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Resolvent Estimates for Time-Harmonic Maxwell's Equations in the Partially Anisotropic Case

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Abstract
We prove resolvent estimates in L^p-spaces for time-harmonic Maxwell's equations in two spatial dimensions and in three dimensions in the partially anisotropic case. In the two-dimensional case the estimates are sharp up to endpoints. We consider anisotropic permittivity and permeability, which are both taken to be time-independent and spatially homogeneous. For the proof we diagonalize time-harmonic Maxwell's equations to equations involving Half-Laplacians. We apply these estimates to infer a Limiting Absorption Principle in intersections of L^p-spaces and to localize eigenvalues for perturbations by potentials.

Keywords Resolvent estimates · Maxwell's equations · Limiting Absorption Principle

Mathematics Subject Classification Primary 47A10 · Secondary 35Q61

1 Introduction and Main Results

Maxwell's equations describe electromagnetic waves and consequently the propagation of light. We refer to the physics' literature for further query (cf. [9, 20]). Time-dependent Maxwell's equations in media in three spatial dimensions relate *electric and magnetic field* $(\mathcal{E}, \mathcal{B}) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^3 \times \mathbb{C}^3$ with *displacement and magnetizing fields* $(\mathcal{D}, \mathcal{H}) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^3 \times \mathbb{C}^3$, the *electric and magnetic current* $(\mathcal{J}_e, \mathcal{J}_m) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^3 \times \mathbb{C}^3$, and *electric and magnetic charges*

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$(\rho_e, \rho_m) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C} \times \mathbb{C}$:

$$\begin{cases} \partial_t \mathcal{D} = \nabla \times \mathcal{H} + \mathcal{J}_e, & \nabla \cdot \mathcal{D} = \rho_e, \quad \nabla \cdot \mathcal{B} = \rho_m, \\ \partial_t \mathcal{B} = -\nabla \times \mathcal{E} + \mathcal{J}_m. \end{cases} \tag{1}$$

In physical contexts, fields, currents and charges are real-valued, and the magnetic charge and current vanish. We consider possibly non-vanishing magnetic charge and current to highlight symmetry between the electric and magnetic field. Moreover, $\mathcal{J}_e$ and $\mathcal{J}_m$ are typically taken with opposite signs.

In the following we consider the time-harmonic, monochromatic ansatz

$$\begin{aligned} \mathcal{D}(t,x) &= e^{i\omega t} D(x), \quad \mathcal{H}(t,x) = e^{i\omega t} H(x), \\ \mathcal{J}_e(t,x) &= e^{i\omega t} J_e(x), \quad \mathcal{J}_m(t,x) = e^{i\omega t} J_m(x) \end{aligned} \tag{2}$$

with $\omega \in \mathbb{R}$. We supplement (1) with the material laws

$$\mathcal{D}(t,x) = \varepsilon \mathcal{E}(t,x), \quad \mathcal{B}(t,x) = \mu \mathcal{H}(t,x), \tag{3}$$

where $\varepsilon = \operatorname{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathbb{R}^{3\times 3}$, ε_i, $\mu \in \mathbb{R}_{>0}$. Requiring ε and μ to be symmetric and positive definite is a physically natural assumption. The fully anisotropic case

$$\varepsilon = \operatorname{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3), \quad \mu = \operatorname{diag}(\mu_1, \mu_2, \mu_3) \text{ with } \frac{\varepsilon_1}{\mu_1} \neq \frac{\varepsilon_2}{\mu_2} \neq \frac{\varepsilon_3}{\mu_3} \neq \frac{\varepsilon_1}{\mu_1}$$

is analyzed in joint work with Mandel [22], where we argue in detail how the analysis reduces in the general case to scalar μ (see also [21, p. 63]). Material laws with scalar μ are frequently used in optics (cf. [23, Section 2]). Then (1) becomes under (2) and (3) to relate E with D and H with B:

$$P(\omega, D) \begin{pmatrix} D \\ B \end{pmatrix} = \begin{pmatrix} J_e \\ J_m \end{pmatrix}, \quad P(\omega, D) = \begin{pmatrix} i\omega & -\mu^{-1}\nabla\times \\ \nabla \times (\varepsilon^{-1}\cdot) & i\omega \end{pmatrix}. \tag{4}$$

(2) can be explained by considering (1) under Fourier transforms in time: Letting

$$\mathcal{D}(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega t} D(\omega, x) d\omega, \quad \mathcal{H}(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega t} H(\omega, x) d\omega, \dots,$$

we find a solution to (1) provided that $D(\omega, \cdot)$,... solve (4). We focus on solenoidal currents, but shall also consider the effect of non-vanishing divergence. We deduce from the continuity equation for electric charges $\partial_t \rho_e(t,x) - \nabla \cdot \mathcal{J}_e(t,x) = 0$ the following relation between $J_e(\omega, \cdot)$ and the time-dependent charges:

$$\nabla \cdot J_e(\omega, x) = i\omega \int_{\mathbb{R}} e^{-i\omega t} \rho_e(t,x) dt.$$

Since ω will be fixed in the following analysis of the time-harmonic equation, we let

$$\rho_e(x) = \nabla \cdot J_e(x) \text{ and } \rho_m(x) = \nabla \cdot J_m(x). \tag{5}$$

We consider Maxwell's equations in two spatial dimensions and the partially anisotropic case in three dimensions. The time-dependent form of Maxwell's equations in two dimensions corresponds to electric and magnetic fields and currents of the form

$$\begin{aligned} \mathcal{E}_i(t,x) &= \mathcal{E}_i(t,x_1,x_2), \quad i=1,2; \quad \mathcal{E}_3 = 0; \\ \mathcal{B}_i &= 0, \quad i=1,2; \quad \mathcal{B}_3(t,x) = \mathcal{B}_3(t,x_1,x_2); \\ \mathcal{J}_{ei}(t,x) &= \mathcal{J}_{ei}(t,x_1,x_2), \quad i=1,2; \quad \mathcal{J}_{e3} = 0; \\ \mathcal{J}_{mi}(t,x) &= 0, \quad i=1,2; \quad \mathcal{J}_{m3}(t,x) = \mathcal{J}_{m3}(t,x_1,x_2). \end{aligned}$$

(1) simplifies to (cf. [3]):

$$\begin{cases} \partial_t \mathcal{D} = \nabla_\perp \mathcal{H} + \mathcal{J}_e, \quad \nabla \cdot \mathcal{D} = \rho_e, \\ \partial_t \mathcal{B} = -\nabla \times \mathcal{E} + \mathcal{J}_m, \end{cases} \tag{6}$$

where $\mathcal{D}, \mathcal{E}, \mathcal{J}_e : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}^2$, $\mathcal{B}, \mathcal{H}, \mathcal{J}_m : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$, $\nabla_\perp = (\partial_2, -\partial_1)^t$, and we assume (3) with $\mu > 0$, and $(\varepsilon^{ij})_{i,j} \in \mathbb{R}^{2\times 2}$ denoting a symmetric, positive definite matrix. We can rewrite (6) under (2) and (3) as

$$P(\omega, D) \begin{pmatrix} D \\ B \end{pmatrix} = \begin{pmatrix} J_e \\ J_m \end{pmatrix}, \quad P(\omega, D) = \begin{pmatrix} i\omega & 0 & -\mu^{-1}\partial_2 \\ 0 & i\omega & \mu^{-1}\partial_1 \\ \partial_1 \varepsilon_{21} - \partial_2 \varepsilon_{11} & \partial_1 \varepsilon_{22} - \partial_2 \varepsilon_{12} & i\omega \end{pmatrix} \tag{7}$$

denoting with ε_{ij} the components of the inverse of ε. In two dimensions, we let

$$\rho_e = \partial_1 J_e + \partial_2 J_e \text{ and } \rho_m = 0. \tag{8}$$

In the following let $d \in \{2,3\}$, $m(2) = 3$, $m(3) = 6$, and

$$\begin{aligned} L^p_0(\mathbb{R}^2) &= \{(f_1, f_2, f_3) \in L^p(\mathbb{R}^2)^3 \,:\, \partial_1 f_1 + \partial_2 f_2 = 0 \text{ in } \mathcal{S}'(\mathbb{R}^2)\}, \\ L^p_0(\mathbb{R}^3) &= \{(f_1, \dots, f_6) \in L^p(\mathbb{R}^3)^6 \,:\, \nabla \cdot (f_1, f_2, f_3) = \nabla \cdot (f_4, f_5, f_6) \\ &= 0 \text{ in } \mathcal{S}'(\mathbb{R}^3)\}. \end{aligned}$$

