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Isomorphic Copies of ℓ^∞ in the Weighted Hardy Spaces on the Unit Disc

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Abstract

It is still unclear whether the density of analytic polynomials in an H-admissible space is sufficient to the minimality of the space? This question has a purely foundational background, relating fundamental concepts from the theory of H^p spaces. We hypothesize that there is no general relationship between the density of analytic polynomials and the R-admissibility of an H-admissible space. We solve this problem by finding suitable counterexamples of Hardy spaces built upon some weighted Lebesgue spaces. In particular, we provide a direct construction of weights from Szegő class, which guarantees the existence of isomorphic copies of the space of bounded sequences in weighted Hardy spaces on the unit disc.

Keywords Hardy spaces · Isomorphic theory of Banach spaces · Nonseparable Banach spaces · Banach spaces of analytic functions

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1 Introduction

Let $H(\mathbb{D})$ denote the space of all analytic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let f be a complex-valued function on $\mathbb{D}$ and $0 \le r < 1$. We write f_r for the function $f_r(z) := f(rz)$, $z \in \mathbb{T} = \partial\mathbb{D}$ and f_* for its radial limit. The class $H(\overline{\mathbb{D}})$ consists of functions which are analytic in a neighbourhood of the closed disc $\overline{\mathbb{D}}$. Let $u_n : \mathbb{D} \to \mathbb{C}$ denote the monomials $u_n(z) = z^n$, $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$. The set $\mathcal{P}$ of all analytic polynomials, is defined to be the linear span of $\{u_n\}_{n\in\mathbb{N}_0}$.

Following Pavlović [10, 2nd rev.] let us recall that a quasinormed vector space $X \subset H(\mathbb{D})$ (i.e., equipped with a q-norm for some $0 < q \le 1$), is called H-admissible if it is complete, $H(\overline{\mathbb{D}}) \subset X$, and the inclusion $X \subset H(\mathbb{D})$ is continuous. If an H-admissible space X satisfies

$$\sup_{0\le r<1} \|f_r\|_X \le C_X \|f\|_X, \quad f \in X,$$

then we call it an R-admissible space. An H-admissible space is said to be minimal if

$$\lim_{r\to 1^-} \|f_r - f\|_X = 0.$$

The analytic polynomials always form a dense subset in a minimal space. The R-admissibility of a minimal space follows from the Banach–Steinhaus principle (see, e.g., [10, 11] for its more general variants). Moreover, if X is an R-admissible space, then the closure $X_{\mathcal{P}}$ of $\mathcal{P}$ in X equals

$$\left\{ f : \lim_{r\to 1^-} \|f - f_r\|_X = 0 \right\}$$

(see, e.g., [10, Lemma 1.21]). However, it is still unclear whether *the density of analytic polynomials in an H-admissible space is related to the R-admissibility of this space?*

We hypothesize that there is no general relationship between the density of analytic polynomials and the R-admissibility of an H-admissible space. We shall prove that neither the R-admissibility nor the density of analytic polynomials alone is sufficient for minimality to hold. Let us note here, that in the second revision of his book [10, p. 15], Pavlović posed a closely related problem whether *the density of harmonic polynomials in an h-admissible space is sufficient to the minimality of the space?*

The purpose of this paper is to study the weighted Hardy spaces $H^p_\phi(\mathbb{D})$ and $H^p_\phi(\mathbb{T})$ for Szegő's weights. Here $0 < p < \infty$ and $\mathcal{B}$ is the σ-algebra consisting of Borel sets of $\mathbb{T}$ and ℓ is the normalized Haar measure on $\mathbb{T}$. By a weight function we shall always mean a non-negative function ϕ on the unit circle $\mathbb{T}$. The Szegő class of weights, denoted by W, is the collection of all weights on $\mathbb{T}$ satisfying $\phi \in L^1(\mathbb{T})$ and $\log\phi \in L^1(\mathbb{T})$. We adopt the convention that (a class of ℓ-a.e. equal measurable functions) f belongs to $L^p_\phi(\mathbb{T})$, for $0 < p < \infty$, exactly when

$$\|f\|_{L^p_\phi(\mathbb{T})} = \left(\frac{1}{2\pi}\int_{\mathbb{T}} |f(\xi)|^p\,\phi(\xi)\,|d\xi|\right)^{1/p} < \infty,$$

and $f \in L^\infty(\mathbb{T})$ whenever its essential supremum satisfies

$$\|f\|_{L^\infty(\mathbb{T})} = \operatorname*{ess\,sup}_{(\mathbb{T},\mathcal{B},\ell)} |f| < \infty.$$

Here, every contour integral along $\mathbb{T}$ will be taken counter-clockwise. For $L^p(\mathbb{T})$ we take $\phi \equiv 1$. Moreover, the following functionals from $H(\mathbb{D})$ give rise to the quasi-norms of the weighted Hardy spaces $H^p_\phi(\mathbb{D})$, $H^\infty(\mathbb{D})$ and $H^p_\phi(\mathbb{T})$, namely

$$f \mapsto \sup_{0\le r<1} \|f_r\|_{L^p_\phi(\mathbb{T})}, \quad f \mapsto \sup_{0\le r<1} \|f_r\|_{L^\infty(\mathbb{T})} \quad \text{and} \quad f \mapsto \|f_*\|_{L^p_\phi(\mathbb{T})}.$$

If $\phi \equiv 1$, then we recover the classical case $H^p(\mathbb{D}) \cong H^p(\mathbb{T})$.

