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Fourier Characterizations and Non-triviality of Gelfand–Shilov Spaces, with Applications to Toeplitz Operators

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Abstract
We find growth estimates on functions and their Fourier transforms in the one-parameter Gelfand–Shilov spaces S_s, S^σ, Σ_s and Σ^σ. We obtain characterizations for these spaces and their duals in terms of estimates of short-time Fourier transforms. We determine conditions on the symbols of Toeplitz operators under which the operators are continuous on the one-parameter spaces. Lastly, it is determined that Σ_s^σ is nontrivial if and only if $s + \sigma > 1$.

Keywords Gelfand-Shilov spaces · Ultradistributions · Short-time Fourier transform · Toeplitz operators

1 Introduction

The Gelfand–Shilov spaces were first introduced as a useful set of functions for the study of Cauchy problems in partial differential equations. These functions are convenient in this setting because of their smoothness, and because of the conditions of regularity imposed on them. For instance, the initial value problem for some partial differential equations is ill-posed in the Schwartz space $\mathscr{S}$ or its dual $\mathscr{S}'$, the space of tempered distributions, but is well-posed in suitable Gelfand–Shilov spaces. One such example is the "inverse heat equation" $\partial_t u = a\partial_x^2 u$, $a < 0$, which among other examples may be found in [2, 9]. This fact exemplifies the need to determine properties of functions in those spaces. Gelfand–Shilov spaces can also be useful in the study

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of pseudo-differential operators [1], which in turn have uses in, for instance, quantum theory [10] and signal processing [7].

The Gelfand–Shilov spaces S_s^σ, S_s and S^σ of Roumieu type (cf. [3, 5, 8]) and Σ_s^σ, Σ_s and Σ^σ of Beurling type (cf. [15]) can be considered as refinements of the Schwartz space $\mathscr{S}$, where we impose analyticity-like smoothness and/or decay conditions. The strength of these conditions depend on the parameters s and σ. The smaller s is the faster the functions must vanish at infinity, and smaller σ impose stronger conditions on the growth of the derivatives (meaning the Fourier transform vanishes faster). In the one-parameter spaces, functions have exponential decay and their Fourier transforms tend to zero faster than the reciprocal of any polynomial, or vice versa. In the two-parameter spaces, both the functions and their Fourier transforms have exponential decay. If $s + \sigma$ is sufficiently small, the only function found in S_s^σ or Σ_s^σ is $f(x) \equiv 0$, and the spaces are trivial. There are more general Gelfand–Shilov spaces, such as the $S_{M_p}^{N_p}$-spaces whose properties are explored in [3], for instance.

In this paper, we are mostly interested in discussing the properties of the one-parameter spaces S_s and S^σ, their duals $(S_s)'$ and $(S^\sigma)'$, as well as the corresponding spaces Σ_s and Σ^σ and their duals. More specifically, we establish growth estimates on elements in these spaces and their Fourier transforms. Additionally, we find estimates involving the short-time Fourier transform which provide an alternative characterization of Gelfand–Shilov spaces. Such estimates exist for the two-parameter spaces (cf. [12]) and for more general Gelfand–Shilov spaces as well (cf. [4]). Here we extend characterizations of this type to one-parameter spaces. We find that the short-time Fourier transform admits exponential decay in one parameter, and tends to zero faster than reciprocals of polynomials in the other. Corresponding estimates are found for the duals of one-parameter spaces as well. We also examine Toeplitz operators on these one-parameter spaces, where the symbol $a(x, \xi)$ of the operator lies in different one-parameter spaces in each variable. Toeplitz operators are important in different fields of mathematics and physics. For example, in [13], they are applied for obtaining estimates of kinetic energy in quantum systems. We find conditions such that the Toeplitz operator is continuous on S_s, S^σ and their respective duals.

We also determine when the two-parameter spaces are nontrivial. These results are well-known for S_s^σ-spaces, but for the Σ_s^σ-spaces, we find that the space is nontrivial if and only if $s + \sigma > 1$, as opposed to the condition $s + \sigma \geq 1$, $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$ often cited in other works (cf. [18]). This result, which was initially suggested by Andreas Debrouwere, directly contradicts versions of this result in previous works.

The paper is structured as follows. In Sect. 2, we introduce notations, definitions and preliminary propositions regarding Gelfand–Shilov spaces necessary to obtain results in subsequent sections. These preliminary results can either be found in [3, 5, 8] or are simple enough to be left as an exercise for the reader. In Sect. 3, we obtain growth estimates for the short-time Fourier transform of functions in S_s, S^σ and Σ_s, Σ^σ. In Sect. 4, we show how these results can be used to characterize the duals of these one-parameter spaces via the short-time Fourier transform as well. In Sect. 5, we find conditions on the symbol of Toeplitz operators so that the operator is continuous on one-parameter spaces and their duals. Lastly, in Sect. 6, we determine for which s and σ the space Σ_s^σ is nontrivial.

2 Preliminaries

We begin by defining the spaces we will deal with in this paper. These are the so-called Gelfand–Shilov spaces. There is a clear and intuitive correspondence between the spaces of Roumieu type and those of Beurling type and the order in which the definitions are listed is meant to highlight this correspondence. Here, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, where $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, $j = 1, \dots, n$.

Definition 2.1 Suppose $s, \sigma > 0$.

(i) $S_s(\mathbb{R}^n)$ consists of all $f \in C^\infty(\mathbb{R}^n)$ for which there is an $h > 0$ such that

$$\sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| \le C_\beta h^{|\alpha|} \alpha!^s, \quad \forall \alpha, \beta \in \mathbb{N}^n, \tag{2.1}$$

where C_β is a constant depending only on β.

(ii) $\Sigma_s(\mathbb{R}^n)$ consists of all $f \in C^\infty(\mathbb{R}^n)$ such that (2.1) holds for every $h > 0$, where $C_\beta = C_{h,\beta}$ depends on h and β.

(iii) $S^\sigma(\mathbb{R}^n)$ consists of all $f \in C^\infty(\mathbb{R}^n)$ for which

$$\sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| \le C_\alpha h^{|\beta|} \beta!^\sigma, \quad \forall \alpha, \beta \in \mathbb{N}^n, \tag{2.2}$$

holds for some $h > 0$, where C_α is a constant depending only on α.

(iv) $\Sigma^\sigma(\mathbb{R}^n)$ consists of all $f \in C^\infty(\mathbb{R}^n)$ such that (2.2) holds for every $h > 0$, where $C_\alpha = C_{h,\alpha}$ depends on h and α.

(v) $S_s^\sigma(\mathbb{R}^n)$ consists of all $f \in C^\infty(\mathbb{R}^n)$ for which there are constants $h > 0$ and $C > 0$ such that

$$\sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| \le C h^{|\alpha+\beta|} \alpha!^s \beta!^\sigma, \quad \forall \alpha, \beta \in \mathbb{N}^n. \tag{2.3}$$

(vi) $\Sigma_s^\sigma(\mathbb{R}^n)$ consists of all $f \in C^\infty(\mathbb{R}^n)$ such that (2.3) holds for all $h > 0$, where $C = C_h$ depends only on h.

With $s, \sigma > 0$, $s < s_1$ and $\sigma < \sigma_1$, we see that

$$\Sigma_s^\sigma \subseteq S_s^\sigma \subseteq \Sigma_{s_1}^{\sigma_1}, \tag{2.4}$$

$S_s^\sigma \subseteq S_s \cap S^\sigma$ and $\Sigma_s^\sigma \subseteq \Sigma_s \cap \Sigma^\sigma$. In fact, we have $S_s^\sigma = S_s \cap S^\sigma$ (cf. [5]) and $\Sigma_s^\sigma = \Sigma_s \cap \Sigma^\sigma$ (this is a well-known result that follows by analogous arguments, but an explicit proof can be found in [14] for instance).

We now list some basic properties of Gelfand–Shilov spaces in the form of two propositions. The first proposition establishes exponential decay of derivatives in Gelfand–Shilov spaces, and the second establishes how Fourier transforms work in Gelfand–Shilov spaces. We shall denote by $\mathscr{F}$ the Fourier transform given by

$$(\mathscr{F} f)(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i\langle x, \xi\rangle}\, dx$$

for $f \in \mathscr{S}(\mathbb{R}^n)$, and similarly, we denote by $\mathscr{F}^{-1}$ the corresponding inverse Fourier transform

$$(\mathscr{F}^{-1} f)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\xi) e^{i\langle x,\xi\rangle}\, d\xi$$

for $f \in \mathscr{S}(\mathbb{R}^n)$. For both of the following two propositions, (a) can be found in [3, 5, 8], and (b) follows by analogous arguments.

Proposition 2.2 *Suppose $s > 0$ and $f \in C^\infty(\mathbb{R}^n)$. Then*

(a) *$f \in S_s(\mathbb{R}^n)$ if and only if there are constants $C_\beta, r > 0$ such that*

$$|D^\beta f(x)| \leq C_\beta e^{-r|x|^{1/s}} \tag{2.5}$$

for all multi-indices β;

(b) *$f \in \Sigma_s(\mathbb{R}^n)$ if and only if for every $r > 0$*

$$|D^\beta f(x)| \leq C_{r,\beta} e^{-r|x|^{1/s}} \tag{2.6}$$

holds for all multi-indices β, where $C_{r,\beta} > 0$ depends only on r and β.

