Journal of Fourier Analysis and Applications (2022) 28:54
https://doi.org/10.1007/s00041-022-09947-1

Interpolation of Generalized Gamma Spaces in a Critical Case

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Received: 6 September 2021 / Revised: 25 February 2022 / Accepted: 28 April 2022

Abstract

We establish some interpolation formulae for generalized gamma spaces with double weights in a critical case. Our approach is based on identifying generalized gamma spaces as appropriate K-interpolation spaces with general weights and then applying the reiteration technique for K-interpolation spaces.

Keywords Generalized gamma spaces · Small and grand Lebesgue spaces · K-interpolation spaces · Weighted inequalities

Mathematics Subject Classification 46E30 · 46B70 · 26D15

Dedicated to the 80th anniversary of Professor Stefan Samko.

Communicated by Vladislav Kravchenko.

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Published online: 10 June 2022

1 Introduction

The scale of generalized gamma spaces with double weights (see Sect. 2 for definitions) was introduced in [18] in order to characterize the following real interpolation spaces

$$(L^{p),\alpha}, L^{(q,\beta})_{\theta,r}$$

between grand Lebesgue spaces $L^{p),\alpha}$ (with $\alpha = 1$) and small Lebesgue spaces $L^{(q,\beta}$ (with $\beta = 1$) in the critical case $p = q$. Later on, it turned out (see [2, 4]) that the following real interpolation spaces (with appropriate conditions on α and β)

$$(L^{p)}, L^{q})_{\theta,r}, \quad (L^{p}, L^{(q})_{\theta,r}, \quad (L^{p),\alpha}, L^{q),\beta})_{\theta,r}, \quad (L^{(p,\alpha}, L^{(q,\beta})_{\theta,r}$$

also coincided with appropriate $G\Gamma$-spaces in the critical case $p = q$. Thus, it becomes imperative to investigate the interpolation properties of $G\Gamma$-spaces themselves in the critical case. The aim of the present paper is to pursue this goal. The main finding of our investigation is this: in our special critical case, the scale of $G\Gamma$-spaces remains stable under real interpolation method. We emphasize that this is not the case in non-critical cases as it is clear from the results in [3, 15–18].

Let us illustrate our special case critical. Consider the following real interpolation spaces

$$(\Lambda^{p}(w_0), G\Gamma(q, m; v, w_1))_{\theta,r}$$

between classical Lorentz spaces $\Lambda^{p}(w_0)$ and $G\Gamma$-spaces $G\Gamma(q, m; v, w_1)$. We characterize these interpolation spaces in the critical case $p = q$ with an extra restriction $w_0 = w_1$ (see Theorem 6.1 below).

The key feature of our approach is to identify $G\Gamma$-spaces as K-interpolation spaces (with general weights) between the classical Lorentz and L^{∞} spaces. This is done in Sect. 3. Then, in order to apply the reiteration technique, we formulate appropriate reiteration theorems for K-interpolation spaces involving general weights (see Sect. 5). The proofs of these reiteration theorems are essentially based on certain Holmstedt-type estimates (from [1]) and weighted Hardy-type inequalities (presented in Sect. 4). The interpolation formulae for $G\Gamma$-spaces (our main results) are contained in Sect. 6. Finally, in Sect. 7, we single out some special cases from Sect. 6 in order to illustrate how our obtained results generalize/complement the existing results in previous papers [2–4, 9, 25].

2 Preliminaries

2.1 Notation

Throughout the paper we will stick to the following notations. We write $A \lesssim B$ or $B \gtrsim A$ for two non-negative quantities A and B to mean that $A \leq cB$ for some positive constant c which is independent of appropriate parameters involved in A and B. If both the estimates $A \lesssim B$ and $B \lesssim A$ hold, we simply put $A \approx B$. We let $\|\cdot\|_{q,(a,b)}$ denote the standard L^q-quasi-norm on an interval $(a, b) \subset \mathbb{R}$. We write $X \hookrightarrow Y$ for two quasi-normed spaces X and Y to mean that X is continuously embedded in Y. By a weight w on $(0, 1)$, we always mean a positive locally integrable function on $(0, 1)$. We let Ω denote a bounded Lebesgue measurable domain in $\mathbb{R}^n$ with measure 1. Finally, the symbol f^* will denote the non-increasing rearrangement of a real-valued Lebesgue measurable function f on Ω (see, for instance, [7]).

2.2 Slowly Varying Functions

Following [22], we say a weight b is slowly varying on $(0, 1)$ if for every $\varepsilon > 0$, there are positive functions g_ε and $g_{-\varepsilon}$ on $(0, 1)$ such that g_ε is non-decreasing and $g_{-\varepsilon}$ is non-increasing, and we have

$$t^\varepsilon b(t) \approx g_\varepsilon(t) \quad \text{and} \quad t^{-\varepsilon} b(t) \approx g_{-\varepsilon}(t) \quad \text{for all} \quad t \in (0, 1).$$

We denote the class of all slowly varying functions by SV. The class SV contains, for example, positive constant functions, and the functions $t \mapsto (1 - \ln t)$ and $t \mapsto 1 + \ln(1 - \ln t)$. We collect in the next Proposition some properties of slowly varying functions. The proofs of these assertions can be carried out as in [22, Lemma 2.1] or [11, Proposition 3.4.33].

Proposition 2.1 *Given $b, b_1, b_2 \in SV$, the following are true:*

(i) *$b_1 b_2 \in SV$ and $b^r \in SV$ for each $r \in \mathbb{R}$.*
(ii) *If $0 < k < 1$, then $b(kt) \approx b(t)$, $0 < t < 1$.*
(iii) *For $\alpha > 0$, set $\tilde{b}(t) = b(t^\alpha)$, $0 < t < 1$. Then $\tilde{b} \in SV$.*
(iv) *If $\alpha > 0$, then*

$$\int_0^t u^\alpha b(u) \frac{du}{u} \approx t^\alpha b(t), \qquad 0 < t < 1.$$

(v) *If $\alpha > 0$, then*

$$1 + \int_t^1 u^{-\alpha} b(u) \frac{du}{u} \approx t^{-\alpha} b(t), \qquad 0 < t < 1.$$

(vi) *Set*

$$\tilde{b}(t) = 1 + \int_t^1 b(u) \frac{du}{u}, \qquad 0 < t < 1.$$

Then $\tilde{b} \in SV$, and $b(t) \lesssim \tilde{b}(t)$, $0 < t < 1$.

(vii) *Set*

$$\tilde{b}(t) = \sup_{0<u<t} b(u), \qquad 0 < t < 1.$$

Then $\tilde{b} \in SV$.

2.3 *K*-Interpolation Spaces

Let A_0 and A_1 be two quasi-normed spaces. We say (A_0, A_1) is a compatible couple if A_0 and A_1 are continuously embedded in the same Hausdorff topological vector space. For each $f \in A_0 + A_1$ and $t > 0$, the Peetre K-functional is defined by

$$\begin{aligned} K(t, f) &= K(t, f; A_0, A_1) \\ &= \inf\{\|f_0\|_{A_0} + t\|f_1\|_{A_1} : \ f_0 \in A_0, \ f_1 \in A_1, \ f = f_0 + f_1\}. \end{aligned}$$

Note that $K(t, f)$ is, as a function of t, non-decreasing on $(0, \infty)$. In the sequel, we will refer to this fact simply as monotonicity of K-functional.