Recall that by Beurling's theorem (see, e.g., [5, Theorem 7.4]), the analytic polynomials $\mathcal{P}$ are always dense in $H^p_\phi(\mathbb{T})$ for every $\phi \in W$. Specifically, we plan to solve the aforementioned problem by addressing somewhat simpler claims:

$*$ There exists a weight $\phi \in W$ such that $H^p_\phi(\mathbb{D}) \neq H^p_\phi(\mathbb{T})$ for every $0 < p < \infty$.
$**$ There exists a weight $\phi \in W$ such that the set of analytic polynomials $\mathcal{P}$ is not dense in $H^p_\phi(\mathbb{D})$ for every $0 < p < \infty$.

In contrast to the case of the Hardy spaces $H^p_\phi(\mathbb{T})$, the subspace structure of the corresponding disc spaces $H^p_\phi(\mathbb{D})$ is still not satisfactorily understood. Nonetheless, we also show that for the weighted Hardy spaces on the disc:

$***$ There exists a weight $\phi \in W$ such that $H^p_\phi(\mathbb{D})$ contains an isomorphic copy of ℓ^∞.

Finally, let us recall that the question of how to construct copies of ℓ_p in H^p for $1 \le p \le \infty$ was already considered in the literature (see, e.g. the survey [1]).

2 Notation, Definitions and Auxiliary Results

Note that $H(\mathbb{D})$ is a Fréchet space equipped with the F-norm generated by the family of seminorms $\{\|\cdot\|_n : n \in \mathbb{N}\}$ given by

$$\sup\Big\{|f(z)| : |z| \le 1 - \frac{1}{n}\Big\}, \quad f \in H(\mathbb{D}).$$

In the case where $f \in H(\mathbb{D})$ has non-tangential limits ℓ-a.e. on $\mathbb{T}$ (which gives rise to boundary function ℓ-a.e. on $\mathbb{T}$), we write f_* for the ℓ-a.e. defined radial limit $f_* = \lim_{r\to 1^-} f_r$.

A function $f \in H(\mathbb{D})$ is said to be of the Nevanlinna class N if the integrals

$$\int_{\mathbb{T}} \log^+ |f_r|\, d\ell$$

are bounded for $0 \le r < 1$. Since $\log^+ f$ is subharmonic, it follows that these integrals increase with r. The theorem due to F. and R. Nevanlinna says that a function belongs to N if and only if it is the quotient of two bounded analytic functions (see, e.g., [5, Theorem 2.1]). Therefore, for each non-zero function $f \in N$, there exists non-tangential limit f_* and $\log |f_*|$ is integrable (see, e.g., [5, Theorem 2.2]).

Let us recall that an outer function for the class N is a function F of the form $e^{ic} F_\phi$ for some real constant c, where

$$F_\phi(z) = \exp\left(\frac{1}{2\pi}\int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \log \phi(\xi)\, |d\xi|\right), \quad z \in \mathbb{D}$$

and the weight ϕ satisfies $\log \phi \in L^1(\mathbb{T})$. If ϕ is additionally assumed to satisfy $\phi \in L^p(\mathbb{T})$ then $F = e^{ic} F_\phi$, $c \in \mathbb{R}$ is an outer function for the class $H^p(\mathbb{T})$. The outer function satisfies the equality $|F_*| = \phi$. A function $f \in H(\mathbb{D})$ satisfying

(i) $|f| \le 1$ on $\mathbb{D}$, and
(ii) $|f_*| = 1$ a.e. on $\mathbb{T}$,

is called an inner function. In the case where an inner function has no zeros, it is called a singular inner function. It is well known that $S \in H(\mathbb{D})$ is a singular inner function if and only if there exists a non-negative singular measure σ on $\mathbb{T}$ satisfying

$$S(z) = e^{ic} \exp\left(-\frac{1}{2\pi}\int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\sigma(\xi)\right), \quad z \in \mathbb{D},$$

where c is a real constant. Let us remark that both functions S and F do not vanish in $\mathbb{D}$.

The zeros of a non-zero function f from $H(\mathbb{D})$ cannot cluster inside its domain of analyticity $\mathbb{D}$. In the case where the set of zeros is empty, we have $f \in N$ by Jensen's formula and we set $B \equiv 1$ (the auxiliary function B will be defined below for the other cases). If this is not the case, we arrange the zeros $\{a_n\}$ in an order of non-decreasing absolute values, where each zero is counted according to its algebraic multiplicity. If f has infinitely many zeros $\{a_n\}$, then $\lim_{n\to\infty} |a_n| = 1$. In the case where the function satisfies a growth condition, the zeros must tend more rapidly to the boundary. Let us recall the theorem due to Blaschke (see, e.g., [5, Theorem 2.3 and 2.4] or [10]). The zeros $\{a_n\}$ of a non-zero function $f \in H(\mathbb{D})$ satisfy the Blaschke condition

$$\sum_n \big(1 - |a_n|\big) < \infty$$

if and only if $f \in N$. If a sequence $\{a_n\} \subset \mathbb{D}$ satisfies the Blaschke condition, then the product

$$B(z) = \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a}_n z}$$

converges in $H(\mathbb{D})$ (i.e., uniformly on each disc $\{z \in \mathbb{C} : |z| \le R < 1\}$). By convention, we set $|a_n|/a_n = -1$ if $a_n = 0$.

(i) Each a_n is a zero of B, with multiplicity equal to the number of times it occurs in the sequence, and B has no other zeros in $\mathbb{D}$.
(ii) $|B| \leq 1$ on $\mathbb{D}$.
(iii) $|B_*| = 1$ a.e. on $\mathbb{T}$.

The function B is called a Blaschke product. A function $f \in N$ is said to be of the Smirnov class N^+ if

$$\lim_{r\to 1^-} \int_{\mathbb{T}} \log^+ |f_r|\, d\ell = \int_{\mathbb{T}} \log^+ |f_*|\, d\ell.$$

Every non-zero function of the class N^+ can be expressed in the form $f = BSF$, where B is a Blaschke product, S is a singular inner function, and F is an outer function for the class N (see, e.g., [5, Theorem 2.10]). The Smirnov class N^+ is the natural limit space of $H^p(\mathbb{D})$ as $p \to 0$ and the inclusions $H^p(\mathbb{D}) \subset N^+ \subset N$ are proper for every $p > 0$.