Proposition 2.3 *Suppose $s, \sigma > 0$.*

(a) *If $s + \sigma \geq 1$, then $f \in S_s^\sigma(\mathbb{R}^n)$ if and only if $\hat{f} \in S_\sigma^s(\mathbb{R}^n)$. Moreover, $f \in S_s(\mathbb{R}^n)$ if and only if $\hat{f} \in S^s(\mathbb{R}^n)$.*
(b) *If $s+\sigma > 1$, then $f \in \Sigma_s^\sigma(\mathbb{R}^n)$ if and only if $\hat{f} \in \Sigma_\sigma^s(\mathbb{R}^n)$. Moreover, $f \in \Sigma_s(\mathbb{R}^n)$ if and only if $\hat{f} \in \Sigma^s(\mathbb{R}^n)$.*

We will now discuss the topology of Gelfand–Shilov spaces. Since spaces involving inductive and projective limits are frequent, we recall their definitions, see [17].

Definition 2.4 Suppose V_j, $j = 0, 1, 2, \ldots$, are Banach spaces,

$$V = \bigcap_{j \geq 0} V_j$$

and

$$W = \bigcup_{j \geq 0} V_j.$$

(a) Let $i_j : V \to V_j$ be inclusion maps. We say that the *projective limit* is the space V with the smallest possible topology such that i_j is continuous for all j. We write this as

$$V = \operatorname*{proj\,lim}_{j \geq 0} V_j.$$

(b) Suppose further that $V_j \hookrightarrow V_{j+1}$, meaning that V_j is continuously embedded in V_{j+1}, and let $\tilde{i}_j : V_j \to V_{j+1}$ be inclusion maps. We say that the *inductive limit* is the space W with the greatest possible topology such that $\tilde{i}_j$ is continuous for all j. We write this as

$$W = \operatorname*{ind\,lim}_{j \geq 0} V_j.$$

With these definitions and propositions in mind, we can construct topologies on S_s, S^σ, Σ_s and Σ^σ and define their duals. For more information on topological vector spaces, see for instance [17].

Definition 2.5 (1) Let $V_{s,r,N}(\mathbb{R}^n)$ consist of all $f \in C^\infty(\mathbb{R}^n)$ such that

$$||f||_{s,r,N} = \sup_{x \in \mathbb{R}^n, |\alpha| \leq N} \left| D^\alpha f(x) e^{r|x|^{1/s}} \right| < \infty.$$

(2) Let $V^\sigma_{r,M}(\mathbb{R}^n)$ consist of all $f \in C^\infty(\mathbb{R}^n)$ such that

$$||f||^\sigma_{r,M} = \sup_{\xi \in \mathbb{R}^n, |\beta| \leq M} \left| D^\beta \hat{f}(\xi) e^{r|\xi|^{1/\sigma}} \right| < \infty.$$

We see that $V_{s,r,N}(\mathbb{R}^n)$ and $V^\sigma_{r,M}(\mathbb{R}^n)$ are Banach spaces, that

$$S_s(\mathbb{R}^n) = \operatorname*{ind\,lim}_{r>0} \left(\operatorname*{proj\,lim}_{N \geq 0} V_{s,r,N}(\mathbb{R}^n) \right)$$

and that

$$S^\sigma(\mathbb{R}^n) = \operatorname*{ind\,lim}_{r>0} \left(\operatorname*{proj\,lim}_{M \geq 0} V^\sigma_{r,M}(\mathbb{R}^n) \right),$$

which implies

$$S_s(\mathbb{R}^n) = \bigcup_{r>0} \left(\bigcap_{N \geq 0} V_{s,r,N}(\mathbb{R}^n) \right), \quad S^\sigma(\mathbb{R}^n) = \bigcup_{r>0} \left(\bigcap_{M \geq 0} V^\sigma_{r,M}(\mathbb{R}^n) \right).$$

For the Σ_s- and Σ^σ-spaces, we obtain

$$\Sigma_s(\mathbb{R}^n) = \operatorname*{proj\,lim}_{r>0} \left(\operatorname*{proj\,lim}_{N \geq 0} V_{s,r,N}(\mathbb{R}^n) \right)$$

and

$$\Sigma^{\sigma}(\mathbb{R}^n) = \operatorname{proj}\lim_{r>0} \left(\operatorname{proj}\lim_{M\geq 0} V^{\sigma}_{r,M}(\mathbb{R}^n) \right)$$

which implies

$$\Sigma_s(\mathbb{R}^n) = \bigcap_{r>0} \left(\bigcap_{N\geq 0} V_{s,r,N}(\mathbb{R}^n) \right), \quad \Sigma^{\sigma}(\mathbb{R}^n) = \bigcap_{r>0} \left(\bigcap_{M\geq 0} V^{\sigma}_{r,M}(\mathbb{R}^n) \right).$$

Remark 2.6 While Σ_s and Σ^{σ} are Fréchet spaces for all $s, \sigma > 0$, the same is not known to be true for S_s and S^{σ} in current literature.

This leads us to define the dual spaces of S_s and S^{σ} in the following way. Here we denote a functional u of such a dual space being applied to a test function f in the appropriate corresponding space by $u(f) = \langle u, f\rangle$.

Definition 2.7 (i) We say that $u \in (S_s)'(\mathbb{R}^n)$ if for every $r > 0$ there exist $N \geq 0$ and $C > 0$ such that

$$|\langle u, f\rangle| \leq C \sum_{|\alpha|\leq N} ||D^{\alpha} f e^{r|\cdot|^{1/s}}||_{\infty},$$

for any $f \in S_s(\mathbb{R}^n)$.

(ii) We say that $u \in (S^{\sigma})'(\mathbb{R}^n)$ if for every $r > 0$ there exist $N \geq 0$ and $C > 0$ such that

$$|\langle u, f\rangle| \leq C \sum_{|\alpha|\leq N} ||D^{\alpha} \hat{f} e^{r|\cdot|^{1/\sigma}}||_{\infty},$$

for any $f \in S^{\sigma}(\mathbb{R}^n)$.

Similarly, we define the dual spaces of Σ_s and Σ^{σ} as follows.

Definition 2.8 (i) We say that $u \in (\Sigma_s)'(\mathbb{R}^n)$ if there exist $r_0 > 0$, $N \geq 0$ and $C > 0$ such that

$$|\langle u, f\rangle| \leq C \sum_{|\alpha|\leq N} ||D^{\alpha} f e^{r_0|\cdot|^{1/s}}||_{\infty},$$

for any $f \in \Sigma_s(\mathbb{R}^n)$.

(ii) We say that $u \in (\Sigma^{\sigma})'(\mathbb{R}^n)$ if there exist $r_0 > 0$, $N \geq 0$ and $C > 0$ such that

$$|\langle u, f\rangle| \leq C \sum_{|\alpha|\leq N} ||D^{\alpha} \hat{f} e^{r_0|\cdot|^{1/\sigma}}||_{\infty},$$

for any $f \in \Sigma^{\sigma}(\mathbb{R}^n)$.

Remark 2.9 Since $S_0(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$, the space of compactly supported smooth functions, (cf. [8, p. 170]) we have $(S_0)'(\mathbb{R}^n) = \mathscr{D}'(\mathbb{R}^n)$. Since $S_s(\mathbb{R}^n)$ is continuously embedded and dense in $S_{s'}(\mathbb{R}^n)$ for $s \le s'$, we thus have

$$(S_s)'(\mathbb{R}^n) \subseteq \mathscr{D}'(\mathbb{R}^n)$$

for all positive s. By Proposition 2.3, we therefore have

$$(S^\sigma)'(\mathbb{R}^n) \subseteq \mathscr{F}\mathscr{D}'(\mathbb{R}^n)$$

for all positive σ, where $\mathscr{F}\mathscr{D}'(\mathbb{R}^n)$ is the space of continuous linear forms on $\mathscr{F}C_c^\infty(\mathbb{R}^n) = \{\hat{f} : f \in C_c^\infty(\mathbb{R}^n)\}$.

In Definitions 2.7 and 2.8, we can replace the $L^\infty(\mathbb{R}^n)$-norm with the $L^2(\mathbb{R}^n)$-norm by the arguments of [5, p. 134]. We can extend this further with Hölder's inequality to obtain the following equivalent definitions of the dual spaces.

Proposition 2.10 *Suppose* $1 \le p \le \infty$.

(i) $u \in (S_s)'(\mathbb{R}^n)$ *if and only if for every* $r > 0$ *there exist* $N \ge 0$ *and* $C > 0$ *such that*

$$|\langle u, f\rangle| \le C \sum_{|\alpha|\le N} ||D^\alpha f e^{r|\cdot|^{1/s}}||_p,$$

for any $f \in S_s(\mathbb{R}^n)$.