In this paper we are concerned with the resolvent estimates

$$\|(D,B)\|_{L^q_0(\mathbb{R}^d)} = \|P(\omega,D)^{-1}(J_e,J_m)\|_{L^q_0(\mathbb{R}^d)} \lesssim \kappa_{p,q}(\omega) \|(J_e,J_m)\|_{L^p_0(\mathbb{R}^d)}. \tag{9}$$

However, as will be clear from perceiving $P(\omega, D)$ as a Fourier multiplier, $P(\omega, D)^{-1}$ cannot even be understood in the distributional sense for $\omega \in \mathbb{R}$. The remedy will be to consider $\omega \in \mathbb{C}\backslash\mathbb{R}$ and prove estimates independent of the distance to the real axis. Then we can consider limits $\Im(\omega) \downarrow 0$ and $\Im(\omega) \uparrow 0$. This is presently referred to as Limiting Absorption Principle (LAP) in the L^p-L^q-topology. Moreover, the analysis yields explicit formulae for the resulting limits. It appears that this is the first contribution to resolvent estimates for the Maxwell operator in anisotropic media in the L^p-L^q-topology.

Recently, Cossetti–Mandel analyzed the isotropic[1], possibly spatially inhomogeneous case $\varepsilon, \mu \in W^{1,\infty}(\mathbb{R}^3; \mathbb{R}_{>0})$ in [5]. In the isotropic case, iterating (1) and using the divergence conditions yields Helmholtz-like equations for D and H. This approach was carried out in [5]. In the anisotropic case this strategy becomes less straight-forward. Instead we choose to diagonalize the Fourier multiplier to get into the position to use resolvent estimates for the fractional Laplacian. Kwon–Lee–Seo [19] previously used a diagonalization to prove resolvent estimates for the Lamé operator. However, there are degenerate components in the diagonalization of time-harmonic Maxwell's operators, which do not occur for the Lamé operator. We use the divergence condition to ameliorate the contribution of the degeneracies. In case the currents have non-vanishing divergence, we can quantify this contribution with the charges.

We digress for a moment to elaborate on L^p-L^q-estimates for the fractional Laplacian and applications. Let $s \in (0, d)$. For $\omega \in \mathbb{C}\backslash[0, \infty)$ we consider the resolvents as Fourier multiplier:

$$((-\Delta)^{s/2} - \omega)^{-1} f = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{f}(\xi)}{\|\xi\|^s - \omega} e^{ix.\xi} d\xi \tag{10}$$

for $f : \mathbb{R}^d \to \mathbb{C}$ in some suitable a priori class, e.g., $f \in \mathcal{S}(\mathbb{R}^d)$. In the present context, resolvent estimates for the Half-Laplacian $\|((-\Delta)^{\frac{1}{2}} - \omega)^{-1}\|_{p \to q}$ are most important. There is a huge body of literature on resolvent estimates for the Laplacian $(-\Delta - \omega)^{-1} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$. This is due to versatile applications to uniform Sobolev estimates and unique continuation (cf. [17]), the localization of eigenvalues for Schrödinger operators with complex potential (cf. [6, 10, 11]), or LAPs in L^p-spaces (cf. [14]). Kenig–Ruiz–Sogge [17] showed that uniform resolvent estimates in $\omega \in \mathbb{C}\backslash[0, \infty)$ for $d \geq 3$ hold if and only if

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{d} \text{ and } \frac{2d}{d+3} < p < \frac{2d}{d+1}. \tag{11}$$

By homogeneity and scaling, we find

$$\|(-\Delta - \omega)^{-1}\|_{p \to q} = |\omega|^{-1+\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\big(-\Delta - \frac{\omega}{|\omega|}\big)^{-1}\|_{p \to q} \quad \forall \omega \in \mathbb{C}\backslash[0, \infty). \tag{12}$$

Thus, it suffices to consider $|\omega| = 1$ to discuss boundedness. Kwon–Lee [18] showed the currently widest range of resolvent estimates for the fractional Laplacian outside the uniform boundedness range (see [15] for a previous contribution). To state the

[1] In the isotropic case we identify $\varepsilon = \lambda 1_{3\times3}$ with $\lambda \in \mathbb{R}_{>0}$ and do likewise for μ.

range of admissible L^p-L^q-estimates, we shall use notations from [18]. Let $I^2 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1\}$, and let $(x, y)' = (1 - x, 1 - y)$ for $(x, y) \in I^2$. For $\mathcal{R} \subseteq I^2$ we set $\mathcal{R}' = \{(x, y)' \mid (x, y) \in \mathcal{R}\}$.

The resolvent of the fractional Laplacian $((-\Delta)^{\frac{s}{2}} - z)^{-1}$ is bounded for fixed $z \in \mathbb{C}\backslash[0, \infty)$ if and only if $(1/p, 1/q) \in \mathcal{R}_0^{\frac{s}{2}}$ with

$$\mathcal{R}_0^{\frac{s}{2}} = \mathcal{R}_0^{\frac{s}{2}}(d) = \{(x, y) \in I^2 \mid 0 \leq x - y \leq \frac{s}{d}\}\backslash\{(1, \frac{d-s}{d}), (\frac{s}{d}, 0)\};$$

see, e.g., [18, Proposition 6.1]. Gutiérrez showed in [14] that uniform estimates for $\omega \in \{z \in \mathbb{C} : |z| = 1, z \neq 1\}$ hold if and only if $(1/p, 1/q)$ lies in the set

$$\mathcal{R}_1 = \mathcal{R}_1(d) = \{(x, y) \in \mathcal{R}_0^1(d) : \frac{2}{d+1} \leq x - y \leq \frac{2}{d}, \ x > \frac{d+1}{2d}, \ y < \frac{d-1}{2d}\}. \tag{13}$$

Failure outside this range was known before (cf. [4, 17]) due to the connection to Bochner-Riesz operators with negative index. Clearly, there are more estimates available outside $\mathcal{R}_1$ if one allows for dependence on ω, e.g.,

$$\|(-\Delta - \omega)^{-1}\|_{L^2 \to L^2} \sim \mathrm{dist}(\omega, [0, \infty))^{-1}.$$

Kwon–Lee [18] analyzed estimates outside the uniform boundedness range in detail and covered a wide range. Estimates with dependence on ω can be used to localize eigenvalues for Schrödinger operators with complex potentials (cf. [6]), which is done for Maxwell operators in Sect. 4.