If ϕ is a weight and $0 < p < \infty$, then the weighted Hardy space $H^p_\phi(\mathbb{D})$, consists of all functions f which are analytic in $\mathbb{D}$ and satisfy $f_r \in L^p_\phi(\mathbb{T})$ for all $0 \leq r < 1$, where the corresponding $L^p_\phi(\mathbb{T})$ bounds are uniform with

$$\|f\|_{H^p_\phi(\mathbb{D})} = \sup_{0\leq r<1} \|f_r\|_{L^p_\phi(\mathbb{T})} < \infty.$$

The classical Hardy space $H^p(\mathbb{D})$ is obtained by taking $\phi \equiv 1$. At the other endpoint, the Hardy space $H^\infty(\mathbb{D})$ is the collection of all analytic functions $f \in H(\mathbb{D})$ that satisfy

$$\|f\|_{H^\infty(\mathbb{D})} = \sup_{0\leq r<1} \|f_r\|_{L^\infty(\mathbb{T})} < \infty.$$

By $\mathcal{A}(\mathbb{D})$ we denote the disc algebra, that is the set of all functions from $H(\mathbb{D})$ which are continuous on $\overline{\mathbb{D}}$. We also have $H(\overline{\mathbb{D}}) \subset \mathcal{A}(\mathbb{D}) \subset H^\infty(\mathbb{D})$. Moreover, the assumption that $\phi \in L^1(\mathbb{T})$ is equivalent to the inclusion $\mathcal{A}(\mathbb{D}) \subset H^p_\phi(\mathbb{D})$. The Poisson integral of a function $f \in L^1(\mathbb{T})$ is the harmonic function $\mathcal{P}[f]$ defined by

$$\mathcal{P}[f](z) = \int_{\mathbb{T}} P(z\xi^{-1}) f(\xi)\, |d\xi|, \quad z \in \mathbb{D},$$

where P is the Poisson kernel, given by

$$P(z) = \frac{1-|z|^2}{|1-z|^2} = \operatorname{Re}\frac{1+z}{1-z}, \quad z \in \mathbb{D}.$$

It is a well-known fact that if $f \in H^1(\mathbb{D})$, then $f_* \in L^1(\mathbb{T})$ and $f = \mathcal{P}[f_*]$. For more details concerning classical Hardy spaces on the unit disc we refer the reader to [5, 10].

Assuming $\log\phi \in L^1(\mathbb{T})$ and $0 < p < \infty$, let $H^p_\phi(\mathbb{T})$ denote the space of all analytic functions $f \in H(\mathbb{D})$ that satisfy $fF_\phi^{1/p} \in H^p(\mathbb{D})$ with

$$\|f\|_{H^p_\phi(\mathbb{T})} = \left\|fF_\phi^{1/p}\right\|_{H^p(\mathbb{D})}.$$

Let us remark here that we always have $H^p_\phi(\mathbb{T}) \subset N^+$. Indeed, taking any non-zero $f \in H^p_\phi(\mathbb{T})$ we have $fF_\phi^{1/p} = BSF_\psi$ for some ψ satisfying $\log\psi \in L^1(\mathbb{T})$. Since both $\log\psi$ and $\log\phi$ are members of $L^1(\mathbb{T})$, it follows that $\log\varphi \in L^1(\mathbb{T})$ for $\varphi = \psi/\phi^{1/p}$, and the function

$$f = \frac{BSF_\psi}{F_{\phi^{1/p}}} = BSF_\varphi$$

belongs to N^+. Therefore, each $f \in H^p_\phi(\mathbb{T})$ has a boundary function f_* (ℓ-a.e. on $\mathbb{T}$) and

$$\|f\|_{H^p_\phi(\mathbb{T})} = \left\|f_*\phi^{1/p}\right\|_{L^p(\mathbb{T})} = \|f_*\|_{L^p_\phi(\mathbb{T})}. \tag{2.1}$$

We make the following provisional definition. If for $\log\psi \in L^1(\mathbb{T})$ we have that $\psi \in L^p_\phi(\mathbb{T})$, then we say that $F = e^{ic}F_\psi$, $c \in \mathbb{R}$, is an outer function for the class $H^p_\phi(\mathbb{T})$.

Thus, we have proved the canonical factorization theorem for weighted Hardy spaces on the torus.

Theorem 2.1 *If $\log\phi \in L^1(\mathbb{T})$ and $0 < p < \infty$, then every non-zero function f from $H^p_\phi(\mathbb{T})$ admits a unique factorization of the form $f = BSF$, where B is a Blaschke product, S is a singular inner function, and F is an outer function for the class $H^p_\phi(\mathbb{T})$. Conversely, every product BSF for the outer function F for the class $H^p_\phi(\mathbb{T})$ is a member of $H^p_\phi(\mathbb{T})$.*

In fact, (2.1) allows us to treat $H^p_\phi(\mathbb{T})$ as a subspace of $L^p_\phi(\mathbb{T})$. If also we have $\phi \in L^1(\mathbb{T})$, then $A(\mathbb{D}) \subset H^p_\phi(\mathbb{T})$, and moreover, $H^p_\phi(\mathbb{T})$ is isometrically isomorphic to the closure of $\mathcal{P}$ in $L^p_\phi(\mathbb{T})$ by Beurling's theorem (see, e.g., [5, Theorem 7.4] or [4]). In the case where $\phi \equiv 1$, we have $H^p(\mathbb{D}) \cong H^p(\mathbb{T})$.

