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New Atomic Decomposition for Besov Type Space $\dot{B}^0_{1,1}$ Associated with Schrödinger Type Operators

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Abstract

Let (X, d, μ) be a space of homogeneous type. Let L be a nonnegative self-adjoint operator on $L^2(X)$ satisfying certain conditions on the heat kernel estimates which are motivated from the heat kernel of the Schrödinger operator on $\mathbb{R}^n$. The main aim of this paper is to prove a new atomic decomposition for the Besov space $\dot{B}^{0,L}_{1,1}(X)$ associated with the operator L. As a consequence, we prove the boundedness of the Riesz transform associated with L on the Besov space $\dot{B}^{0,L}_{1,1}(X)$.

Keywords Heat semigroup · Besov space $B^0_{1,1}$ · Atomic decomposition · Riesz transform

Mathematics Subject Classification 42B35 · 42B15

1 Introduction

Let (X, d, μ) be a metric spaces endowed with a nonnegative Borel measure μ. Denote $B(x, r) := \{y \in X : d(x, y) < r\}$. In this paper we assume that the measure satisfies the doubling property condition, i.e., there exists a constant $C_1 > 0$ such that

$$\mu(B(x, 2r)) \le C_1 \mu(B(x, r)) \tag{1}$$

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for all $x \in X$ and $r > 0$. This condition implies that there exist constants $C_2, D \geq 0$ such that

$$V(x, \lambda r) \leq C_2 \lambda^D V(x, r) \tag{2}$$

for all $x \in X, r > 0$ and $\lambda \geq 1$. See [11].

We also assume further that (X, d, μ) satisfies the noncollapsing condition, i.e., there exists $c_0 > 0$ such that

$$V(x, 1) \geq c_0 \tag{3}$$

for all $x \in X$.

From now on, for any measurable subset $E \subset X$, we denote $V(E) := \mu(E)$. For all $x \in X$ and $r > 0$, we also denote $V(x, r) = \mu(B(x, r))$.

Note that the classical Hardy space $H^1(X)$ is a suitable substitution for the space $L^1(X)$ when we work with Calderón–Zygmund operators but the classical Hardy space might not be suitable for the study of certain operators that lie beyond the Calderon Zygmund class. This observation highlights the need for the development of new function spaces that adapt well to these operators. In recent times, there has been a remarkable progress in the field of function spaces associated with operators, reflecting the growing interest in understanding the behaviour of these operators and their associated function spaces. See for example [1, 5, 15, 21, 23, 29] and the references therein.

Motivated by this ongoing research, we aim to study new atomic decomposition of Besov spaces associated to Schrödinger type operators. Throughout this paper, we assume that H is a non-negative self-adjoint operator on $L^2(X)$ which generates the analytic semigroup $\{e^{-tH}\}_{t>0}$. Denote by $\widetilde{p}_t(x, y)$ and $\widetilde{q}_t(x, y)$ the kernels of e^{-tH} and tHe^{-tH}, respectively, we assume that the kernels $\widetilde{p}_t(x, y)$ satisfy the following conditions:

(H1) There exist positive constants C and c such that

$$|\widetilde{p}_t(x, y)| + |\widetilde{q}_t(x, y)| \leq \frac{C}{V(x, \sqrt{t})} \exp\Big(-c\frac{d(x, y)^2}{t}\Big)$$

for all $x, y \in X$ and $t > 0$;

(H2) There exist positive constants δ_1, c and C such that

$$\begin{aligned}&|\widetilde{p}_t(x, y) - \widetilde{p}_t(\overline{x}, y)| + |\widetilde{q}_t(x, y) - \widetilde{q}_t(\overline{x}, y)|\\ &\leq \frac{C}{V(x, \sqrt{t})}\Big[\frac{d(x, \overline{x})}{d(x, y)}\Big]^{\delta_1} \exp\Big(-c\frac{d(x, y)^2}{t}\Big)\end{aligned}$$

whenever $d(x, \overline{x}) \leq \sqrt{t}$ and $t > 0$;

(H3) $\displaystyle\int_X \widetilde{p}_t(x, y)\,d\mu(x) = 1$ for $y \in X$.

In fact, the assumptions (H1) and (H2) can be assumed only for the kernel $\widetilde{p}_t(x, y)$ since the estimates in (H1) and (H2) for $\widetilde{p}_t(x, y)$ imply similar estimates for $\widetilde{q}_t(x, y)$. However, for the sake of simplicity, we make the assumptions for both $\widetilde{p}_t(x, y)$ and $\widetilde{q}_t(x, y)$.

Standard examples of operators which satisfy conditions $(H1)$, $(H2)$ and $(H3)$ include the Laplacians Δ on the Euclidean spaces $\mathbb{R}^n$, the Laplace-Beltrami operators on non-compact Riemannian manifolds with doubling property, the Bessel operators on $(0, \infty)^n$, the sub-Laplacians on stratified Lie groups and certain degenerate elliptic operators on doubling spaces and domains.

Our motivation is to study the Schrödinger operator $L = H + V$ which is a non-negative self-adjoint operator on $L^2(X)$. Under suitable conditions, the potential V induces a critical function ρ which appears on the upper bounds and regularity estimates of the heat kernels of L and its time derivative. We refer the reader to Sect. 2.1 for a general definition of critical functions and further details.

In this paper, without the assumption $L = H + V$, we assume that L is a non-negative self-adjoint operator on $L^2(X)$. Denote by $p_t(x, y)$ and $q_t(x, y)$ the kernels of e^{-tL} and tLe^{-tL}, respectively. Suppose that ρ is a critical function defined on X. See Sect. 2.1 for the precise definition of critical functions. We assume that the kernels $p_t(x, y)$ and $q_t(x, y)$ satisfy the following conditions:

(L1) For all $N > 0$, there exist positive constants c and C so that

$$|p_t(x, y)| \le \frac{C}{V(x, \sqrt{t})} \exp\Big(-c\frac{d(x, y)^2}{t}\Big)\Big(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\Big)^{-N}$$

for all $x, y \in X$ and $t > 0$;

(L2) There is a positive constant δ_2 so that for all $N > 0$, there exist positive constants c and C which satisfy

$$|q_t(x, y) - q_t(\overline{x}, y)| \le \frac{C}{V(x, \sqrt{t})}\Big[\frac{d(x, \overline{x})}{d(x, y)}\Big]^{\delta_2} \exp\Big(-c\frac{d(x, y)^2}{t}\Big)\Big(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\Big)^{-N}$$

whenever $d(x, \overline{x}) \le \sqrt{t}$ and $t > 0$;

(L3) There is a positive constant δ_3 such that

$$|p_t(x, y) - \widetilde{p}_t(x, y)| + |q_t(x, y) - \widetilde{q}_t(x, y)| \le \frac{C}{V(x, \sqrt{t})}\Big(\frac{\sqrt{t}}{\sqrt{t} + \rho(x)}\Big)^{\delta_3} \exp\Big(-c\frac{d(x, y)^2}{t}\Big)$$

for all $x, y \in X$ and $t > 0$.

Remark 1.1 (a) If we set $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, then $(H2)$, (L2) and (L3) are satisfied with the exponent δ. For this reason, we might assume that $\delta_1 = \delta_2 = \delta_3 = \delta$.

(b) Note that the condition (L1) implies that for all $N > 0$, there exist positive constants c and C so that

$$|q_t(x,y)| \le \frac{C}{V(x,\sqrt{t})}\exp\Big(-c\frac{d(x,y)^2}{t}\Big)\Big(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\Big)^{-N} \tag{4}$$

for all $x, y \in X$ and $t > 0$.

Since the proof of (4) is standard, we leave it to the interested reader.

(c) As mentioned above, an example of the pairs of operators (H, L) which satisfy our assumptions are the operators H mentioned above and $L = H + V$ for suitable potentials V. See Sect. 2.1, also [9, Section 6] and [34]. We remark that our work on the operator L in this paper only relies on the assumptions $(L1)$, $(L2)$, $(L3)$ and does not use the representation $L = H + V$.

Our aim is to study the homogeneous Besov space $\dot{B}^{0,L}_{1,1}(X)$ associated with the operator L.

Definition 1.2 The homogeneous Besov space $\dot{B}^{0,L}_{1,1}(X)$ is defined as the set of $f \in L^1(X)$ such that

$$\|f\|_{\dot{B}^{0,L}_{1,1}(X)} := \int_0^\infty \|tLe^{-tL}f\|_1 \frac{dt}{t} < \infty.$$

When $L = -\Delta$ the Laplacian on $\mathbb{R}^n$, the Besov space $\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$ coincides with the classical Besov space $\dot{B}^{0}_{1,1}(\mathbb{R}^n)$. It is well known that the Besov space $\dot{B}^{0}_{1,1}(\mathbb{R}^n)$ is contained in the Hardy space $H^1(\mathbb{R}^n)$ and is used in proving the dispersive estimates of the wave equations (see for example [3, 8, 13]) and the regularity of the Green functions on domains (see for example [20]). See also [17, 18, 24–26] and the references therein for further discussion on the Besov space type $\dot{B}^{0}_{1,1}$ and the Besov spaces on spaces of homogeneous type. It is worth noticing that in the definition above we define the Besov space a subset of $L^1(X)$. This is more advantageous than the approach using new distributions as in [5, 26].

We are interested in atomic decompositions of the Besov space $\dot{B}^{0,L}_{1,1}(X)$. Note that atomic decompositions of Besov spaces associated to non-negative self-adjoint operators satisfying Gaussian upper bounds were obtained in [5] for homogeneous Besov spaces and in [27] for inhomogeneous Besov spaces. Adapting ideas in [5, 27], we can define atoms for the Besov spaces $\dot{B}^{0,L}_{1,1}(X)$ as follows.

Definition 1.3 Let $M \in \mathbb{N}_+$. A function a is said to be an (L, M) atom if there exists a ball B so that

(i) $a = L^M b$ with $b \in D(L^M)$, where $D(L^M)$ is the domain of L^M;
(ii) $\operatorname{supp} L^k b \subset B$, $k = 0, \dots, 2M$;
(iii) $|L^k b(x)| \le r_B^{2(M-k)} V(B)^{-1}$, $k = 0, \dots, 2M$, where r_B denotes the radius of ball B..

Note that the atoms defined in [5, 27] are supported in balls associated to dyadic cubes. See Lemma 2.2 for the definition of dyadic cubes. In this paper, we do not need the dyadic cubes in Definition 1.3 and we are able to prove the following result.

Theorem 1.4 *Let $M \in \mathbb{N}_+$. Assume $f \in \dot{B}^{0,L}_{1,1}(X)$. Then there exist a sequence of (L, M) atoms $\{a_j\}$ and a sequence of coefficients $\{\lambda_j\} \in \ell^1$ so that*

$$f = \sum_j \lambda_j a_j \ \ in\ L^1(X),$$

and

$$\sum_j |\lambda_j| \lesssim \|f\|_{\dot{B}^{0,L}_{1,1}(X)}.$$

Conversely, if

$$f = \sum_j \lambda_j a_j \ \ in\ L^1(X),$$

where $\{a_j\}$ is a sequence of (L, M)-atoms and $\{\lambda_j\} \in \ell^1$, then

$$\|f\|_{\dot{B}^{0,L}_{1,1}(X)} \lesssim \sum_j |\lambda_j|.$$

The proof of Theorem 1.4 will be presented later. In comparison with the atomic decomposition in Theorems 4.2 and 4.3 in [5], the main difference is that in Theorem 1.4, the convergence used in the atomic decomposition is in $L^1(X)$ instead of in the space of new distributions associated with the operator L; moreover, Theorem 1.4 uses the atoms associated with balls rather than the dyadic cubes as in Theorems 4.2 and 4.3 in [5].

We now consider new atoms associated with the critical function ρ which will be defined in Sect. 2.1. Note that the idea of the atomic decomposition associated to the critical functions was used in the setting of Hardy spaces. In [16], the atomic decomposition associated to the critical functions was studied for the Hardy spaces associated to Schrödinger operators with potential satisfying certain reverse Hölder inequality. Then the results were extended to encompass a broader scope, incorporating Schrödinger operators in various contexts such as stratified Lie groups and doubling manifolds. See for example [9, 34]. However, this is the first time the atomic decomposition associated to the critical functions was established for the Besov spaces.

Definition 1.5 Let $\epsilon > 0$ and ρ be a critical function. A function a is said to be an $(\epsilon, \rho(\cdot))$-atom if there exists a ball B such that

(i) supp $a \subset B$;
(ii) $|a(x)| \le V(B)^{-1}$;

(iii) $|a(x) - a(y)| \le V(B)^{-1}\left(\frac{d(x,y)}{r_B}\right)^{\epsilon}, \quad x, y \in X;$

(iv) $\int a(x)d\mu(x) = 0$ if $r_B < \rho(x_B)$.

It is interesting that the atoms in Definition 1.5 depend on the critical function ρ only. This type of atoms can be viewed as an extended version of the atoms used for the inhomogeneous Besov type. In fact, in the particular case $\rho =$ constant, the atoms in Definition 1.5 turn out to be the atoms which characterize the inhomogeneous Besov spaces. See for example [26]. Our main result is the following theorem.

Theorem 1.6 *If $f \in \dot{B}^{0,L}_{1,1}(X)$, then there exist a sequence of $(\epsilon, \rho(\cdot))$-atoms $\{a_j\}$ for some $\epsilon > 0$ and a sequence of coefficients $\{\lambda_j\} \in \ell^1$ so that*

$$f = \sum_j \lambda_j a_j \ \ in\ L^1(X),$$

and

$$\sum_j |\lambda_j| \lesssim \|f\|_{\dot{B}^{0,L}_{1,1}(X)}.$$

Conversely, if

$$f = \sum_j \lambda_j a_j \ \ in\ L^1(X),$$

where $\{a_j\}$ is a sequence of $(\epsilon, \rho(\cdot))$-atoms with $\epsilon > 0$ and $\{\lambda_j\} \in \ell^1$, then

$$\|f\|_{\dot{B}^{0,L}_{1,1}(X)} \lesssim \sum_j |\lambda_j|.$$

The organization of the paper is as follows. In Sect. 2, we recall the definitions of critical functions and dyadic cubes, and prove some kernel estimates of the spectral multipliers of H. In Sect. 3, we will set up the theory of the inhomogeneous Besov space $B^0_{1,1}(X)$ including atomic decomposition results. The proofs of the main results will be given in Sect. 4. Finally, Sect. 5 is devoted in the proof of the boundedness of the Riesz transform associated with L in Besov spaces.

Throughout the paper, we always use C and c to denote positive constants that are independent of the main parameters involved but whose values may differ from line to line. We write $A \lesssim B$ if there is a universal constant C so that $A \le CB$ and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. Given a $\lambda > 0$ and a ball $B := B(x, r)$, we write λB for the λ-dilated ball, which is the ball with the same center as B and with radius λr. For each ball $B \subset X$, we set

$$S_0(B) = B \text{ and } S_j(B) = 2^j B \backslash 2^{j-1} B \text{ for } j \in \mathbb{N}.$$

2 Preliminaries

2.1 Critical Functions

A function $\rho : X \to (0, \infty)$ is called a *critical function* if there exist positive constants C_ρ and k_0 so that

$$\rho(y) \le C_\rho \rho(x) \left(1 + \frac{d(x,y)}{\rho(x)}\right)^{\frac{k_0}{k_0+1}} \tag{5}$$

for all $x, y \in X$.