(ii) $u \in (S^\sigma)'(\mathbb{R}^n)$ *if and only if for every* $r > 0$ *there exist* $N \ge 0$ *and* $C > 0$ *such that*

$$|\langle u, f\rangle| \le C \sum_{|\alpha|\le N} ||D^\alpha \hat{f} e^{r|\cdot|^{1/\sigma}}||_p,$$

for any $f \in S^\sigma(\mathbb{R}^n)$.

Replacing the L^∞-norm with L^p-norms, $1 \le p < \infty$, is possible for the Σ_s and Σ^σ duals by similar arguments.

For $u \in (S_s)'$, $f \in S_s$ we will also consider (u, f), which we denote to mean the continuous extension of the regular inner product of $L^2(\mathbb{R}^n)$ given by

$$(u, f)_2 = \int_{\mathbb{R}^n} u(y)\overline{f(y)}\, dy.$$

The fact that the $L^2(\mathbb{R}^n)$ inner product can be extended continuously in this way follows from the analysis of [8]. (This is essentially because $(S_s, L^2, (S_s)')$ forms a *Gelfand triple*, cf. [6].) The same is true with S^σ, Σ_s or Σ^σ in place of S_s at each occurrence.

We also recall the following definition of the short-time Fourier transform, which serves a pivotal role in several of the characterizations in this paper.

Definition 2.11 The *short-time Fourier transform* of $u \in (S_s)'(\mathbb{R}^n)$ with window function $\phi \in S_s(\mathbb{R}^n)$ is given by

$$V_\phi u(x, \xi) = (2\pi)^{-n/2}(u, \phi(\cdot - x)e^{i\langle \cdot, \xi \rangle}).$$

For u belonging to $(S^\sigma)'(\mathbb{R}^n)$, $(\Sigma_s)'(\mathbb{R}^n)$ or $(\Sigma^\sigma)'(\mathbb{R}^n)$, we define the short-time Fourier transform by replacing each occurrence of S_s above with S^σ, Σ_s and Σ^σ, respectively.

3 Characterizations by Short-Time Fourier Transform

In this section, we characterize S_s- and S^σ-spaces in terms of their short-time Fourier transforms. This characterization is detailed in the following theorem, which is the main result of this section.

Theorem 3.1 *Suppose $s, \sigma > 0$.*

(i) *Let $\phi \in S_s(\mathbb{R}^n) \setminus \{0\}$. Then $f \in S_s(\mathbb{R}^n)$ if and only if there is an $r > 0$ such that*

$$|V_\phi f(x, \xi)| \leq C_N (1 + |\xi|^2)^{-N} e^{-r|x|^{1/s}} \tag{3.1}$$

for every $N \geq 0$.

(ii) *Let $\phi \in S^\sigma(\mathbb{R}^n)\setminus\{0\}$. Then $f \in S^\sigma(\mathbb{R}^n)$ if and only if there is an $r > 0$ such that*

$$|V_\phi f(x, \xi)| \leq C_N (1 + |x|^2)^{-N} e^{-r|\xi|^{1/\sigma}} \tag{3.2}$$

for every $N \geq 0$.

For the proof, we will need the following three lemmas. These follow by basic computations[1] and are left for the reader to prove.

Lemma 3.2 *For $f \in S_s(\mathbb{R}^n)$ ($f \in \Sigma_s(\mathbb{R}^n)$) and $\phi_1, \phi_2, \phi_3 \in S_s(\mathbb{R}^n)$ ($\phi_1, \phi_2, \phi_3 \in \Sigma_s(\mathbb{R}^n)$), we have*

$$(\phi_3, \phi_1) V_{\phi_2} f(x, \xi) = \frac{1}{(2\pi)^{n/2}} \iint V_{\phi_1} f(x - y, \xi - \eta) V_{\phi_2} \phi_3(y, \eta) e^{-i\langle x - y, \eta \rangle} \, dy \, d\eta.$$

Lemma 3.3 *If $s > 0$ then there is a constant $C \geq 1$ such that*

$$C^{-1}(|x|^{1/s} + |y|^{1/s}) \leq |y|^{1/s} + |y - x|^{1/s} \leq C(|x|^{1/s} + |y|^{1/s}). \tag{3.3}$$

for every $x, y \in \mathbb{R}^n$.

[1] For Lemma 3.2, see the computations leading to (11.29) in [11].

Lemma 3.4 *For any $\xi, \eta \in \mathbb{R}^n$ and any $N \geq 0$, there is a constant*[2] $C > 0$ *such that*

$$(1 + |\xi - \eta|^2)^{-N} \leq C(1 + |\xi|^2)^{-N}(1 + |\eta|^2)^N.$$

Proof of Theorem 3.1 Suppose that $f \in S_s$. For every $N \geq 0$, we have

$$\begin{aligned}
|(1+|\xi|^2)^N V_\phi f(x,\xi)| &= \frac{1}{(2\pi)^{n/2}} \left| \int_{\mathbb{R}^n} f(y)\overline{\phi(y-x)}(1+|\xi|^2)^N e^{-i\langle y,\xi\rangle}\, dy \right| \\
&= \frac{1}{(2\pi)^{n/2}} \left| \int_{\mathbb{R}^n} f(y)\overline{\phi(y-x)}(1-\Delta)^N e^{-i\langle y,\xi\rangle}\, dy \right| \\
&= \frac{1}{(2\pi)^{n/2}} \left| \sum_{\gamma_0+|\gamma|=N} \frac{N!}{\gamma_0!\gamma!} \int_{\mathbb{R}^n} f(y)\overline{\phi(y-x)} D^{2\gamma} e^{-i\langle y,\xi\rangle}\, dy, \right|
\end{aligned}$$

where $\gamma' = (\gamma_0, \gamma) \in \mathbb{N}^{1+n}$ and the derivatives are taken with respect to y. Integration by parts together with Leibniz formula yields

$$\begin{aligned}
&|(1+|\xi|^2)^N V_\phi f(x,\xi)| \\
&= \frac{1}{(2\pi)^{n/2}} \left| \sum_{\gamma_0+|\gamma|=N} \frac{N!}{\gamma_0!\gamma!} \int_{\mathbb{R}^n} D^{2\gamma}\left[f(y)\overline{\phi(y-x)}\right] e^{-i\langle y,\xi\rangle}\, dy \right| \\
&= \frac{1}{(2\pi)^{n/2}} \left| \sum_{\gamma_0+|\gamma|=N} \frac{N!}{\gamma_0!\gamma!} \sum_{\alpha\leq 2\gamma} \binom{2\gamma}{\alpha} \int D^\alpha f(y) D^{2\gamma-\alpha}\overline{\phi(y-x)} e^{-i\langle y,\xi\rangle} dy \right| \\
&\leq \frac{1}{(2\pi)^{n/2}} \sum_{\gamma_0+|\gamma|=N} \frac{N!}{\gamma_0!\gamma!} \sum_{\alpha\leq 2\gamma} \binom{2\gamma}{\alpha} \int \left| D^\alpha f(y) D^{2\gamma-\alpha}\overline{\phi(y-x)} \right| dy.
\end{aligned}$$

By Proposition 2.2 there are $C_{\gamma,\alpha}, a > 0$ such that

$$\int \left| D^\alpha f(y) D^{2\gamma-\alpha}\overline{\phi(y-x)} \right| dy \leq C_{\gamma,\alpha} \int e^{-a(|y|^{1/s}+|y-x|^{1/s})}\, dy,$$

and by Lemma 3.3 there is a $c > 0$ such that

$$\int \left| D^\alpha f(y) D^{2\gamma-\alpha}\overline{\phi(y-x)} \right| dy \leq C'_{\gamma,\alpha} e^{-ac|x|^{1/s}}.$$

Hence, with $r = ac > 0$ and $C_{N,\gamma',\alpha} = \frac{N!}{\gamma_0!\gamma!}\binom{2\gamma}{\alpha} C'_{\gamma,\alpha} > 0$ we obtain

[2] The best such constant is given by $C = \left(\frac{2}{\sqrt{3}}\right)^N$, according to [16].

$$
\begin{aligned}
|(1+|\xi|^2)^N V_\phi f(x,\xi)| &\le \frac{1}{(2\pi)^{n/2}} \sum_{\gamma_0+|\gamma|=N} \sum_{\alpha\le 2\gamma} C_{N,\gamma',\alpha} e^{-r|x|^{1/s}} \\
&\le C_N e^{-r|x|^{1/s}},
\end{aligned}
$$

where $C_N = \frac{1}{(2\pi)^{n/2}} \sum_{\gamma_0+|\gamma|=N} \sum_{\alpha\le 2\gamma} C_{N,\gamma',\alpha}$. Thus (3.1) holds for every $N \ge 0$.

Now suppose that (3.1) holds for every $N \ge 0$. This condition implies that $f \in \mathscr{S}$ (cf. [12]). In particular $f \in C^\infty$, hence by Proposition 2.2, the result follows if there is an $r > 0$ such that (2.5) holds for every multi-index β.