The Szegő class of weights, denoted by W, is the collection of all weights on $\mathbb{T}$ satisfying $\phi \in L^1(\mathbb{T})$ and $\log\phi \in L^1(\mathbb{T})$.

$$W = \left\{\phi \in L^1(\mathbb{T}) \ : \ \log\phi \in L^1(\mathbb{T})\right\}$$

McCarthy showed in [9] (see also [8]) that N^+, can be realized as a union of weighted Hardy spaces, namely

$$N^+ = \bigcup_{\phi\in W} H^2_\phi(\mathbb{T}).$$

Let us note that since a non-zero function $f \in N^+$ has a factorization $f = BSF$ and $g = F^{p/2} \in N^+$ for all $p > 0$, it follows that

$$N^+ = \bigcup_{\phi \in W} H^p_\phi(\mathbb{T}). \tag{2.2}$$

Let us first discuss the relation between the weighted Hardy spaces $H^p_\phi(\mathbb{D})$ and $H^p_\phi(\mathbb{T})$ for $\phi \in W$. We state the following technical result, which is a more general version of Stoll's lemma given in [3]. The proof that $fF_\phi^{1/p} \in H^p(\mathbb{D})$ whenever $f \in H^p_\phi(\mathbb{D})$ appeared first in [3, Lemma 2.3] and [7, Theorem 3]. Here is a simple direct proof.

Lemma 2.2 *Let $0 < p < \infty$. Assume that $\phi \in W$ and $f \in H^p_\phi(\mathbb{D})$. Then*

$$\|f\|_{H^p_\phi(\mathbb{T})} \le \|f\|_{H^p_\phi(\mathbb{D})} \,. \tag{2.3}$$

Moreover if $f \in H^p_\phi(\mathbb{T})$ with $\limsup_{r\to 1^-} \|f_r\|_{L^p_\phi(\mathbb{T})} < \infty$, then $f \in H^p_\phi(\mathbb{D})$.

Proof In the proof below, we shall use the mean convergence theorem and Hardy's convexity theorem (see, e.g., [5, Theorems 1.5 and 2.6]). Fix $f \in H^p_\phi(\mathbb{D})$. Since

$$\sup_{0\le r<1} \left\| \left(f_\rho F_\phi^{1/p}\right)_r \right\|_{L^p(\mathbb{T})} = \left\| f_\rho \phi^{1/p} \right\|_{L^p(\mathbb{T})} < \infty, \quad 0 \le \rho < 1$$

where $f_\rho \phi^{1/p} = (f_\rho F_\phi^{1/p})_*$, it follows that

$$\begin{aligned}
\left\|fF_\phi^{1/p}\right\|_{H^p(\mathbb{D})} &= \sup_{0\le r<1} \left\|\left(fF_\phi^{1/p}\right)_r\right\|_{L^p(\mathbb{T})} = \sup_{0\le r<1} \left\| \liminf_{\rho\to 1^-} f_{r\rho}\left(F_\phi^{1/p}\right)_r \right\|_{L^p(\mathbb{T})} \\
&\le \sup_{0\le r<1} \liminf_{\rho\to 1^-} \left\| f_{r\rho}\left(F_\phi^{1/p}\right)_r \right\|_{L^p(\mathbb{T})} \le \liminf_{\rho\to 1^-} \left\| f_\rho \phi^{1/p} \right\|_{L^p(\mathbb{T})} \\
&\le \|f\|_{H^p_\phi(\mathbb{D})}
\end{aligned}$$

by Fatou's lemma.

We now prove the remaining inequality (cf. [3, Remark 2.4]). By assumption, $\phi \in L^1(\mathbb{T})$ and $\log\phi \in L^1(\mathbb{T})$, and moreover $F_\phi^{1/p}$ is an outer function for the class $H^p(\mathbb{D})$. Let $f \in H^p_\phi(\mathbb{T})$. Then $f \in H(\mathbb{D})$ with $fF_\phi^{1/p} \in H^p(\mathbb{D})$. Since $f_*\phi^{1/p} \in L^p(\mathbb{T})$ and

$$\limsup_{r\to 1^-} \|f_r\phi^{1/p}\|_{L^p(\mathbb{T})} < \infty,$$

and since moreover

$$\sup_{0\le r\le R} \|f_r\|_{L^p_\phi(\mathbb{T})} \le \|f_R\|_{H^\infty(\mathbb{D})} \|\phi\|_{L^1(\mathbb{T})} < \infty$$

for every $0 < R < 1$, it follows that $f \in H^p_\phi(\mathbb{D})$. □

Lemma 2.3 *If $0 < p < \infty$ and $\phi \in W$, then both $H^p_\phi(\mathbb{D})$ and $H^p_\phi(\mathbb{T})$ are H-admissible.*

Proof For the sake of rigor, we first show that both $H^p_\phi(\mathbb{D})$ and $H^p_\phi(\mathbb{D})$ are complete. Fix $0 < p < \infty$. Let $\{f_n\}$ be a Cauchy sequence in $H^p_\phi(\mathbb{T})$. Since $\{f_n F_\phi^{1/p}\}$ is a Cauchy sequence in $H^p(\mathbb{T})$, it follows that there is $g := \lim_{n\to\infty} f_n F_\phi^{1/p}$ in $H^p(\mathbb{T})$. On the other hand, $\{(f_n)_*\phi^{1/p}\}$ is a Cauchy sequence in $L^p(\mathbb{T})$ with $g_* = \lim_{n\to\infty}(f_n)_*\phi^{1/p}$ in $L^p(\mathbb{T})$. By Theorem 2.1,

$$f = \frac{g}{F_{\phi^{1/p}}} = \frac{BSF_{g_*}}{F_{\phi^{1/p}}}$$

is a limit $\lim_{n\to\infty} f_n$ in $H^p_\phi(\mathbb{T})$ with $f_* \in L^p_\phi(\mathbb{T})$.