Note that the concept of critical functions was introduced in the setting of Schrödinger operators on $\mathbb{R}^D$ in [19] (see also [30]) and then was extended to the spaces of homogeneous type in [34].

A simple example of a critical function is $\rho \equiv 1$. Moreover, one of the most important classes of the critical functions is the one involving the weights satisfying the reverse Hölder inequality. Recall that a non-negative locally integrable function w is said to be in the reverse Hölder class $RH_q(X)$ with $q > 1$ if there exists a constant $C > 0$ so that

$$\left(\frac{1}{V(B)} \int_B (w(x))^q d\mu(x)\right)^{1/q} \le \frac{C}{V(B)} \int_B w(x) d\mu(x)$$

for all balls $B \subset X$. Note that if $w \in RH_q(X)$ then w is a Muckenhoupt weight. See [32].

Now suppose $V \in RH_q(X)$ for some $q > \max\{1, D/2\}$ and, following [30, 34], set

$$\rho(x) = \sup\left\{r > 0 : \frac{r^2}{\mu(B(x,r))} \int_{B(x,r)} V(y) d\mu(y) \le 1\right\}. \tag{6}$$

Then it was proved in [30, 34] that ρ is a critical function provided $q > \max\{1, D/2\}$. The following result will be useful in the sequel which is taken from Lemma 2.3 and Lemma 2.4 of [34].

Lemma 2.1 *Let ρ be a critical function on X. Then there exist a sequence of points $\{x_\alpha\}_{\alpha\in\mathcal{I}} \subset X$ and a family of functions $\{\psi_\alpha\}_{\alpha\in\mathcal{I}}$ satisfying the following:*

(i) $\bigcup_{\alpha\in\mathcal{I}} B(x_\alpha, \rho(x_\alpha)) = X$.

(ii) *For every $\lambda \ge 1$ there exist constants C and N_1 such that* $\sum_{\alpha\in\mathcal{I}} 1_{B(x_\alpha,\lambda\rho(x_\alpha))} \le C\lambda^{N_1}$.

(iii) $\operatorname{supp} \psi_\alpha \subset B_\alpha := B(x_\alpha, \epsilon_0\rho(x_\alpha))$ *and* $0 \le \psi_\alpha(x) \le 1$ *for all $x \in X$, where ϵ_0 is a fixed constant such that* $C_\rho\epsilon_0(1+\epsilon_0)^{\frac{k_0}{k_0+1}} < 1$.

(iv) $|\psi_\alpha(x) - \psi_\alpha(y)| \le C d(x,y)/\rho(x_\alpha)$;

(v) $\sum_{\alpha \in \mathcal{I}} \psi_\alpha(x) = 1$ *for all* $x \in X$.

2.2 Dyadic Cubes

We now recall an important covering lemma in [10].

Lemma 2.2 *There exists a collection of open sets* $\{Q_\tau^k \subset X : k \in \mathbb{Z}, \tau \in I_k\}$, *where* I_k *denotes certain (possibly finite) index set depending on* k, *and constants* $\eta \in (0, 1)$, $a_0 \in (0, 1]$ *and* $\kappa_0 \in (0, \infty)$ *such that*

(i) $\mu(X \setminus \cup_\tau Q_\tau^k) = 0$ *for all* $k \in \mathbb{Z}$;
(ii) *if* $i \geq k$, *then either* $Q_\tau^i \subset Q_\beta^k$ *or* $Q_\tau^i \cap Q_\beta^k = \emptyset$;
(iii) *for every* (k, τ) *and each* $i < k$, *there exists a unique* τ' *such* $Q_\tau^k \subset Q_{\tau'}^i$;
(iv) *the diameter* $\mathrm{diam}\,(Q_\tau^k) \leq \kappa_0 \eta^k$;
(v) *each* Q_τ^k *contains certain ball* $B(x_{Q_\tau^k}, a_0 \eta^k)$.

Remark 2.3 Since the constants η and a_0 are not essential in the paper, without loss of generality, we may assume that $\eta = a_0 = 1/2$. We then fix a collection of open sets in Lemma 2.2 and denote this collection by $\mathcal{D}$. We call open sets in $\mathcal{D}$ the dyadic cubes in X and $x_{Q_\tau^k}$ the center of the cube $Q_\tau^k \in \mathcal{D}$. We also denote

$$\mathcal{D}_\nu := \{Q_\tau^{\nu+1} \in \mathcal{D} : \tau \in I_{\nu+1}\}$$

for each $\nu \in \mathbb{Z}$. Then for $Q \in \mathcal{D}_\nu$, we have $B(x_Q, c_0 2^{-\nu}) \subset Q \subset B(x_Q, \kappa_0 2^{-\nu}) =: B_Q$, where c_0 is a constant independent of Q. For the sake of simplicity we might assume that $\kappa_0 = 1$.

2.3 Kernel Estimates

Denote by $E_H(\lambda)$ a spectral decomposition of H. Then by spectral theory, for any bounded Borel funtion $F : [0, \infty) \to \mathbb{C}$ we can define

$$F(H) = \int_0^\infty F(\lambda) dE_H(\lambda)$$

as a bounded operator on $L^2(X)$. It is well-known that the kernel $\cos(t\sqrt{H})(\cdot, \cdot)$ of $\cos(t\sqrt{H})$ satisfies the finite propagation speed

$$\mathrm{supp}\cos(t\sqrt{H}) \subset \{(x, y) \in X \times X : d(x, y) \leq \tilde{c}_0 t\} \tag{7}$$

for some $\tilde{c}_0 > 0$. See for example [31].

In what follows, without loss of generality we may assume that $\tilde{c}_0 = 1$.

We have the following useful lemma.

Lemma 2.4 ([23]) *Let $\phi \in C_0^\infty(\mathbb{R})$ be an even function with* $\operatorname{supp}\phi \subset (-1,1)$ *and* $\int \phi = 2\pi$. *Denote by Φ the Fourier transform of ϕ, i.e.,*

$$\Phi(\xi) := \mathcal{F}\phi(\xi) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\xi}\phi(x)dx. \tag{8}$$

Then for every $k \in \mathbb{N}$, the operator $(t^2H)^k\Phi(t\sqrt{H})$ is an integral operator with kernel denoted by $(t^2H)^k\Phi(t\sqrt{H})(x,y)$ satisfying the following

$$\operatorname{supp}(t^2H)^k\Phi(t\sqrt{H})(\cdot,\cdot) \subset \{(x,y) \in X \times X : d(x,y) \le t\}, \tag{9}$$

and

$$|(t^2H)^k\Phi(t\sqrt{H})(x,y)| \le \frac{C}{V(x,t)} \tag{10}$$

for all $t > 0$ and $x, y \in X$.

Lemma 2.5 ([7]) *Let $\lambda > 0$. Then we have:*

(a) *For any $N > 0$ and $s = N + 2D + 1$, there exists $C = C(N)$ so that*

$$|F(\lambda\sqrt{H})(x,y)| \le \frac{C}{V(x,\lambda)}\Big(1 + \frac{d(x,y)}{\lambda}\Big)^{-N}\|F\|_{W_s^2} \tag{11}$$

for all $x, y \in X$, and all functions F supported in $[1/2, 2]$.

(b) *For any $N > 0$ and $s = 2(N + 2D + 1)$ there exists $C = C(N)$ so that*

$$|F(\lambda\sqrt{H})(x,y)| \le \frac{C}{V(x,\lambda)}\Big(1 + \frac{d(x,y)}{\lambda}\Big)^{-N}\|F\|_{W_s^\infty} \tag{12}$$

for all $x, y \in X$, and for all functions F supported in $[0, 2]$ *with* $F^{(2\nu+1)}(0) = 0$ *for all $\nu \in \mathbb{N}$.*
Here, $\|F\|_{W_s^q} = \|(I - d^2/dx^2)F\|_q$ for $s > 0$ and $q \in [1, \infty]$.

Lemma 2.6 *Let $\lambda > 0$. Then we have:*

(a) *For any $N > 0$ and $s = N + 3D + 2$, there exists $C = C(N)$ so that*

$$\begin{aligned} &|F(\lambda\sqrt{H})(x,y) - F(\lambda\sqrt{H})(x,y')| \\ &\le C\Big(\frac{d(y,y')}{\lambda}\Big)^{\delta}\frac{1}{V(x,\lambda)}\Big(1 + \frac{d(x,y)}{\lambda}\Big)^{-N}\|F\|_{W_s^2} \end{aligned} \tag{13}$$

for all $x, y, y' \in X$ with $d(y,y') < \lambda$, and all functions supported in $[1/2, 2]$.

(b) *For any* $N > 0$ *and* $s = 2(N + 3D + 2)$ *there exists* $C = C(N)$ *so that*

$$|F(\lambda\sqrt{H})(x, y) - F(\lambda\sqrt{H})(x, y')| \leq C\Big(\frac{d(y, y')}{\lambda}\Big)^{\delta} \frac{1}{V(x, \lambda)}\Big(1 + \frac{d(x, y)}{\lambda}\Big)^{-N} \|F\|_{W_s^\infty} \tag{14}$$

for all $x, y, y' \in X$ *with* $d(y, y') < \lambda$, *and for all functions* F *supported in* $[0, 2]$ *with* $F^{(2\nu+1)}(0) = 0$ *for all* $\nu \in \mathbb{N}$.

Proof (a) We write $F(\lambda) = G(\lambda)e^{-\lambda^2}$, where $G(\lambda) = F(\lambda)e^{\lambda^2}$. Then we have

$$F(\lambda\sqrt{H})(x, y) = \int_X G(\lambda\sqrt{H})(x, z)\widetilde{p}_{\lambda^2}(z, y)d\mu(z).$$

This, along with Lemma 2.5, (H2) and the fact $\|G\|_{W_s^2} \lesssim \|F\|_{W_s^2}$ for every $s > 0$, yields that, for $x, y, y' \in X$ with $d(y, y') < \lambda$, $N > 0$ and $s = \tilde{N} + D + 1$ with $\tilde{N} = N + D + 1$,

$$\begin{aligned}
|F(\lambda\sqrt{H})(x, y) &- F(\lambda\sqrt{H})(x, y')| \\
&\leq \int_X |G(\lambda\sqrt{H})(x, z)||\widetilde{p}_{\lambda^2}(z, y) - \widetilde{p}_{\lambda^2}(z, y')|d\mu(z) \\
&\lesssim \|G\|_{W_s^2}\Big[\frac{d(y, y')}{\lambda}\Big]^{\delta} \int_X \frac{1}{V(x, \lambda)}\Big(1 \\
&\quad + \frac{d(x, z)}{\lambda}\Big)^{-\tilde{N}} \frac{1}{V(z, \lambda)} \exp\Big(-c\frac{d(y, z)^2}{\lambda^2}\Big)d\mu(z) \\
&\lesssim \|F\|_{W_s^2}\Big[\frac{d(y, y')}{\lambda}\Big]^{\delta} \int_X \frac{1}{V(x, \lambda)}\Big(1 \\
&\quad + \frac{d(x, z)}{\lambda}\Big)^{-\tilde{N}} \frac{1}{V(z, \lambda)} \exp\Big(-c\frac{d(y, z)^2}{\lambda^2}\Big)d\mu(z).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Big(1 + \frac{d(x, z)}{\lambda}\Big)^{-\tilde{N}} \exp\Big(-c\frac{d(y, z)^2}{\lambda^2}\Big) &\lesssim \Big(1 + \frac{d(x, z)}{\lambda}\Big)^{-\tilde{N}}\Big(1 \\
&\quad + \frac{d(y, z)}{\lambda}\Big)^{-\tilde{N}} \exp\Big(-c\frac{d(y, z)^2}{2\lambda^2}\Big) \\
&\lesssim \Big(1 + \frac{d(x, y)}{\lambda}\Big)^{-\tilde{N}} \exp\Big(-c\frac{d(y, z)^2}{2\lambda^2}\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}\int_X \frac{1}{V(x,\lambda)}\Big(1+\frac{d(x,z)}{\lambda}\Big)^{-\tilde{N}}\frac{1}{V(z,\lambda)}\exp\Big(-c\frac{d(y,z)^2}{\lambda^2}\Big)d\mu(z)\\ &\lesssim \frac{1}{V(x,\lambda)}\Big(1+\frac{d(x,y)}{\lambda}\Big)^{-\tilde{N}}\int_X \frac{1}{V(z,\lambda)}\\ &\quad\exp\Big(-c\frac{d(y,z)^2}{2\lambda^2}\Big)d\mu(z)\\ &\lesssim \frac{1}{V(x,\lambda)}\Big(1+\frac{d(x,y)}{\lambda}\Big)^{-\tilde{N}}\\ &\lesssim \frac{1}{V(x,\lambda)}\Big(1+\frac{d(x,y)}{\lambda}\Big)^{-N},\end{aligned}$$

which implies (13).

The estimate (14) can be proved similarly.

This completes our proof. □

Lemma 2.7 *Let $\varphi \in \mathcal{S}(\mathbb{R})$ be an even function. Then for any $N > 0$ there exists C_N such that*

$$|\varphi(t\sqrt{H})(x,y)| \le \frac{C_N}{V(x,t)}\Big(1+\frac{d(x,y)}{t}\Big)^{-N}, \tag{15}$$

and

$$|\varphi(t\sqrt{H})(x,y)-\varphi(t\sqrt{H})(x,y')| \le C_N\Big[\frac{d(y,y')}{t}\Big]^{\delta}\frac{1}{V(x,t)}\Big(1+\frac{d(x,y)}{t}\Big)^{-N} \tag{16}$$

for all $t > 0$ and $x, y, y' \in X$ with $d(y,y') < t$.

Consequently, $\varphi(t\sqrt{H})$ is bounded on $L^1(X)$.

Proof The inequality (15) was proved in [7]. Taking $N > D$, it follows that $\varphi(t\sqrt{H})$ is bounded on $L^1(X)$ since

$$\int_X \frac{1}{V(x,t)}\Big(1+\frac{d(x,y)}{t}\Big)^{-N}d\mu(x) \lesssim 1,$$

as long as $N > D$.

We need only to prove (16).

Let $\psi_0 \in C^\infty(\mathbb{R})$ supported in $[0,2]$ such that $\psi_0 = 1$ on $[0,1]$ and $0 \le \psi_0 \le 1$. Set $\psi(\lambda) = \psi_0(\lambda) - \psi_0(2\lambda)$ and $\psi_j(\lambda) = \psi(2^{-j}\lambda)$ for $j \ge 1$. Then we have

$$\sum_{j\ge 0}\psi_j(\lambda) = 1, \lambda > 0.$$

Hence,

$$\varphi(t\sqrt{H}) = \sum_{j\geq 0} \psi_j(t\sqrt{H})\varphi(t\sqrt{H}). \tag{17}$$

By (14), for $N > 0$ we have

$$\begin{aligned}
&|\psi_0(t\sqrt{H})\varphi(t\sqrt{H})(x, y) - \psi_0(t\sqrt{H})\varphi(t\sqrt{H})(x, y')| \\
&\lesssim \Big[\frac{d(y, y')}{t}\Big]^{\delta} \frac{1}{V(x, t)}\Big(1 + \frac{d(x, y)}{t}\Big)^{-N}
\end{aligned} \tag{18}$$

for all $t > 0$ and $x, y, y' \in X$ with $d(y, y') < t$.