Consider $V_\phi[D^\beta f](x,\xi)$. Integrating by parts and applying Leibniz formula gives

$$
\begin{aligned}
V_\phi[D^\beta f](x,\xi) &= \frac{(-1)^{|\beta|}}{(2\pi)^{n/2}} \int f(y) D^\beta \left(\overline{\phi(y-x)} e^{-i\langle y,\xi\rangle} \right) dy \\
&= \sum_{\alpha\le\beta} \frac{C_{\alpha,\beta}}{(2\pi)^{n/2}} \int f(y) D^\alpha \overline{\phi(y-x)} \xi^{\beta-\alpha} e^{-i\langle y,\xi\rangle}\, dy \\
&= \sum_{\alpha\le\beta} C_{\alpha,\beta} \xi^{\beta-\alpha} V_{D^\alpha\phi} f(x,\xi),
\end{aligned}
$$

where $C_{\alpha,\beta} = (-1)^{|\alpha|}\binom{\beta}{\alpha}$. By Lemma 3.2 we therefore have

$$
\begin{aligned}
&V_\phi[D^\beta f](x,\xi) \\
&\quad = \sum_{\alpha\le\beta} C'_{\alpha,\beta} \xi^{\beta-\alpha} \iint V_\phi f(x-y,\xi-\eta) V_{D^\beta\phi}\phi(y,\eta) e^{-i\langle x-y,\eta\rangle} dy\, d\eta, \quad (3.4)
\end{aligned}
$$

where $C'_{\alpha,\beta} = (2\pi)^{-n/2}(\phi,\phi)^{-1} C_{\alpha,\beta}$.

For now, we consider only the double integral

$$
I = \iint V_\phi f(x-y,\xi-\eta) V_{D^\beta\phi}\phi(y,\eta) e^{-i\langle x-y,\eta\rangle} dy\, d\eta
$$

from the right-hand side of the previous equation. Note that

$$
V_{D^\beta\phi}\phi(x,\xi) = e^{-i\langle x,\xi\rangle} V_{\overline{\phi}}[D^\beta\overline{\phi}](-x,\xi),
$$

and since $D^\beta\overline{\phi}, \overline{\phi} \in S_s\backslash\{0\}$ the first part of this theorem now implies that there is an $r_1 > 0$ such that

$$
|V_{D^\beta\phi}\phi(x,\xi)| \le C_{\beta,N_1}(1+|\xi|^2)^{-N_1} e^{-r_1|x|^{1/s}}
$$

for every $N_1 \geq 0$ and every β. (Note that all the derivatives of $\overline{\phi}$ fulfill Proposition 2.2 with the same exponent, hence we can use the same $r_1 > 0$ for every β.) By assumption, (3.1) holds for all $N \geq 0$. For any given N, pick $N_1 > N + n$. We now obtain

$$|I| \leq A_{\beta,N} \iint (1 + |\xi - \eta|^2)^{-N} e^{-r|x-y|^{1/s}} (1 + |\eta|^2)^{-N_1} e^{-r_1|y|^{1/s}} dy d\eta = A_{\beta,N} I_1 \cdot I_2$$

where $A_{\beta,N} = C_N C_{\beta,N_1}$,

$$I_1 = \int e^{-r|x-y|^{1/s}} e^{-r_1|y|^{1/s}} dy$$

and

$$I_2 = \int (1 + |\xi - \eta|^2)^{-N} (1 + |\eta|^2)^{-N_1} d\eta.$$

In order to estimate I_1, we let $r_2 = \min\{r, r_1\}$ and apply Lemma 3.3 to obtain $c > 0$ such that

$$\begin{aligned} I_1 &\leq \int e^{-r_2(|y-x|^{1/s}+|y|^{1/s})} dy \\ &\leq e^{-r_2 c|x|^{1/s}} \int e^{-r_2 c|y|^{1/s}} dy \\ &= B e^{-r_2 c|x|^{1/s}}, \end{aligned}$$

where $B = \int e^{-r_2 c|y|^{1/s}} dy < \infty$. Since $N - N_1 < -n$, Lemma 3.4 gives

$$\begin{aligned} I_2 &= \int (1 + |\xi - \eta|^2)^{-N} (1 + |\eta|^2)^{-N_1} d\eta \\ &\leq C(1 + |\xi|^2)^{-N} \int (1 + |\eta|^2)^{N-N_1} d\eta \\ &= B_N (1 + |\xi|^2)^{-N}, \end{aligned}$$

where $B_N = C \int (1 + |\eta|^2)^{N-N_1} d\eta < \infty$. Combining these estimates, we get

$$I \leq B_{\beta,N} (1 + |\xi|^2)^{-N} e^{-r_2 c|x|^{1/s}}$$

for every $N \geq 0$, where $B_{\beta,N} = A_{\beta,N} B B_N$. Combining this with (3.4), we obtain

$$|V_\phi[D^\beta f](x, \xi)| \leq \sum_{\alpha \leq \beta} B_{\alpha,\beta,N} |\xi^{\beta-\alpha}| (1 + |\xi|^2)^{-N} e^{-cr|x|^{1/s}} \tag{3.5}$$

for every $N \geq 0$, where $B_{\alpha,\beta,N} = C'_{\alpha,\beta} B_{\beta,N}$.

We now integrate both sides of (3.5) with respect to ξ. Note that

$$|V_\phi f(x,\xi)| = (2\pi)^{-n/2}\left|\left(\hat{f}_{-x} * \psi\right)(\xi)\right|,$$

where $\psi = \mathscr{F}\left[\overline{\phi}\right]$, and since $\mathscr{F}[f_a](\eta) = e^{-i\langle a,\eta\rangle}\hat{f}(\eta)$,

$$\begin{aligned}(2\pi)^{-n/2}\int_{\mathbb{R}^n}\left|\left(\hat{f}_{-x} * \psi\right)(\xi)\right| d\xi &\geq (2\pi)^{-n/2}\left|\int_{\mathbb{R}^n}\left(\hat{f}_{-x} * \psi\right)(\xi)\, d\xi\right| \\ &= (2\pi)^{-n/2}\left|\iint \hat{f}_{-x}(\eta)\psi(\xi-\eta)\, d\eta\, d\xi\right| \\ &= (2\pi)^{-n/2}\left|\int e^{i\langle x,\eta\rangle}\hat{f}(\eta)\, d\eta \int \psi(\xi-\eta)\, d\xi\right| \\ &= |f(x)|\left|\int \psi(\xi-\eta)\, d\xi\right|.\end{aligned}$$

Since $\int \psi(\xi-\eta)\, d\xi < \infty$, we therefore obtain

$$|D^\beta f(x)| \leq C_\phi \int |V_\phi[D^\beta f](x,\xi)| d\xi, \tag{3.6}$$

for some constant $C_\phi > 0$. Moreover, if we fix $N > |\beta| + n$, then

$$\int |\xi^{\beta-\alpha}|(1+|\xi|^2)^{-N} d\xi = D_{\alpha,\beta} < \infty$$

for each $\alpha \leq \beta$ and thus, with $r' = r_2 c$,

$$\int |V_\phi[D^\beta f](x,\xi)| d\xi \leq \sum_{\alpha\leq\beta} B_{\alpha,\beta,N} D_{\alpha,\beta} e^{-r'|x|^{1/s}}. \tag{3.7}$$

Finally let $C_\beta = C_\phi^{-1}\sum_{\alpha\leq\beta} B_{\alpha,\beta,N} D_{\alpha,\beta}$. Then combining (3.6) with (3.7) now yields

$$|D^\beta f(x)| \leq C_\beta e^{-r'|x|^{1/s}}$$

for every multi-index β. This completes the proof of (i).

To prove (ii), we first note that by Proposition 2.3, $f \in S^\sigma$ and $\phi \in S^\sigma \setminus \{0\}$ if and only if $\hat{f} \in S_\sigma$ and $\hat{\phi} \in S_\sigma \setminus \{0\}$. By (i), we therefore have $f \in S^\sigma$ if and only if

$$|V_{\hat{\phi}}\hat{f}(x,\xi)| \leq C_N(1+|\xi|^2)^{-N} e^{-r|x|^{1/\sigma}}$$

for every $N \geq 0$. Since $V_{\hat{\phi}}\hat{f}(x,\xi) = e^{-i\langle x,\xi\rangle} V_\phi f(-\xi, x)$, this condition can be rewritten as

$$|V_\phi f(-\xi,x)| \leq C_N(1+|\xi|^2)^{-N} e^{-r|x|^{1/\sigma}}.$$

Performing a change of variables now yields (3.2). This completes the proof. □

Utilizing Proposition 2.2(b) instead of Proposition 2.2(a), we obtain the following characterizations of short-time Fourier transforms in Σ_s and Σ^σ by analogous arguments.