In what follows, we always assume that the couple (A_0, A_1) is ordered in the sense that $A_1 \hookrightarrow A_0$.

Let $0 < q \le \infty$, and let w be a positive weight on $(0, 1)$ satisfying the following condition

$$\|t^{1-1/q} w(t)\|_{q,(0,1)} < \infty. \tag{2.1}$$

Then the K-interpolation space $\bar{A}_{w,q} = (A_0, A_1)_{w,q}$ is formed of those $f \in A_0$ for which the quasi-norm

$$\|f\|_{\bar{A}_{w,q}} = \|t^{-1/q} w(t) K(t, f)\|_{q,(0,1)}$$

is finite; see, for instance, [1]. If $0 < q < \infty$ and $w(t) = t^{-\theta}$ with $0 < \theta < 1$, then we recover the classical real interpolation spaces $\bar{A}_{\theta,q}$ (see [7, 8, 24, 27]).

Note that, thanks to the condition (2.1), the spaces $\bar{A}_{w,q}$ are intermediate for the couple (A_0, A_1), that is,

$$A_1 \hookrightarrow \bar{A}_{w,q} \hookrightarrow A_0.$$

Next let $f \in \bar{A}_{w,q}$. By monotonicity of K-functional and $K(1, f) \approx \|f\|_{A_0}$, we have

$$\|f\|_{\bar{A}_{w,q}} \lesssim \|f\|_{A_0} \|t^{-1/q} w(t)\|_{q,(0,1)}.$$

Thus we can conclude that we always have to work under the following condition on w

$$\|t^{-1/q} w(t)\|_{q,(0,1)} = \infty, \tag{2.2}$$

so that the trivial case $\bar{A}_{w,q} = A_0$ is excluded. If $w \in SV$, then the condition (2.1) is met thanks to Proposition 2.1 (iv) (if $0 < q < \infty$) or to the very definition of a slowly varying function (if $q = \infty$).

2.4 Classical Lorentz Spaces

Let $0 < q \leq \infty$ and let w be weight on $(0, 1)$. Assume that

(c1) $w(2t) \lesssim w(t), \quad 0 < t < 1/2.$
(c2) $\|t^{-1/q} w(t)\|_{q,(0,1)} < \infty.$

The classical Lorentz spaces $\Lambda^q(w) = \Lambda^q(w)(\Omega)$ consists of those real-valued Lebesgue measurable functions f on Ω, for which the quasi-norm

$$\|f\|_{\Lambda^q(w)} = \|t^{-1/q} w(t) f^*(t)\|_{q,(0,1)}$$

is finite; see [26]. Thanks to the condition (c2), we always have $\Lambda^q(w) \neq \{0\}$; more precisely, we have the embedding $L^\infty \hookrightarrow \Lambda^q(w)$. The classical Lorentz spaces cover many well-known spaces: for instance, when $w(t) = t^{1/p} b(t)$ (with $0 < p \leq \infty$ and $b \in SV$) the spaces $\Lambda^q(w)$ become the Lorentz–Karamata spaces $L_{p,q;b}$ (see, for instance, [20]). In particular, when $b(t) = (1 - \ln t)^\alpha$, $\alpha \in \mathbb{R}$, we put $L^{p,q}(\log L)^\alpha = L_{p,q;b}$. The space $L^{p,q}(\log L)^\alpha$ is called the Lorent–Zygmund space and it was introduced by Bennett and Rudnick [6]. If $\alpha = 0$, the Lorentz–Zygmund space $L^{p,q}(\log L)^\alpha$ coincides with the Lorentz space $L^{p,q}$ which becomes the Lebesgue space L^p if $p = q$.

Remark 2.2 Let $f \in \Lambda^\infty(w)$. Since f^* is non-increasing, we can verify easily that

$$\sup_{0<t<1} w(t) f^*(t) = \sup_{0<t<1} \left[\sup_{0<s<t} w(s) \right] f^*(t).$$

Thus, in the case $q = \infty$ we can assume that w is non-decreasing.

2.5 Generalized Gamma Spaces

We first introduce a notation. For $0 < m, q \leq \infty$, we say a pair (w, v) of weights is admissible if the following conditions are met:

(d1) For all $0 < t < 1/2$, $w(2t) \lesssim w(t)$ and $v(2t) \lesssim v(t)$.
(d2) $\|t^{-1/q} w(t)\|_{q,(0,1)} < \infty.$
(d3) $\|t^{-1/m} v(t)\|_{m,(0,1)} = \infty.$
(d4) $\left\| t^{-1/m} v(t) \|\tau^{-1/q} w(\tau)\|_{q,(0,t)} \right\|_{m,(0,1)} < \infty.$

Definition 2.3 [18] Let $0 < m, q \leq \infty$ and (w, v) be a pair of admissible weights. The generalized gamma space $G\Gamma(q, m; v, w) = G\Gamma(q, m; v, w)(\Omega)$ consists of all those real-valued Lebesgue measurable functions f on Ω, for which the quasi-norm

$$\|f\|_{G\Gamma(q,m;v,w)} = \left\| t^{-1/m} v(t) \|\tau^{-1/q} w(\tau) f^*(\tau)\|_{q,(0,t)} \right\|_{m,(0,1)}$$

is finite.

Remark 2.4 Let $f \in G\Gamma(q, m; v, w)$. Since $t \mapsto f^*(t)$ is non-increasing, we can check that the following function

$$t \mapsto \frac{1}{\|\tau^{-1/q} w(\tau)\|_{q,(0,t)}} \|\tau^{-1/q} w(\tau) f^*(\tau)\|_{q,(0,t)}$$

is equivalent to a non-increasing function. Consequently (thanks to the Condition (d4)), it follows that

$$G\Gamma(q, m; v, w) \hookrightarrow \Lambda^q(w).$$

Moreover, the Condition (d3) guarantees that the converse embedding

$$\Lambda^q(w) \hookrightarrow G\Gamma(q, m; v, w)$$

does not hold. Thus, the trivial case $G\Gamma(q, m; v, w) = \Lambda^q(w)$ is excluded. However, note that for $q = m$ the spaces $G\Gamma(q, m; v, w)$ again coincide with $\Lambda^m(\tilde{w})$ for an appropriate weight $\tilde{w}$.

Remark 2.5 The scale of $G\Gamma(q, m; v, w)$ spaces is very general and covers many well-known scales of spaces. If we take $q = 1$ and $w(t) = t$, then we recover the classical gamma spaces $\Gamma^m(\tilde{v})$ (see [26]) for an appropriate weight $\tilde{v}$. Let $0 < m, p, q < \infty$, $w(t) = t^{1/p}$ and $v \in SV$, then the spaces $G\Gamma(q, m; v, w)$ coincide with the small Lorentz spaces $L_v^{(p,q,m}$ from [3]. As a still more special case, if $\alpha > 0$, $1 < q < \infty$, $v(t) = (1 - \ln t)^{-\frac{\alpha}{q}+\alpha-1}$, $w(t) = t^{1/q}$, $m = 1$, the spaces $G\Gamma(q, m; v, w)$ become the small Lebesgue spaces $L^{(q,\alpha}$; see [18, 19]. Finally, since we also allow the case $m = \infty$ in our definition in contrast to [18], we observe that the spaces $S_{p,\alpha}$ considered in [12] are also a special case of the spaces $G\Gamma(q, m; v, w)$.