Since $\operatorname{supp}\psi \subset [1/2, 2]$, using (11) and (13), we have, for $j \geq 1, t > 0, x, y, y' \in X$ with $d(y, y') < t$,

$$\begin{aligned}
|\psi_j(t\sqrt{H})\varphi(t\sqrt{H})(x, y) - \psi_j(t\sqrt{H})\varphi(t\sqrt{H})(x, y')| \\
&\lesssim \Big[\frac{d(y, y')}{2^{-j}t}\Big]^{\delta} \frac{1}{V(x, 2^{-j}t)}\Big[\Big(1 + \frac{d(x, y)}{2^{-j}t}\Big)^{-N} \\
&\quad + \Big(1 + \frac{d(x, y')}{2^{-j}t}\Big)^{-N}\Big]\|h_j\|_{W_s^2} \\
&\lesssim \Big[\frac{d(y, y')}{t}\Big]^{\delta} \frac{2^{j(n+\delta)}}{V(x, t)}\Big[\Big(1 + \frac{d(x, y)}{t}\Big)^{-N} \\
&\quad + \Big(1 + \frac{d(x, y')}{t}\Big)^{-N}\Big]\|h_j\|_{W_s^2} \\
&\lesssim \Big[\frac{d(y, y')}{t}\Big]^{\delta} \frac{2^{j(n+\delta)}}{V(x, t)}\Big(1 + \frac{d(x, y)}{t}\Big)^{-N} \|h_j\|_{W_s^2},
\end{aligned}$$

where $s = N + 3n + 2$ and $h_j(\lambda) = \psi(\lambda)\varphi(2^{-j}\lambda)$.

Since $\varphi \in \mathcal{S}(\mathbb{R})$, $\|h_j\|_{W_s^2} \leq C_s 2^{-j(n+\delta+1)}$ for every $s > 0$. As a consequence,

$$\begin{aligned}
&|\psi_j(t\sqrt{H})\varphi(t\sqrt{H})(x, y) - \psi_j(t\sqrt{H})\varphi(t\sqrt{H})(x, y')| \\
&\lesssim 2^{-j}\Big[\frac{d(y, y')}{t}\Big]^{\delta} \frac{1}{V(x, t)}\Big(1 + \frac{d(x, y)}{t}\Big)^{-N},
\end{aligned}$$

whenever $d(y, y') < t$.

This, along with (17) and (18), implies that for each $N > 0$ there exists C such that

$$|\varphi(t\sqrt{H})(x, y) - \varphi(t\sqrt{H})(x, y')| \lesssim \Big[\frac{d(y, y')}{t}\Big]^{\delta} \frac{1}{V(x, t)}\Big(1 + \frac{d(x, y)}{t}\Big)^{-N}$$

for all $t > 0$ and $x, y, y' \in X$ with $d(y, y') < t$.

This completes the proof. □

Remark 2.8 The results in Lemmas 2.5, 2.6 and 2.7 hold true if we replace H by L since we do not use the assumption (H3) in the proofs.

Lemma 2.9 *Assume that $\varphi(\lambda) = \lambda^2\phi(\lambda)$, where $\phi \in \mathcal{S}(\mathbb{R})$ is an even function. Then we have*

$$\int_X \varphi(t\sqrt{H})(x, y)d\mu(y) = \int_X \varphi(t\sqrt{H})(y, x)d\mu(y) = 0$$

for all $x \in X$ and $t > 0$.

Proof Let ψ_j be the function as in the proof of Lemma 2.7 for $j = 0, 1, 2, \dots$. Then we have

$$\varphi(t\sqrt{H})f = \sum_{j\geq 0} \psi_j(t\sqrt{H})\varphi(t\sqrt{H})f \quad \text{in } L^2(X)$$

for $f \in L^2(X)$.

Let $B_R = B(x_0, R)$ for a fixed $x_0 \in X$ and $R > 0$. Taking $f = 1_{B_R}$, then it follows that

$$\int_{B_R} \varphi(t\sqrt{H})(x, y)d\mu(y) = \sum_{j\geq 0} \int_{B_R} \psi_j(t\sqrt{H})\varphi(t\sqrt{H})(x, y)d\mu(y) \quad \text{in } L^2(X). \tag{19}$$

Arguing similarly to the proof of Lemma 2.7, we also yield that for any $N > n$ and $j = 0, 1, 2, \dots$,

$$|\psi_j(t\sqrt{H})\varphi(t\sqrt{H})(x, y)| \lesssim 2^{-j} \frac{1}{V(x, t)}\Big(1 + \frac{d(x, y)}{t}\Big)^{-N}.$$

Consequently,

$$\begin{aligned}\sum_{j\geq 0} \int_{B_R} |\psi_j(t\sqrt{H})\varphi(t\sqrt{H})(x, y)|d\mu(y) &\lesssim \sum_{j\geq 0} 2^{-j} \int_X \frac{1}{V(x, t)}\Big(1 + \frac{d(x, y)}{t}\Big)^{-N} d\mu(y)\\ &\lesssim 1.\end{aligned} \tag{20}$$

This, together with (19), implies that

$$\int_{B_R} \varphi(t\sqrt{H})(x, y)d\mu(y) = \sum_{j\geq 0} \int_{B_R} \psi_j(t\sqrt{H})\varphi(t\sqrt{H})(x, y)d\mu(y)$$

for $x \in X$.

Using (20), and letting $R \to \infty$, the above identity deduces that

$$\int_X \varphi(t\sqrt{H})(x, y)d\mu(y) = \sum_{j\geq 0} \int_X \psi_j(t\sqrt{H})\varphi(t\sqrt{H})(x, y)d\mu(y)$$

for $x \in X$.

It now suffices to prove

$$\int_X \psi_j(t\sqrt{H})\varphi(t\sqrt{H})(x,y)d\mu(y) = 0$$

for $x \in X$ and $j = 0, 1, 2, \ldots$.

Indeed, since $\varphi(\lambda) = \lambda^2\phi(\lambda)$, we have

$$\psi_j(t\sqrt{H})\varphi(t\sqrt{H}) = G_{j,t}(H) \circ \left[t^2 H e^{-t^2 H}\right],$$

where $G_{j,t}(\lambda) = e^{t^2\lambda^2}\psi_j(t\lambda)\phi(t\lambda)$.

Therefore, due to Lemma 2.5, the upper bound of $\widetilde{q}_t(x,y)$ and Fubini's theorem,

$$\begin{aligned}\int_X \psi_j(t\sqrt{H})\varphi(t\sqrt{H})(x,y)d\mu(y) &= \int_X\int_X G_{j,t}(H)(x,z)\widetilde{q}_{t^2}(z,y)d\mu(z)d\mu(y)\\ &= \int_X G_{j,t}(H)(x,z)\int_X \widetilde{q}_{t^2}(z,y)d\mu(y)d\mu(z).\end{aligned}$$

In addition, from the conservation property (H3), we immediately have

$$\int_X \widetilde{q}_{t^2}(z,y)d\mu(y) = 0,$$

which implies

$$\int_X \psi_j(t\sqrt{H})\varphi(t\sqrt{H})(x,y)d\mu(y) = 0.$$

This completes our proof. □

3 Inhomogeneous Besov Spaces $B^0_{1,1}(X)$ and Atomic Decomposition

In this section, we will introduce the Besov space $B^0_{1,1}(X)$. Our approach relies on the function spaces associated to the "Laplace-like" operator. This is motivated from the classical case in which the classical Besov spaces can be viewed as Besov spaces associated with the Laplacian. In our setting, under the three conditions (H1), (H2) and (H3), the operator H satisfies important properties which are similar to the Laplacian on the Euclidean space.

3.1 Inhomogeneous Besov Spaces $B^0_{1,1}(X)$

Definition 3.1 The (inhomogeneous) Besov space $B^0_{1,1}(X)$ is defined as the set of $f \in L^1(X)$ such that

$$\|f\|_{B^0_{1,1}(X)} := \|e^{-H} f\|_1 + \int_0^1 \|tHe^{-tH} f\|_1 \frac{dt}{t} < \infty.$$

In the sequel we will show that the Besov space $B^0_{1,1}(X)$ is independent of the operator H. This is a reason why we do not include the operator H in the notation of the Besov space.

Lemma 3.2 *The inhomogeneous Besov space $B^0_{1,1}(X)$ is complete.*

In order to prove Lemma 3.2 we need the following technical lemmas.

Lemma 3.3 *For each $1 \le p < \infty$, the space $L^p(X)$ is dense in inhomogeneous Besov space $B^0_{1,1}(X)$. In fact, for each $f \in B^0_{1,1}(X)$ and each $1 \le p < \infty$, there exists a sequence $\{f_k\} \subset L^1(X) \cap L^p(X)$ such that*

$$\|f_k - f\|_1 + \|f_k - f\|_{B^0_{1,1}(X)} \to 0 \ \ as\ k \to \infty.$$

Proof We first recall the following fact in [4]

$$\lim_{s\to 0} \|e^{-sH} f - f\|_1 = 0 \ \text{ for } f \in L^1(X). \tag{21}$$

Assume that $f \in B^0_{1,1}(X)$. It follows that $f \in L^1(X)$. For each $n \in \mathbb{N}$, define

$$f_k = e^{-H/k} f.$$

From the Gaussian upper bound condition (H1) and (3),

$$\|e^{-H/k} f\|_p \lesssim c_0 k^{n/p'} \|f\|_1, \quad p \in [1, \infty),$$

which implies $f_k \in L^p(X)$ for each $1 \le p < \infty$.

Hence,

$$\|f - f_k\|_{B^0_{1,1}(X)} = \|e^{-H}(f_k - f)\|_1 + \int_0^1 \|tHe^{-tH}(f_k - f)\|_1 \frac{dt}{t}.$$

By (21),

$$\|e^{-H}(f_k - f)\|_1 \lesssim \|f_k - f\|_1 = \|e^{-H/k} f - f\|_1 \to 0 \ \ \text{as } k \to \infty.$$

Similarly,

$$\|tLe^{-tH}(f_k - f)\|_1 \to 0 \quad \text{as } k \to \infty.$$

On the other hand, since e^{-sH} is bounded on $L^1(X)$, we have

$$\begin{aligned}\|tHe^{-tH}(f_k - f)\|_1 &\le \|tHe^{-tH} f\|_1 + \|tHe^{-tH} f_k\|_1 \\ &= \|tHe^{-tH} f\|_1 + \left\|e^{-H/k}\big(tHe^{-tH} f\big)\right\|_1 \\ &\lesssim \|tHe^{-tH} f\|_1.\end{aligned}$$

In addition,

$$\int_0^1 \|tHe^{-tH} f\|_1 \frac{dt}{t} \le \|f\|_{B^0_{1,1}(X)}.$$

By the Dominated Convergence Theorem,

$$\int_0^1 \|tHe^{-tH}(f_k - f)\|_1 \frac{dt}{t} \to 0 \quad \text{as } k \to \infty.$$

It follows that

$$\|f - f_k\|_{B^0_{1,1}(X)} \to 0 \quad \text{as } k \to \infty.$$

This, along with the fact that $f_k \in L^p(X)$ for each $n \in \mathbb{N}$ and $p \in [1, \infty)$, implies that $L^p(X)$ is dense in $B^0_{1,1}(X)$ for each $p \in [1, \infty)$.

This completes our proof. □

Lemma 3.4 *Let* ψ_0, ψ *be even functions such that* $\operatorname{supp} \psi_0 \subset \{\lambda : |\lambda| \le 2\}$ *and* $\operatorname{supp} \psi \subset \{\lambda : 1/2 \le |\lambda| \le 2\}$, *and*

$$\sum_{j=0}^{\infty} \psi_j(\lambda) = 1, \quad \lambda \in \mathbb{R},$$

where $\psi_j(\lambda) = \psi(2^{-j}\lambda),\ j = 1, 2, \ldots$.

Then we have

$$\sum_{j=0}^{\infty} \psi_j(\sqrt{H}) f = f \ \text{ in } L^1(X)$$

for $f \in B^0_{1,1}(X)$.

Proof Let $f \in B^0_{1,1}(X)$. By Lemma 2.7, we have

$$\|\psi_0(\sqrt{H})f\|_1 = \|\psi_0(\sqrt{H})e^H(e^{-H}f)\|_1 \lesssim \|e^{-H}f\|_1,$$

and for $j \geq 1$,

$$\begin{aligned}\|\psi_j(\sqrt{H})f\|_1 &= \left\|\widetilde{\psi}_j(\sqrt{H})\big(2^{-2j}He^{-2^{-2j}H}f\big)\right\|_1 \\ &\lesssim \|2^{-2j}He^{-2^{-2j}H}f\|_1,\end{aligned}$$

where $\widetilde{\psi}_j(\lambda) = (2^{-2j}\lambda^2)^{-1}e^{2^{-2j}\lambda^2}\psi_j(\lambda)$.

Note that for $t \in [2^{-2j-2}, 2^{-2j}]$,

$$\begin{aligned}\|2^{-2j}He^{-2^{-2j}H}f\|_1 &= \frac{2^{-2j}}{t}\left\|e^{-(2^{-2j}-t)H}\big(tHe^{-tH}f\big)\right\|_1 \\ &\lesssim \|tHe^{-tH}f\|_1,\end{aligned}$$

which implies

$$\|\psi_j(\sqrt{H})f\|_1 \lesssim \int_{2^{-2j-2}}^{2^{-2j}} \|tHe^{-tH}f\|_1 \frac{dt}{t}.$$

Therefore,

$$\begin{aligned}\sum_{j=0}^{\infty}\|\psi_j(\sqrt{H})f\|_1 &\lesssim \|e^{-H}f\|_1 + \sum_{j\geq 1}\int_{2^{-2j-2}}^{2^{-2j}} \|tHe^{-tH}f\|_1\frac{dt}{t} \\ &\lesssim \|e^{-H}f\|_1 + \sum_{j\geq 1}\int_0^1 \|tHe^{-tH}f\|_1\frac{dt}{t} \\ &\lesssim \|f\|_{B^0_{1,1}(X)}.\end{aligned} \tag{22}$$

It follows that there exists $g \in L^1(X)$ such that

$$g = \sum_{j=0}^{\infty}\psi_j(\sqrt{H})f \ \text{ in } L^1(X).$$

If $f \in L^2(X)$, then by the spectral theory,

$$\sum_{j=0}^{\infty}\psi_j(\sqrt{H})f = f \ \text{ in } L^2(X).$$

Consequently, $f = g$ for a.e.. Hence,

$$f = \sum_{j=0}^{\infty} \psi_j(\sqrt{H}) f \ \text{ in } L^1(X).$$

In general, for $f \in B^0_{1,1}(X)$, by Lemma 3.3 there exists a sequence $\{f_k\} \subset L^2(X)$ such that

$$\|f - f_k\|_1 + \|f - f_k\|_{B^0_{1,1}} \to 0 \quad \text{as } k \to \infty.$$

Similarly to (22),

$$\Big\| \sum_{j=0}^{\infty} \psi_j(\sqrt{H})(f_k - f) \Big\|_1 \lesssim \|f_k - f\|_{B^0_{1,1}(X)}.$$

Hence,

$$\lim_{k\to\infty} \Big\| \sum_{j=0}^{\infty} \psi_j(\sqrt{H})(f_k - f) \Big\|_1 = 0. \tag{23}$$

We now write

$$\sum_{j=0}^{\infty} \psi_j(\sqrt{H}) f = \sum_{j=0}^{\infty} \psi_j(\sqrt{H})(f - f_k) + \Big[\sum_{j=0}^{\infty} \psi_j(\sqrt{H}) f_k - f_k \Big] + \big[f_k - f \big] + f.$$

From (23),

$$\sum_{j=0}^{\infty} \psi_j(\sqrt{H})(f - f_k) \to 0 \ \text{ in } L^1(X) \text{ as } k \to \infty.$$

Since $f_k \in L^2(X) \cap B^0_{1,1}(X)$, we have proved that

$$\sum_{j=0}^{\infty} \psi_j(\sqrt{H}) f_k - f_k = 0 \ \text{ in } L^1(X) \text{ for } k \in \mathbb{N}.$$

In addition,

$$\|f_k - f\|_1 \to 0 \quad \text{as } k \to \infty.$$

Consequently,

$$\sum_{j=0}^{\infty} \psi_j(\sqrt{L})f = f \ \text{ in } L^1(X)$$

for all $f \in B^0_{1,1}(X)$.