Theorem 3.5 *Suppose $s, \sigma > 0$.*

(i) *Let $\phi \in \Sigma_s(\mathbb{R}^n)\backslash\{0\}$. Then $f \in \Sigma_s(\mathbb{R}^n)$ if and only if for every $r > 0$ and every $N \geq 0$,*

$$|V_\phi f(x, \xi)| \leq C_{r,N}(1 + |\xi|^2)^{-N} e^{-r|x|^{1/s}}.$$

(ii) *Let $\phi \in \Sigma^\sigma(\mathbb{R}^n)\backslash\{0\}$. Then $f \in \Sigma^\sigma(\mathbb{R}^n)$ if and only if for every $r > 0$ and every $N \geq 0$,*

$$|V_\phi f(x, \xi)| \leq C_{r,N}(1 + |x|^2)^{-N} e^{-r|\xi|^{1/\sigma}}.$$

Using these short-time Fourier transform characterizations, we can obtain characterizations for the one-parameter spaces similar to those of [3] for the two-parameter spaces.

Theorem 3.6 *Suppose $s, \sigma > 0$ and $f \in C^\infty(\mathbb{R}^n)$.*

(a) *$f \in S_s(\mathbb{R}^n)$ if and only if there is an $r > 0$ such that*

$$|f(x)| \leq Ce^{-r|x|^{1/s}}, \quad |\hat{f}(\xi)| \leq C_N(1 + |\xi|^2)^{-N} \tag{3.8}$$

for every $N \geq 0$.

(b) *$f \in S^\sigma(\mathbb{R}^n)$ if and only if there is an $r > 0$ such that*

$$|f(x)| \leq C_N(1 + |x|^2)^{-N}, \quad |\hat{f}(\xi)| \leq Ce^{-r|\xi|^{1/\sigma}} \tag{3.9}$$

for every $N \geq 0$.

Proof Suppose first that $f \in S_s$. Then the first inequality of (3.8) holds by (2.5). For the second inequality, we apply Fubini's theorem and integrate both sides of (3.1) with respect to x to obtain

$$\begin{aligned} |\hat{f}(\xi)| &\leq C'_\phi \left| (2\pi)^{-n/2} \int f(y) e^{-i\langle y, \xi\rangle} \int \overline{\phi(y - x)}\, dx\, dy \right| \\ &\leq C'_\phi \int |V_\phi f(x, \xi)|\, dx \\ &\leq C''_N (1 + |\xi|^2)^{-N}, \end{aligned}$$

where $C'_\phi = (2\pi)^{n/2} \left| \int \overline{\phi(x)}\, dx \right|^{-1}$ and $C''_N = C'_\phi C_N \int e^{-r|x|^{1/s}}\, dx$. Hence

$$|\hat{f}(\xi)| \leq C''_N (1 + |\xi|^2)^{-N}$$

holds for every $N \geq 0$.

Suppose instead that f fulfills (3.8) for some $r > 0$ and every $N \geq 0$. We have

$$|V_\phi(x,\xi)| \leq (2\pi)^{-n/2} \int |f(y)||\phi(y-x)|\, dy.$$

By assumption, there are $C_0, r_1, r_2 > 0$ such that

$$\int |f(y)||\phi(y-x)|\, dy \leq C_0 \int e^{-r_1|y|^{1/s}} e^{-r_2|y-x|^{1/s}}\, dy \leq C_1 e^{-r|x|^{1/s}}$$

for some $r > 0$, where we use Lemma 3.3 for the last inequality. Hence

$$|V_\phi f(x,\xi)| \leq C e^{-r|x|^{1/s}}.$$

Using the same strategy once more but starting from the fact that

$$|V_\phi f(x,\xi)| \leq (2\pi)^{-n/2} \int |\hat{f}(\eta)||\hat{\phi}(\eta-\xi)|\, d\eta,$$

and this time utilizing Lemma 3.4, we obtain

$$|V_\phi f(x,\xi)| \leq C_N(1+|\xi|^2)^{-N}$$

for every $N \geq 0$. Combining both these inequalities we see that for every $N \geq 0$,

$$|V_\phi f(x,\xi)|^2 \leq C_N(1+|\xi|^2)^{-N} e^{-r|x|^{1/s}},$$

and in particular for $N = 2k$, $k \geq 0$,

$$|V_\phi f(x,\xi)|^2 \leq C_{2k}(1+|\xi|^2)^{-2k} e^{-r|x|^{1/s}},$$

thus

$$|V_\phi f(x,\xi)| \leq C'_k(1+|\xi|^2)^{-k} e^{-r'|x|^{1/s}}$$

for all $k \geq 0$, where $C'_k = \sqrt{C_{2k}}$ and $r' = r/2$. This completes the proof of (a).

To prove (b), simply perform Fourier transforms in light of Proposition 2.3 and apply (a). □

As for the other results, we state the corresponding theorem for the Σ_s- and Σ^σ-spaces but omit its proof as it follows by analogous arguments.

Theorem 3.7 *Suppose $s, \sigma > 0$ and $f \in C^\infty(\mathbb{R}^n)$.*

(a) *$f \in \Sigma_s(\mathbb{R}^n)$ if and only if for every $r > 0$ and $N \geq 0$,*

$$|f(x)| \leq C_r e^{-r|x|^{1/s}}, \quad |\hat{f}(\xi)| \leq C_N(1+|\xi|^2)^{-N}. \tag{3.10}$$

(b) $f \in \Sigma^\sigma(\mathbb{R}^n)$ *if and only if for every* $r > 0$ *and every* $N \geq 0$,

$$|f(x)| \leq C_N(1+|x|^2)^{-N}, \quad |\hat{f}(\xi)| \leq C_r e^{-r|\xi|^{1/\sigma}}. \tag{3.11}$$

4 Characterizations of Dual Spaces

We now move on to the characterization of duals to one-parameter spaces S_s and S^σ. Note here that when u is a generalized function, $\mathscr{F}(u) = \hat{u}$ denotes the adjoint operator of the Fourier transform described in Sect. 2. The duals of one-parameter spaces were defined in Sect. 2 via the topologies detailed in Definition 2.5 and using the results of Sect. 3, we now arrive at the following equivalent topologies.

Proposition 4.1 *Suppose* $s, \sigma > 0$.

(i) *Let* $\phi \in S_s(\mathbb{R}^n) \setminus \{0\}$, *let*

$$p^\phi_{s,r,N}(f) = \sup_{x,\xi\in\mathbb{R}^n} \left| V_\phi f(x,\xi)(1+|\xi|^2)^N e^{r|x|^{1/s}} \right|$$

and let $B_{s,r,N}(\mathbb{R}^n)$ *be the Banach space consisting of all* $f \in C^\infty(\mathbb{R}^n)$ *such that* $p^\phi_{s,r,N}(f)$ *is finite. Then*

$$S_s(\mathbb{R}^n) = \operatorname*{ind\,lim}_{r>0} \left(\operatorname*{proj\,lim}_{N\geq 0} B_{s,r,N}(\mathbb{R}^n) \right)$$

where the equality holds in a topological sense as well.

(ii) *Let* $\phi \in S^\sigma(\mathbb{R}^n)\setminus\{0\}$, *let*

$$q^{\phi,\sigma}_{r,M}(f) = \sup_{x,\xi\in\mathbb{R}^n} \left| V_\phi f(x,\xi)(1+|x|^2)^M e^{r|\xi|^{1/\sigma}} \right|$$

and let $B^\sigma_{r,M}(\mathbb{R}^n)$ *be the Banach space consisting of all* $f \in C^\infty(\mathbb{R}^n)$ *such that* $q^{\phi,\sigma}_{r,M}(f)$ *is finite. Then*

$$S^\sigma(\mathbb{R}^n) = \operatorname*{ind\,lim}_{r>0} \left(\operatorname*{proj\,lim}_{M\geq 0} B^\sigma_{r,M}(\mathbb{R}^n) \right)$$

where the equality holds in a topological sense as well.

Proof The equivalence of the semi-norms $p^\phi_{s,r,N}$ and $||\cdot||_{s,r,N}$, as well as that of $q^{\phi,\sigma}_{r,M}$ and $||\cdot||^\sigma_{r,M}$ is established implicitly in the proof of Theorem 3.1. □

Proposition 4.1 gives us the following equivalent definitions for the dual spaces $(S_s)'$ and $(S^\sigma)'$.

Corollary 4.2 *Suppose $s, \sigma > 0$.*

(i) *Let $\phi \in S_s(\mathbb{R}^n) \setminus \{0\}$ and let $u \in \mathscr{D}'(\mathbb{R}^n)$. Then $u \in (S_s)'(\mathbb{R}^n)$ if and only if for every $r > 0$ there is an $N \geq 0$ such that*

$$|\langle u, f\rangle| \leq C_N p^{\phi}_{s,r,N}(f)$$

for all $f \in S_s(\mathbb{R}^n)$.

(ii) *Let $\phi \in S^\sigma(\mathbb{R}^n)\setminus\{0\}$ and $u \in \mathscr{F}\mathscr{D}'(\mathbb{R}^n)$. Then $u \in (S^\sigma)'(\mathbb{R}^n)$ if and only if for every $r > 0$ there is an $M \geq 0$ such that*

$$|\langle u, f\rangle| \leq C_M q^{\phi,\sigma}_{r,M}(f)$$

for all $f \in S^\sigma(\mathbb{R}^n)$.