Take an absolutely convergent series $\sum_{n=1}^\infty f_n$ in $H^p_\phi(\mathbb{D})$. Lemma 2.2 shows that this series converges in $H^p_\phi(\mathbb{T})$ to some analytic function, say $f = \sum_{n=1}^\infty f_n$. Observe that the series $\sum_{n=1}^\infty (f_n)_r$ is absolutely convergent in $L^p_\phi(\mathbb{T})$ to f_r for every $0 \le r < 1$, where

$$\|f_r\|_{L^p_\phi(\mathbb{T})} \le \sum_{n=1}^\infty \|(f_n)_r\|_{L^p_\phi(\mathbb{T})} \le \sum_{n=1}^\infty \|f_n\|_{H^p_\phi(\mathbb{D})} < \infty.$$

That

$$\mathcal{A}(\mathbb{D}) \subset H^p_\phi(\mathbb{D}) \subset H^p_\phi(\mathbb{T}), \tag{2.4}$$

where the inclusions are continuous, follows immediately from $\phi \in L^1(\mathbb{T})$ and Lemma 2.2.

We will make use of the classical inequality

$$|f(z)|^p\,(1-|z|^2) \le \|f\|^p_{H^p(\mathbb{T})}, \quad z \in \mathbb{D}$$

(see, e.g., [10]) which we apply to the case of weighted Hardy spaces. Take $f \in H^p_\phi(\mathbb{T})$ and $0 \le r < 1$. Then

$$\begin{aligned} c_{r,\phi}\,|f(z)|^p\,(1-|z|^2) &\le |f(z)|^p\,\left|F_\phi(z)\right|(1-|z|^2) \\ &\le \|fF_\phi^{1/p}\|^p_{H^p(\mathbb{T})} = \|f\|^p_{H^p_\phi(\mathbb{T})} \end{aligned} \tag{2.5}$$

is valid for $|z| \le r$, where $c_{r,\phi} = \min_{|z|\le r}\left|F_\phi(z)\right|$ is positive because of the fact that an outer function cannot vanish in $\mathbb{D}$. By (2.4) and (2.5), $H^p_\phi(\mathbb{D}) \subset H^p_\phi(\mathbb{T}) \subset H(\mathbb{D})$, where the last inclusion is continuous, too. □

3 Main Results

Let us point out that the equality $H_\phi^p(\mathbb{D}) = H_\phi^p(\mathbb{T})$ as sets, first announced in [6] without proof, was discussed in a series of papers [2, 3, 7] in the context of Muckenhoupt's weights.

While the Hardy spaces $H_\phi^p(\mathbb{D})$ and $H_\phi^p(\mathbb{T})$ resemble the classical Hardy spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$, they in general need not be equal for weights from W.

Theorem 3.1 *If $0 < p < \infty$, then there exists a weight $\phi \in W$ such that $H_\phi^p(\mathbb{D}) \neq H_\phi^p(\mathbb{T})$.*

Proof Fix $0 < p < \infty$. We give a direct construction of both the weight $\phi \in W$ and the function $f \in H_\phi^p(\mathbb{T})$ that satisfy $\limsup_{r\to 1^-} \|f_r\|_{L_\phi^p(\mathbb{T})} = \infty$. Let $\{I_n\}_{n\in\mathbb{N}_0}$ be a partition of $\mathbb{T}$ given by

$$\begin{aligned} I_0 &:= \{\xi \in \mathbb{T} : 2 < \arg \xi \leq 2\pi\} \\ I_{2n+1} &:= \left\{\xi \in \mathbb{T} : 3/2^{n+1} < \arg \xi \leq 2/2^n\right\} \quad \text{and} \\ I_{2n+2} &:= \left\{\xi \in \mathbb{T} : 1/2^n < \arg \xi \leq 3/2^{n+1}\right\}. \end{aligned}$$

Consider $1 < c < 2$. Then the following weight

$$\phi := \chi_{I_0} + \sum_{n=0}^{\infty} \left(e^{-c^n} \chi_{I_{2n+1}} + \chi_{I_{2n+2}}\right) \in W.$$

Indeed,

$$\begin{aligned} 2\pi \int_{\mathbb{T}} \phi \, d\ell &= (2\pi - 2) + \sum_{n=0}^{\infty} \frac{e^{-c^n} + 1}{2^{n+1}} < \infty, \quad \text{and} \\ 2\pi \int_{\mathbb{T}} |\log \phi| \, d\ell &= \sum_{n=0}^{\infty} \frac{c^n}{2^{n+1}} < \infty. \end{aligned}$$

By the above, we also have $\log \phi^{-1} \in L^1(\mathbb{T})$. Now f is defined by setting an appropriate outer function, where $\psi := \phi^{-1}$, namely

$$\psi = \chi_{I_0} + \sum_{n=0}^{\infty} \left(e^{c^n} \chi_{I_{2n+1}} + \chi_{I_{2n+2}}\right)$$

and

$$f(z) := F_\psi^{1/p} = \exp\left(\frac{1}{2p\pi} \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \log \psi(\xi) \, |d\xi|\right), \quad z \in \mathbb{D}.$$