This completes our proof. □

Corollary 3.5 *We have the following continuous embedding*

$$B^0_{1,1} \hookrightarrow L^1(X).$$

Proof Let ψ_0, ψ be even functions such that $\operatorname{supp}\psi_0 \subset \{\lambda : |\lambda| \le 2\}$ and $\operatorname{supp}\psi \subset \{\lambda : 1/2 \le |\lambda| \le 2\}$, and

$$\sum_{j=0}^{\infty} \psi_j(\lambda) = 1, \quad \lambda \in \mathbb{R},$$

where $\psi_j(\lambda) = \psi(2^{-j}\lambda),\ j = 1, 2, \ldots$.

By Lemma 3.4,

$$\sum_{j=0}^{\infty} \psi_j(\sqrt{H})f = f \ \text{ in } L^1(X)$$

for $f \in B^0_{1,1}(X)$.

It follows that

$$\|f\|_1 \le \sum_{j=0}^{\infty} \|\psi_j(\sqrt{H})f\|_1.$$

This, along with (22), implies that

$$\|f\|_1 \lesssim \|f\|_{B^0_{1,1}}.$$

This completes our proof. □

We are now ready to prove Lemma 3.2.

Proof of Lemma 3.2 Assume that $\{f_k\}$ is a Cauchy sequence in $B^0_{1,1}(X)$. Hence, this is also a Cauchy sequence in $L^1(X)$ since $B^0_{1,1}(X) \hookrightarrow L^1(X)$. As a consequence, $f_k \to f \in L^1(X)$ for some $f \in L^1(X)$. On the other hand, we have

$$\|e^{-H}\|_{1\to 1} + \|tHe^{-tH}\|_{1\to 1} \lesssim 1$$

uniformly in $t > 0$.

Therefore,

$$\|e^{-H} f_k\|_1 \to \|e^{-H} f\|_1 \ \text{ as } k \to \infty,$$

and

$$\|tHe^{-tH} f_k\|_1 \to \|tHe^{-tH} f\|_1 \ \text{ as } k \to \infty.$$

Since $\{f_k\}$ is a Cauchy sequence in $B^0_{1,1}(X)$, for any $\epsilon > 0$ there exists N such that for $m, k \geq N$,

$$\|e^{-H}(f_k - f_m)\|_1 + \int_0^1 \|tLe^{-tH}(f_k - f_m)\|_1 \frac{dt}{t} < \epsilon.$$

Fixing k, then using Fatou's Lemma we have

$$\begin{aligned}
&\|e^{-H}(f_k - f)\|_1 + \int_0^1 \|tLe^{-tH}(f_k - f)\|_1 \frac{dt}{t} \\
&\quad \leq \lim_{m\to\infty} \|e^{-H}(f_k - f_m)\|_1 + \liminf_{m\to\infty} \int_0^1 \|tLe^{-tH}(f_k - f_m)\|_1 \frac{dt}{t} \\
&\quad < \epsilon.
\end{aligned}$$

It follows that

$$f_k \to f \quad \text{in } B^0_{1,1}(X).$$

This completes our proof. □

3.2 Atomic Decomposition

In order to establish atomic decomposition for the Besov space, we need another Calderón reproducing formula.

Proposition 3.6 *Let φ be as in Lemma 2.4. Let $\psi \in C_0^\infty(\mathbb{R})$ be an even function with* $\operatorname{supp}\psi \subset (-1, 1)$ *and $\int \psi = 2\pi$. Let Φ and Ψ be the Fourier transforms of φ and ψ, respectively. Then we have, for $f \in B^0_{1,1}(X)$,*

$$\begin{aligned}
f = \Phi(2^{-2}\sqrt{H})\Psi(2^{-2}\sqrt{H})f - \int_0^{1/4} (t\sqrt{H})\Phi'(t\sqrt{H})\Psi(t\sqrt{H})f \frac{dt}{t} \\
- \int_0^{1/4} (t\sqrt{H})\Psi'(t\sqrt{H})\Phi(t\sqrt{H})f \frac{dt}{t}
\end{aligned} \tag{24}$$

in $L^1(X)$.

Proof Similarly to the proof of Lemma 3.4, it suffices to prove the proposition for $f \in L^2(X) \cap B^0_{1,1}(X)$. Observe that

$$\begin{aligned}\int_0^{1/4} (tz)(\Phi\Psi)^{'}(tz)\frac{dt}{t} &= \int_0^{z/4} (\Phi\Psi)^{'}(u)du \\ &= \Phi(z/4)\Psi(z/4) - \Phi(0)\Psi(0) \\ &= \Phi(z/4)\Psi(z/4) - 1,\end{aligned}$$

which implies that

$$\begin{aligned}1 &= \Phi(z/4)\Psi(z/4) - \int_0^{1/4} (tz)(\Phi\Psi)^{'}(tz)\frac{dt}{t} \\ &= \Phi(z/4)\Psi(z/4) - \int_0^{1/4} (tz)\Phi'(tz)\Psi(tz)\frac{dt}{t} - \int_0^{1/4} (tz)\Psi'(tz)\Phi(tz)\frac{dt}{t}.\end{aligned}$$

This, along with spectral theory, yields

$$\begin{aligned}f = \Phi(2^{-2}\sqrt{H})\Psi(2^{-2}\sqrt{H})f - \int_0^{1/4} (t\sqrt{H})\Phi'(t\sqrt{H})\Psi(t\sqrt{H})f\frac{dt}{t} \\ - \int_0^{1} (t\sqrt{H})\Psi'(t\sqrt{H})\Phi(t\sqrt{H})f\frac{dt}{t}\end{aligned} \tag{25}$$

in $L^2(X)$.

Set

$$\begin{aligned}\mathcal{F}(\sqrt{H}) = \Phi(\sqrt{H})\Psi(\sqrt{H}) - \textstyle\int_0^{1/4}(t\sqrt{H})\Phi'(t\sqrt{H})\Psi(t\sqrt{H})\frac{dt}{t} \\ - \textstyle\int_0^{1/4}(t\sqrt{H})\Psi'(t\sqrt{H})\Phi(t\sqrt{H})\frac{dt}{t}.\end{aligned}$$

Then, by Lemma 3.9 and Corollary 3.5,

$$\begin{aligned}\|\mathcal{F}(\sqrt{H})f\|_1 &\lesssim \|f\|_1 + \|f\|_{B^0_{1,1}(X)} \\ &\lesssim \|f\|_{B^0_{1,1}(X)}.\end{aligned}$$

This implies that

$$\mathcal{F}(\sqrt{H})f = g \quad \text{in } L^1(X)$$

for some $g \in L^1(X)$.

This, in combination with (25), implies that $f = g$ for a.e.. Therefore,

$$f = \Phi(\sqrt{H})\Psi(\sqrt{H})f - \int_0^{1} (t\sqrt{H})\Phi'(t\sqrt{H})\Psi(t\sqrt{H})f\frac{dt}{t}$$

$$- \int_0^1 (t\sqrt{H})\Psi'(t\sqrt{H})\Phi(t\sqrt{H}) f \frac{dt}{t} \quad \text{in } L^1(X)$$

for $f \in L^2(X) \cap B^0_{1,1}(X)$.

This completes our proof. □

For any bounded Borel function φ defined on $[0, \infty)$. We now define, for $\lambda > 0$,

$$\varphi^*_\lambda(t\sqrt{H}) f(x) = \sup_{y \in X} \frac{|\varphi(t\sqrt{H}) f(y)|}{(1 + d(x, y)/t)^\lambda}$$

for all $f \in L^1(X)$, $x \in X$ and $t > 0$.

Definition 3.7 ([27]) Let (φ, φ_0) be a pair of even functions in $\mathcal{S}(\mathbb{R})$. We say that the pair (φ, φ_0) belongs to the class $\mathcal{A}(\mathbb{R})$ if

$$|\varphi_0(\lambda)| > 0 \text{ for } |\lambda| < 4\epsilon, \quad |\varphi(\lambda)| > 0 \text{ for } \epsilon/4 < |\lambda| < 4\epsilon \tag{26}$$

for some $\epsilon > 0$, and

$$\lambda^{-2}\varphi(\lambda) \in \mathcal{S}([0, \infty)).$$

Arguing similarly to the proof of Therem 1.2 in [28], we have:

Lemma 3.8 *Let (φ, φ_0) be a pair of even functions in $\mathcal{A}(\mathbb{R})$. Then, for $\lambda > 2n$, we have*

$$\|f\|_{B^0_{1,1}(X)} \sim \|(\varphi_0)^*_\lambda(\sqrt{H}) f\|_1 + \sum_{j=1}^{\infty} \|\varphi^*_\lambda(2^{-j}\sqrt{H}) f\|_1$$

for all $f \in B^0_{1,1}(X)$.

Lemma 3.9 *Let $\varphi(\lambda) = \lambda^2\phi(\lambda)$ be an even function in $\mathcal{S}(\mathbb{R})$. Then, for $\lambda > 2n$, we have*

$$\int_0^1 \|\varphi^*_\lambda(t\sqrt{H}) f\| \frac{dt}{t} \lesssim \|f\|_{B^0_{1,1}(X)}$$

for all $f \in B^0_{1,1}(X)$.

Proof Let ψ_0, ψ be even functions such that supp $\psi_0 \subset \{\lambda : |\lambda| \le 2\}$, supp $\psi \subset \{\lambda : 1/2 \le |\lambda| \le 2\}$, and

$$\sum_{j=0}^{\infty} \psi_j(\lambda) = 1, \quad \lambda \in \mathbb{R},$$

where $\psi_j(\lambda) = \psi(2^{-j}\lambda),\ \ j = 1, 2, \ldots$.

Then we have

$$\varphi(t\sqrt{H})f = \sum_{j=0}^{\infty} \psi_j(\sqrt{H})\varphi(t\sqrt{H})f \tag{27}$$

for all $t \in (0, 1)$.

Let $\lambda > 0$ and let $t \in [2^{-j_0-1}, 2^{-j_0}]$ for some $j_0 \geq 0$ and $M > \lambda/2$. We then have

$$\varphi(t\sqrt{H})f = \sum_{j=j_0+1}^{\infty} \varphi(t\sqrt{H})\psi_j(\sqrt{H})f + \sum_{0\leq j\leq j_0} \varphi(t\sqrt{H})\psi_j(\sqrt{H})f.$$

Set $\psi_{j,M}(\lambda) = (2^{-j}\lambda)^{-2M}\psi_j(\lambda)$. This, along with the fact that $\varphi(\lambda) = \lambda^2\phi(\lambda)$, yields

$$\begin{aligned}\varphi(t\sqrt{H})f = &\sum_{j=j_0+1}^{\infty} 2^{-2M(j-j_0)}(t\sqrt{H})^{2M}\varphi(t\sqrt{H})\psi_{j,M}(\sqrt{H})f\\ &+ \sum_{0\leq j\leq j_0} 2^{-2(j_0-j)}\phi(t\sqrt{H})(2^{-j}\sqrt{H})\psi_j(\sqrt{H})f.\end{aligned}$$

By Lemma 2.7, for each $y \in X$ and $N > n$,

$$\begin{aligned}|\varphi(t\sqrt{H})f(y)| = &\sum_{j=j_0+1}^{\infty} 2^{-2M(j-j_0)} \int_X \frac{1}{V(y,t)}\Big(1+\frac{d(y,z)}{t}\Big)^{-N-\lambda}|\psi_{j,M}(\sqrt{H})f(z)|d\mu(z)\\ &+ \sum_{0\leq j\leq j_0} 2^{-2(j_0-j)} \int_X \frac{1}{V(y,t)}\Big(1+\frac{d(y,z)}{t}\Big)^{-N-\lambda}|\psi_j(\sqrt{H})f(z)|d\mu(z).\end{aligned}$$

Using the inequality

$$\Big(1+\frac{d(x,y)}{t}\Big)^{-\lambda}\Big(1+\frac{d(y,z)}{t}\Big)^{-\lambda} \leq \Big(1+\frac{d(x,z)}{t}\Big)^{-\lambda},$$

we obtain, for $x, y \in X$,

$$\begin{aligned}\frac{|\varphi(t\sqrt{H})f(y)|}{(1+d(x,y)/t)^{\lambda}} \lesssim &\sum_{j=j_0+1}^{\infty} 2^{-2M(j-j_0)} \int_X \frac{1}{V(y,t)}\Big(1+\frac{d(y,z)}{t}\Big)^{-N} \frac{|\psi_{j,M}(\sqrt{H})f(z)|}{(1+d(x,z)/t)^{\lambda}}d\mu(z)\\ &+ \sum_{0\leq j\leq j_0} 2^{-2(j_0-j)} \int_X \frac{1}{V(y,t)}\Big(1+\frac{d(y,z)}{t}\Big)^{-N} \frac{|\psi_j(\sqrt{H})f(z)|}{(1+d(x,z)/t)^{\lambda}}d\mu(z)\\ \lesssim &\sum_{j=j_0+1}^{\infty} 2^{-2M(j-j_0)} \sup_{z\in X} \frac{|\psi_{j,M}(\sqrt{H})f(z)|}{(1+d(x,z)/t)^{\lambda}}d\mu(z)\\ &+ \sum_{0\leq j\leq j_0} 2^{-2(j_0-j)} \sup_{z\in X} \frac{|\psi_j(\sqrt{H})f(z)|}{(1+d(x,z)/t)^{\lambda}}d\mu(z).\end{aligned}$$

Since $t \sim 2^{-j_0}$, for $x, y \in X$ we further simplify to that

$$\begin{aligned}\frac{|\varphi(t\sqrt{H})f(y)|}{(1+d(x,y)/t)^\lambda} &\lesssim \sum_{j=j_0+1}^{\infty} 2^{-(2M-\lambda)(j-j_0)} \sup_{z\in X} \frac{|\psi_{j,M}(\sqrt{H})f(z)|}{(1+2^j d(x,z))^\lambda} d\mu(z) \\ &\quad + \sum_{0\le j\le j_0} 2^{-2(j_0-j)} \sup_{z\in X} \frac{|\psi_j(\sqrt{H})f(z)|}{(1+2^j d(x,z))^\lambda} d\mu(z) \\ &\sim \sum_{j=j_0+1}^{\infty} 2^{-(2M-\lambda)(j-j_0)} (\psi_{j,M})^*_\lambda f(x) \\ &\quad + \sum_{0\le j\le j_0} 2^{-2(j_0-j)} (\psi_j)^*_\lambda f(x),\end{aligned}$$

which implies that for each $x \in X$ and $t \sim 2^{-j_0} \in (0, 1)$,

$$\varphi(t\sqrt{H})^*_\lambda f(x) \lesssim \sum_{j=j_0+1}^{\infty} 2^{-(2M-\lambda)(j-j_0)} (\psi_{j,M})^*_\lambda f(x) + \sum_{0\le j\le j_0} 2^{-2(j_0-j)} (\psi_j)^*_\lambda f(x).$$

This, along with Lemma 3.8, implies that

$$\begin{aligned}\int_0^1 \|\varphi^*_\lambda(t\sqrt{H})f\| \frac{dt}{t} &= \sum_{j_0\ge 0} \int_{2^{-j_0-1}}^{2^{-j_0}} \|\varphi^*_\lambda(t\sqrt{H})f\| \frac{dt}{t} \\ &\lesssim \sum_{j_0\ge 0} \sum_{j=j_0+1}^{\infty} 2^{-(2M-\lambda)(j-j_0)} \|(\psi_{j,M})^*_\lambda f\|_1 \\ &\quad + \sum_{j_0\ge 0} \sum_{0\le j\le j_0} 2^{-2(j_0-j)} \|(\psi_j)^*_\lambda f\|_1 \\ &\lesssim \sum_{j\ge 1} \|(\psi_{j,M})^*_\lambda f\|_1 + \sum_{j\ge 0} \|(\psi_j)^*_\lambda f\|_1 \\ &\lesssim \|f\|_{B^0_{1,1}(X)},\end{aligned}$$

provided that $\lambda > 2n$.