Diagonalizing the symbol of (4) to operators involving the Half-Laplacian works in the *partially anisotropic case*, i.e.,

$$\#\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} \leq 2. \tag{14}$$

This includes the isotropic case $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$, for which the results of Cossetti–Mandel [5] are recovered for constant coefficients, albeit via a different approach. It turns out that in the fully anisotropic case

$$\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3 \neq \varepsilon_1,$$

diagonalizing the multiplier introduces singularities, and this case has to be treated differently (cf. [22]). The estimates proved in [22] for the fully anisotropic case are strictly weaker than in the partially anisotropic case. We connect resolvent bounds for the Maxwell operator with resolvent estimates for the Half-Laplacian:

Theorem 1.1 *Let $1 < p, q < \infty$, $d \in \{2, 3\}$, and $\omega \in \mathbb{C}\backslash\mathbb{R}$. Let $\varepsilon \in \mathbb{R}^{d \times d}$ denote a symmetric positive definite matrix, and let $P(\omega, D)$ as in (7) for $d = 2$, and as in (4) for $d = 3$. For $d = 3$, we assume that $\varepsilon = diag(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and satisfies (14).*

Then, $P(\omega, D)^{-1} : L^p_0(\mathbb{R}^d) \to L^q_0(\mathbb{R}^d)$ is bounded if and only if $(1/p, 1/q) \in \mathcal{R}_0^{\frac{1}{2}}(d)$, and we find the estimate

$$\|P(\omega, D)^{-1}\|_{L^p_0 \to L^q_0} \sim \|((-\Delta)^{\frac{1}{2}} - \omega)^{-1}\|_{L^p \to L^q} + \|((-\Delta)^{\frac{1}{2}} + \omega)^{-1}\|_{L^p \to L^q} \tag{15}$$

to hold.
If $1 \leq p \leq \infty$ and $1 < q < \infty$, then we find the estimate

$$\begin{aligned} &\|P(\omega, D)^{-1}(J_e, J_m)\|_{L^q} \\ &\quad \lesssim (\|((-\Delta)^{\frac{1}{2}} - \omega)^{-1}\|_{L^p \to L^q} + \|((-\Delta)^{\frac{1}{2}} + \omega)^{-1}\|_{L^p \to L^q})\|(J_e, J_m)\|_{L^p} \\ &\quad\quad + \|(-\Delta)^{-\frac{1}{2}} \rho_e\|_{L^q} + \|(-\Delta)^{-\frac{1}{2}} \rho_m\|_{L^q} \end{aligned} \tag{16}$$

to hold with ρ_e and ρ_m defined as in (8) for $d = 2$ and (5) for $d = 3$. If $\rho_e = \rho_m = 0$, $1 < p < \infty$, and $q \in \{1, \infty\}$, then (16) also holds.

We cannot allow for $p \in \{1, \infty\}$ or $q \in \{1, \infty\}$ in the proof of

$$\|P(\omega, D)^{-1}\|_{L^p_0 \to L^q_0} \gtrsim \|((-\Delta)^{\frac{1}{2}} - \omega)^{-1}\|_{L^p \to L^q} + \|((-\Delta)^{\frac{1}{2}} + \omega)^{-1}\|_{L^p \to L^q}$$

as multiplier bounds for Riesz transforms are involved. It is well-known that the Riesz transforms are bounded on $L^p(\mathbb{R}^d)$, $1 < p < \infty$, but neither on L^1 nor on L^∞. In the proof of (16) for $\rho_e = \rho_m = 0$, which covers the reverse estimate of the above display, we can overcome this possibly technical issue by arranging the Riesz transforms acting on a reflexive L^p-space. Hence, we can allow for either $p \in \{1, \infty\}$ or $q \in \{1, \infty\}$. For the sake of simplicity, in Corollary 1.2 we only consider $1 < p, q < \infty$ although (16) partially extends to $p \in \{1, \infty\}$ or $q \in \{1, \infty\}$.

Coming back to resolvent estimates for the Half-Laplacian, for $d \in \{2, 3\}$ and $(1/p, 1/q) \in I^2$, define

$$\gamma_{p,q} = \gamma_{p,q}(d) = \max\{0, 1 - \frac{d+1}{2}(\frac{1}{p} - \frac{1}{q}), \frac{d+1}{2} - \frac{d}{p}, \frac{d}{q} - \frac{d-1}{2}\}.$$

Set

$$\begin{aligned} \kappa^{(\frac{1}{2})}_{p,q}(\omega) &= |\omega|^{-1+d\left(\frac{1}{p}-\frac{1}{q}\right)+\gamma_{p,q}} \operatorname{dist}(\omega, [0, \infty))^{-\gamma_{p,q}}, \\ \kappa_{p,q}(\omega) &= |\omega|^{-1+d\left(\frac{1}{p}-\frac{1}{q}\right)+\gamma_{p,q}} \operatorname{dist}(\omega, \mathbb{R})^{-\gamma_{p,q}}. \end{aligned}$$

Kwon–Lee [18, Conjecture 3, p. 1462] conjectured for $(1/p, 1/q) \in \mathcal{R}_0^{1/2}(d)$

$$\kappa^{(\frac{1}{2})}_{p,q}(\omega) \sim_{p,q,d} \|((-\Delta)^{1/2} - \omega)^{-1}\|_{p \to q}. \tag{17}$$

They verified the conjecture for $d = 2$ and for $d = 3$ in the restricted range in $\tilde{\mathcal{R}}_0^{1/2}(3)$ [18, Theorem 6.2, p. 1462]. We refer to [18] for the precise description. For notational convenience, let $\tilde{\mathcal{R}}_0^{1/2}(2) = \mathcal{R}_0^{1/2}(2)$. By invoking the results from [18], we find the following:

Corollary 1.2 *Let $1 < p, q < \infty$, $d \in \{2, 3\}$, and $\omega \in \mathbb{C}\backslash\mathbb{R}$. Let $\varepsilon \in \mathbb{R}^{d\times d}$ and $P(\omega, D)$ be as in Theorem 1.1. Then we find the following:*

1. *If $d = 2$, then*

$$\|P(\omega, D)^{-1}\|_{L_0^p(\mathbb{R}^d)\to L_0^q(\mathbb{R}^d)} \sim \kappa_{p,q}(\omega) \tag{18}$$

   is true for $(1/p, 1/q) \in \mathcal{R}_0^{\frac{1}{2}}(2)$.
2. *If $d = 3$ with ε satisfying (14), then (18) is true for $(1/p, 1/q) \in \tilde{\mathcal{R}}_0^{\frac{1}{2}}(3)$.*

Turning to LAPs, we work with the following notions:

Definition 1.3 Let $d \in \{2, 3\}$, $1 \leq p, q \leq \infty$, $\omega \in \mathbb{R}\backslash 0$, and $0 < \delta < 1/2$. We say that a global L_0^p-L_0^q-LAP holds if $P(\omega \pm i\delta, D)^{-1} : L_0^p(\mathbb{R}^d) \to L_0^q(\mathbb{R}^d)$ are bounded uniformly in $\delta > 0$, and there are operators $P_\pm(\omega) : L_0^p(\mathbb{R}^d) \to L_0^q(\mathbb{R}^d)$ such that

$$P(\omega \pm i\delta, D)^{-1} f \to P_\pm(\omega) f \text{ as } \delta \to 0 \text{ in } (\mathcal{S}'(\mathbb{R}^d))^{m(d)}. \tag{19}$$

We say that a local L_0^p-L_0^q-LAP holds if for any $\beta \in C_c^\infty(\mathbb{R}^d)$, $P(\omega \pm i\delta, D)^{-1}\beta(D) : L_0^p(\mathbb{R}^d) \to L_0^q(\mathbb{R}^d)$ are bounded uniformly in $\delta > 0$, and there are operators $P_\pm^{loc}(\omega) : L_0^p(\mathbb{R}^d) \to L_0^q(\mathbb{R}^d)$ such that

$$P(\omega \pm i\delta, D)^{-1}\beta(D) f \to P_\pm^{loc}(\omega) f \text{ in } \mathcal{S}'(\mathbb{R}^d)^{m(d)}. \tag{20}$$

Remark 1.4 By the explicit formulae for $P(\omega, D)^{-1}$ for $\omega \in \mathbb{C}\backslash\mathbb{R}$ we can also handle currents with non-vanishing divergence as in Theorem 1.1. We omitted this discussion for the sake of brevity.