Since $|f_*|^p \equiv \psi$ ℓ-a.e., it follows that $f \in H^p_\phi(\mathbb{T})$. Let us also define a family $\{J_n\}_{n\in\mathbb{N}_0}$ of subsets of $\mathbb{T}$ by

$$J_n := \left\{\xi \in \mathbb{T} : -1/2^{n+1} \le \arg\xi \le 0\right\}.$$

It is routine to verify that for $r_n := 1 - 1/2^{n+1}$ we have $P(r_n\xi) \ge 2^n$ for all $\xi \in J_n$ for each $n \in \mathbb{N}$, where P is the Poisson kernel. Thus,

$$\begin{aligned} \|f_{r_n}\|^p_{L^p_\phi(\mathbb{T})} &= \int_{\mathbb{T}} |f_{r_n}|^p \,\phi\, d\ell \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \exp\left(\frac{1}{2\pi}\int_{\mathbb{T}} P(r_n\zeta\xi^{-1})\log\psi(\xi)\,|d\xi|\right)\phi(\zeta)\,|d\zeta| \\ &\ge \frac{1}{2\pi}\int_{I_{2n+2}} \exp\left(\frac{1}{2\pi}\int_{\zeta J_n} 2^n\log\psi(\xi)\,|d\xi|\right)\phi(\zeta)\,|d\zeta| \qquad (3.1)\\ &\ge \frac{1}{2\pi}\exp\left(\frac{1}{2\pi}\frac{c^n}{4}\right)\frac{1}{2^{n+1}} \\ &= \frac{1}{4\pi}\exp\left(\frac{c^n}{8\pi} - n\log 2\right) \to \infty \quad \text{as} \quad n\to\infty, \end{aligned}$$

and so $H^p_\phi(\mathbb{D}) \ne H^p_\phi(\mathbb{T})$, as required. □

Actually, based on the ideas presented in Theorem 3.1 one can also find a pair consisting of the weight and the function satisfying more refined estimates.

Theorem 3.2 *If $0 < p < \infty$, then there exists a weight $\phi \in W$ such that $H^p_\phi(\mathbb{D})$ contains an isomorphic copy of ℓ^∞.*

Proof We use the notation from the proof of Theorem 3.1, and the definitions of the intervals and functions given there $(I_n, \phi, \ldots)$. The purpose of this proof is to give the construction of a weight ϕ and a linear embedding $T: \ell^\infty \to H^p_\phi(\mathbb{D})$. This will give rise to the countable family $\mathcal{F}$ of analytic functions $h_n := T(e_n)$, $n \in \mathbb{N}_0$ on the disc satisfying

$$\left\|\sum_{n\in I} h_n\right\|_{H^p_\phi(\mathbb{D})} \asymp 1 \quad \text{and} \quad \inf_{\substack{I\ne J\\ J\subset\mathbb{N}_0}} \left\|\sum_{n\in I} h_n - \sum_{n\in J} h_n\right\|_{H^p_\phi(\mathbb{D})} \asymp 1$$

for all $\emptyset \ne I \subset \mathbb{N}_0$. Our proof breaks into three parts.

As a first step we put

$$f_n(z) := F^{1/p}_{\psi_n} = \exp\left(\frac{1}{2p\pi}\int_{\mathbb{T}} \frac{\xi+z}{\xi-z}\log\psi_n(\xi)\,|d\xi|\right), \quad z\in\mathbb{D},$$

where

$$\psi_n := \chi_{\mathbb{T}\setminus I_{2n+1}} + e^{c^n}\chi_{I_{2n+1}}, \quad n\in\mathbb{N}_0.$$

It is also clear that $|(f_n)_*|^p = \psi_n$ ℓ-a.e. and $\log \psi_n \in L^1(\mathbb{T})$. Since $\|f\|_{L^p_\phi(\mathbb{T})} \leq \|f\|_{L^p(\mathbb{T})}$ for all $f \in L^p(\mathbb{T})$ and $\|g\|_{H^p(\mathbb{D})} \leq \|g\|_{H^\infty(\mathbb{D})}$ for all $g \in H^\infty(\mathbb{T})$, and since moreover $\psi_n \leq e^{c^n}$ ℓ-a.e., it follows that $\|f_n\|_{H^p_\phi(\mathbb{D})} < \infty$, $n \in \mathbb{N}$. On the other hand, the key estimate (3.1) in the proof of Theorem 3.1 also gives

$$\lim_{n\to\infty} \|f_n\|_{H^p_\phi(\mathbb{D})} = \infty. \tag{3.2}$$

Indeed, a close inspection of the above proof reveals that

$$\|f_n\|^p_{H^p_\phi(\mathbb{D})} \geq \frac{1}{4\pi} \exp\left(\frac{c^n}{8\pi} - n\log 2\right), \quad n \in \mathbb{N}_0. \tag{3.3}$$

By the mean convergence theorem for $H^p(\mathbb{D})$ (see, e.g., [5, Theorem 2.6]) one can quickly check that

$$\lim_{r\to 1^-} \|(f_n)_r - (f_n)_*\|_{L^p_\phi(\mathbb{T})} = 0, \quad n \in \mathbb{N}_0. \tag{3.4}$$

A standard verification shows that for all $0 \leq r < 1$,

$$\lim_{n\to\infty} \|(f_n)_r\|_{H^\infty(\mathbb{D})} = \lim_{n\to\infty} \sup_{|z|\leq r} |f_n(z)| = 1, \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|(f_n)_*\|^p_{L^p_\phi(\mathbb{T})} < 1. \tag{3.5}$$

Indeed, this follows immediately from

$$\begin{aligned} |f_n(rz)|^p &= \exp\left(\frac{1}{2\pi}\int_{\mathbb{T}} P(rz\xi^{-1}) \log\psi_n(\xi)\,|d\xi|\right) \\ &\leq \exp\left(\frac{1}{2\pi}\frac{1}{1-r}\int_{\mathbb{T}} \log\psi_n(\xi)\,|d\xi|\right) \\ &\leq \exp\left(\frac{1}{2\pi}\frac{1}{1-r}\frac{c^n}{2^{n+1}}\right) \to 1 \quad \text{as} \quad n\to\infty, \quad z\in\mathbb{D}, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \|(f_n)_*\|^p_{L^p_\phi(\mathbb{T})} &= \int_{\mathbb{T}} \psi_n \phi\, d\ell = \frac{1}{2\pi}\left(2\pi - 2 + \sum_{\substack{k=0\\k\neq n}}^{\infty} \frac{e^{-c^k}+1}{2^{k+1}} + \frac{2}{2^{n+1}}\right) \\ &\leq \frac{1}{2\pi}\left(2\pi - 2 + \sum_{k=1}^{\infty} \frac{e^{-c^k}+1}{2^{k+1}} + 1\right) < 1, \end{aligned}$$

respectively.

