. Regarding the proof given in [1, p. 21], the constant C that appears in the first three occurrences for the estimation of Mf_1 may depend on x via the bound of $\left[|x - \cdot|^{-\lambda(\cdot)p(\cdot)}\right]_{A_1}$ (cf. (1.8)). Consequently, we should take the supremum with respect to x on the set Ω instead on $\mathbb{R}^n$ everywhere in the proof. Full details of such proof, for the case of Riesz potentials, can be found in [3].

We take the opportunity to correct a misprint in [1, Lemma 2.1]: instead of $\mathcal{P}^{\log}(\mathbb{R}^n)$ it should be $\mathcal{P}(\mathbb{R}^n)$.

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References

1. Almeida, A., Rafeiro, H.: Maximal operator in variable Stummel spaces. J. Fourier Anal. Appl. **28**(3) Id/No 50, 24 (2022)
2. Almeida, A., Rafeiro, H.: A note on Muckenhoupt weights with nonstandard growth. Trans. Razmadze Math. Inst. **177**(1), 127–130 (2023)
3. Almeida, A., Rafeiro, H.: Fractional integral operators in variable exponent Stummel spaces (submitted), 17 pp (2023)
4. Cruz-Uribe, D., Diening, L., Hästö, P.: The maximal operator on weighted variable Lebesgue spaces. Fract. Calc. Appl. Anal. **14**(3), 361–374 (2011)