We observe that $\gamma_{p,q} > 0$ for p and q as in Corollary 1.2:

Corollary 1.5 *Let $d \in \{2, 3\}$. For $1 < p, q < \infty$, $(1/p, 1/q) \in \tilde{\mathcal{R}}_0^{\frac{1}{2}}(d)$, there is no global L_0^p-L_0^q-LAP for (7) or (4).*

We show a local L_0^p-L_0^q-LAP for the Maxwell operator in Proposition 3.2. Roughly speaking, for low frequencies the resolvent estimates are equivalent to resolvent estimates for the Laplacian, and uniform estimates $L^{p_1} \to L^q$ are possible for $(1/p_1, 1/q) \in \mathcal{P}(d)$ (see Sect. 3). For the high frequencies, away from the singular set, the multiplier is smooth, but provides merely the smoothing of the Half-Laplacian. We use different $L^{p_2} \to L^q$-estimates for this region. This gives $L^{p_1} \cap L^{p_2} \to L^q$-estimates, which are uniform in ω in a compact set away from the origin, and an LAP in the same spaces. The necessity of considering currents in intersections of L^p-spaces is shown in Corollary 1.5. Below for $s \geq 0$ and $1 < q < \infty$, $W^{s,q}(\mathbb{R}^d)$ denotes the L^q-based Sobolev space:

$$W^{s,q}(\mathbb{R}^d) = \{f \in L^q(\mathbb{R}^d) : (1-\Delta)^{s/2} f \in L^q\} \text{ and } \|f\|_{W^{s,q}} := \|(1-\Delta)^{s/2} f\|_{L^q}.$$

Theorem 1.6 (LAP for Time-Harmonic Maxwell's equations) *Let $1 \leq p_1, p_2, q \leq \infty$, and let $d \in \{2,3\}$. If $(1/p_1, 1/q) \in \mathcal{P}(d)$, $(1/p_2, 1/q) \in \mathcal{R}_0^{\frac{1}{2}}(d)$, then $P(\omega, D)^{-1} : L_0^{p_1}(\mathbb{R}^d) \cap L_0^{p_2}(\mathbb{R}^d) \to L_0^q(\mathbb{R}^d)$ is bounded uniformly for $\omega \in \mathbb{C}\backslash\mathbb{R}$ in a compact set away from the origin. Furthermore, for $\omega \in \mathbb{R}\backslash 0$ there are limiting operators $P_\pm(\omega) : L_0^{p_1}(\mathbb{R}^d) \cap L_0^{p_2}(\mathbb{R}^d) \to L_0^q(\mathbb{R}^d)$ with*

$$P(\omega \pm i\delta, D)^{-1}(J_e, J_m) \to P_\pm(\omega)(J_e, J_m) \text{ in } (\mathcal{S}'(\mathbb{R}^d))^{m(d)} \text{ as } \delta \downarrow 0$$

such that $(D, B) = P_\pm(\omega)(J_e, J_m)$ satisfy

$$P(\omega, D)(D, B) = (J_e, J_m) \text{ in } (\mathcal{S}'(\mathbb{R}^d))^{m(d)}. \tag{21}$$

Additionally, if $q < \infty$, and $s \in [1, \infty)$, then

$$\|(D, B)\|_{(W^{s,q}(\mathbb{R}^d))^{m(d)}} \lesssim \|(J_e, J_m)\|_{(W^{s-1,q}(\mathbb{R}^d))^{m(d)} \cap L_0^{p_1}(\mathbb{R}^d)}. \tag{22}$$

Previously, Picard–Weck–Witsch [26] showed an LAP in weighted L^2-spaces (cf. [1]). Since the results in [26] are proved via Fredholm's Alternative, the frequencies $\omega \in \mathbb{R}\backslash 0$ are assumed not to belong to a discrete set of eigenvalues. In [26] ε and μ are assumed to be positive-definite and isotropic, but allowed to depend on x as in [5]. Pauly [25] proved similar results as Picard–Weck–Witsch [26] in weighted L^2-spaces in the anisotropic case; see also [2, 24]. Much earlier, Eidus [8] already proved non-existence of eigenvalues of the Maxwell operator provided that ε and μ are sufficiently smooth short-range perturbations of the identity and satisfy a repulsivity condition. Recently, D'Ancona–Schnaubelt [7] proved global-in-time Strichartz estimates from resolvent estimates in weighted L^2-spaces.

It appears that in the present work the role of the Half-Laplacian is explicitly identified for the analysis of the Maxwell operator the first time. We note that in [27, 28], in joint work with R. Schnaubelt, we apply a similar diagonalization to show Strichartz estimates for time-dependent Maxwell's equations with rough coefficients. In these works, due to variable permittivity and permeability, the diagonalization is carried out with pseudo-differential operators, and the present role of the Half-Laplacian is played by the Half-Wave operator. Provided that suitable estimates for the Half-Laplacian with variable coefficients were at disposal, of which the author is not aware, it seems possible that the present approach extends to variable permittivity and permeability as well.

Outline of the Paper In Sect. 2 we diagonalize time-harmonic Maxwell's equations in Fourier space to reduce the resolvent estimates to estimates for the Half-Laplacian. We also give examples for lower resolvent bounds in terms of the Half-Laplacian. In Sect. 3 we argue how an LAP fails in L^p-spaces, but can be salvaged in intersections of L^p-spaces. In Sect. 4 we show how the ω-dependent resolvent estimates lead to localization of eigenvalues in the presence of potentials. We postpone technical computations to the Appendix, where we also give explicit solution formulae.

2 Reduction to Resolvent Estimates for the Half-Laplacian

Let $\omega \in \mathbb{C}\backslash\mathbb{R}$. We diagonalize $P(\omega, D)$ defined in (7) or in (4) in the partially anisotropic case. We shall see that the transformation matrices are essentially Riesz transforms. This allows to bound the resolvents with estimates for the Half-Laplacian. We will make repeated use of the Mikhlin–Hörmander multiplier theorem (cf. [12, Theorem 6.2.7, p. 446]):

Theorem 2.1 (Mikhlin–Hörmander) *Let $1 < p < \infty$ and $m : \mathbb{R}^n\backslash 0 \to \mathbb{C}$ be a bounded function that satisfies*

$$|\partial^\alpha m(\xi)| \leq D_\alpha |\xi|^{-|\alpha|} \qquad (\xi \in \mathbb{R}^n\backslash 0) \tag{23}$$

for $|\alpha| \leq \lfloor \frac{n}{2} \rfloor + 1$. Then, $\mathfrak{m}_p : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ given by $f \mapsto (m\hat{f})^\vee$ defines a bounded mapping with

$$\|\mathfrak{m}_p\|_{L^p \to L^p} \leq C_n \max(p, (p-1)^{-1})(A + \|m\|_{L^\infty}), \tag{24}$$

where

$$A = \max(D_\alpha,\ |\alpha| \leq \lfloor \frac{n}{2} \rfloor + 1).$$

As pointed out in [12], $m \in C^k(\mathbb{R}^n\backslash 0)$, $k \geq \lfloor \frac{n}{2} \rfloor + 1$ is an L^p-multiplier for $1 < p < \infty$, if it is zero-homogeneous, i.e., there is $\tau \in \mathbb{R}$ such that for any $\lambda > 0$ and $\xi \neq 0$, we have

$$m(\lambda \xi) = \lambda^{i\tau} m(\xi). \tag{25}$$

Differentiating the above display with respect to ξ, we obtain for $\lambda > 0$

$$\lambda^{|\alpha|} (\partial_\xi^\alpha m)(\lambda \xi) = \lambda^{i\tau} \partial_\xi^\alpha m(\xi)$$

and (23) is satisfied with $D_\alpha = \sup_{|\theta|=1} |\partial^\alpha m(\theta)|$.