3 Generalized Gamma Spaces as *K*-Interpolation Spaces

In this section we characterize the generalized gamma spaces as K-interpolation spaces with general weights. To this end, we first need the following computation of K-functional for the couple $(\Lambda^q(w), L^\infty)$. While this computation is a special case of a far more general formula in [13, p. 84], we present a simple proof for reader's convenience.

Lemma 3.1 *Let $0 < q \le \infty$. Then, for all $f \in \Lambda^q(w)$, we have*

$$K(\tilde{w}(t), f; \Lambda^q(w), L^\infty) \approx \|\tau^{-1/q} w(\tau) f^*(\tau)\|_{q,(0,t)}, \quad 0 < t < 1, \tag{3.1}$$

where

$$\tilde{w}(t) = \|\tau^{-1/q} w(\tau)\|_{q,(0,t)}, \quad 0 < t < 1.$$

Proof Let $f = f_0 + f_1$ be an arbitrary decomposition of f with $f_0 \in \Lambda^q(w)$ and $f_1 \in L^\infty$. Using the elementary inequality

$$f^*(\tau) \leq f_0^*(\tau) + f_1^*(0), \quad 0 < \tau < 1,$$

we get

$$\|\tau^{-1/q} w(\tau) f^*(\tau)\|_{q,(0,t)} \lesssim \|f_0\|_{\Lambda^q(w)} + \tilde{w}(t)\|f_1\|_{L^\infty},$$

whence we get the estimate "$\gtrsim$" in (3.1), by taking the infimum over all decompositions of f. To prove the converse estimate "$\lesssim$", we fix $0 < t < 1$ and take the following particular decomposition of f:

$$g = (f - f^*(t) sgn f)\chi_E, \quad h = f - g,$$

where $E = \{x \in \Omega : |f(x)| > f^*(t)\}$. Then $g^* = (f^* - f^*(t))\chi_{(0,t)}$ and $h^* = f^*(t)\chi_{(0,t)} + f^*\chi_{(t,1)}$. Therefore, we can check easily that

$$\|g\|_{\Lambda^q(w)} \leq \|\tau^{-1/q} w(\tau) f^*(\tau)\|_{q,(0,t)},$$

and

$$\|h\|_{L^\infty} \leq 2f^*(t) \leq \frac{2}{\tilde{w}(t)} \|\tau^{-1/q} w(\tau) f^*(\tau)\|_{q,(0,t)}.$$

Thus, we arrive at

$$\|g\|_{\Lambda^q(w)} + \tilde{w}(t)\|h\|_{L^\infty} \lesssim \|\tau^{-1/q} w(\tau) f^*(\tau)\|_{q,(0,t)}, \quad 0 < t < 1,$$

from which follows the estimate "$\lesssim$". The proof is complete. □

The next two results describe the characterization of $G\Gamma(q, m; v, w)$ spaces as K-interpolation spaces.

Theorem 3.2 *Let $0 < m \leq \infty$, $0 < q < \infty$ and (w, v) be a pair of admissible weights. Let ϕ be the inverse of the following function*

$$\psi(t) = c \left(\int_0^t w^q(\tau) \frac{d\tau}{\tau} \right)^{1/q}, \quad 0 < t < 1,$$

where

$$1/c = \left(\int_0^1 w^q(\tau) \frac{d\tau}{\tau} \right)^{1/q}.$$

Moreover, define

$$\rho(t) = v(\phi(t)) \left[\frac{t}{w(\phi(t))} \right]^{q/m}, \quad 0 < t < 1.$$

Then

$$G\Gamma(q,m;v,w)=\begin{cases}(\Lambda^q(w),L^\infty)_{\rho,m}, & m<\infty,\\ (\Lambda^q(w),L^\infty)_{v(\phi),m}, & m=\infty.\end{cases}$$

Proof We give the argument only in the case $m<\infty$ since the other case $m=\infty$ is analogous. Set temporarily $X=(\Lambda^q(w),L^\infty)_{\rho,m}$, and let $f\in\Lambda^q(w)$. In view of the simple fact that

$$K(c\psi,f;\Lambda^q(w),L^\infty)\approx K(\psi,f;\Lambda^q(w),L^\infty),$$

an application of Lemma 3.1 yields

$$\|f\|_X\approx\left(\int_0^1\rho^m(t)\left(\int_0^{\phi(t)}\left[w(\tau)f^*(\tau)\right]^q\frac{d\tau}{\tau}\right)^{m/q}\frac{dt}{t}\right)^{1/m},$$

now making a change of variable $t=\psi(s)$, it turns out that

$$\|f\|_X\approx\left(\int_0^1\rho^m(\psi(s))\left(\int_0^{s}\left[w(\tau)f^*(\tau)\right]^q\frac{d\tau}{\tau}\right)^{m/q}\frac{\psi'(s)}{\psi(s)}ds\right)^{1/m},$$

finally, the following simple computation

$$\rho^m(\psi(s))\frac{\psi'(s)}{\psi(s)}\approx s^{-1}v^m(s),\quad 0<s<1,$$

completes the proof. □

We omit the proof of the next result since it can be carried out by using the same argument as in the proof of the previous theorem.

Theorem 3.3 *Let* $0<m\le\infty$. *Suppose that* (w,v) *is a pair of admissible weights such that* w *is strictly increasing on* $(0,1)$ *with* $\lim_{t\to0^+}w(t)=0$ *and* $\lim_{t\to1^-}w(t)=1$. *Then*

$$G\Gamma(\infty,\infty;v,w)=(\Lambda^\infty(w),L^\infty)_{v(w^{-1}),\infty}.$$

If we assume additionally that w *is differentiable on* $(0,1)$, *then*

$$G\Gamma(\infty,m;v,w)=(\Lambda^\infty(w),L^\infty)_{\rho,m},\quad m\neq\infty,$$

where

$$\rho(t)=v(w^{-1}(t))\left[\frac{t}{w^{-1}(t)w'\left(w^{-1}(t)\right)}\right]^{1/m},\quad 0<t<1.$$

4 Weighted Hardy-Type Inequalities

The weighted Hardy-type inequalities presented in this section will be the key ingredients in the proofs of our reiteration theorems in the next section.

Theorem 4.1 [1, Lemma 3.2] *Let* $1 < \alpha < \infty$, *and assume that* g *and* ϕ *are non-negative functions on* $(0, \infty)$. *Put*

$$v_1(t) = (g(t))^{1-\alpha} \left(\phi(t) \int_t^\infty g(u)du \right)^\alpha .$$

Then

$$\int_0^\infty \left(\int_0^t \phi(u)h(u)du \right)^\alpha g(t)dt \lesssim \int_0^\infty h^\alpha(t)v_1(t)dt$$

holds for all non-negative functions h *on* $(0, \infty)$.

We also have the following variant of the previous result; see [3, Theorem 3.3].