This brings us to the following characterization of the duals via short-time Fourier transforms, which is the main result of this section.

Theorem 4.3 (i) *Let $\phi \in S_s(\mathbb{R}^n) \setminus \{0\}$ and $u \in \mathscr{D}'(\mathbb{R}^n)$. Then $u \in (S_s)'(\mathbb{R}^n)$ if and only if for every $r > 0$ there is an $N_0 \geq 0$ such that*

$$|V_\phi u(x, \xi)| \leq C_r(1 + |\xi|^2)^{N_0} e^{r|x|^{1/s}}. \tag{4.1}$$

(ii) *Let $\phi \in S^\sigma(\mathbb{R}^n)\setminus\{0\}$ and $u \in \mathscr{F}\mathscr{D}'(\mathbb{R}^n)$. Then $u \in (S^\sigma)'(\mathbb{R}^n)$ if and only if for every $r > 0$ there is an $N_0 \geq 0$ such that*

$$|V_\phi u(x, \xi)| \leq C_r(1 + |x|^2)^{N_0} e^{r|\xi|^{1/\sigma}}. \tag{4.2}$$

Proof Suppose $u \in (S_s)'$, $\phi \in S_s\setminus\{0\}$. Then by Proposition 2.10 and Leibniz formula, for every $r > 0$ there are $N_0 > 0$ and $C > 0$ such that

$$\begin{aligned}
|V_\phi u(x, \xi)| &= |(u, \phi(\cdot - x)e^{i\langle\cdot,\xi\rangle})| \\
&\leq C \sum_{|\alpha|\leq N_0} ||D^\alpha \left(\phi(\cdot - x)e^{i\langle\cdot,\xi\rangle}\right) e^{r|\cdot|^{1/s}}||_2 \\
&= C \sum_{|\alpha|\leq N_0} \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} ||(D^\gamma\phi)(\cdot - x)\xi^{\alpha-\gamma} e^{i\langle\cdot,\xi\rangle} e^{r|\cdot|^{1/s}}||_2 \\
&\leq C'(1 + |\xi|^2)^{N_0} \sum_{|\alpha|\leq N_0} \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} ||(D^\gamma\phi)(\cdot - x)e^{r|\cdot|^{1/s}}||_2,
\end{aligned}$$

where C' depends on r only.

By Theorem 2.2, there is an $r_0 > 0$ such that $|D^\gamma\phi(y - x)| \leq C_\gamma e^{-r_0|y-x|^{1/s}}$, hence

$$|V_\phi u(x, \xi)| \leq C''(1 + |\xi|^2)^{N_0} \sum_{|\alpha|\leq N_0} \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} ||e^{-r_0|\cdot-x|^{1/s}} e^{r|\cdot|^{1/s}}||_2.$$

By Lemma 3.3, there is a $c \geq 1$ such that

$$-r_0|y-x|^{1/s} \leq r_0|x|^{1/s} - r_0/c \cdot |y|^{1/s}.$$

Let $r \in (0, r_0/(2c))$. Then

$$-r_0|y-x|^{1/s} \leq -2cr|y-x|^{1/s} \leq 2cr|x|^{1/s} - 2r|y|^{1/s},$$

and

$$\begin{aligned}
|V_\phi u(x,\xi)| &\leq C''(1+|\xi|^2)^{N_0} \sum_{|\alpha|\leq N_0} \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} ||e^{2cr|x|^{1/s}-2r|\cdot|^{1/s}+r|\cdot|^{1/s}}||_2 \\
&= C''(1+|\xi|^2)^{N_0} e^{2cr|x|^{1/s}} \sum_{|\alpha|\leq N_0} \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} ||e^{-r|\cdot|^{1/s}}||_2 \\
&\leq C'''(1+|\xi|^2)^{N_0} e^{2cr|x|^{1/s}},
\end{aligned}$$

where the constants C'' and C''' depend on r only.

Clearly, the inequality still holds if we let $r \geq r_0/(2c)$ (the right hand side only becomes larger), hence we have shown that the inequality is valid for all $r > 0$, as was to be shown.

Now suppose that for every $r > 0$ there is an $N_0 \geq 0$ such that (4.1) holds. Then by Moyal's identity, for every $f \in S_s$

$$|(u, f)| = ||\phi||_2^{-2} |(V_\phi u, V_\phi f)|,$$

hence

$$|(u, f)| \leq ||\phi||_2^{-2} \int\int |V_\phi u(x,\xi)| \cdot |V_\phi f(x,\xi)| \, dx \, d\xi.$$

By assumption combined with Theorem 3.1, for every $r > 0$ there is an $N_0 \geq 0$ such that

$$|(u, f)| \leq C_r ||\phi||_2^{-2} \int\int (1+|\xi|^2)^{N_0} e^{r|x|^{1/s}} |V_\phi f(x,\xi)| \, dx \, d\xi.$$

Clearly, for any $r_1 > 0$ and $N_1 \geq 0$, we therefore have

$$\begin{aligned}
|(u, f)| &\leq C_r ||\phi||_2^{-2} \int\int (1+|\xi|^2)^{(N_0-N_1)} e^{(r-r_1)|x|^{1/s}} \\
&\quad \cdot \left| V_\phi f(x,\xi)(1+|\xi|^2)^{N_1} e^{r_1|x|^{1/s}} \right| \, dx \, d\xi \\
&\leq C_r ||\phi||_2^{-2} p^{\phi,s}_{r_1,N_1}(f) \int\int (1+|\xi|^2)^{-(N_1-N_0)} e^{-(r_1-r)|x|^{1/s}} \, dx \, d\xi.
\end{aligned}$$

Now pick r such that $r < r_1$, and pick N_1 such that $N_1 > N_0$. We obtain

$$|(u, f)| \leq C_{N_1,r} ||\phi||_2^{-2} p^{\phi,s}_{r_1,N_1}(f)$$

for all $r_1 > r$. By picking $r > 0$ arbitrarily small, we thus obtain

$$|(u, f)| \leq C'_{N_1,r_1} \cdot p^{\phi,s}_{r_1,N_1}(f)$$

for all $r_1 > 0$. This completes the proof of (i). The proof of (ii) is very similar, utilizing the fact that $\phi \in S^\sigma$ is equivalent to $\hat{\phi} \in S_\sigma$, and the fact that $V_\phi u(x, \xi) = (\hat{u}, \hat{\phi}(\cdot - \xi)e^{-i\langle \cdot, x\rangle})$. □

Lastly we include the corresponding result for the dual spaces of Σ_s and Σ^σ, which follows by analogous arguments.

Theorem 4.4 (i) *Let $\phi \in \Sigma_s(\mathbb{R}^n)\backslash\{0\}$ and let $u \in \mathscr{D}'(\mathbb{R}^n)$. Then $u \in (\Sigma_s)'(\mathbb{R}^n)$ if and only if there exist $r_0 > 0$, $C > 0$ and $N_0 \geq 0$ such that*

$$|V_\phi u(x, \xi)| \leq C(1 + |\xi|^2)^{N_0} e^{r_0|x|^{1/s}}.$$

(ii) *Let $\phi \in \Sigma^\sigma(\mathbb{R}^n)\backslash\{0\}$ and let $u \in \mathscr{F}\mathscr{D}'(\mathbb{R}^n)$. Then $u \in (\Sigma^\sigma)'(\mathbb{R}^n)$ if and only if there exist $r_0 > 0$, $C > 0$ and $N_0 \geq 0$ such that*

$$|V_\phi u(x, \xi)| \leq C(1 + |x|^2)^{N_0} e^{r_0|\xi|^{1/\sigma}}.$$

5 Continuity of Toeplitz Operators

We will now look at Toeplitz operators on one-parameter Gelfand–Shilov spaces. To analyze these, we will need to consider functions in $2n$ dimensions which belong to different one-parameter Gelfand–Shilov spaces in different (n-dimensional) variables. To give them a sense, we begin by examining the case when the functions belong to different two-parameter Gelfand–Shilov spaces with respect to each variable. These spaces are defined as follows.

Definition 5.1 Suppose $s_1, s_2, \sigma_1, \sigma_2 > 0$. Then $S^{\sigma_1,s_2}_{s_1,\sigma_2}(\mathbb{R}^{2n})$ consists of every $f \in C^\infty(\mathbb{R}^{2n})$ for which there is an $h > 0$ such that

$$\sup \left(\frac{\left| x^{\alpha_1} \xi^{\alpha_2} D_x^{\beta_1} D_\xi^{\beta_2} f(x, \xi) \right|}{h^{|\alpha_1+\alpha_2+\beta_1+\beta_2|} (\alpha_1!)^{s_1} (\alpha_2!)^{\sigma_2} (\beta_1!)^{\sigma_1} (\beta_2!)^{s_2}} \right) < \infty$$

for every $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}^n$, where the supremum is taken over $x, \xi \in \mathbb{R}^n$.