In the second part, we define a new sequence of functions by setting

$$g_n := \frac{f_n}{\|f_n\|_{H^p_\phi(\mathbb{D})}}, \quad n \in \mathbb{N}_0.$$

By (3.2), (3.4) and (3.5), $\|g_n\|_{H^p_\phi(\mathbb{D})} = 1$ and

$$\lim_{r\to 1^-} \|(g_n)_r - (g_n)_*\|_{L^p_\phi(\mathbb{T})} = 0, \quad \lim_{n\to\infty} \|(g_n)_*\|_{L^p_\phi(\mathbb{T})} = 0, \quad \lim_{n\to\infty} \|(g_n)_r\|_{H^p_\phi(\mathbb{D})} = 0$$

for all $n \in \mathbb{N}_0$ and $0 \le r < 1$. Fix $0 < \varepsilon < 1$ and $0 < \varepsilon_n$, $n \in \mathbb{N}_0$ satisfying

$$\sum_{n=0}^{\infty} \varepsilon_n < \varepsilon \quad \text{and} \quad \sum_{n=0}^{\infty} \varepsilon_n^p < \varepsilon.$$

Now that we have the above claim, we can select a subsequence of $\{g_n\}_{n\in\mathbb{N}_0}$ of appropriately separated functions (in the sense of $H^p_\phi(\mathbb{D})$ norm). For the base step of the induction, we take $r_0 := 0$ and choose $n_0 \in \mathbb{N}_0$ to satisfy

$$\sup_{0\le r\le r_0} \left\|(g_{n_0})_r\right\|_{L^p_\phi(\mathbb{D})} < \varepsilon_0 \quad \text{and} \quad \left\|(g_{n_0})_*\right\|_{L^p_\phi(\mathbb{T})} < \varepsilon_0.$$

There exists $r_0 < r_1 < 1$ such that

$$\sup_{r_1\le r<1} \left\|(g_{n_0})_r\right\|_{L^p_\phi(\mathbb{T})} < \varepsilon_0.$$

The next step is to find $n_0 < n_1 \in \mathbb{N}$ satisfying

$$\sup_{0\le r\le r_1} \left\|(g_{n_1})_r\right\|_{L^p_\phi(\mathbb{T})} < \varepsilon_1 \quad \text{and} \quad \left\|(g_{n_1})_*\right\|_{L^p_\phi(\mathbb{T})} < \varepsilon_1.$$

There is $r_1 < r_2 < 1$ that satisfy

$$\sup_{r_2\le r<1} \left\|(g_{n_1})_r\right\|_{L^p_\phi(\mathbb{T})} < \varepsilon_1.$$

Let us prove the induction step. Suppose that for all $0 \le k \le m-1$ the following three statements hold

$$\begin{aligned} &\sup_{0\le r\le r_k} \left\|(g_{n_k})_r\right\|_{L^p_\phi(\mathbb{T})} < \varepsilon_k, \quad \left\|(g_{n_k})_*\right\|_{L^p_\phi(\mathbb{T})} < \varepsilon_k \quad \text{and} \\ &\sup_{r_{k+1}\le r<1} \left\|(g_{n_k})_r\right\|_{L^p_\phi(\mathbb{T})} < \varepsilon_k, \end{aligned} \tag{3.7}$$

where $0 = r_0 < r_1 < \ldots < r_{m-1} < r_m < 1$ and $0 \le n_0 < n_1 < \ldots < n_{m-2} < n_{m-1}$. Now $n_{m-1} < n_m \in \mathbb{N}$ is chosen to satisfy

$$\sup_{0\le r\le r_m} \left\|(g_{n_m})_r\right\|_{L^p_\phi(\mathbb{T})} < \varepsilon_m \quad \text{and} \quad \left\|(g_{n_m})_*\right\|_{L^p_\phi(\mathbb{T})} < \varepsilon_m.$$

There exists $r_m < r_{m+1} < 1$ that satisfy

$$\sup_{r_{m+1} \le r < 1} \left\| (g_{n_m})_r \right\|_{L^p_\phi(\mathbb{T})} < \varepsilon_m .$$

By the induction hypothesis, it follows that (3.7) holds for all $0 \le k \le m$. We proceed by induction.

Let us move to the final step of the proof. We check first that

$$g_\lambda := \sum_{k=0}^{\infty} \lambda_k g_{n_k}$$

is a member of $H^p_\phi(\mathbb{D})$ for every sequence $\lambda := \{\lambda_k\}_{k \in \mathbb{N}_0} \in \ell^\infty$.