This completes our proof. □

We now introduce the notion of atoms for the Besov space $B^0_{1,1}(X)$.

Definition 3.10 Let $\epsilon > 0$. A function a is said to be an ϵ-atom if there exists a ball B with $r_B \le 1$ such that

(i) supp $a \subset B$;
(ii) $|a(x)| \le V(B)^{-1}$;
(iii) $|a(x) - a(y)| \le V(B)^{-1} \left(\frac{d(x,y)}{r_B}\right)^\epsilon$;

(iv) $\int a(x)d\mu(x) = 0$ if $r_B < 1$.

Theorem 3.11 *(a) Let $f \in L^1(X)$. Then $f \in B^0_{1,1}(X)$ if and only if there exist a sequence of ϵ-atoms $\{a_j\}$ for some $\epsilon > 0$ and a sequence of numbers $\{\lambda_j\} \in l^1$ such that*

$$f = \sum_j \lambda_j a_j \quad \text{in } L^1(X), \tag{28}$$

and

$$\|f\|_{B^0_{1,1}(X)} \sim \sum_j |\lambda_j|. \tag{29}$$

(b) In particular, if $f \in B^0_{1,1}(X)$ supported in a ball B with $r_B = 1$, then there exist a sequence of ϵ-atoms $\{a_j\}$ supported in $3B$ for some $\epsilon > 0$ and a sequence of numbers $\{\lambda_j\}$ such that (28) and (29) hold true.

Proof (a) Let Φ, Ψ be as in Lemma 3.6 such that

$$\begin{aligned}
f &= \Phi(2^{-2}\sqrt{H})\Psi(2^{-2}\sqrt{H})f - \int_0^{1/4} \widetilde{\Phi}(t\sqrt{H})\Psi(t\sqrt{H})f\frac{dt}{t} \\
&\quad - \int_0^{1/4} \widetilde{\Psi}(t\sqrt{H})\Phi(t\sqrt{H})f\frac{dt}{t} \\
&= \Phi(2^{-2}\sqrt{H})\Psi(2^{-2}\sqrt{H})f + \sum_{j=3}^{\infty}\Big[- \int_{2^{-j}}^{2^{-j+1}} \widetilde{\Phi}(t\sqrt{H})\Psi(t\sqrt{H})f\frac{dt}{t} \\
&\quad - \int_{2^{-j}}^{2^{-j+1}} \widetilde{\Psi}(t\sqrt{H})\Phi(t\sqrt{H})f\frac{dt}{t}\Big] \\
&=: f_1 + f_2
\end{aligned}$$

in $L^1(X)$, where $\widetilde{\Phi}(\lambda) = \lambda\Phi'(\lambda)$ and $\widetilde{\Psi}(\lambda) = \lambda\Psi'(\lambda)$.

Moreover, according to Lemma 2.4, we have, for $t > 0$ and $x, y \in X$,

$$\operatorname{supp} F(t\sqrt{H})(\cdot,\cdot) \subset \{(x, y) \in X \times X : d(x, y) < t\}, \tag{30}$$

and

$$|F(t\sqrt{H})(x, y)| \le \frac{C}{V(x, t)}, \tag{31}$$

where $F \in \{\Phi, \Psi, \widetilde{\Phi}, \widetilde{\Psi}\}$.

We first decompose f_1 as follows:

$$f_1 = \sum_{Q\in\mathcal{D}_2} \Phi(2^{-2}\sqrt{H})\big[\Psi(2^{-2}\sqrt{H}) f \cdot 1_Q\big].$$

For each $Q \in \mathcal{D}_2$ as in Remark 2.3, we set

$$s_Q = V(Q)\Big(\sup_{y\in Q} |\Psi(2^{-2}\sqrt{H}) f(y)|\Big)$$

and

$$a_Q = \frac{1}{s_Q}\Phi(2^{-2}\sqrt{H})\big[(\Psi(2^{-2}\sqrt{H}) f) \cdot 1_Q\big]. \tag{32}$$

It is clear that

$$f_1 = \sum_{Q\in\mathcal{D}_2} s_Q a_Q.$$

For the part f_2, we write

$$\begin{aligned} f_2 = \sum_{j=3}^{\infty}\sum_{Q\in\mathcal{D}_j}\Big[&-\int_{2^{-j}}^{2^{-j+1}} \widetilde{\Phi}(t\sqrt{H})\big(\Psi(t\sqrt{H}) f \cdot 1_Q\big)\frac{dt}{t} \\ &-\int_{2^{-j}}^{2^{-j+1}} \widetilde{\Psi}(t\sqrt{H})\big(\Phi(t\sqrt{H}) f \cdot 1_Q\big)\frac{dt}{t}\Big]. \end{aligned}$$

For each $Q \in \mathcal{D}_j$ with $j \geq 3$, we set

$$s_Q = -V(Q)\sup_{y\in Q}\Big[\int_{2^{-j}}^{2^{-j+1}} |\Psi(t\sqrt{H}) f(y)|\frac{dt}{t} + \int_{2^{-j}}^{2^{-j+1}} |\Phi(t\sqrt{H}) f(y)|\frac{dt}{t}\Big],$$

and

$$\begin{aligned} a_Q = \frac{1}{s_Q}\Big[&\int_{2^{-j}}^{2^{-j+1}} \widetilde{\Phi}(t\sqrt{H})\big(\Psi(t\sqrt{H}) f \cdot 1_Q\big)\frac{dt}{t} \\ &+\int_{2^{-j}}^{2^{-j+1}} \widetilde{\Psi}(t\sqrt{H})\big(\Phi(t\sqrt{H}) f \cdot 1_Q\big)\frac{dt}{t}\Big]. \end{aligned} \tag{33}$$

Then we have

$$f_2 = \sum_{j\geq 3}\sum_{Q\in\mathcal{D}_j} s_Q a_Q.$$

Therefore,

$$f = f_1 + f_2 = \sum_{j\geq 2}\sum_{Q\in\mathcal{D}_j} s_Q a_Q.$$

We next claim that a_Q is an atom for each $Q\in\mathcal{D}_j$, $j\geq 2$. Indeed, for $j=2$ we have

$$a_Q(x) = \frac{1}{s_Q}\int_Q \Phi(2^{-2}\sqrt{H})(x,y)\Psi(2^{-2}\sqrt{H})f(y)d\mu(y).$$

It follows, by (30) and Remark 2.3, that $\operatorname{supp} a_Q \subset 3B(x_Q, 2^{-2}) \subset B_Q := B(x_Q, 1)$. Moreover, owing to (31),

$$\begin{aligned}
|a_Q(x)| &\leq \frac{1}{s_Q}\int_Q |\Phi(2^{-2}\sqrt{H})(x,y)|\,|\Psi(2^{-2}\sqrt{H})f(y)|d\mu(y)\\
&\leq \frac{1}{V(Q)}\int_Q |\Phi(2^{-2}\sqrt{H})(x,y)|d\mu(y)\\
&\lesssim \frac{1}{V(Q)} \sim \frac{1}{V(B_Q)}.
\end{aligned}$$

On the other hand, by Lemma 2.7,

$$\begin{aligned}
|a_Q(x) - a_Q(x')| &\leq \frac{1}{s_Q}\int_Q |\Phi(2^{-2}\sqrt{H})(x,y)\\
&\qquad - \Phi(2^{-2}\sqrt{H})(x',y)|\,|\Psi(2^{-2}\sqrt{H})f(y)|d\mu(y)\\
&\leq \frac{1}{V(Q)}\int_Q |\Phi(2^{-2}\sqrt{H})(x,y) - \Phi(2^{-2}\sqrt{H})(x',y)|d\mu(y)\\
&\lesssim \frac{1}{V(B_Q)}\Big(\frac{d(x,x')}{r_{B_Q}}\Big)^{\delta},
\end{aligned}$$

whenever $d(x,x') < r_{B_Q} = 1$.

Hence, a_Q is a multiple of an ϵ-atom associated to the ball B_Q for each $Q\in\mathcal{D}_j$ with $j=2$.

Arguing similarly to above, we can verify that for $Q\in\mathcal{D}_j$, $j\geq 3$, a_Q satisfies (i)-(iii) in Definition 3.10 with the corresponding ball defined by $\widetilde{B}_Q = B(x_Q, 2^{-j})$. The condition $\displaystyle\int a_Q(x)d\mu(x) = 0$ follows directly from Lemma 2.9 and the fact that $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are even and $\widetilde{\Phi}(0) = \widetilde{\Psi}(0) = 0$. Hence, a_Q is a multiple of an ϵ-atom associated to B_Q with $\epsilon = \delta$ for each $Q\in\mathcal{D}_j$, $j\geq 3$.

It remains to show that

$$\sum_{j\geq 2}^{\infty}\sum_{Q\in D_j} |s_Q| \lesssim \|f\|_{\dot{B}^0_{1,1}(X)}.$$

Indeed, from the definition of $\{s_Q\}$, we have, for $\lambda > 2D$

$$\begin{aligned}\sum_{Q\in\mathcal{D}_2}|s_Q| &\lesssim \sum_{Q\in\mathcal{D}_0} V(Q)\inf_{x\in Q}\Psi^*_\lambda f(x)\\ &\lesssim \sum_{Q\in\mathcal{D}_0}\int_Q \Psi^*_\lambda f(x)d\mu(x)\\ &\sim \|\Psi^*_\lambda f\|_1.\end{aligned}$$

It follows, by using Lemma 3.8, that

$$\sum_{Q\in\mathcal{D}_0}|s_Q| \lesssim \|f\|_{\dot{B}^0_{1,1}(X)}.$$

We now show that

$$\sum_{j\geq 3}\sum_{Q\in\mathcal{D}_j}|s_Q| \lesssim \|f\|_{\dot{B}^0_{1,1}(X)}.$$

Indeed, for $Q\in\mathcal{D}_j$ with $j\geq 3$,

$$\begin{aligned}s_Q &\leq V(Q)\Big[\int_{2^{-j}}^{2^{-j+1}}\inf_{x\in Q}\Psi(t\sqrt{H})^*_\lambda f(x)|\frac{dt}{t} + \int_{2^{-j}}^{2^{-j+1}}\inf_{x\in Q}\Phi(t\sqrt{H})^*_\lambda f(x)|\frac{dt}{t}\Big]\\ &\leq \Big[\int_{2^{-j}}^{2^{-j+1}}\|\Psi(t\sqrt{H})^*_\lambda f\|_{L^1(Q)}\frac{dt}{t} + \int_{2^{-j}}^{2^{-j+1}}\|\Phi(t\sqrt{H})^*_\lambda f\|_{L^1(Q)}\frac{dt}{t}\Big].\end{aligned}$$

This, together with Lemma 3.9, implies that

$$\begin{aligned}\sum_{j\geq 1}\sum_{Q\in\mathcal{D}_j}|s_Q| &\lesssim \int_0^1\|\Psi(t\sqrt{H})^*_\lambda f\|_1 f dtt + \int_0^1\|\Phi(t\sqrt{H})^*_\lambda f\|_1 f dtt\\ &\lesssim \|f\|_{\dot{B}^0_{1,1}(X)}.\end{aligned}$$

For the reverse direction, it suffices to prove that there exists $C > 0$ such that

$$\|e^{-H}a\|_1 + \int_0^1 \|tHe^{-tH}a\|_1\frac{dt}{t} \leq C$$

for every ϵ-atom a.

Assume that a is an ϵ-atom associated to a ball B. Since $\|a\|_1\leq 1$, we have

$$\|e^{-H}a\|_1 \lesssim \|a\|_1 \lesssim 1.$$

It remains to prove that

$$\int_0^1 \|tHe^{-tH}a\|_1 \frac{dt}{t} \lesssim 1.$$

To do this, we write

$$\begin{aligned}\|a\|_{B^0_{1,1}(X)} &= \int_0^{4r_B^2} \|tHe^{-tH}a\|_{L^1(4B)} \frac{dt}{t} + \int_0^{4r_B^2} \|tHe^{-tH}a\|_{L^1(X\backslash 4B)} \frac{dt}{t} \\ &\quad + \int_{\min\{(4r_B^2),1\}}^1 \|tHe^{-tH}a\|_1 \frac{dt}{t} \\ &:= E_1 + E_2 + E_3.\end{aligned}$$

For the second term E_2, using the Gaussian upper bound of $\widetilde{q}_t(x, y)$,

$$\begin{aligned}\|tHe^{-tH}a\|_{L^1(X\backslash 4B)} &\lesssim \exp\Big(-c\frac{d(B, X\backslash 4B)^2}{t}\Big)\|a\|_1 \\ &\lesssim \frac{\sqrt{t}}{r_B},\end{aligned}$$

which implies $E_2 \leq C$.

To estimate the term E_1, using the fact that

$$\int_X \widetilde{q}_t(x, y)d\mu(y) = 0,$$

we obtain

$$\|tHe^{-tH}a\|_{L^1(4B)} \leq \int_{4B} \Big| \int_B \widetilde{q}_t(x, y)(a(y) - a(x))d\mu(y)\Big| d\mu(x).$$

By the smoothness condition of the atom a and the Gaussian upper bound of $\widetilde{q}_t(x, y)$, we have

$$\begin{aligned}\Big|\int_B \widetilde{q}_t(x, y)(a(y) - a(x))d\mu(y)\Big| &\lesssim \frac{1}{V(B)} \int_B \frac{1}{V(x,t)} \exp\Big(-c\frac{d(x, y)^2}{t}\Big)\Big(\frac{d(x, y)}{r_B}\Big)^{\epsilon} d\mu(y) \\ &\lesssim \frac{1}{V(B)}\Big(\frac{\sqrt{t}}{r_B}\Big)^{\epsilon},\end{aligned}$$

which implies

$$\int_{4B} \Big| \int_B \widetilde{q}_t(x, y)(a(y) - a(x))d\mu(y)\Big| d\mu(x) \lesssim \Big(\frac{\sqrt{t}}{r_B}\Big)^{\epsilon}.$$

It follows that $E_1 \lesssim 1$.