2.1 Proof of Theorem 1.1 for $d = 2$

Let $u = (D_1, D_2, B)$. We denote $(\varepsilon^{-1})_{ij} = (\varepsilon_{ij})_{i,j}$. To reduce to estimates for the Half-Laplacian, we diagonalize the symbol associated with the operator defined in (7). We write $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$:

$$(P(\omega, D)u)\widehat{(\xi)} = p(\omega, \xi)\hat{u}(\xi) = i \begin{pmatrix} \omega & 0 & -\xi_2\mu^{-1} \\ 0 & \omega & \xi_1\mu^{-1} \\ \xi_1\varepsilon_{12} - \xi_2\varepsilon_{11} & \xi_1\varepsilon_{22} - \xi_2\varepsilon_{12} & \omega \end{pmatrix} \hat{u}(\xi). \tag{26}$$

Let $\|\xi\|^2_{\varepsilon'} = \langle \xi, \mu^{-1} \det(\varepsilon)^{-1}\varepsilon\xi \rangle$, $\xi' = \xi / \|\xi\|_{\varepsilon'}$, and define

$$e_{\pm}(\omega, D) : L^p(\mathbb{R}^2) \to L^q(\mathbb{R}^2), \quad (e_{\pm} f)\widehat{(\xi)} = \frac{1}{\omega \pm \|\xi\|_{\varepsilon'}} \hat{f}(\xi). \tag{27}$$

We have the following lemma on diagonalization:

Lemma 2.2 *For almost all $\xi \in \mathbb{R}^2$ there is a matrix $m(\xi) \in \mathbb{C}^{3\times 3}$ such that*

$$p(\omega, \xi) = m(\xi) d(\omega, \xi) m^{-1}(\xi)$$

with

$$d(\omega, \xi) = i diag(\omega, \omega - \|\xi\|_{\varepsilon'}, \omega + \|\xi\|_{\varepsilon'}). \tag{28}$$

Furthermore, the operators $m_{ij}(D)$ and $m^{-1}_{ij}(D)$ are L^p-bounded for $1 < p < \infty$.

Proof It is straight-forward to check that the eigenvalues are as in (28) with the eigenvectors at hand. We align the corresponding eigenvectors as columns to

$$m(\xi) = \begin{pmatrix} \varepsilon_{22}\xi'_1 - \varepsilon_{12}\xi'_2 & -\xi'_2\mu^{-1} & \xi'_2\mu^{-1} \\ \varepsilon_{11}\xi'_2 - \varepsilon_{12}\xi'_1 & \xi'_1\mu^{-1} & -\xi'_1\mu^{-1} \\ 0 & -1 & -1 \end{pmatrix} \tag{29}$$

and note that $\det m(\xi) = -1$ for $\xi \neq 0$. For the inverse matrix we compute

$$m^{-1}(\xi) = \begin{pmatrix} \mu^{-1}\xi'_1 & \mu^{-1}\xi'_2 & 0 \\ \frac{\xi'_1\varepsilon_{21} - \xi'_2\epsilon_{11}}{2} & \frac{\varepsilon_{22}\xi'_1 - \varepsilon_{21}\xi'_2}{2} & -\frac{1}{2} \\ \frac{\xi'_2\varepsilon_{11} - \xi'_1\varepsilon_{12}}{2} & \frac{\xi'_2\varepsilon_{12} - \xi'_1\varepsilon_{22}}{2} & -\frac{1}{2} \end{pmatrix}. \tag{30}$$

L^p-boundedness is immediate from Theorem 2.1 because the components of m and m^{-1} are zero-homogeneous and smooth away from the origin. □

In Proposition 5.2 we compute $p^{-1}(\omega,\xi)$ via this diagonalization. The diagonalization allows us to separate

$$p^{-1}(\omega,\xi) = M^2(A,B) + M_c^2 \tag{31}$$

with $M_c^2 v = 0$ for $\xi_1 v_1 + \xi_2 v_2 = 0$ and

$$A = \frac{1}{i(\omega - \|\xi\|_{\varepsilon'})}, \quad B = \frac{1}{i(\omega + \|\xi\|_{\varepsilon'})}. \tag{32}$$

We can finish the proof of Theorem 1.1 for $d = 2$:

Proof of Theorem 1.1, d=2 We begin with the lower bound in (15). For u with $\partial_1 u_1 + \partial_2 u_2 = 0$, we have

$$p^{-1}(\omega,\xi)\hat{u}(\xi) = M^2(A,B)\hat{u}(\xi).$$

The entries of $M^2(A,B)$ are linear combinations of $e_{\pm}(\omega,\xi)$ and ξ_i'. The operators

$$(\mathcal{R}_i^{\varepsilon'} f)\widehat{(\xi)} = \xi_i' \hat{f}(\xi)$$

are L^p-bounded for $1 < p < \infty$ with a constant only depending on p, ε, μ as the symbols are linear combinations of Riesz symbols after changes of variables. We find (see (27) for notations)

$$\|P(\omega,D)^{-1}\|_{L_0^p \to L_0^q} \lesssim \|e_+(\omega,D)\|_{L^p \to L^q} + \|e_-(\omega,D)\|_{L^p \to L^q} \tag{33}$$

for $1 \leq p,q \leq \infty$ with ($1 < p < \infty$ or $1 < q < \infty$). The reason we are not required to take $1 < p < \infty$ and $1 < q < \infty$ is that, if there is one reflexive L^p-space, then we can commute the Fourier multipliers after multiplying out the matrices such that the Riesz transforms act on a reflexive L^p-space.[2] This shows the lower bound in (15) for $d = 2$.

We turn to show the upper bound in (15), which is

$$\|P(\omega,D)^{-1}\|_{L_0^p \to L_0^q} \gtrsim \|e_+(\omega,D)\|_{L^p \to L^q} + \|e_-(\omega,D)\|_{L^p \to L^q} \tag{34}$$

for $1 < p,q < \infty$.

The operators $\mathcal{R}_j^{\varepsilon'}$ satisfy for $1 < p < \infty$

$$\|f\|_{L^p(\mathbb{R}^2)} \sim_{p,\varepsilon,\mu} \|\mathcal{R}_1^{\varepsilon'} f\|_{L^p(\mathbb{R}^2)} + \|\mathcal{R}_2^{\varepsilon'} f\|_{L^p(\mathbb{R}^2)}. \tag{35}$$

In fact, as already used above, $\|\mathcal{R}_j^{\varepsilon'} f\|_{L^p} \lesssim_{p,\varepsilon,\mu} \|f\|_{L^p}$ for $1 < p < \infty$ as a consequence of Theorem 2.1. Let $\chi_1, \chi_2 : \mathbb{R}/(2\pi\mathbb{Z}) \to [0,1]$ be a smooth partition

[2] I thank the referee for pointing this out.

of unity of the unit circle such that

$$\begin{cases} \chi_1(\theta) = 1 \text{ for } \theta \in [-\frac{\pi}{8}, \frac{\pi}{8}] \cup [\frac{7\pi}{8}, \frac{9\pi}{8}], \\ \chi_2(\theta) = 1 \text{ for } \theta \in [\frac{3\pi}{8}, \frac{5\pi}{8}] \cup [\frac{11\pi}{8}, \frac{13\pi}{8}]. \end{cases}$$

We extend χ_i to $\mathbb{R}^2\backslash 0$ by zero-homogeneity.