We can interpret this as a space where functions belong to $S^{\sigma_1}_{s_1}(\mathbb{R}^n)$ in the x-variable and $S^{s_2}_{\sigma_2}(\mathbb{R}^n)$ in the ξ-variable. Note that $S^{\sigma,\sigma}_{s,s}(\mathbb{R}^{2n}) = S^\sigma_s(\mathbb{R}^{2n})$. With the notations

$S_s^\infty = S_s$ and $S_\infty^\sigma = S^\sigma$, we can construct similar spaces where the functions belong to the one-parameter spaces in single variables instead. These are the spaces we will focus on in this section.

Definition 5.2 Suppose $s, t > 0$. Then $S_{s,\infty}^{\infty,t}(\mathbb{R}^{2n})$ consists of every $f \in C^\infty(\mathbb{R}^{2n})$ for which there is an $h > 0$ such that

$$\sup \frac{\left|x^{\alpha_1}\xi^{\alpha_2} D_x^{\beta_1} D_\xi^{\beta_2} f(x,\xi)\right|}{h^{|\alpha_1+\beta_2|}(\alpha_1!)^s(\beta_2!)^t} \le C_{\beta_1,\alpha_2} \tag{5.1}$$

for every $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}^n$, where the supremum is taken over $x, \xi \in \mathbb{R}^n$ and where C_{β_1,α_2} is a constant depending only on β_1 and α_2.

In similar ways with $\sigma, \tau > 0$, we let $S_{\infty,\tau}^{\sigma,\infty}(\mathbb{R}^{2n})$ consist of every $f \in C^\infty(\mathbb{R}^{2n})$ for which there is an $h > 0$ such that

$$\sup \frac{\left|x^{\alpha_1}\xi^{\alpha_2} D_x^{\beta_1} D_\xi^{\beta_2} f(x,\xi)\right|}{h^{|\beta_1+\alpha_2|}(\beta_1!)^\sigma(\alpha_2!)^\tau} \le C_{\alpha_1,\beta_2} \tag{5.2}$$

for every $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}^n$. Here, the supremum is taken over $x, \xi \in \mathbb{R}^n$ and C_{α_1,β_2} is a constant depending only on α_1 and β_2. We also consider the duals of these spaces, which we construct as follows.

Let $||f||_{s,h,N,M}^t$ be the supremum in (5.1) taken over $x, \xi \in \mathbb{R}^n$, $\alpha_2, \beta_1 \in \mathbb{N}^n$, but only $|\alpha_1| \le N$ and $|\beta_2| \le M$. With

$$V_{s,h,N,M}^t(\mathbb{R}^{2n}) = \{f \in C^\infty(\mathbb{R}^{2n}) : ||f||_{s,h,N,M}^t < \infty\},$$

we observe that

$$S_{s,\infty}^{\infty,t}(\mathbb{R}^{2n}) = \operatorname*{ind\,lim}_{h>0}\left(\operatorname*{proj\,lim}_{N,M\ge 0} V_{h,N,M}(\mathbb{R}^{2n})\right). \tag{5.3}$$

It is therefore natural to set

$$(S_{s,\infty}^{\infty,t})'(\mathbb{R}^{2n}) = \operatorname*{proj\,lim}_{h>0}\left(\operatorname*{ind\,lim}_{N,M\ge 0}(V_{h,N,M})'(\mathbb{R}^{2n})\right). \tag{5.4}$$

We construct the space $(S_{\infty,\tau}^{\sigma,\infty})'(\mathbb{R}^{2n})$ analogously.

Similar to the characterizations of S_s and S^σ via Fourier transform in Theorem 3.6, we can characterize the double spaces $S_{s,\infty}^{\infty,t}$ and $S_{\infty,\tau}^{\sigma,\infty}$ as follows.

Proposition 5.3 *Suppose $s, t, \sigma, \tau > 0$ and $f \in C^\infty(\mathbb{R}^{2n})$.*

(a) $f \in S_{s,\infty}^{\infty,t}(\mathbb{R}^{2n})$ *if and only if there is an $r > 0$ such that*

$$|f(x,\xi)| \le C_N(1+|\xi|^2)^{-N}e^{-r|x|^{1/s}}, \quad |\hat{f}(\eta,y)| \le (1+|\eta|^2)^{-N}e^{-r|y|^{1/t}} \tag{5.5}$$

for every $N \ge 0$.

(b) $f \in S_{\infty,\tau}^{\sigma,\infty}(\mathbb{R}^{2n})$ *if and only if there is an* $r > 0$ *such that*

$$|f(x,\xi)| \le C_N(1+|x|^2)^{-N}e^{-r|\xi|^{1/\tau}}, \quad |\hat{f}(\eta,y)| \le (1+|y|^2)^{-N}e^{-r|\eta|^{1/\sigma}} \tag{5.6}$$

for every $N \ge 0$.

Proof This result follows directly from Theorem 3.6. □

With this in mind we now consider Toeplitz operators on one-parameter Gelfand–Shilov spaces.

Definition 5.4 Let $s, \sigma > 0$, $\phi_1, \phi_2 \in S_s(\mathbb{R}^n)$ and $a \in S_{s,\infty}^{\infty,s}(\mathbb{R}^{2n})$. The Toeplitz operator $Tp_{\phi_1,\phi_2}(a)$ is given by

$$(Tp_{\phi_1,\phi_2}(a)f, u)_{L^2(\mathbb{R}^{2n})} = (a, \overline{V_{\phi_1}f} \cdot V_{\phi_2}u)_{L^2(\mathbb{R}^{2n})} \tag{5.7}$$

for every $f \in S_s(\mathbb{R}^n)$ and $u \in (S_s)'(\mathbb{R}^n)$.

If instead $\phi_1, \phi_2 \in S^\sigma(\mathbb{R}^n)$ and $a \in S_{\infty,\sigma}^{\sigma,\infty}(\mathbb{R}^{2n})$, the Toeplitz operator is given by (5.7) for every $f \in S^\sigma(\mathbb{R}^n)$ and $u \in (S^\sigma)'(\mathbb{R}^n)$.

We observe that the Toeplitz operator in (5.7) can be expressed as

$$Tp_{\phi_1,\phi_2}(a)f = V_{\phi_2}^*(a \cdot V_{\phi_1}f)$$

and that the estimates on a, $V_{\phi_1}f$ and $V_{\phi_2}u$ imply that this is a continuous operator from $S_s(\mathbb{R}^n)$ to $S_s(\mathbb{R}^n)$ when $a \in S_{s,\infty}^{\infty,s}(\mathbb{R}^{2n})$, and from $S^\sigma(\mathbb{R}^n)$ to $S^\sigma(\mathbb{R}^n)$ when $a \in S_{\infty,\sigma}^{\sigma,\infty}(\mathbb{R}^{2n})$. We now want to show that we can loosen the restriction on a to instead be in the duals of $S_{s,\infty}^{\infty,s}(\mathbb{R}^{2n})$ and $S_{\infty,\sigma}^{\sigma,\infty}(\mathbb{R}^{2n})$. To do this, we need the following lemma.

Lemma 5.5 *Let* $\phi_1, \phi_2, f \in S_s(\mathbb{R}^n)$ *and* $u \in (S_s)'(\mathbb{R}^n)$. *Then*

$$\mathscr{F}[\overline{V_{\phi_1}f} \cdot V_{\phi_2}u](\eta, y) = e^{i\langle y,\eta\rangle}V_{\phi_2}\phi_1(y,-\eta) \cdot V_f u(-y,\eta).$$

Proof We prove this in the case that $u \in S_s(\mathbb{R}^n)$. The case that $u \in (S_s)'(\mathbb{R}^n)$ follows by similar arguments. We have

$$\begin{aligned}
&\mathscr{F}[\overline{V_{\phi_1}f} \cdot V_{\phi_2}u](\eta, y)\\
&\quad = (2\pi)^{-2n}\iiiint \overline{f(z)}\phi_1(z-x)u(w)\overline{\phi_2(w-x)}e^{i\langle z-w-y,\xi\rangle - i\langle x,\eta\rangle}\,dz dw dx d\xi\\
&\quad = (2\pi)^{-n}\iint \overline{f(w+y)}\phi_1(w+y-x)u(w)\overline{\phi_2(w-x)}e^{-i\langle x,\eta\rangle}\,dw\,dx\\
&\quad = (2\pi)^{-n}\iint \overline{f(s)}\phi_1(s-x)u(s-y)\overline{\phi_2(s-y-x)}e^{-i\langle x,\eta\rangle}\,ds\,dx\\
&\quad = (2\pi)^{-n}\iint \overline{f(s)}\phi_1(t)u(s-y)\overline{\phi_2(t-y)}e^{-i\langle s-t,\eta\rangle}\,ds\,dt
\end{aligned}$$

$$
\begin{aligned}
&= (2\pi)^{-n} \iint \overline{f(z+y)}\phi_1(t)u(z)\overline{\phi_2(t-y)}e^{-i\langle z+y-t,\eta\rangle}\,dz\,dt \\
&= e^{-i\langle y,\eta\rangle} V_{\phi_2}\phi_1(y,-\eta)\cdot V_f u(-y,\eta),
\end{aligned}
$$

where we apply the Fourier inversion theorem in the second step, apply the variable substitution $s = w + y$ in the third step, $t = s - x$ in the fourth step and $z = s - y$ in the fifth step. □

With this lemma in mind we move on to the main result of this section.