Theorem 4.2 *Let* $1 < \alpha < \infty$, *and assume that* g *and* ϕ *are non-negative functions on* $(0, \infty)$. *Put*

$$v_2(t) = (g(t))^{1-\alpha} \left(\phi(t) \int_0^t g(u)du \right)^\alpha .$$

Then

$$\int_0^\infty \left(\int_t^\infty \phi(u)h(u)du \right)^\alpha g(t)dt \lesssim \int_0^\infty h^\alpha(t)v_2(t)dt$$

holds for all non-negative functions h *on* $(0, \infty)$.

The next result is a simple consequence of [1, Lemma 3.3].

Theorem 4.3 *Let* $0 < \alpha < 1$, *and assume that* g *and* ϕ *are non-negative functions on* $(0, \infty)$. *Put*

$$v_3(t) = \phi(t) \left(\int_t^\infty \phi(u)du \right)^{\alpha-1} \int_0^t g(u)du.$$

Then

$$\int_0^\infty \left(\int_t^\infty \phi(u)h(u)du \right)^\alpha g(t)dt \lesssim \int_0^\infty h^\alpha(t)v_3(t)dt$$

holds for all non-negative and non-decreasing functions h *on* $(0, \infty)$.

Theorem 4.4 [23, Theorem 3.3 (b)] *Let* $0 < \alpha < 1$*. Assume that* g *and* v *are non-negative functions on* $(0, 1)$*, and* ψ *is a non-negative function on* $(0, 1) \times (0, 1)$*. Then*

$$\int_0^1 \left(\int_0^1 \psi(t,u)h(u)du\right)^\alpha g(t)dt \lesssim \int_0^\infty h^\alpha(t)v(t)dt \tag{4.1}$$

holds for all non-negative and non-decreasing functions h *on* $(0, 1)$ *if and only if*

$$\int_0^1 \left(\int_x^1 \psi(t,u)du\right)^\alpha g(t)dt \lesssim \int_x^1 v(t)dt \tag{4.2}$$

holds for all $0 < x < 1$.

5 Reiteration

First of all, we recall (from Sect. 2.3) that a weight w appearing in the K-interpolation space $\bar{A}_{w,q}$ has to satisfy the conditions (2.1) and (2.2) so that both the trivial cases $\bar{A}_{w,q} = \{0\}$ and $\bar{A}_{w,q} = A_0$ are excluded.

For convenience we introduce a further notation: for $0 < m < \infty$, we say a weight w satisfies the condition (H_m) if the following estimate holds:

$$t^{-1}\left(\int_0^t u^m w^m(u)\frac{du}{u}\right)^{1/m} \lesssim \left(1 + \int_t^1 w^m(u)\frac{du}{u}\right)^{1/m}, \quad 0 < t < 1.$$

Moreover, we say a weight w satisfies the condition (H_∞) if the following estimate holds:

$$t^{-1} \sup_{0<u<t} uw(u) \lesssim w(t), \quad 0 < t < 1.$$

Remark 5.1 Let $w \in SV$. Then, by Proposition 2.1 (iv)–(vi), w satisfies (H_m). Clearly, by the very definition of a slowly varying function, w also satisfies (H_∞).

Theorem 5.2 *Let* $0 < m, r < \infty$, $0 < \theta < 1$*, and let* w *satisfy* (H_m)*. Then*

$$(A_0, \bar{A}_{w,m})_{\theta,r} = \bar{A}_{\tilde{w},r},$$

where

$$\tilde{w}(t) = \left(1 + \int_t^1 w^m(u)\frac{du}{u}\right)^{\theta/m - 1/r} w^{m/r}(t), \quad 0 < t < 1.$$

Proof Set $X = (A_0, \bar{A}_{w,m})_{\theta,r}$, $Y = \bar{A}_{\tilde{w},r}$ and

$$\rho(t) = \left(1 + \int_t^1 w^m(u)\frac{du}{u}\right)^{-1/m}, \quad 0 < t < 1.$$

Note that ρ is increasing with $\lim_{t\to 0^+} \rho(t) = 0$ (thanks to (2.2)) and $\lim_{t\to 1^-} \rho(t) = 1$. Next define

$$W(t) = \begin{cases} w(t),\ 0 < t < 1, \\ t^{-1}, \qquad t \geq 1, \end{cases}$$

and note that

$$\left(\int_t^\infty W^m(u)\frac{du}{u}\right)^{1/m} \approx \left(1 + \int_t^1 w^m(u)\frac{du}{u}\right)^{1/m}, \quad 0 < t < 1.$$

Let $f \in A_0$. Since w satisfies (H_m), we can apply the estimate (2.19) in [1] to obtain

$$K(\rho(t), f; A_0, \bar{A}_{w,m}) \approx \rho(t)\left(\int_t^\infty W^m(u)K^m(u, f)\frac{du}{u}\right)^{1/m}, \quad 0 < t < 1,$$

whence, by an appropriate change of variable, we get

$$\|f\|_X^r \approx \int_0^1 \rho^{(1-\theta)r}(t)\left(\int_t^\infty W^m(u)K^m(u, f)\frac{du}{u}\right)^{r/m} \frac{\rho'(t)}{\rho(t)}dt. \tag{5.1}$$

In view of monotonicity of K-functional, it follows immediately from (5.1) that

$$\|f\|_X^r \gtrsim \int_0^1 \rho^{(1-\theta)r}(t)K^r(t, f)\left(\int_t^\infty W^m(u)\frac{du}{u}\right)^{r/m} \frac{\rho'(t)}{\rho(t)}dt,$$

from which it follows that $\|f\|_X \gtrsim \|f\|_Y$ since

$$\rho'(t) \approx t^{-1}w^m(t)\rho^{1+m}(t), \quad 0 < t < 1/2.$$

Next we establish the converse estimate $\|f\|_X \lesssim \|f\|_Y$. To this end, we note that, from (5.1), we have

$$\|f\|_X^r \approx I_1 + I_2,$$

where

$$I_1 = \int_0^1 \rho^{(1-\theta)r}(t)\left(\int_t^1 w^m(u)K^m(u, f)\frac{du}{u}\right)^{r/m} \frac{\rho'(t)}{\rho(t)}dt,$$

and

$$I_2 = \int_0^1 \rho^{(1-\theta)r}(t) \left(\int_1^\infty u^{-m} K^m(u, f) \frac{du}{u} \right)^{r/m} \frac{\rho'(t)}{\rho(t)} dt.$$

In view of

$$K(t, f) \approx \|f\|_{A_0}, \quad t \geq 1,$$

and

$$\int_0^1 \rho^{(1-\theta)r}(t) \frac{\rho'(t)}{\rho(t)} dt < \infty,$$

we get that $I_2 \approx \|f\|_{A_0}^r$. Since $Y \hookrightarrow A_0$, it follows that $I_2 \lesssim \|f\|_Y^r$. Thus, it remains to establish that $I_1 \lesssim \|f\|_Y^r$. The case $r = m$ immediately follows from Fubini's theorem. For the case $r \neq m$, we take $\alpha = r/m$, $h(t) = K^m(t, f)$, $\phi(t) = t^{-1} W^m(t)$ and $g = \rho^{(1-\theta)r-1}\rho' \chi_{(0,1)}$, and apply Theorem 4.2 (if $r > m$) or Theorem 4.3 (if $r < m$). It is not hard to verify that

$$v_2(t) \approx v_3(t) \approx t^{-1}[\tilde{w}(t)]^r, \quad 0 < t < 1,$$

and consequently, the estimate $I_1 \lesssim \|f\|_Y^r$ holds. The proof is complete. □

Remark 5.3 If we take $w(t) = t^{-\theta_1}$, $0 < \theta_1 < 1$, then we get back the classical result from [24]. If we take $w \equiv 1$ and $m = 1$, then we recover the first assertion in [21, Theorem 3.21]. The case when $w \in SV$ also follows from [5, Theorem 11]. The particular case when w is a logarithmic function has earlier been considered in [10, Theorem 4 (a)].