For the reverse bound in (35), we decompose $f = f_1 + f_2$ as $f_i = \chi_i(D) f$. Set $((\mathcal{R}_i^{\varepsilon'})^{-1} f)\widehat{}(\xi) = \frac{\|\xi\|_{\varepsilon'}}{\xi_i} \hat{f}(\xi)$. Note that $|\xi_i| \gtrsim \|\xi\|_{\varepsilon'}$ for $\xi \in \text{supp}(\hat{f_i})$. By Theorem 2.1, we find the estimate

$$\|\big(\mathcal{R}_i^{\varepsilon'}\big)^{-1} f_i\|_{L^p} \lesssim_{p,\varepsilon,\mu} \|f_i\|_{L^p}.$$

Consequently,

$$\|f\|_{L^p} \leq \|f_1\|_{L^p} + \|f_2\|_{L^p} \leq \sum_{i=1}^{2} \|\big(\mathcal{R}_i^{\varepsilon'}\big)^{-1} \mathcal{R}_i^{\varepsilon'} f_i\|_{L^p} \lesssim_{p,\varepsilon,\mu} \sum_{i=1}^{2} \|\mathcal{R}_i^{\varepsilon'} f_i\|_p.$$

With (35) in mind, we show (34) by considering the data

$$v = \Big(-2\mathcal{R}_2^{\varepsilon'} f \;\; 2\mathcal{R}_1^{\varepsilon'} f \;\; 0\Big)^t. \tag{36}$$

Clearly, $\partial_1 v_1 + \partial_2 v_2 = 0$. We compute

$$m^{-1}(D) v = \mu \left(0 \;\; 1 \;\; -1\right)^t f.$$

We further compute

$$P(\omega, D)^{-1} v = \Big(-\mathcal{R}_2^{\varepsilon'}(e_- + e_+) \;\; \mathcal{R}_1^{\varepsilon'}(e_- + e_+) \;\; \mu(-e_- + e_+)\Big)^t f,$$

and it follows by (35)

$$\begin{aligned} \|P(\omega, D)^{-1} v\|_{L^q} &\sim \|(e_-(\omega, D) + e_+(\omega, D)) f\|_{L^q} + \mu \|(e_-(\omega, D) \\ &\quad - e_+(\omega, D)) f\|_{L^q} \sim \|e_-(\omega, D) f\|_{L^q} + \|e_+(\omega, D) f\|_{L^q} \end{aligned}$$

as claimed. Since $\|v\|_{L^p} \sim \|f\|_{L^p}$, by choosing f suitably, we find

$$\|P(\omega, D)^{-1}\|_{L_0^p \to L_0^q} \gtrsim \max(\|e_-\|_{L^p \to L^q}, \|e_+\|_{L^p \to L^q}) \sim \|e_-\|_{L^p \to L^q} + \|e_+\|_{L^p \to L^q}.$$

Finally, we turn to (16), which reads for $d = 2$

$$\begin{aligned} \|P(\omega, D)^{-1}(J_e, J_m)\|_{L^q} &\lesssim (\|e_-(\omega, D)\|_{L^p \to L^q} + \|e_+(\omega, D)\|_{L^p \to L^q}) \|(J_e, J_m)\|_{L^p} \\ &\quad + \|(-\Delta)^{-\frac{1}{2}} \rho_e\|_{L^q}. \end{aligned} \tag{37}$$

We decompose writing $J = (J_e, J_m)$

$$P(\omega, D)^{-1} J = (M^2(A, B)\hat{J})^\vee + (M_c \hat{J})^\vee$$

as in (31). The arguments from above estimate the contribution of $(M^2(A, B)\hat{J})^\vee$. A computation yields

$$(M_c \hat{J})(\xi) = \begin{pmatrix} \varepsilon_{12}\xi_2' - \varepsilon_{22}\xi_1' \\ \varepsilon_{12}\xi_1' - \varepsilon_{11}\xi_2' \\ 0 \end{pmatrix} \frac{\hat{\rho}_e(\xi)}{\mu\omega\|\xi\|_{\varepsilon'}}$$

with $\rho_e = \partial_1 J_{e1} + \partial_2 J_{e2}$. From this follows

$$\|(M_c \hat{J})^\vee\|_{L^q} \lesssim \|(-\Delta)^{1/2}\rho_e\|_{L^q}$$

by L^q-boundedness of $\mathcal{R}_i^{\varepsilon'}$ for $1 < q < \infty$ and $\|\xi\|/\|\xi\|_{\varepsilon'}$ zero-homogeneous and smooth away from the origin. The proof is complete. □

2.2 Proof of Theorem 1.1 for $d = 3$

We consider $P(\omega, D)$ as in (4) with $\varepsilon = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $\mu > 0$. Here we consider the partially anisotropic case $a^{-1} = \varepsilon_1$; $\varepsilon_2 = \varepsilon_3 = b^{-1}$ and suppose that $\mu = 1$ without loss of generality, to which we can reduce by linear substitution. The computation also covers the isotropic case $a = b$, which was considered in [5]. For $\xi \in \mathbb{R}^3$ we denote

$$\begin{aligned} \|\xi\|^2 &= \xi_1^2 + \xi_2^2 + \xi_3^2, \quad & \|\xi\|_\varepsilon^2 &= b\xi_1^2 + a\xi_2^2 + a\xi_3^2, \\ \xi' &= \xi/\|\xi\|, & \tilde{\xi} &= \xi/\|\xi\|_\varepsilon. \end{aligned}$$

We write further

$$(\nabla \times u)\widehat{\ }(\xi) = -i\mathcal{B}(\xi)\hat{u}(\xi), \quad \mathcal{B}(\xi) = \begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}.$$

We have the following lemma on diagonalization:

Lemma 2.3 *For almost all $\xi \in \mathbb{R}^3$ there is a matrix $\tilde{m}(\xi) \in \mathbb{C}^{6\times 6}$ such that*

$$p(\omega, \xi) = \tilde{m}(\xi) d(\omega, \xi) \tilde{m}^{-1}(\xi)$$

with

$$d(\omega, \xi) = i\, diag(\omega, \omega, \omega - \sqrt{b}\|\xi\|, \omega + \sqrt{b}\|\xi\|, \omega - \|\xi\|_\varepsilon, \omega + \|\xi\|_\varepsilon).$$

Furthermore, the components of $\tilde{m}$ and $\tilde{m}^{-1}$ are L^p-bounded Fourier multipliers for $1 < p < \infty$.

Proof To verify that the diagonal entries of d are truly the eigenvalues of p, we record eigenvectors, which are normalized to zero-homogeneous entries. Eigenvectors to $i\omega$ are

$$v_1^t = \big(0, 0, 0, \xi_1', \xi_2', \xi_3'\big),$$
$$v_2^t = \Big(\frac{\tilde{\xi}_1}{a}, \frac{\tilde{\xi}_2}{b}, \frac{\tilde{\xi}_3}{b}, 0, 0, 0\Big).$$

Eigenvectors to $i\omega \mp i\sqrt{b}\|\xi\|$ are given by

$$v_3^t = \big(0, -\frac{\xi_3'}{\sqrt{b}}, \frac{\xi_2'}{\sqrt{b}}, -((\xi_2')^2 + (\xi_3')^2), \xi_1'\xi_2', \xi_1'\xi_3'\big),$$
$$v_4^t = \Big(0, \frac{\xi_3'}{\sqrt{b}}, -\frac{\xi_2'}{\sqrt{b}}, -((\xi_2')^2 + (\xi_3')^2), \xi_1'\xi_2', \xi_1'\xi_3'\Big).$$

Eigenvectors to $i\omega \mp i\|\xi\|_\varepsilon$ are given by

$$v_5^t = \big(\tilde{\xi}_2^2 + \tilde{\xi}_3^2, -\tilde{\xi}_1\tilde{\xi}_2, -\tilde{\xi}_1\tilde{\xi}_3, 0, -\tilde{\xi}_3, \tilde{\xi}_2\big),$$
$$v_6^t = \big(-(\tilde{\xi}_2^2 + \tilde{\xi}_3^2), \tilde{\xi}_1\tilde{\xi}_2, \tilde{\xi}_1\tilde{\xi}_3, 0, -\tilde{\xi}_3, \tilde{\xi}_2\big).$$