Theorem 5.6 *Suppose $\phi_1, \phi_2 \in S_s(\mathbb{R}^n)$. Then the following is true.*

(a) *The definition of $Tp_{\phi_1,\phi_2}(a)$ is uniquely extendable to any $a \in (S_{s,\infty}^{\infty,s})'(\mathbb{R}^{2n})$ and is then continuous on $S_s(\mathbb{R}^n)$.*
(b) *If $a \in (S_{s,\infty}^{\infty,s})'(\mathbb{R}^{2n})$ then $Tp_{\phi_1,\phi_2}(a)$ is uniquely extendable to a continuous operator on $(S_s)'(\mathbb{R}^n)$.*

Proof For (a), it is sufficient to show that $H = \overline{V_{\phi_1} f} \cdot V_{\phi_2}u \in S_{s,\infty}^{\infty,s}(\mathbb{R}^{2n})$ whenever $f \in S_s(\mathbb{R}^n)$ and $u \in (S_s)'(\mathbb{R}^n)$ and for (b), the same statement is sufficient but with $f \in (S_s)'(\mathbb{R}^n)$ and $u \in S_s(\mathbb{R}^n)$ instead.

We begin by proving (a). By Theorems 3.1 and 4.3, there exist $r_0 > 0$ and $N_0 \geq 0$ such that

$$
\begin{aligned}
|H(x,\xi)| &= (2\pi)^{-n}|V_{\phi_1} f(x,\xi)|\cdot|V_{\phi_2}u(x,\xi)| \\
&\leq C_{N,r}(1+|\xi|^2)^{-(N-N_0)}e^{-(r_0-r)|x|^{1/s}}
\end{aligned}
$$

for every $r > 0$ and $N \geq 0$ and for some constant $C_{N,r} > 0$. Choosing $r < r_0$ and noting that $N_0 \geq 0$ gives the first inequality of Proposition 5.3.

By Lemma 5.5,

$$
|\hat{H}(\eta,y)| = (2\pi)^{-n}|V_{\phi_2}\phi_1(y,-\eta)\cdot V_f u(-y,\eta)|.
$$

Applying Theorems 3.1 and 4.3 exactly as before, we now obtain the second inequality of Proposition 5.3. This completes the proof of (a). To prove (b), simply reverse the roles of f and u and the result follows. □

We also state the corresponding result for $a \in (S_{\infty,\sigma}^{\sigma,\infty})'(\mathbb{R}^{2n})$, which follows by similar arguments.

Theorem 5.7 *Suppose $\phi_1, \phi_2 \in S^\sigma(\mathbb{R}^n)$. Then the following is true.*

(a) *The definition of $Tp_{\phi_1,\phi_2}(a)$ is uniquely extendable to any $a \in (S_{\infty,\sigma}^{\sigma,\infty})'(\mathbb{R}^{2n})$ and is then continuous on $S^\sigma(\mathbb{R}^n)$.*
(b) *If $a \in (S_{\infty,\sigma}^{\sigma,\infty})'(\mathbb{R}^{2n})$ then $Tp_{\phi_1,\phi_2}(a)$ is uniquely extendable to a continuous operator on $(S^\sigma)'(\mathbb{R}^n)$.*

We can define the spaces $\Sigma_{s,\infty}^{\infty,t}(\mathbb{R}^{2n})$ and $\Sigma_{\infty,\tau}^{\sigma,\infty}(\mathbb{R}^{2n})$ as the spaces consisting of every $f \in C^\infty(\mathbb{R}^{2n})$ such that for every $h > 0$, (5.1) and (5.2) hold, respectively. We define the dual spaces as before, but for $(\Sigma_{s,\infty}^{\infty,t})'(\mathbb{R}^{2n})$ we replace $\operatorname{proj\,lim}_{h>0}$ with $\operatorname{ind\,lim}_{h>0}$ in (5.4), and analogously for the construction of $(\Sigma_{\infty,\tau}^{\sigma,\infty})'(\mathbb{R}^{2n})$. For these spaces, we can state the following theorems, which follow by similar arguments to those in the proof of Theorem 5.6.

Theorem 5.8 *Suppose $\phi_1, \phi_2 \in \Sigma_s(\mathbb{R}^n)$. Then the following is true.*

(a) *The definition of $Tp_{\phi_1,\phi_2}(a)$ is uniquely extendable to any $a \in (\Sigma_{s,\infty}^{\infty,s})'(\mathbb{R}^{2n})$ and is then continuous on $\Sigma_s(\mathbb{R}^n)$.*
(b) *If $a \in (\Sigma_{s,\infty}^{\infty,s})'(\mathbb{R}^{2n})$ then $Tp_{\phi_1,\phi_2}(a)$ is uniquely extendable to a continuous operator on $(\Sigma_s)'(\mathbb{R}^n)$.*

Theorem 5.9 *Suppose $\phi_1, \phi_2 \in \Sigma^\sigma(\mathbb{R}^n)$. Then the following is true.*

(a) *The definition of $Tp_{\phi_1,\phi_2}(a)$ is uniquely extendable to any $a \in (\Sigma_{\infty,\sigma}^{\sigma,\infty})'(\mathbb{R}^{2n})$ and is then continuous on $\Sigma^\sigma(\mathbb{R}^n)$.*
(b) *If $a \in (\Sigma_{\infty,\sigma}^{\sigma,\infty})'(\mathbb{R}^{2n})$ then $Tp_{\phi_1,\phi_2}(a)$ is uniquely extendable to a continuous operator on $(\Sigma^\sigma)'(\mathbb{R}^n)$.*

6 Non-triviality of Σ_s^σ-Spaces

In this final section, we divert our attention to the two-parameter spaces Σ_s^σ, and determine when they are nontrivial. Similar results have already been established for S_s^σ-spaces (cf. [8]). By nontrivial we mean that the space contains a function which is not constantly equal to zero. To establish non-triviality conditions, we will need two propositions.

The following proposition follows by similar arguments to those in [8, p. 172–175].

Proposition 6.1 *If $s, \sigma > 0$, $\sigma < 1$ and $f \in \Sigma_s^\sigma(\mathbb{R}^n)$, then f can be continued analytically as an entire function in the (n-dimensional) complex plane. Moreover, for every $a, b > 0$,*

$$|f(x+iy)| \le C \exp\left(-a|x|^{1/s} + b|y|^{1/(1-\sigma)}\right) \tag{6.1}$$

for some constant $C = C_{a,b}$.

Remark 6.2 The converse of the above proposition is also true: If $s, \sigma > 0$, $\sigma < 1$ and an entire function f fulfills (6.1) for every $a, b > 0$, then $f \in \Sigma_s^\sigma$. This follows by analogous arguments to [8, p. 219–220]. Hence this is another characterization of Σ_s^σ in the case that $\sigma < 1$.

We also find the following result in [8, p. 228-233].

Proposition 6.3 *For positive s and σ, the space $S_s^\sigma(\mathbb{R}^n)$ is nontrivial if and only if $s + \sigma \ge 1$.*

With these propositions in mind, we prove the main result of this section. In previous works the condition "$s+\sigma \geq 1, (s,\sigma) \neq (1/2, 1/2)$" is employed instead of "$s+\sigma > 1$". Here, we prove that the correct condition is $s+\sigma > 1$.

Theorem 6.4 *Suppose $s, \sigma > 0$. Then the space $\Sigma_s^\sigma(\mathbb{R}^n)$ is nontrivial if and only if $s+\sigma > 1$.*

Proof Since $\Sigma_s^\sigma \subseteq S_s^\sigma$, it follows by Proposition 6.3 that Σ_s^σ is trivial whenever $s+\sigma < 1$. Furthermore, Proposition 6.3 with (2.4) implies that Σ_s^σ is nontrivial when $s+\sigma > 1$. Thus we need only consider the case $s+\sigma = 1$. Since s and σ are both assumed to be positive, we must have $\sigma < 1$. By Proposition 6.1, it is then true that

$$|f(z)| \leq C_{a,b} \exp\left(-a|x|^{1/s} + b|y|^{1/s}\right)$$

for every $a, b > 0$, where $z = x + iy$. Moreover

$$|f(iz)| \leq C_{a,b} \exp\left(-a|y|^{1/s} + b|x|^{1/s}\right)$$

and therefore

$$|f(z) \cdot f(iz)| \leq C_{a,b}^2 \exp\left((b-a)(|x|^{1/s} + |y|^{1/s})\right).$$

Since this inequality holds for all $a, b > 0$, then by picking $a > b$ we see that $g(z) = f(z) \cdot f(iz)$ is bounded and tends to zero as $|x|, |y| \to \infty$. By Proposition 6.1, f is an entire function, thus so is g. Hence Liouville's theorem implies that $g \equiv 0$. But this implies that $f \equiv 0$ as well, completing the proof. $\square$

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