Next we treat the case $m = \infty$. In this regard, an elementary but important observation is made in the next remark.

Remark 5.4 Let (A_0, A_1) be a compatible couple of quasi-normed spaces. Using monotonicity of K-functional, we observe that the following identity

$$\sup_{0<t<1} w(t)K(t, f) = \sup_{0<t<1} \left[\sup_{t<s<1} w(s) \right] K(t, f),$$

holds for every $f \in A_0$. Therefore, while working with $\bar{A}_{w,\infty}$, we can always assume, without loss of generality, that w is non-increasing.

Theorem 5.5 *Let* $0 < r < \infty$, $0 < \theta < 1$, *and suppose* w *is strictly decreasing and differentiable on* $(0, 1)$ *and satisfies* (H_∞). *Put* $\rho = 1/w$, *and assume that* $\lim_{t \to 1^-} \rho(t) = 1$. *Then we have*

$$(A_0, \bar{A}_{w,\infty})_{\theta,r} = \bar{A}_{\tilde{w},r},$$

where

$$\tilde{w}(t) = w^{\theta}(t)\left[t w(t)\rho'(t)\right]^{1/r}, \quad 0 < t < 1.$$

Proof Put $X = (A_0, \bar{A}_{w,\infty})_{\theta,r}$ and $Y = \bar{A}_{\tilde{w},r}$. Next, in view of (2.2), we observe that $\lim_{t\to 0^+} \rho(t) = 0$. Let $f \in A_0$. Since w satisfies (H_∞), we can apply the estimate (2.19) in [1] to obtain

$$K(\rho(t), f; A_0, \bar{A}_{w,\infty}) \approx \rho(t) \sup_{t\le u<1} w(u)K(u, f), \quad 0 < t < 1,$$

whence we arrive at

$$\|f\|_X^r \approx \int_0^1 \rho^{(1-\theta)r}(t)\left[\sup_{t\le u<1} w^r(u)K^r(u, f)\right]\frac{\rho'(t)}{\rho(t)}dt. \tag{5.2}$$

Now the estimate $\|f\|_X \gtrsim \|f\|_Y$ follows immediately from (5.2). Next we establish the converse estimate $\|f\|_X \lesssim \|f\|_Y$. Put

$$I = \int_0^1 \rho^{(1-\theta)r}(t)\left[\sup_{t\le u<1} w^r(u)K^r(u, f)\right]\frac{\rho'(t)}{\rho(t)}dt,$$

and noting

$$\int_t^1 w^r(u)\frac{\rho'(u)}{\rho(u)}du = \frac{1}{r}\left(w^r(t) - 1\right), \quad 0 < t < 1,$$

we can write

$$I \lesssim I_1 + I_2$$

where

$$I_1 = \int_0^1 \rho^{(1-\theta)r}(t) \sup_{t\le u<1} K^r(u, f)\left[\int_u^1 w^r(\tau)\frac{\rho'(\tau)}{\rho(\tau)}d\tau\right]\frac{\rho'(t)}{\rho(t)}dt,$$

and

$$I_2 = \int_0^1 \rho^{(1-\theta)r}(t) \sup_{t\le u<1} K^r(u, f)\frac{\rho'(t)}{\rho(t)}dt.$$

Now by monotonicity of K-functional, we obtain

$$I_1 \le \int_0^1 \rho^{(1-\theta)r}(t)\left[\sup_{t\le u<1}\int_u^1 w^r(\tau)\frac{\rho'(\tau)}{\rho(\tau)}K^r(\tau, f)d\tau\right]\frac{\rho'(t)}{\rho(t)}dt,$$

and

$$I_2 = \int_0^1 \rho^{(1-\theta)r}(t) K^r(1, f) \frac{\rho'(t)}{\rho(t)} dt,$$

whence we get

$$I_1 \leq \int_0^1 \rho^{(1-\theta)r}(t) \left[\int_t^1 w^r(\tau) \frac{\rho'(\tau)}{\rho(\tau)} K^r(\tau, f) d\tau \right] \frac{\rho'(t)}{\rho(t)} dt,$$

and

$$I_2 \approx \|f\|_{A_0}^r.$$

Now an application of Fubini's theorem gives

$$I_1 \lesssim \int_0^1 w^r(\tau) \frac{\rho'(\tau)}{\rho(\tau)} K^r(\tau, f) \rho^{(1-\theta)r}(\tau) d\tau,$$

which shows that $I_1 \lesssim \|f\|_Y^r$. Since $Y \hookrightarrow A_0$, we also have $I_2 \lesssim \|f\|_Y^r$. Altogether, we arrive at $\|f\|_X^r \approx I \lesssim \|f\|_Y^r$ which completes the proof. □

Remark 5.6 To the best of our knowledge, the assertion of Theorem 5.5 is new. We note that the particular case when w is a general slowly varying function is entirely missing from [1, 5, 20], and also not covered by [14, Theorem 5.5].

Remark 5.7 Let $0 < m < \infty$. Suppose that w_0 and w_1 are two weights such that w_0/w_1 is non-decreasing. Then it is not hard to check that $\bar{A}_{w_1,m} \hookrightarrow \bar{A}_{w_0,m}$. If we assume, additionally, that

$$\frac{w_0(t)}{w_1(t)} \leq 1, \quad 0 < t < 1,$$

then we also have

$$\frac{w_0(t)}{w_1(t)} \leq \left(\frac{1 + \int_t^1 w_0^m(u) \frac{du}{u}}{1 + \int_t^1 w_1^m(u) \frac{du}{u}} \right)^{1/m}, \quad 0 < t < 1.$$

Theorem 5.8 *Let $0 < m, r < \infty$ and $0 < \theta < 1$. Suppose that w_0 and w_1 are two weights such that $\rho = w_0/w_1$ is strictly increasing on $(0, 1)$ with $\lim_{t \to 0^+} \rho(t) = 0$ and $\lim_{t \to 1^-} \rho(t) = 1$. Assume further that w_1 satisfies (H_m) and that there exists $c_1 \in (1, \infty)$ and $c_2 \in (0, 1)$ such that*

$$\left(\frac{1 + \int_t^1 w_0^m(u) \frac{du}{u}}{1 + \int_t^1 w_1^m(u) \frac{du}{u}} \right)^{1/m} < c_1 \rho(t), \quad 0 < t < 1, \tag{5.3}$$

and

$$\rho(t) < c_2 \left(\frac{1 + \int_t^1 w_0^m(u) \frac{du}{u}}{1 + \int_t^1 w_1^m(u) \frac{du}{u}} \right)^{1/m}, \quad 0 < t < 1/2. \tag{5.4}$$