Set

$$m(\xi) = (v_1, \dots, v_6) \tag{38}$$

and

$$\alpha(\xi) = \frac{(\xi_2^2 + \xi_3^2)^{1/2}}{(\|\xi\|\|\xi\|_\varepsilon)^{\frac{1}{2}}} \text{ and } \delta = \frac{\|\xi\|}{\|\xi\|_\varepsilon}. \tag{39}$$

The determinant of $m(\xi)$ is computed in Lemma 5.1 in the Appendix. We have

$$|\det m(\xi)| \sim \alpha^4(\xi).$$

Furthermore, we find for $\alpha \neq 0$:

$$m^{-1}(\xi) =$$
$$\begin{pmatrix} 0 & 0 & 0 & \xi_1' & \xi_2' & \xi_3' \\ ab\tilde{\xi}_1 & ab\tilde{\xi}_2 & ab\tilde{\xi}_3 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{b}\|\xi\|}{2\|\xi\|_\varepsilon}\frac{\tilde{\xi}_3}{\tilde{\xi}_2^2+\tilde{\xi}_3^2} & \frac{\sqrt{b}\|\xi\|}{2\|\xi\|_\varepsilon}\frac{\tilde{\xi}_2}{\tilde{\xi}_2^2+\tilde{\xi}_3^2} & -1/2 & \frac{\xi_1'\xi_2'}{2(\xi_2'^2+\xi_3'^2)} & \frac{\xi_1'\xi_3'}{2(\xi_2'^2+\xi_3'^2)} \\ 0 & \frac{\sqrt{b}\|\xi\|}{2\|\xi\|_\varepsilon}\frac{\tilde{\xi}_3}{\tilde{\xi}_2^2+\tilde{\xi}_3^2} & -\frac{\sqrt{b}\|\xi\|}{2\|\xi\|_\varepsilon}\frac{\tilde{\xi}_2}{\tilde{\xi}_2^2+\tilde{\xi}_3^2} & -1/2 & \frac{\xi_1'\xi_2'}{2(\xi_2'^2+\xi_3'^2)} & \frac{\xi_1'\xi_3'}{2(\xi_2'^2+\xi_3'^2)} \\ a/2 & -\frac{b\tilde{\xi}_1\tilde{\xi}_2}{2(\tilde{\xi}_2^2+\tilde{\xi}_3^2)} & -\frac{b\tilde{\xi}_1\tilde{\xi}_3}{2(\tilde{\xi}_2^2+\tilde{\xi}_3^2)} & 0 & -\frac{\xi_3'\|\xi\|_\varepsilon}{2\|\xi\|(\xi_2'^2+\xi_3'^2)} & \frac{\|\xi\|_\varepsilon\xi_2'}{2\|\xi\|(\xi_2'^2+\xi_3'^2)} \\ -a/2 & \frac{b\tilde{\xi}_1\tilde{\xi}_2}{2(\tilde{\xi}_2^2+\tilde{\xi}_3^2)} & \frac{b\tilde{\xi}_1\tilde{\xi}_3}{2(\tilde{\xi}_2^2+\tilde{\xi}_3^2)} & 0 & -\frac{|\xi|_\varepsilon}{2\|\xi\|}\frac{\xi_3'}{(\xi_2'^2+\xi_3'^2)} & \frac{\|\xi\|_\varepsilon\xi_2'}{2\|\xi\|(\xi_2'^2+\xi_3'^2)} \end{pmatrix}.$$

Since $\alpha(\xi) \to 0$ as $|\xi_2| + |\xi_3| \to 0$, m becomes singular along the ξ_1-axis, and the entries of $m^{-1}(\xi)$ are no L^p-bounded Fourier multipliers anymore. This suggests to renormalize $v_3, \dots, v_6$ with $1/\alpha(\xi)$. We let

$$\tilde{m}(\xi) =$$
$$\begin{pmatrix} 0 & \frac{\tilde{\xi}_1}{a} & 0 & 0 & (\delta(\tilde{\xi}_2^2 + \tilde{\xi}_3^2))^{\frac{1}{2}} & -(\delta(\tilde{\xi}_2^2 + \tilde{\xi}_3^2))^{\frac{1}{2}} \\ 0 & \frac{\tilde{\xi}_2}{b} & -\frac{\xi_3'}{\sqrt{b}(\delta(\xi_2'^2+\xi_3'^2))^{\frac{1}{2}}} & \frac{\xi_3'}{\sqrt{b}(\delta(\xi_2'^2+\xi_3'^2))^{\frac{1}{2}}} & -\frac{\delta^{\frac{1}{2}}\tilde{\xi}_1\tilde{\xi}_2}{(\tilde{\xi}_2^2+\tilde{\xi}_3^2)^{1/2}} & \frac{\delta^{\frac{1}{2}}\tilde{\xi}_1\tilde{\xi}_2}{(\tilde{\xi}_2^2+\tilde{\xi}_3^2)^{\frac{1}{2}}} \\ 0 & \frac{\tilde{\xi}_3}{b} & \frac{\xi_2'}{\sqrt{b}(\delta(\xi_2'^2+\xi_3'^2))^{\frac{1}{2}}} & -\frac{\xi_2'}{\sqrt{b}(\delta(\xi_2'^2+\xi_3'^2))^{\frac{1}{2}}} & -\frac{\delta^{\frac{1}{2}}\tilde{\xi}_1\tilde{\xi}_3}{(\tilde{\xi}_2^2+\tilde{\xi}_3^2)^{\frac{1}{2}}} & \frac{\delta^{\frac{1}{2}}\tilde{\xi}_1\tilde{\xi}_3}{(\tilde{\xi}_2^2+\tilde{\xi}_3^2)^{\frac{1}{2}}} \\ \xi_1' & 0 & -\frac{(\xi_2'^2+\xi_3'^2)^{\frac{1}{2}}}{\delta^{\frac{1}{2}}} & -\frac{(\xi_2'^2+\xi_3'^2)^{\frac{1}{2}}}{\delta^{\frac{1}{2}}} & 0 & 0 \\ \xi_2' & 0 & \frac{\xi_1'\xi_2'}{(\delta(\xi_2'^2+\xi_3'^2))^{\frac{1}{2}}} & \frac{\xi_1'\xi_2'}{(\delta(\xi_2'^2+\xi_3'^2))^{\frac{1}{2}}} & -\frac{\delta^{\frac{1}{2}}\tilde{\xi}_3}{(\tilde{\xi}_2^2+\tilde{\xi}_3^2)^{\frac{1}{2}}} & -\frac{\delta^{\frac{1}{2}}\tilde{\xi}_3}{(\tilde{\xi}_2^2+\tilde{\xi}_3^2)^{\frac{1}{2}}} \\ \xi_3' & 0 & \frac{\xi_1'\xi_3'}{(\delta(\xi_2'^2+\xi_3'^2))^{\frac{1}{2}}} & \frac{\xi_1'\xi_3'}{(\delta(\xi_2'^2+\xi_3'^2))^{\frac{1}{2}}} & \frac{\tilde{\xi}_2\delta^{\frac{1}{2}}}{(\tilde{\xi}_2^2+\tilde{\xi}_3^2)^{\frac{1}{2}}} & \frac{\tilde{\xi}_2\delta^{\frac{1}{2}}}{(\tilde{\xi}_2^2+\tilde{\xi}_3^2)^{\frac{1}{2}}} \end{pmatrix}.$$