It remains to estimate E_3. Note that if $r_B = 1$, then $E_3 = 0$. Hence, we need only to consider the case $r_B < 1$. Due to the cancellation property of the atom a, we have

$$\begin{aligned}
\|tHe^{-tH}a\|_1 &= \int_{X\backslash 4B}\Big|\int_B \big(\widetilde{q}_t(x,y)-\widetilde{q}_t(x,x_B)\big)a(y)d\mu(y)\Big|d\mu(x)\\
&\lesssim \int_{X\backslash 4B}\Big|\int_B \Big(\frac{d(y,x_B)}{\sqrt{t}}\Big)^{\delta}\frac{1}{V(x,\sqrt{t})}\exp\Big(-c\frac{d(x,y)^2}{t}\Big)|a(y)|d\mu(y)d\mu(x)\\
&\lesssim \Big(\frac{r_B}{\sqrt{t}}\Big)^{\delta}\|a\|_1\int_X \frac{1}{V(x,\sqrt{t})}\exp\Big(-c\frac{d(x,y)^2}{t}\Big)d\mu(x)\\
&\lesssim \Big(\frac{r_B}{\sqrt{t}}\Big)^{\delta}.
\end{aligned}$$

It follows that $E_3 \lesssim 1$.

This completes our proof of (a).

(b) Assume that $\operatorname{supp} f \subset B$ with $r_B = 1$. Recall that in (a) we have proved that

$$f = \sum_{j=2}^{\infty}\sum_{Q\in\mathcal{D}_j} s_Q a_Q,$$

where $\{s_Q\}$ is a sequence of numbers satisfying (29) and $\{a_Q\}$ is a sequence of ϵ-atoms defined by (32) and (33). From (30), (32) and (33), we have

$$f = \sum_{j=0}^{\infty}\sum_{Q\in\mathcal{D}_j : Q\cap\frac{3}{2}B\neq\emptyset} s_Q a_Q$$

and

$$\operatorname{supp} a_Q \subset 3B \quad \text{whenever } Q\cap \tfrac{3}{2}B \neq \emptyset.$$

This completes the proof of (b). □

We now introduce a new variant of the inhomogeneous Besov spaces. For $\ell > 0$, the Besov space $B^{0,\ell}_{1,1}(X)$ is defined as the set of functions $f \in L^1(X)$ such that

$$\|f\|_{B^{0,\ell}_{1,1}(X)} := \|e^{-\ell^2 H} f\|_1 + \int_0^{\ell^2}\|tHe^{-tH}f\|_1\frac{dt}{t}.$$

When $\ell = 1$, we simply write $B^0_{1,1}(X)$.

Definition 3.12 Let $\epsilon > 0$ and $\ell > 0$. A function a is said to be an (ϵ, ℓ)-atom if there exists a ball B such that

(i) $\operatorname{supp} a \subset B$;
(ii) $|a(x)| \leq V(B)^{-1}$;

(iii) $|a(x) - a(y)| \le V(B)^{-1} \left(\frac{d(x, y)}{r_B} \right)^{\epsilon}$;

(iv) $\int a(x) d\mu(x) = 0$ if $r_B < \ell$.

Using the approach in the proof of Theorem 3.11 and the scaling argument, we are also able to prove the following theorem.

Theorem 3.13 *Let $\ell > 0$ and $f \in L^1(X)$. Then $f \in B^{0,\ell}_{1,1}(X)$ if and only if there exist a sequence of (ϵ, ℓ)-atoms $\{a_j\}$ for some $\epsilon > 0$ and a sequence of numbers $\{\lambda_j\}$ such that*

$$f = \sum_j \lambda_j a_j, \tag{34}$$

and

$$\|f\|_{B^{0,\ell}_{1,1}(X)} \sim \sum_j |\lambda_j|. \tag{35}$$

In particular, if $f \in B^{0,\ell}_{1,1}(X)$ supported in a ball B with $r_B = \ell$, then there exist a sequence of (ϵ, ℓ)-atoms $\{a_j\}_j$ supported in $3B$ for some $\epsilon > 0$ and a sequence of numbers $\{\lambda_j\}$ such that (34) and (35) hold true.

4 Proofs of Main Results

4.1 Proof of Theorem 1.4

We state the following results in which the proofs of Lemma 4.1 and Proposition 4.2 below are similar to those of Lemmas 3.4, 3.2, 3.3 and Corollary 3.5.

Lemma 4.1 *Let ψ be an even function in $\mathcal{S}(\mathbb{R})$ such that* $\operatorname{supp}\psi \subset \{\lambda : 1/2 \le |\lambda| \le 2\}$*, and*

$$\sum_{j\in\mathbb{Z}} \psi_j(\lambda) = 1, \quad \lambda > 0,$$

where $\psi_j(\lambda) = \psi(2^{-j}\lambda),\ j \in \mathbb{Z}$.

Then we have

$$\sum_{j\in\mathbb{Z}} \psi_j(\sqrt{L}) f = f \ \text{ in } L^1(X)$$

for $f \in \dot{B}^{0,L}_{1,1}(X)$.

Proposition 4.2 *The following properties hold true for the homogeneous Besov space* $\dot{B}^{0,L}_{1,1}(X)$.

(i) *The homogeneous Besov space* $\dot{B}^{0,L}_{1,1}(X)$ *is complete.*
(ii) *The inclusion* $\dot{B}^{0,L}_{1,1}(X) \hookrightarrow L^1(X)$ *is continuous.*
(iii) *For each* $p \in [1, \infty)$*, the space* $L^p(X)$ *is dense in* $\dot{B}^{0,L}_{1,1}(X)$.

Proposition 4.3 *Let* φ *be as in Lemma 2.4 and let* Φ *be the Fourier transforms of* φ. *For each* $m \in \mathbb{N}$,

$$f = c\int_0^\infty (t^2L)^m e^{-t^2L}\Phi(t\sqrt{L})f\frac{dt}{t} \quad \text{in } L^1(X) \tag{36}$$

for $f \in B^{0,L}_{1,1}(X)$*, where* $c = \Big[\int_0^\infty z^{2m}e^{-z^2}\Phi(z)\frac{dz}{z}\Big]^{-1}$.

Proof Similarly to the proof of Lemma 3.4, it suffices to prove the proposition for $f \in L^2(X) \cap \dot{B}^{0,L}_{1,1}(X)$. By spectral theory,

$$f = c\int_0^\infty (t^2L)^m e^{-t^2L}\Phi(t\sqrt{L})f\frac{dt}{t} \tag{37}$$

in $L^2(X)$.

On the other hand, from Lemma 2.7,

$$\begin{aligned}\Big\|\int_0^\infty (t^2L)^m e^{-t^2L}\Phi(t\sqrt{L})f\frac{dt}{t}\Big\|_1 &\lesssim \int_0^\infty \|(t^2L)^m e^{-t^2L}\Phi(t\sqrt{L})f\|_1\frac{dt}{t}\\ &\lesssim \int_0^\infty \|t^2Le^{-t^2L}f\|_1\frac{dt}{t}\\ &\lesssim \|f\|_{\dot{B}^{0,L}_{1,1}(X)}.\end{aligned}$$

This implies that

$$\int_0^\infty (t^2L)^m e^{-t^2L}\Phi(t\sqrt{L})f\frac{dt}{t} = g \quad \text{in } L^1(X)$$

for some $g \in L^1(X)$.

This, in combination with (37), implies that $f = g$ for a.e.. Therefore,

$$f = c\int_0^\infty (t^2L)^m e^{-t^2L}\Phi(t\sqrt{L})f\frac{dt}{t} \quad \text{in } L^1(X)$$

for $f \in L^2(X) \cap \dot{B}^{0,L}_{1,1}(X)$.

This completes our proof. □

Proof of Theorem 1.4: The proof of the atomic decomposition for functions $f \in \dot{B}^{0,L}_{1,1}(X)$ is similar to that of Theorem 4.2 in [5] and the proof of Theorem 3.11. Hence, we leave it to the interested reader.

For the reverse direction, it suffices to show that there exists $C > 0$ such that

$$\int_0^\infty \|tLe^{-tL}a\|_1 \frac{dt}{t} \le C$$

for every (L, M)-atom a.

Suppose that a is an (L, M)-atom associated with a ball B. Then we have

$$\int_0^\infty \|tLe^{-tL}a\|_1 \frac{dt}{t} = \int_0^{r_B^2} \|tLe^{-tL}a\|_1 \frac{dt}{t} + \int_{r_B^2}^\infty \|tLe^{-tL}a\|_1 \frac{dt}{t}.$$

For the first term, we have

$$\begin{aligned}\int_0^{r_B^2} \|tLe^{-tL}a\|_1 \frac{dt}{t} &= \int_0^{r_B^2} t\|e^{-tL}La\|_1 \frac{dt}{t}\\ &\lesssim \int_0^{r_B^2} t\|a\|_1 \frac{dt}{t}\\ &\lesssim \int_0^{r_B^2} tr_B^{-2} \frac{dt}{t}\\ &\lesssim 1.\end{aligned}$$

For the second term, using $a = L^M b$,

$$\begin{aligned}\int_{r_B^2}^\infty \|tLe^{-tL}a\|_1 \frac{dt}{t} &= \int_{r_B^2}^\infty \|(tL)^{M+1}e^{-tL}b\|_1 \frac{dt}{t^{M+1}}\\ &\lesssim \int_{r_B^2}^\infty \|b\|_1 \frac{dt}{t^{M+1}}\\ &\lesssim \int_{r_B^2}^\infty r_B^{2M} \frac{dt}{t^{M+1}}\\ &\lesssim 1.\end{aligned}$$

This completes our proof. □

4.2 Proof of Theorem 1.6

We refer the reader to Sect. 2.1 for the index set $\mathcal{I}$, the family functions $\{\psi_\alpha\}_\alpha$ and the family of balls $\{B_\alpha\}_\alpha$ which will be used in this section.

Lemma 4.4 *For each $\alpha \in \mathcal{I}$ and $f \in L^1(X)$ we have*

$$\int_X \int_0^{\rho(x_\alpha)^2} \Big|\big[tHe^{-tH} - tLe^{-tL}\big](f\psi_\alpha)(x)\Big| \frac{dt}{t} d\mu(x) \lesssim \|f\psi_\alpha\|_1. \tag{38}$$

Proof By (L2), we have

$$\begin{aligned}
&\int_X \int_0^{\rho(x_\alpha)^2} \Big|\big[tHe^{-tH} - tLe^{-tL}\big](f\psi_\alpha)(x)\Big| \frac{dt}{t} d\mu(x) \\
&\quad \lesssim \int_X \int_0^{\rho(x_\alpha)^2} \int_{B_\alpha} \frac{1}{V(x,\sqrt{t})} \Big(\frac{\sqrt{t}}{\sqrt{t}+\rho(y)}\Big)^\delta \\
&\qquad \exp\Big(-c\frac{d(x,y)^2}{t}\Big)|(f\psi_\alpha)(y)| d\mu(y) \frac{dt}{t} d\mu(x).
\end{aligned}$$

Since $\rho(y) \sim \rho(x_\alpha)$ for all $y \in B_\alpha$, we have

$$\begin{aligned}
&\int_X \int_0^{\rho(x_\alpha)^2} \Big|\big[tHe^{-tH} - tLe^{-tL}\big](f\psi_\alpha)(x)\Big| \frac{dt}{t} d\mu(x) \\
&\quad \lesssim \int_{B_\alpha} \int_0^{\rho(x_\alpha)^2} \int_X \frac{1}{V(x,\sqrt{t})} \exp\Big(\\
&\qquad -c\frac{d(x,y)^2}{t}\Big)\Big(\frac{\sqrt{t}}{\rho(x_\alpha)}\Big)^\delta |(f\psi_\alpha)(y)| d\mu(x) \frac{dt}{t} d\mu(y).
\end{aligned}$$

Using the fact that

$$\int_X \frac{1}{V(x,\sqrt{t})} \exp\Big(-c\frac{d(x,y)^2}{t}\Big) d\mu(x) \lesssim 1, \tag{39}$$

we obtain that

$$\begin{aligned}
&\int_X \int_0^{\rho(x_\alpha)^2} \Big|\big[tHe^{-tH} - tLe^{-tL}\big](f\psi_\alpha)(x)\Big| \frac{dt}{t} d\mu(x) \\
&\lesssim \int_{B_\alpha} \int_0^{\rho(x_\alpha)^2} \Big(\frac{\sqrt{t}}{\rho(x_\alpha)}\Big)^\delta |(f\psi_\alpha)(y)| \frac{dt}{t} d\mu(y) \\
&\lesssim \int_{B_\alpha} |(f\psi_\alpha)(y)| d\mu(y) \\
&\lesssim \|f\psi_\alpha\|_1.
\end{aligned}$$

This completes our proof. □

Lemma 4.5 *For each $f \in L^1(X)$ we have*

$$\sum_{\alpha\in\mathcal{I}}\int_X\int_0^{\rho(x_\alpha)^2}\Big|tLe^{-tL}f(x)\psi_\alpha(x)-tLe^{-tL}(f\psi_\alpha)(x)\Big|\frac{dt}{t}d\mu(x)\lesssim\|f\|_1. \tag{40}$$

Proof Denote

$$\mathcal{I}_{1,\alpha}=\{\beta\in\mathcal{I}:B_\alpha^*\cap B_\beta\neq\emptyset\}\quad\text{and}\quad\mathcal{I}_{2,\alpha}=\{\beta\in\mathcal{I}:B_\alpha^*\cap B_\beta=\emptyset\},$$

where $B_\alpha^*=4B_\alpha$, $\alpha\in\mathcal{I}$.