Then we have

$$\left(\bar{A}_{w_0,m}, \bar{A}_{w_1,m}\right)_{\theta,r} = \bar{A}_{\tilde{w},r},$$

where

$$\tilde{w}(t) = [\rho(t)]^{(1-\theta)} w_1^{m/r}(t) \left(1 + \int_t^1 w_1^m(u) \frac{du}{u}\right)^{1/m-1/r}, \quad 0 < t < 1.$$

Proof Set $X = \left(\bar{A}_{w_0,m}, \bar{A}_{w_1,m}\right)_{\theta,r}$, $Y = \bar{A}_{\tilde{w},r}$ and

$$W_1(t) = \begin{cases} w_1(t), \; 0 < t < 1, \\ t^{-1}, \qquad t \geq 1. \end{cases}$$

Let $f \in A_0$, and put

$$\sigma(t) = \left(\frac{1 + \int_t^1 w_0^m(u) \frac{du}{u}}{1 + \int_t^1 w_1^m(u) \frac{du}{u}} \right)^{1/m}, \quad 0 < t < 1.$$

In view of Remark 5.7 and (5.3), we have $\rho \approx \sigma$ on $(0, 1)$. Moreover, since ρ is strictly increasing, we have in fact $\rho < \sigma$ on $(0, 1)$. As a consequence, we obtain $\sigma' > 0$ on $(0, 1)$, that is, σ is also strictly increasing on $(0, 1)$. Now, according to the estimates (2.30) and (2.35) in [1], for all $0 < t < 1$ we have

$$\begin{aligned} K\left(\sigma(t), f, \bar{A}_{w_0,m}, \bar{A}_{w_1,m}\right) &\lesssim I(t, f) + \sigma(t) J(t, f) \\ &\quad + \frac{\sigma(t)}{\sigma_1(t)} K(t, f) + \frac{\rho_0(t)}{t} K(t, f), \end{aligned} \tag{5.5}$$

and

$$K\left(\sigma(t), f, \bar{A}_{w_0,m}, \bar{A}_{w_1,m}\right) \gtrsim I(t, f) + \sigma(t) J(t, f), \tag{5.6}$$

where

$$I(t, f) = \left(\int_0^t w_0^m(u) K^m(u, f) \frac{du}{u}\right)^{\frac{1}{m}},$$

$$J(t, f) = \left(\int_t^\infty W_1^m(u) K^m(u, f) \frac{du}{u}\right)^{\frac{1}{m}},$$

$$\sigma_1(t) = t \left(\int_0^t u^m w_1^m(u) \frac{du}{u} \right)^{-1/m},$$

and

$$\rho_0(t) = t \left(1 + \int_t^1 w_0^m(u) \frac{du}{u} \right)^{1/m}.$$

By monotonicity of K-functional, we get

$$J(t, f) \geq K(t, f) \left(1 + \int_t^1 w_1^m(u) \frac{du}{u} \right)^{1/m},$$

from which it follows that

$$\sigma(t) J(t, f) \gtrsim \frac{\rho_0(t)}{t} K(t, f).$$

Since w_1 satisfies (H_m), we also have

$$J(t, f) \gtrsim \frac{1}{\sigma_1(t)} K(t, f).$$

Altogether, (5.5) reduces to

$$K\left(\sigma(t), f, \bar{A}_{w_0,m}, \bar{A}_{w_1,m}\right) \lesssim I(t, f) + \sigma(t) J(t, f). \tag{5.7}$$

Thus, from (5.6) and (5.7), we have the following two-sided Holmstedt-type estimate

$$K\left(\sigma(t), f, \bar{A}_{w_0,m}, \bar{A}_{w_1,m}\right) \approx I(t, f) + \sigma(t) J(t, f), \quad 0 < t < 1,$$

whence it turns out that

$$\|f\|_X^r \approx I_1 + I_2,$$

where

$$I_1 = \int_0^1 [\rho(t)]^{-\theta r} \left(\int_0^t w_0^m(u) K^m(u, f) \frac{du}{u} \right)^{r/m} \frac{\sigma'(t)}{\rho(t)} dt,$$

and

$$I_2 = \int_0^1 [\rho(t)]^{(1-\theta) r} \left(\int_t^\infty W_1^m(u) K^m(u, f) \frac{du}{u} \right)^{\frac{r}{m}} \frac{\sigma'(t)}{\rho(t)} dt.$$

Next, using (5.4), we can compute that

$$\frac{\sigma'(t)}{\sigma(t)} \approx t^{-1} \frac{w_1^m(t)}{1 + \int_t^1 w_1^m(u) \frac{du}{u}}, \quad 0 < t < 1/2. \tag{5.8}$$

Now following the same line of argument which we used while estimating the quantity on right hand side of (5.1), we can show that $\|f\|_Y^r \approx I_2$. Thus it remains to establish the estimate $I_1 \lesssim \|f\|_Y^r$. In the case when $r \geq m$, this desired estimate follows from Fubini's theorem (if $r = m$) or from Theorem 4.1 (if $r > m$). For the remaining case $r < m$, we apply Theorem 4.4 with $\alpha = r/m$, $h(t) = K^m(t, f)$, $g = \rho^{-\theta r-1}\sigma'$, $\psi(t,u) = u^{-1}w_0^m(u)\chi_{(0,t)}(u)$ and $v(t) = t^{-1}[\tilde{w}(t)]^r$. Observe that (4.2) holds trivially for $1/2 < x < 1$, and for $0 < x < 1/2$ we have

$$\begin{aligned}\int_0^1 \left(\int_x^1 \psi(t,u)du\right)^{\alpha} g(t)dt &= \int_x^1 \left(\int_x^t w_0^m(u)\frac{du}{u}\right)^{r/m} g(t)dt \\ &\leq \left(\int_x^1 w_0^m(u)\frac{du}{u}\right)^{r/m} \int_x^1 g(t)dt \\ &\lesssim \left(\int_x^1 w_0^m(u)\frac{du}{u}\right)^{r/m} [\rho(x)]^{-\theta r},\end{aligned}$$

and

$$\begin{aligned}\int_x^1 v(t)dt &\gtrsim [\rho(x)]^{r(1-\theta)} \int_x^1 [w_1(t)]^m \left(\int_t^1 w_1^m(u)\frac{du}{u}\right)^{r/m-1} \frac{dt}{t} \\ &\approx \left(\int_x^1 w_0^m(u)\frac{du}{u}\right)^{r/m} [\rho(x)]^{-\theta r}.\end{aligned}$$

Thus, (4.2) is valid. Hence, the estimate $I_1 \lesssim \|f\|_Y^r$ follows from Theorem 4.4 in the case $r < m$. This completes the proof. □

Remark 5.9 The particular case, when $w_j(t) = (1 - \ln t)^{-\alpha_j}$ $(j = 0, 1)$ with $\alpha_1 < \alpha_0 < 0$, has earlier been considered in [10, Corollary 1].