That $g_\lambda \in H(\mathbb{D})$ follows from (3.1) and (3.6). Indeed, if $0 \le r < 1$, then by (3.3)

$$\sum_{k=0}^{\infty} \left\| (g_{n_k})_r \right\|_\infty \le \sum_{n=0}^{\infty} \frac{\|(f_n)_r\|_\infty}{\|f_n\|_{H^p_\phi(\mathbb{D})}} < \sum_{n=0}^{N_r} \frac{\|(f_n)_r\|_\infty}{\|f_n\|_{H^p_\phi(\mathbb{D})}} + \sum_{n=N_r+1}^{\infty} \frac{2}{e^n} < \infty$$

for some $N_r \in \mathbb{N}_0$. We also have

$$\begin{aligned} \|g_\lambda\|^q_{H^p_\phi(\mathbb{D})} &= \sup_{m \in \mathbb{N}_0} \sup_{r_m \le r \le r_{m+1}} \|(g_\lambda)_r\|^q_{L^p_\phi(\mathbb{T})} \\ &\le \sup_{m \in \mathbb{N}_0} \left(|\lambda_m|^q + \sum_{m \ne k \in \mathbb{N}_0}^{\infty} \varepsilon_k^q \, |\lambda_k|^q \right) \le (1+\varepsilon)(\|\lambda\|_\infty)^q, \end{aligned} \tag{3.8}$$

where $q = \min\{1, p\}$. We define a mapping $T : \ell^\infty \to H^p_\phi(\mathbb{D})$ by

$$T : \lambda \mapsto g_\lambda . \tag{3.9}$$

By the above, T is well defined, linear and bounded with $\|T\| \le (1+\varepsilon)^{1/q}$. Moreover, T is bounded below. To see this, we shall bound $\|g_\lambda\|_{H^p_\phi(\mathbb{D})}$ from below by $\|\lambda\|_\infty$ up to a constant. Indeed,

$$|\lambda_m|^q = \left\| \lambda_m g_{n_m} \right\|^q_{H^p_\phi(\mathbb{D})} \le \|g_\lambda\|^q_{H^p_\phi(\mathbb{D})} + \sum_{m \ne k \in \mathbb{N}_0}^{\infty} \varepsilon_k^q \, |\lambda_k|^q , \quad m \in \mathbb{N}_0$$

and therefore

$$(1-\varepsilon)(\|\lambda\|_\infty)^q \le \|g_\lambda\|^q_{H^p_\phi(\mathbb{D})} .$$

□

A close inspection of the proof reveals that with the new weight

$$\varphi := \chi_K + \sum_{k=0}^{\infty} e^{-c^{n_k}} \chi_{I_{2n_k+1}} + \chi_{I_{2n_k+2}} \quad \text{for} \quad K := \mathbb{T} \setminus \bigcup_{k=0}^{\infty} (I_{2n_k+1} \cup I_{2n_k+2})$$

this procedure can be repeated with previously chosen family of functions $\{g_{n_k}\}$.

Since minimal spaces are separable, we actually proved that $H^p_\phi(\mathbb{D})$ is not minimal.

Corollary 3.3 *If* $0 < p < \infty$*, then the set of analytic polynomials* $\mathcal{P}$ *is not dense in* $H^p_\phi(\mathbb{D})$.

Proof This is a consequence of Theorem 3.2. □

We finish with the following result which follows directly from the proof of Theorem 3.2.

Proposition 3.4 *If* $0 < p < \infty$*, then there exists a weight* $\phi \in W$ *such that the closure of the set* $\mathcal{P}$ *of all analytic polynomials in* $H^p_\phi(\mathbb{D})$ *contains an isomorphic copy of* c_0.

Proof The proof proceeds along the same lines as the proof of Theorem 3.2 and will only be indicated briefly. Fix $0 < p < \infty$ and $0 < \eta < 1/2$. There is the weight $\phi \in W$ and the mapping $T : \ell^\infty \to H^p_\phi(\mathbb{D})$ satisfying

$$(1 - \eta)(\|\lambda\|_\infty)^q \leq \|T(\lambda)\|^q_{H^p_\phi(\mathbb{D})} \leq (1 + \eta)(\|\lambda\|_\infty)^q \tag{3.10}$$

where $q = \min\{1, p\}$ and $\lambda := \{\lambda_k\}_{k\in\mathbb{N}_0} \in \ell^\infty$. Since $T(e_n) \in H^\infty(\mathbb{D})$ for each $n \in \mathbb{N}_0$, it follows that there are analytic polynomials $p_n \in \mathcal{P}$, $n \in \mathbb{N}_0$ satisfying

$$\|T(e_n) - p_n\|^q_{H^p_\phi(\mathbb{D})} < \eta/2^{n+1}, \quad n \in \mathbb{N}_0. \tag{3.11}$$

By the above,

$$p_\lambda := \sum_{n=0}^{\infty} \lambda_n p_n \in H^p_\phi(\mathbb{D})$$

for every $\lambda \in c_0$. In a similar fashion we define a mapping $S : c_0 \to H^p_\phi(\mathbb{D})$ by

$$S : \lambda \mapsto p_\lambda. \tag{3.12}$$

Then (3.10) combined with (3.11) gives

$$(1 - 2\eta)(\|\lambda\|_\infty)^q \leq \|S(\lambda)\|^q_{H^p_\phi(\mathbb{D})} \leq (1 + 2\eta)(\|\lambda\|_\infty)^q,$$

which completes the proof. □

4 Conclusions

Recall that by Beurling's theorem (see, e.g., [5, Theorem 7.4]), the analytic polynomials $\mathcal{P}$ are always dense in $H^p_\phi(\mathbb{T})$ for every $\phi \in W$. With Lemma 2.3 at hand, Theorem 3.1 gives an example of an H-admissible space which is not R-admissible but yet analytic polynomials are dense in it.

That the weighted Hardy space $H^p_\phi(\mathbb{D})$ on the disc is R-admissible follows from the definition and Lemma 2.3. Indeed,

$$\|f_\rho\|_{H^p_\phi(\mathbb{D})} = \sup_{0\le r<1} \|f_{r\rho}\|_{L^p_\phi(\mathbb{D})} \le \sup_{0\le r<1} \|f_r\|_{L^p_\phi(\mathbb{D})} = \|f\|_{H^p_\phi(\mathbb{D})}, \quad 0 \le \rho < 1.$$

On the other hand, Corollary 3.3 shows that there exists an H-admissible space which is R-admissible but not minimal. In conclusion, there is no general relationship between the density of analytic polynomials and the R-admissibility of an H-admissible space.

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