Observe that

$$\begin{aligned}&tLe^{-tL}f(x)\psi_\alpha(x)-tLe^{-tL}(f\psi_\alpha)(x)\\&=\sum_{\beta\in\mathcal{I}}\int_{B_\beta}q_t(x,y)\big(\psi_\alpha(x)-\psi_\alpha(y)\big)(f\psi_\beta)(y)d\mu(y).\end{aligned}$$

Then we write

$$\begin{aligned}&\sum_{\alpha\in\mathcal{I}}\int_X\int_0^{\rho(x_\alpha)^2}\Big|tLe^{-tL}f(x)\psi_\alpha(x)-tLe^{-tL}(f\psi_\alpha)(x)\Big|\frac{dt}{t}d\mu(x)\\&=\sum_{\alpha\in\mathcal{I}}\int_X\int_0^{\rho(x_\alpha)^2}\Big|\sum_{\beta\in\mathcal{I}}\int_{B_\beta}q_t(x,y)\big(\psi_\alpha(x)\\&\quad-\psi_\alpha(y)\big)(f\psi_\beta)(y)d\mu(y)\Big|\frac{dt}{t}d\mu(x)\\&\le\sum_{\alpha\in\mathcal{I}}\int_X\int_0^{\rho(x_\alpha)^2}\Big|\sum_{\beta\in\mathcal{I}_{1,\alpha}}\int_{B_\beta}q_t(x,y)\big(\psi_\alpha(x)\\&\quad-\psi_\alpha(y)\big)(f\psi_\beta)(y)d\mu(y)\Big|\frac{dt}{t}d\mu(x)\\&\quad+\sum_{\alpha\in\mathcal{I}}\int_X\int_0^{\rho(x_\alpha)^2}\Big|\sum_{\beta\in\mathcal{I}_{2,\alpha}}\int_{B_\beta}q_t(x,y)\big(\psi_\alpha(x)\\&\quad-\psi_\alpha(y)\big)(f\psi_\beta)(y)d\mu(y)\Big|\frac{dt}{t}d\mu(x)\\&=:E_1+E_2.\end{aligned}$$

We estimate E_1 first. Owing to Lemma 2.1 and the upper bound of $q_t(x, y)$, we have

$$\sum_{\beta\in\mathcal{I}_{1,\alpha}} \int_{B_\beta} \frac{1}{V(x,\sqrt{t})} \exp\Big(-c\frac{d(x,y)^2}{t}\Big)\frac{d(x,y)}{\rho(x_\alpha)}|(f\psi_\beta)(y)|d\mu(y)$$
$$\lesssim \sum_{\beta\in\mathcal{I}_{1,\alpha}} \int_{B_\beta} \frac{1}{V(x,\sqrt{t})} \exp\Big(-c'\frac{d(x,y)^2}{t}\Big)\frac{\sqrt{t}}{\rho(x_\alpha)}|(f\psi_\beta)(y)|d\mu(y).$$

This implies that

$$E_1 \lesssim \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}_{1,\alpha}} \int_X \int_0^{\rho(x_\alpha)^2} \int_{B_\beta} \frac{1}{V(x,\sqrt{t})} \exp$$
$$\Big(-c'\frac{d(x,y)^2}{t}\Big)\frac{\sqrt{t}}{\rho(x_\alpha)}|(f\psi_\beta)(y)|d\mu(y)\frac{dt}{t}d\mu(x)$$
$$\lesssim \sum_{\beta\in\mathcal{I}} \int_{B_\beta} \sum_{\alpha\in\mathcal{J}_{1,\beta}} \int_0^{\rho(x_\alpha)^2} \int_X \frac{1}{V(x,\sqrt{t})} \exp$$
$$\Big(-c'\frac{d(x,y)^2}{t}\Big)d\mu(x)\frac{\sqrt{t}}{\rho(x_\alpha)}|(f\psi_\beta)(y)|\frac{dt}{t}d\mu(y),$$

where

$$\mathcal{J}_{1,\beta} = \{\alpha \in \mathcal{I} : B_\alpha^* \cap B_\beta \neq \emptyset\}.$$

Since $\sharp\mathcal{J}_{1,\beta}$ is uniformly bounded in $\beta \in \mathcal{I}$, using (39) we obtain

$$\sum_{\alpha\in\mathcal{I}} \int_X \int_0^{\rho(x_\alpha)^2} \Big|tLe^{-tL}f(x)\psi_\alpha(x) - tLe^{-tL}(f\psi_\alpha)(x)\Big|\frac{dt}{t}d\mu(x)$$
$$\lesssim \sum_{\beta\in\mathcal{I}} \int_{B_\beta} \sum_{\alpha\in\mathcal{J}_{1,\beta}} \int_0^{\rho(x_\alpha)^2} \frac{\sqrt{t}}{\rho(x_\alpha)}|(f\psi_\beta)(y)|\frac{dt}{t}d\mu(y)$$
$$\lesssim \sum_{\beta\in\mathcal{I}} \|f\psi_\beta\|_1 \sim \|f\|_1.$$

If $\beta \in \mathcal{I}_{2,\alpha}$, then $\psi_\alpha(y) = 0$ for all $y \in B_\beta$. Therefore,

$$E_2 = \sum_{\alpha\in\mathcal{I}} \int_{B_\alpha} \int_0^{\rho(x_\alpha)^2} \Big|\sum_{\beta\in\mathcal{I}_{2,\alpha}} \int_{B_\beta} q_t(x,y)\psi_\alpha(x)(f\psi_\beta)(y)d\mu(y)\Big|\frac{dt}{t}d\mu(x).$$

By the upper bound of $q_t(x,y)$ and the fact that $d(x,y) > \rho(x_\alpha)$ whenever $x \in B_\alpha$, $y \in B_\beta$ with $\beta \in \mathcal{I}_{2,\alpha}$, we further simplify to that

$$\begin{aligned} E_2 &\lesssim \sum_{\alpha\in\mathcal{I}} \int_{B_\alpha} \int_0^{\rho(x_\alpha)^2} \sum_{\beta\in\mathcal{I}_{2,\alpha}} \int_{B_\beta} \frac{1}{V(x,\sqrt{t})} \exp \\ &\qquad \Big(-c\frac{d(x,y)^2}{t}\Big)\psi_\alpha(x)|(f\psi_\beta)(y)|d\mu(y)\frac{dt}{t}d\mu(x) \\ &\lesssim \sum_{\alpha\in\mathcal{I}} \int_{B_\alpha} \int_0^{\rho(x_\alpha)^2} \sum_{\beta\in\mathcal{I}_{2,\alpha}} \int_{B_\beta} \frac{1}{V(x,d(x,y))} \exp \\ &\qquad \Big(-c'\frac{d(x,y)^2}{t}\Big)\frac{\sqrt{t}}{d(x,y)}|(f\psi_\beta)(y)|d\mu(y)\frac{dt}{t}d\mu(x) \\ &\lesssim \sum_{\alpha\in\mathcal{I}} \int_{B_\alpha} \int_0^{\rho(x_\alpha)^2} \sum_{\beta\in\mathcal{I}_{2,\alpha}} \int_{B_\beta} \frac{1}{V(x,d(x,y))} \exp \\ &\qquad \Big(-c'\frac{d(x,y)^2}{t}\Big)\frac{\sqrt{t}}{\rho(x_\alpha)}|(f\psi_\beta)(y)|d\mu(y)\frac{dt}{t}d\mu(x) \\ &\lesssim \sum_{\beta\in\mathcal{I}} \int_{B_\beta} \int_0^{\rho(x_\alpha)^2} \sum_{\alpha\in\mathcal{J}_{2,\beta}} \int_{B_\alpha} \frac{1}{V(x,d(x,y))} \exp \\ &\qquad \Big(-c'\frac{d(x,y)^2}{t}\Big)\frac{\sqrt{t}}{\rho(x_\alpha)}|(f\psi_\beta)(y)|d\mu(x)\frac{dt}{t}d\mu(y), \end{aligned}$$

where

$$\mathcal{J}_{2,\beta} = \{\alpha \in \mathcal{I} : B_\alpha^* \cap B_\beta = \emptyset\}.$$

Note that $d(x,y) \sim d(x,x_\beta) \sim d(x_\alpha,x_\beta)$ whenever $x \in B_\alpha$, $y \in B_\beta$ with $\alpha \in \mathcal{J}_{2,\beta}$. Hence,

$$\begin{aligned} E_2 &\lesssim \sum_{\beta\in\mathcal{I}} \int_{B_\beta} \int_0^{\rho(x_\alpha)^2} \sum_{\alpha\in\mathcal{J}_{2,\beta}} \int_{B_\alpha} \frac{1}{V(x_\beta,d(x,x_\beta))} \exp \\ &\qquad \Big(-c'\frac{d(x_\alpha,x_\beta)^2}{t}\Big)\frac{\sqrt{t}}{\rho(x_\alpha)}|(f\psi_\beta)(y)|d\mu(x)\frac{dt}{t}d\mu(y) \\ &\lesssim \sum_{\beta\in\mathcal{I}} \int_{B_\beta} \int_0^{\rho(x_\alpha)^2} \sum_{\alpha\in\mathcal{J}_{2,\beta}} \int_{B_\alpha} \frac{1}{V(x_\beta,d(x,x_\beta))} \exp \\ &\qquad \Big(-c'\frac{d(x_\alpha,x_\beta)^2}{\rho(x_\alpha)^2}\Big)\frac{\sqrt{t}}{\rho(x_\alpha)}|(f\psi_\beta)(y)|d\mu(x)\frac{dt}{t}d\mu(y). \end{aligned}$$

On the other hand, invoking (5) we have

$$\begin{aligned}\exp\Big(-c'\frac{d(x_\alpha,x_\beta)^2}{\rho(x_\alpha)^2}\Big) &\lesssim \frac{\rho(x_\alpha)}{d(x_\alpha,x_\beta)}\\ &\lesssim \frac{\rho(x_\beta)}{d(x_\alpha,x_\beta)}\Big(1+\frac{d(x_\alpha,x_\beta)}{\rho(x_\beta)}\Big)^{\frac{k_0}{k_0+1}}\\ &\lesssim \Big(\frac{\rho(x_\beta)}{d(x_\alpha,x_\beta)}\Big)^{\frac{1}{k_0+1}}.\end{aligned}$$

Therefore,

$$\begin{aligned}E_2 &\lesssim \sum_{\beta\in\mathcal{I}}\int_{B_\beta}\int_0^{\rho(x_\alpha)^2}\sum_{\alpha\in\mathcal{J}_{2,\beta}}\int_{B_\alpha}\frac{1}{V(x_\beta,d(x,x_\beta))}\\ &\quad\Big(\frac{\rho(x_\beta)}{d(x_\alpha,x_\beta)}\Big)^{\frac{1}{k_0+1}}\frac{\sqrt{t}}{\rho(x_\alpha)}|(f\psi_\beta)(y)|d\mu(x)\frac{dt}{t}d\mu(y)\\ &\sim \sum_{\beta\in\mathcal{I}}\int_{B_\beta}\int_0^{\rho(x_\alpha)^2}\sum_{\alpha\in\mathcal{J}_{2,\beta}}\int_{B_\alpha}\frac{1}{V(x_\beta,d(x,x_\beta))}\\ &\quad\Big(\frac{\rho(x_\beta)}{d(x,x_\beta)}\Big)^{\frac{1}{k_0+1}}\frac{\sqrt{t}}{\rho(x_\alpha)}|(f\psi_\beta)(y)|d\mu(x)\frac{dt}{t}d\mu(y)\\ &\lesssim \sum_{\beta\in\mathcal{I}}\int_{B_\beta}\sum_{\alpha\in\mathcal{J}_{2,\beta}}\int_{B_\alpha}\frac{1}{V(x_\beta,d(x,x_\beta))}\Big(\frac{\rho(x_\beta)}{d(x,x_\beta)}\Big)^{\frac{1}{k_0+1}}|(f\psi_\beta)(y)|d\mu(x)\frac{dt}{t}d\mu(y).\end{aligned}$$

Since $\{B_\beta\}_{\beta\in\mathcal{I}}$ is a finite overlapping family and $\cup_{\alpha\in\mathcal{J}_{2,\beta}}B_\alpha\subset X\backslash B_\beta^*$, we also obtain that

$$\begin{aligned}E_2 &\lesssim \sum_{\beta\in\mathcal{I}}\int_{B_\beta}|(f\psi_\beta)(y)|\int_{X\backslash B_\beta^*}\frac{1}{V(x_\beta,d(x,x_\beta))}\Big(\frac{\rho(x_\beta)}{d(x,x_\beta)}\Big)^{\frac{1}{k_0+1}}d\mu(x)d\mu(y)\\ &\lesssim \sum_{\beta\in\mathcal{I}}\int_{B_\beta}|(f\psi_\beta)(y)|d\mu(y)\\ &\lesssim \|f\|_1.\end{aligned}$$

This completes our proof. □

We are ready to give the proof of Theorem 1.6.

Proof of Theorem 1.6: We first prove that each function $f\in\dot{B}^{0,L}_{1,1}(X)$ admits an atomic decomposition as in the statement of the theorem.

Indeed, we first observe that from Theorem 3.13,

$$\int_0^{\rho(x_\alpha)^2} \|tHe^{-tH}(f\psi_\alpha)\|_1 \frac{dt}{t} \lesssim \int_X \int_0^{\rho(x_\alpha)^2} \left|\left[tHe^{-tH} - tLe^{-tL}\right](f\psi_\alpha)(x)\right| \frac{dt}{t} d\mu(x)$$
$$+ \int_X \int_0^{\rho(x_\alpha)^2} \Big|tLe^{-tL} f(x)\psi_\alpha(x)$$
$$- tLe^{-tL}(f\psi_\alpha)(x)\Big| \frac{dt}{t} d\mu(x)$$
$$+ \int_X \int_0^{\infty} \left|tLe^{-tL} f(x)\psi_\alpha(x)\right| \frac{dt}{t} d\mu(x).$$

By Lemmas 4.4 and 4.5, we have $f\psi_\alpha \in B_{1,1}^{0,\ell_\alpha}(X)$ with $\ell_\alpha = \epsilon_0\rho(x_\alpha)/3$, where ϵ_0 is the constant in Lemma 2.1. Therefore, we can write

$$f\psi_\alpha = \sum_j \lambda_{j,\alpha} a_{j,\alpha},$$

where $a_{j,\alpha}$ is an (ϵ, ℓ_α)-atom associated to a ball $B_{j,\alpha} \subset 3B_\alpha$ for each j, and $\{\lambda_{j,\alpha}\}_j$ is a sequence of numbers satisfying

$$\sum_j |\lambda_{j,\alpha}| \le \int_0^{\rho(x_\alpha)^2} \|tHe^{-tH}(f\psi_\alpha)\|_1 \frac{dt}{t}.$$

Note that $3B_\alpha = B(x_\alpha, \epsilon_0\rho(x_\alpha))$, by (5),

$$\rho(x_{B_{j,\alpha}}) \ge C_\rho^{-1} \rho(x_\alpha)(1+\epsilon_0)^{-\frac{k_0}{k_0+1}},$$

which implies that

$$3\ell_\alpha = \epsilon_0\rho(x_\alpha) < C_\rho\epsilon_0(1+\epsilon_0)^{\frac{k_0}{k_0+1}} \rho(x_{B_{j,\alpha}}).$$

From (iii) in Lemma 2.1, $C_\rho\epsilon_0(1+\epsilon_0)^{\frac{k_0}{k_0+1}} < 1$. Hence,

$$\ell_\alpha \le \rho(x_{B_{j,\alpha}}).$$

Consequently, each $a_{j,\alpha}$ is also an $(\epsilon, \rho(\cdot))$ atom associated to the ball $B_{j,\alpha}$.

Therefore, by Lemmas 4.4, 4.5 and (ii) in Proposition 4.2,

$$f = \sum_\alpha \sum_j \lambda_{j,\alpha} a_{j,\alpha}$$

such that

$$\sum_\alpha \sum_j |\lambda_{j,\alpha}| \lesssim \sum_\alpha \int_X \int_0^{\rho(x_\alpha)^2} \Big|\big[tHe^{-tH} - tLe^{-tL}\big](f\psi_\alpha)(x)\Big| \frac{dt}{t} d\mu(x)$$
$$+ \sum_\alpha \int_X \int_0^{\rho(x_\alpha)^2} \Big|tLe^{-tL} f(x)\psi_\alpha(x) - tLe^{-tL}(f\psi_\alpha)(x)\Big| \frac{dt}{t} d\mu(x)$$
$$+ \sum_\alpha \int_X \int_0^{\infty} \Big|tLe^{-tL} f(x)\psi_\alpha(x)\Big| \frac{dt}{t} d\mu(x)$$
$$\lesssim \sum_\alpha \|f\psi_\alpha\|_1 + \|f\|_1 + \int_X \int_0^\infty \Big|tLe^{-tL} f(x)\Big| \frac{dt}{t} d\mu(x)$$
$$\lesssim \|f\|_1 + \|f\|_{\dot{B}^{0,L}_{1,1}(X)}$$
$$\lesssim \|f\|_{\dot{B}^{0,L}_{1,1}(X)}.$$

This completes the proof of the first direction.