6 Interpolation Formulae

Finally, we are in a position to describe the interpolation properties of generalized gamma spaces. In view of well-known reiteration technique, our interpolation formulae are rather straightforward consequences of reiteration theorems (from previous section) and characterization of generalized gamma spaces as K-interpolation spaces (Theorems 3.2 and 3.3). Thus, we illustrate how the reiteration technique works only in a single case, and omit the proofs of the remaining assertions.

Throughout this section, ψ and ϕ are same as defined in Theorem 3.2.

Theorem 6.1 *Let* $0 < m, q \leq \infty$, $0 < r < \infty$, $0 < \theta < 1$, *and let* (w, v) *be a pair of admissible weights.*

(a) *Let* $0 < q, m < \infty$, *and put*

$$\eta_1(t) = v(\phi(t)) \left[\frac{t}{w(\phi(t))}\right]^{q/m}, \quad 0 < t < 1.$$

Assume that η_1 *satisfies* (H_m)*. Then*

$$(\Lambda^q(w), G\Gamma(q, m; v, w))_{\theta,r} = G\Gamma(q, r; V_1, w),$$

where

$$V_1(t) = \left(1 + \int_{\psi(t)}^1 \eta_1^m(u)\frac{du}{u}\right)^{\theta/m-1/r} v^{m/r}(t), \quad 0 < t < 1.$$

(b) *Let* $0 < m < \infty$ *and* $q = \infty$*. Assume that* w *is strictly increasing and differentiable on* $(0, 1)$ *with* $\lim_{t\to 0^+} w(t) = 0$ *and* $\lim_{t\to 1^-} w(t) = 1$*. Put*

$$\eta_2(t) = v(w^{-1}(t)) \left[\frac{t}{w^{-1}(t)w'\left(w^{-1}(t)\right)}\right]^{1/m}, \quad 0 < t < 1,$$

and assume that η_2 *satisfies* (H_m)*. Then*

$$(\Lambda^\infty(w), G\Gamma(\infty, m; v, w))_{\theta,r} = G\Gamma(\infty, r; V_2, w),$$

where

$$V_2(t) = \left(1 + \int_{w(t)}^1 \eta_2^m(u)\frac{du}{u}\right)^{\theta/m-1/r} v^{m/r}(t), \quad 0 < t < 1.$$

(c) *Let* $m = \infty$ *and* $0 < q < \infty$*. Assume that* $\rho = 1/v$ *is strictly increasing and differentiable on* $(0, 1)$ *with* $\lim_{t\to 0^+} \rho(\phi(t)) = 0$ *and* $\lim_{t\to 1^-} \rho(\phi(t)) = 1$*. Assume further that* $v(\phi)$ *satisfies* (H_∞)*. Then*

$$(\Lambda^q(w), G\Gamma(q, \infty; v, w))_{\theta,r} = G\Gamma(q, r; V_3, w),$$

where

$$V_3(t) = v^\theta(t)\left[t w^q(t)\psi^{-q}(t)v(t)\rho'(t)\right]^{1/r}, \quad 0 < t < 1.$$

(d) *Let* $m = q = \infty$*. Assume that* w *is strictly increasing and differentiable on* $(0, 1)$ *with* $\lim_{t\to 0^+} w(t) = 0$ *and* $\lim_{t\to 1^-} w(t) = 1$*. Assume further that* $v(w^{-1})$ *satisfies* (H_∞) *and that* $\rho = 1/v$ *is strictly increasing and differentiable on* $(0, 1)$ *with* $\lim_{t\to 0^+} \rho(t) = 0$ *and* $\lim_{t\to 1^-} \rho(t) = 1$*. Then*

$$(\Lambda^\infty(w), G\Gamma(\infty, \infty; v, w))_{\theta,r} = G\Gamma(\infty, r; V_4, w),$$

where

$$V_4(t) = v^\theta(t)\left[t^2 v(t)(w(t))^{-1}w'(t)\rho'(t)\right]^{1/r}, \quad 0 < t < 1.$$

Proof We give the argument only in the first case. Using Theorem 3.2, we can write

$$(\Lambda^q(w), G\Gamma(q, m; v, w))_{\theta,r} = (\Lambda^q(w), (\Lambda^q(w), L^\infty)_{\eta_1,m})_{\theta,r},$$

now an application of Theorem 5.2 yields

$$(\Lambda^q(w), G\Gamma(q, m; v, w))_{\theta,r} = (\Lambda^q(w), L^\infty)_{\tilde{\eta}_1,r},$$

where

$$\tilde{\eta}_1(t) = V_1(\phi(t)) \left[\frac{t}{w(\phi(t))}\right]^{q/r}, \quad 0 < t < 1.$$

Temporarily set $X = (\Lambda^q(w), L^\infty)_{\tilde{\eta}_1,r}$ and take $f \in \Lambda^q(w)$. Then

$$\|f\|_X^r = \int_0^1 \tilde{\eta}_1^r(t) K^r(t, f; \Lambda^q(w), L^\infty) \frac{dt}{t},$$

now a change of variable $t = \psi(s)$ gives

$$\|f\|_X^r = \int_0^1 \tilde{\eta}_1^r(\psi(s)) K^r(\psi(s), f; \Lambda^q(w), L^\infty) \frac{\psi'(s)}{\psi(s)} ds,$$

next using Lemma 3.1, we arrive at

$$\|f\|_X^r \approx \int_0^1 \tilde{\eta}_1^r(\psi(s)) \left(\int_0^s [w(\tau) f^*(\tau)]^q \frac{d\tau}{\tau}\right)^{r/q} \frac{\psi'(s)}{\psi(s)} ds,$$

or,

$$\|f\|_X^r \approx \int_0^1 V_1^r(s) \left(\int_0^s [w(\tau) f^*(\tau)]^q \frac{d\tau}{\tau}\right)^{r/q} \frac{ds}{s},$$

whence we get $X = G\Gamma(q, r; V_1, w)$ as desired. □

Theorem 6.2 *Let $0 < m, r < \infty$ and $0 < q \le \infty$, $0 < \theta < 1$, and let (w, v_0) and (w, v_1) be two pairs of admissible weights.*

(a) *Let $0 < q < \infty$. For $j = 0, 1$, put*

$$\sigma_j(t) = v_j(\phi(t)) \left[\frac{t}{w(\phi(t))}\right]^{q/m}, \quad 0 < t < 1,$$

and assume that $\rho = \sigma_0/\sigma_1$ is strictly increasing on $(0, 1)$ with $\lim_{t\to 0^+} \rho(t) = 0$ and $\lim_{t\to 1^-} \rho(t) = 1$. Assume further that σ_1 satisfies (H_m) and that there exists $c_1 \in (1, \infty)$

and $c_2 \in (0,1)$ *such that*

$$\left(\frac{1+\int_t^1 \sigma_0^m(u)\frac{du}{u}}{1+\int_t^1 \sigma_1^m(u)\frac{du}{u}}\right)^{1/m} < c_1\rho(t), \quad 0<t<1, \tag{6.1}$$

and

$$\rho(t) < c_2\left(\frac{1+\int_t^1 \sigma_0^m(u)\frac{du}{u}}{1+\int_t^1 \sigma_1^m(u)\frac{du}{u}}\right)^{1/m}, \quad 0<t<1/2. \tag{6.2}$$