For the reverse direction, it suffices to prove that there exists $C > 0$ such that

$$\|a\|_{\dot{B}^{0,L}_{1,1}(X)} \leq C$$

for every $(\epsilon, \rho(\cdot))$ atom with some $\epsilon > 0$.

To do this, suppose that a is an $(\epsilon, \rho(\cdot))$ atom associated with a ball B. Then we write

$$\|a\|_{\dot{B}^{0,L}_{1,1}(X)} = \int_0^{4r_B^2} \|tLe^{-tL}a\|_{L^1(3B)} \frac{dt}{t} + \int_0^{4r_B^2} \|tLe^{-tL}a\|_{L^1(X\backslash 3B)} \frac{dt}{t}$$
$$+ \int_{4r_B^2}^\infty \|tLe^{-tL}a\|_1 \frac{dt}{t} := A_1 + A_2 + A_3.$$

For the second term A_2, using the Gaussian upper bound of $q_t(x, y)$,

$$\|tLe^{-tL}a\|_{L^1(X\backslash 3B)} \lesssim \exp\Big(-c\frac{d(B, X\backslash 3B)^2}{t}\Big)\|a\|_1$$
$$\lesssim \Big(\frac{\sqrt{t}}{r_B}\Big)^\delta,$$

which implies $A_2 \leq C$.

To estimate the term A_1, observe that

$$\|tLe^{-tL}a\|_{L^1(3B)} \leq \int_{3B} \Big|\int_X q_t(x, y)(a(y) - a(x))d\mu(y)\Big| d\mu(x)$$
$$+ \int_{3B} \Big|\int_X q_t(x, y)a(x)d\mu(y)\Big| d\mu(x).$$

By the smoothness condition of the atom a, we have

$$\begin{aligned}\Big|\int_B q_t(x,y)(a(y)-a(x))d\mu(y)\Big| &\lesssim \frac{1}{V(B)}\int_X \frac{1}{V(x,t)}\exp\Big(-c\frac{d(x,y)^2}{t}\Big)\Big(\frac{d(x,y)}{r_B}\Big)^\epsilon d\mu(y)\\ &\lesssim \frac{1}{V(B)}\Big(\frac{\sqrt{t}}{r_B}\Big)^\epsilon,\end{aligned}$$

which implies

$$\int_{3B}\Big|\int_X q_t(x,y)(a(y)-a(x))d\mu(y)\Big|d\mu(x) \lesssim \Big(\frac{\sqrt{t}}{r_B}\Big)^\epsilon. \tag{41}$$

Invoking the condition (II) to give

$$\Big|\int_X q_t(x,y)a(x)d\mu(y)\Big| \lesssim |a(x)|\Big(\frac{\sqrt{t}}{\rho(x)}\Big)^\delta\Big(1+\frac{\sqrt{t}}{\rho(x)}\Big)^{-K}.$$

This, along with (41), implies that

$$\begin{aligned}A_1 &\lesssim \int_0^{4r_B^2}\Big(\frac{\sqrt{t}}{r_B}\Big)^\epsilon\frac{dt}{t} + \int_0^{4r_B^2}\int_X |a(x)|\Big(\frac{\sqrt{t}}{\rho(x)}\Big)^\delta\Big(1+\frac{\sqrt{t}}{\rho(x)}\Big)^{-K} d\mu(x)\frac{dt}{t}\\ &\lesssim 1+\int_X |a(x)|\int_0^\infty \Big(\frac{\sqrt{t}}{\rho(x)}\Big)^\delta\Big(1+\frac{\sqrt{t}}{\rho(x)}\Big)^{-K}\frac{dt}{t}d\mu(x)\\ &\lesssim 1+\int_X |a(x)|d\mu(x)\\ &\lesssim 1.\end{aligned}$$

It remains to estimate A_3. To do this, we consider two cases.
Case 1: $0 < r_B \le \rho(x_B)$
Due to the cancellation property of the atom a, we have

$$\begin{aligned}\|tLe^{-tL}a\|_1 &= \int_X\Big|\int_{3B}\big(q_t(x,y)-q_t(x,x_B)\big)a(y)d\mu(y)\Big|d\mu(x)\\ &\lesssim \int_X\Big|\int_{3Q}\Big(\frac{d(y,x_B)}{\sqrt{t}}\Big)^\delta\frac{1}{V(x,\sqrt{t})}\exp\Big(-c\frac{d(x,y)^2}{t}\Big)|a(y)|d\mu(y)d\mu(x)\\ &\lesssim \Big(\frac{r_B}{\sqrt{t}}\Big)^\delta \sup_{y\in 3Q}\int_X \frac{1}{V(x,\sqrt{t})}\exp\Big(-c\frac{d(x,y)^2}{t}\Big)d\mu(x)\\ &\lesssim \Big(\frac{r_B}{\sqrt{t}}\Big)^\delta.\end{aligned}$$

It follows that $A_3 \lesssim 1$.
Case 2: $r_B > \rho(x_B)$

Observe that by (5), for $z \in 3B$,

$$\begin{aligned}\rho(z) &\lesssim \rho(x_B)\Big(1+\frac{d(z,x_B)}{\rho(x_B)}\Big)^{\frac{k_0}{k_0+1}}\\ &\lesssim \rho(x_B)\frac{r_B}{\rho(x_B)}=r_B.\end{aligned}$$

This, together with (L1), yields that

$$\begin{aligned}\|tLe^{-tL}a\|_1 &= \int_X\Big|\int_{3B} q_t(x,y)a(y)d\mu(y)\Big|dx\\ &\lesssim \int_X\int_{3B}\Big(\frac{\rho(y)}{\sqrt{t}}\Big)^{\delta}\frac{1}{V(x,\sqrt{t})}\exp\Big(-c\frac{d(x,y)^2}{t}\Big)|a(y)|d\mu(y)d\mu(x)\\ &\int_X\int_{3B}\Big(\frac{r_B}{\sqrt{t}}\Big)^{\delta}\frac{1}{V(x,\sqrt{t})}\exp\Big(-c\frac{d(x,y)^2}{t}\Big)|a(y)|d\mu(y)d\mu(x)\\ &\lesssim \Big(\frac{r_B}{t}\Big)^{\delta}\|a\|_1\\ &\lesssim \Big(\frac{r_B}{t}\Big)^{\delta}.\end{aligned}$$

It follows that $A_3 \lesssim 1$.

This completes our proof. □

5 Application to Boundedness of Riesz Transforms Associated to Schrödinger Operators on $\mathbb{R}^n$

In this section, we show the boundedness of the Riesz transforms associated to Schrödinger operators $L = -\Delta + V$ on $\mathbb{R}^n$ on the new Besov space $\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$. It is worth noticing that although we restrict ourselves to consider the Schrödinger operators on $\mathbb{R}^n$, our approach works well in more general setting including settings listed in Remark 1.1.

Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$ with $V \in RH_{n/2}$. Our main result in this section is the following theorem.

Theorem 5.1 *The Riesz transform $\nabla L^{-1/2}$ is bounded from $\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$ to $\dot{B}^{0}_{1,1}(\mathbb{R}^n)$.*

We would like to remark that in the classical case, the Riesz transform $\nabla(-\Delta)^{-1/2}$ is bounded on the classical Besov spaces $\dot{B}^{0}_{1,1}(\mathbb{R}^n)$. See for example [6, Proposition 2.4]. In the setting of Theorem 5.1, we have a better estimates for the Riesz transform $\nabla L^{-1/2}$ since by Theorem 3.13, $\dot{B}^{0}_{1,1}(\mathbb{R}^n) \hookrightarrow \dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$. Therefore, as a consequence of Theorems 3.13 and 5.1, we have:

Corollary 5.2 *The Riesz transform $\nabla L^{-1/2}$ is bounded on $\dot{B}^{0}_{1,1}(\mathbb{R}^n)$.*

In order to prove Theorem 5.1 we need the following technical lemma.

Lemma 5.3 *Let a be an (L, M) atom associated with a ball B with $M \geq 1$. Then for $\alpha \in (0, 1)$, we have*

$$\|L^\alpha a\|_p \lesssim r_B^{-2\alpha}|B|^{1/p-1}$$

for every $p \in [1, \infty]$.

Proof We have

$$\begin{aligned} L^\alpha a &= c\int_0^\infty s^{1-\alpha}Le^{-sL}a\frac{ds}{s} \\ &= \int_0^{r_B^2} s^{1-\alpha}e^{-sL}La\frac{ds}{s} + \int_{r_B^2}^\infty s^{1-\alpha}Le^{-sL}a\frac{ds}{s}, \end{aligned}$$

which implies

$$\begin{aligned} \|L^\alpha a\|_p &\lesssim \int_0^{r_B^2} s^{1-\alpha}\|e^{-sL}\|_{p\to p}\|La\|_p\frac{ds}{s} + \int_{r_B^2}^\infty s^{-\alpha}\|sLe^{-sL}\|_{p\to p}\|a\|_p\frac{ds}{s} \\ &\lesssim \int_0^{r_B^2} s^{1-\alpha}\|La\|_p\frac{ds}{s} + \int_{r_B^2}^\infty s^{-\alpha}\|sLe^{-sL}\|_{p\to p}\|a\|_p\frac{ds}{s} \\ &\lesssim \int_0^{r_B^2} s^{1-\alpha}r_B^{-2}|B|^{1/p-1}\frac{ds}{s} + \int_{r_B^2}^\infty s^{-\alpha}|B|^{1/p-1}\frac{ds}{s} \\ &\lesssim r_B^{-2\alpha}|B|^{1/p-1}. \end{aligned}$$

□

Proof of Theorem 5.1: Let a be an (L, M) atom associated to a ball B. It suffices to prove that

$$\|\nabla L^{-1/2}a\|_{\dot{B}^0_{1,1}(\mathbb{R}^n)} := \int_0^\infty \|t(-\Delta)e^{t\Delta}\nabla L^{-1/2}a\|_1\frac{dt}{t} \lesssim 1.$$

To do this, we write

$$\begin{aligned} &= \int_0^{4r_B^2} \|t(-\Delta)e^{t\Delta}\nabla L^{-1/2}a\|_{L^1(4B)}\frac{dt}{t} + \int_0^{4r_B^2} \|t(-\Delta)e^{t\Delta}\nabla L^{-1/2}a\|_{L^1(\mathbb{R}^n\setminus 4B)}\frac{dt}{t} \\ &+ \int_{4r_B^2}^\infty \|t(-\Delta)e^{t\Delta}\nabla L^{-1/2}a\|_1\frac{dt}{t} \\ &:= E_1 + E_2 + E_3. \end{aligned}$$

Using the L^r-boundedness of $\nabla^2 L^{-1}$ (see [2]), we have

$$\begin{aligned}\|t(-\Delta)e^{t\Delta}\nabla L^{-1/2}a\|_{L^1(4B)} &\le \|t(-\Delta)e^{t\Delta}\nabla L^{-1/2}a\|_{L^r(4B)}|B|^{1/r'}\\ &\lesssim \|t\nabla e^{t\Delta}\nabla^2 L^{-1/2}a\|_{L^r(4B)}|B|^{1/r'}\\ &\lesssim \sqrt{t}\|t\nabla e^{t\Delta}\|_{r\to r}\|\nabla^2 L^{-1}\|_{r\to r}\|L^{1/2}a\|_r|B|^{1/r'}\\ &\lesssim \frac{\sqrt{t}}{r_B}.\end{aligned}$$

It follows that $E_1 \lesssim 1$.

For the term E_3 we have, for $a = Lb$,

$$\begin{aligned}\|t(-\Delta)e^{t\Delta}\nabla L^{-1/2}a\|_1 &= \|t\nabla(-\Delta)e^{t\Delta}L^{1/2}b\|_1\\ &\le \|t\nabla(-\Delta)e^{t\Delta}\|_{1\to 1}\|L^{1/2}b\|_1\\ &\lesssim \frac{r_B}{\sqrt{t}},\end{aligned}$$

which implies that $E_3 \lesssim 1$.

It remains to estimate E_2. To do this, we use the following formula

$$L^{-1/2} = c\int_0^\infty s^{3/2}Le^{-sL}\frac{ds}{s}$$

so that

$$\begin{aligned}t(-\Delta)e^{t\Delta}\nabla L^{-1/2}a &= c\int_0^\infty ts^{3/2}(-\Delta)e^{t\Delta}\nabla Le^{-sL}a\frac{ds}{s}\\ &= c\int_0^t ts^{3/2}(-\Delta)e^{t\Delta}\nabla e^{-sL}(La)\frac{ds}{s}\\ &\quad + c\int_t^\infty ts^{3/2}\nabla e^{t\Delta}\nabla^2 Le^{-sL}a\frac{ds}{s}.\end{aligned}$$

It follows that

$$\begin{aligned}\|t(-\Delta)e^{t\Delta}\nabla L^{-1/2}a\|_1 &\lesssim \int_0^t ts^{3/2}\|(-\Delta)e^{t\Delta}\nabla e^{-sL}(La)\|_1\frac{ds}{s}\\ &\quad + \int_t^\infty ts^{3/2}\|\nabla e^{t\Delta}\nabla^2 Le^{-sL}a\|_1\frac{ds}{s}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|(-\Delta)e^{t\Delta}\nabla e^{-sL}(La)\|_1 &\le \|(-\Delta)e^{t\Delta}\|_{1\to 1}\|\nabla e^{-sL}\|_{1\to 1}\|La\|_1\\ &\lesssim \frac{1}{\sqrt{s}tr_B^2},\end{aligned}$$

and

$$\begin{aligned}\|\nabla e^{t\Delta}\nabla^2 L e^{-sL} a\|_1 &\le \|\nabla e^{t\Delta}\|_{1\to 1}\|\nabla^2 L^{1-\alpha} e^{-sL}\|_{1\to 1}\|L^\alpha a\|_1\\ &\lesssim \frac{1}{\sqrt{t} s^{2-\alpha} r_B^{2\alpha}}, \quad \alpha \in (0, 1/2).\end{aligned}$$

Therefore,

$$\|t(-\Delta)e^{t\Delta}\nabla L^{-1/2} a\|_1 \lesssim \int_0^t \frac{s}{r_B^2}\frac{ds}{s} + \int_t^\infty \frac{\sqrt{t}}{s^{1/2-\alpha}}\frac{ds}{s} \sim \frac{t}{r_B^2} + \frac{t^\alpha}{r_B^{2\alpha}},$$

which implies that

$$E_2 \lesssim \int_0^{4r_B^2} \Big(\frac{t}{r_B^2} + \frac{t^\alpha}{r_B^{2\alpha}}\Big)\frac{dt}{t} \lesssim 1.$$

It was proved in [12, 22] that there exists $\beta > 0$ such that

$$\int_{\mathbb{R}^n} |\sqrt{t}\nabla p_t(x, y)| e^{\beta\frac{|x-y|^2}{t}} dx + \int_{\mathbb{R}^n} |t\nabla^2 p_t(x, y)| e^{\beta\frac{|x-y|^2}{t}} dx \le 1.$$

This completes our proof. □

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