Then

$$(G\Gamma(q,m;v_0,w), G\Gamma(q,m;v_1,w))_{\theta,r} = G\Gamma(q,r;V_1,w),$$

where

$$V_1(t) = \left[\frac{v_0(t)}{v_1(t)}\right]^{(1-\theta)} v_1^{m/r}(t)\left(1+\int_{\psi(t)}^1 \sigma_1^m(u)\frac{du}{u}\right)^{1/m-1/r}, \quad 0<t<1.$$

(b) *Let* $q=\infty$. *Assume that* w *is strictly increasing and differentiable on* $(0,1)$ *with* $\lim_{t\to 0^+} w(t)=0$ *and* $\lim_{t\to 1^-} w(t)=1$. *For* $j=0,1$, *put*

$$\delta_j(t) = v_j(w^{-1}(t))\left[\frac{t}{w^{-1}(t)w'\left(w^{-1}(t)\right)}\right]^{1/m}, \quad 0<t<1$$

and assume that $\rho=\delta_0/\delta_1$ *is strictly increasing on* $(0,1)$ *with* $\lim_{t\to 0^+}\rho(t)=0$ *and* $\lim_{t\to 1^-}\rho(t)=1$. *Assume further that* δ_1 *satisfies* (H_m) *and that there exists* $c_1\in(1,\infty)$ *and* $c_2\in(0,1)$ *such that*

$$\left(\frac{1+\int_t^1 \delta_0^m(u)\frac{du}{u}}{1+\int_t^1 \delta_1^m(u)\frac{du}{u}}\right)^{1/m} < c_1\rho(t), \quad 0<t<1,$$

and

$$\rho(t) < c_2\left(\frac{1+\int_t^1 \delta_0^m(u)\frac{du}{u}}{1+\int_t^1 \delta_1^m(u)\frac{du}{u}}\right)^{1/m}, \quad 0<t<1/2.$$

Then

$$(G\Gamma(\infty,m;v_0,w), G\Gamma(\infty,m;v_1,w))_{\theta,r} = G\Gamma(\infty,r;V_2,w),$$

where

$$V_2(t) = \left[\frac{v_0(t)}{v_1(t)}\right]^{(1-\theta)} v_1^{m/r}(t)\left(1+\int_{w(t)}^1 \delta_1^m(u)\frac{du}{u}\right)^{1/m-1/r}, \quad 0<t<1.$$

7 Special Cases

Throughout this section, we let $0 < \theta < 1$ and $0 < r < \infty$.

(i) Let $0 < p, q, m < \infty$, and let $v \in SV$. Take $w(t) = t^{1/p}$, then $\psi(t) \approx t^{1/p}$. Now, in view of Proposition 2.1 (vi), we can see easily that η_1 satisfies (H_m). Thus, according to Theorem 6.1 (a), we have

$$(L^{p,q}, L_v^{(p,q,m)})_{\theta,r} = L_V^{(p,q,r}, \tag{7.1}$$

where

$$V(t) = \left(1 + \int_t^1 v^m(u)\frac{du}{u}\right)^{\theta/m-1/r} v^{m/r}(t), \quad 0 < t < 1.$$

Here $L_v^{(p,q,m}$ is the small Lorentz space considered in [3] (see Remark 2.5). Thus, interpolation formula (7.1) provides a limiting version of the interpolation formula contained in [3, Theorem 5.3]. In addition, if we take $1 < p = q < \infty$, $m = 1$, and $v(t) = (1 - \ln t)^{-1/p}$ in (7.1), then we recover the interpolation formula in [2, Corollary 3.2].

(ii) Let $0 < p, q < \infty$ and let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. Take $w(t) = t^{1/p}(1 - \ln t)^\alpha$, $v(t) = (1 - \ln t)^{\beta-\alpha-1/q}$ and $m = q$. Now we have $\psi(t) \approx t^{1/p}(1 - \ln t)^\alpha$. Thus, we can check easily that

$$[\psi(t)]^{-1}\left(\int_0^t [\psi(u)]^q\, v^q(u)\frac{du}{u}\right)^{1/q} \lesssim \left(1 + \int_t^1 v^q(u)\frac{du}{u}\right)^{1/q}, \quad 0 < t < 1.$$

Consequently, η_1 satisfies (H_q). Thus, we can apply Theorem 6.1 (a) to obtain the following description of interpolation spaces between Lorentz–Zygmund spaces (in a limiting case):

$$\big(L^{p,q}(\log L)^\alpha, L^{p,q}(\log L)^\beta\big)_{\theta,r} = G\Gamma(q, r, V, w), \tag{7.2}$$

where

$$V(t) = (1 - \ln t)^{\theta(\beta-\alpha)-1/r}, \quad 0 < t < 1.$$

The interpolation spaces $\big(L^{p,q}(\log L)^\alpha, L^{p,q}(\log L)^\beta\big)_{\theta,r}$ have already been characterized in [25, Theorem 6 (c)], but the description given there is theoretical and complicated. On the other hand, the formula (7.2) provides a concrete description in terms of generalized gamma spaces.

(iii) Assume that w_0 is strictly increasing and differentiable on $(0, 1)$ with $\lim_{t\to 0^+} w_0(t) = 0$ and $\lim_{t\to 1^-} w_0(t) = 1$. Let w_1 be another weight such that $\rho = w_0/w_1$ is strictly increasing and differentiable on $(0, 1)$ with $\lim_{t\to 0^+} \rho(t) = 0$ and $\lim_{t\to 1^-} \rho(t) = 1$. We can check easily that $v(w_0^{-1})$ satisfies (H_∞). Thus, by Theorem 6.1 (d), we have

$$(\Lambda^\infty(w_0), \Lambda^\infty(w_1))_{\theta,r} = G\Gamma(\infty, r; V, w_0), \tag{7.3}$$

where

$$V(t) = \left[\frac{w_1(t)}{w_0(t)}\right]^\theta \left[t^2 w_1(t)(w_0(t))^{-2} w_0'(t)\rho'(t)\right]^{1/r}, \quad 0 < t < 1.$$

The interpolation formula (7.3) complements the diagonal case ($r = \infty$) considered in [9, Theorem 4.4].

(iv) If we take $1 < q < \infty$, $m = 1$, $w(t) = t^{1/q}$ and $v_j(t) = (1 - \ln t)^{-\alpha_j/q+\alpha_j-1}$ ($j = 0, 1$) with $0 < \alpha_0 < \alpha_1 < \infty$ in Theorem 6.2 (a), then we recover the interpolation formula in [4, Theorem 7]. Indeed, all the conditions necessary to apply Theorem 6.2 (a) are trivially met as $\sigma_j \approx v_j$.

Acknowledgements M.R. Formica is partially supported by University of Naples "Parthenope", Dept. of Economic and Legal Studies, project CoRNDiS, DM MUR 737/2021, CUP I55F21003620001. M.R. Formica is member of Gruppo Nazionale per l'Analisi Matematica, la Probabilitá e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and member of the UMI group "Teoria dell'Approssimazione e Applicazioni (T.A.A.)". We are grateful to the anonymous referees for their careful reading of the paper and for the suggestions which improved the original version of this paper.

Funding Open access funding provided by Università degli Studi di Napoli Federico II within the CRUI-CARE Agreement.

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