By Lemma 5.1, we have $\det(\tilde{m}) \sim 1$ if and only if $\xi \neq (\nu, 0, 0)$ for some $\nu \in \mathbb{R}$. Hence, $\tilde{m}$ and $\tilde{m}^{-1}$ are well-defined away from the ξ_1-axis. By Cramer's rule, we obtain $\tilde{m}(\xi)^{-1}$ from $m^{-1}(\xi)$ by modifying the rows 3-6:

$$\tilde{m}^{-1}(\xi) =$$
$$\begin{pmatrix} 0 & 0 & 0 & \xi_1' & \xi_2' & \xi_3' \\ ab\tilde{\xi}_1 & ab\tilde{\xi}_2 & ab\tilde{\xi}_3 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{b}\delta^{\frac{1}{2}}\tilde{\xi}_3}{2(\tilde{\xi}_2^2+\tilde{\xi}_3^2)^{\frac{1}{2}}} & \frac{\sqrt{b}\delta^{\frac{1}{2}}\tilde{\xi}_2}{2(\tilde{\xi}_2^2+\tilde{\xi}_3^2)^{\frac{1}{2}}} & -\frac{(\tilde{\xi}_2^2+\tilde{\xi}_3^2)^{\frac{1}{2}}}{2\delta^{\frac{1}{2}}} & \frac{\xi_1'\xi_2'\delta^{\frac{1}{2}}}{2(\xi_2'^2+\xi_3'^2)^{\frac{1}{2}}} & \frac{\xi_1'\xi_3'\delta^{\frac{1}{2}}}{2(\xi_2'^2+\xi_3'^2)^{\frac{1}{2}}} \\ 0 & \frac{\sqrt{b}\delta^{\frac{1}{2}}\tilde{\xi}_3}{2(\tilde{\xi}_2^2+\tilde{\xi}_3^2)^{\frac{1}{2}}} & -\frac{\sqrt{b}\delta^{\frac{1}{2}}\tilde{\xi}_2}{2(\tilde{\xi}_2^2+\tilde{\xi}_3^2)^{1/2}} & -\frac{(\tilde{\xi}_2^2+\tilde{\xi}_3^2)^{\frac{1}{2}}}{2\delta^{\frac{1}{2}}} & \frac{\delta^{\frac{1}{2}}\xi_1'\xi_2'}{2(\xi_2'^2+\xi_3'^2)^{\frac{1}{2}}} & \frac{\delta^{\frac{1}{2}}\xi_1'\xi_3'}{2(\xi_2'^2+\xi_3'^2)^{\frac{1}{2}}} \\ \frac{a(\tilde{\xi}_2+\tilde{\xi}_3^2)^{\frac{1}{2}}}{2\delta^{\frac{1}{2}}} & -\frac{b\tilde{\xi}_1\tilde{\xi}_2}{2(\delta(\tilde{\xi}_2^2+\tilde{\xi}_3^2))^{\frac{1}{2}}} & -\frac{b\tilde{\xi}_1\tilde{\xi}_3}{2(\delta(\tilde{\xi}_2^2+\tilde{\xi}_3^2))^{\frac{1}{2}}} & 0 & -\frac{\xi_3'}{2(\delta(\xi_2'^2+\xi_3'^2))^{\frac{1}{2}}} & \frac{\xi_2'}{2(\delta(\xi_2'^2+\xi_3'^2))^{\frac{1}{2}}} \\ -\frac{a(\tilde{\xi}_2+\tilde{\xi}_3^2)^{\frac{1}{2}}}{2\delta^{\frac{1}{2}}} & \frac{b\tilde{\xi}_1\tilde{\xi}_2}{2(\delta(\tilde{\xi}_2^2+\tilde{\xi}_3^2))^{\frac{1}{2}}} & \frac{b\tilde{\xi}_1\tilde{\xi}_3}{2(\delta(\tilde{\xi}_2^2+\tilde{\xi}_3^2))^{\frac{1}{2}}} & 0 & -\frac{\xi_3'}{2(\delta(\xi_2'^2+\xi_3'^2))^{\frac{1}{2}}} & \frac{\xi_2'}{2(\delta(\xi_2'^2+\xi_3'^2))^{\frac{1}{2}}} \end{pmatrix}.$$

Also by Cramer's rule, it is enough to check that the Fourier multipliers associated with the entries in $\tilde{m}$ are L^p-bounded, for which we use Theorem 2.1.

For the first and second column this is evident since these are Riesz transforms up to change of variables. We turn to the proof that the entries of $v_i/\alpha(\xi)$, $i = 3, \dots, 6$, are multipliers bounded in L^p for $1 < p < \infty$. This follows by writing them as products of zero-homogeneous functions, which are smooth away from the origin, and Riesz transforms in two variables. We give the details for the entries of $v_3/\alpha(\xi)$:

- $(v_3)_2/\alpha(\xi)$: We have to show that

$$\frac{\xi_3(\|\xi\| \|\xi\|_{\tilde{\varepsilon}})^{1/2}}{\|\xi\|(\xi_2^2 + \xi_3^2)^{1/2}} = \frac{\xi_3}{(\xi_2^2 + \xi_3^2)^{1/2}} \Big(\frac{\|\xi\|_\varepsilon}{\|\xi\|}\Big)^{1/2}$$

is a multiplier. This is the case because $\frac{i\xi_3}{(\xi_2^2+\xi_3^2)^{1/2}}$ is a Riesz transform in (x_2, x_3) and the second factor $\left(\frac{\|\xi\|_\varepsilon}{\|\xi\|}\right)^{1/2}$ is zero-homogeneous and smooth away from the origin, hence, in the scope of Theorem 2.1.

- $(v_3)_3/\alpha(\xi)$ is a multiplier by symmetry in ξ_2 and ξ_3 and the previous considerations.
- $(v_3)_4/\alpha(\xi)$: We find

$$\frac{(\xi_2^2+\xi_3^2)}{\|\xi\|^2(\xi_2^2+\xi_3^2)^{1/2}} \cdot (\|\xi\|\|\xi\|_\varepsilon)^{1/2} = \frac{(\xi_2^2+\xi_3^2)^{1/2}}{\|\xi\|} \cdot \left(\frac{\|\xi\|_\varepsilon}{\|\xi\|}\right)^{1/2}$$

to be a Fourier multiplier as it is zero-homogeneous and smooth away from the origin.

- $(v_3)_5/\alpha(\xi)$: Consider

$$\frac{\xi_1\xi_2}{\|\xi\|^2(\xi_2^2+\xi_3^2)^{1/2}}(\|\xi\|\|\xi\|_\varepsilon)^{1/2} = \frac{\xi_1}{\|\xi\|} \cdot \frac{\xi_2}{(\xi_2^2+\xi_3^2)^{1/2}} \cdot \left(\frac{\|\xi\|_\varepsilon}{\|\xi\|}\right)^{1/2},$$

which is again a Fourier multiplier because the first and third expression are zero-homogeneous and smooth in $\mathbb{R}^n\backslash 0$, the second is again a Riesz transform in two variables.

- $(v_3)_6/\alpha(\xi)$ can be handled like the previous case.

The remaining entries of $\tilde{m}$ are treated similarly, which completes the proof. □

Remark 2.4 To compute the eigenvalues from scratch, it is perhaps easiest to use the block structure of $p(\omega, \xi)$ to find

$$\det(p(\omega, \xi)) = \det(-\omega^2 1_{3\times 3} - \mathcal{B}^2(\xi)\varepsilon^{-1}).$$

Next, we can use the identity $\mathcal{B}^2(\xi) = -\|\xi\|^2 1_{3\times 3} + \xi \otimes \xi$, after which there seems to be no further simplification but to compute the determinant brutely. Note that $\det(i\lambda 1_{6\times 6} - p(\omega, \xi)) = \det(p(\lambda - \omega, \xi))$, which allows to find the eigenvalues from the zero locus of $\det(p(\omega, \xi))$.

We prove Theorem 1.1 for $d = 3$ following along the argument for $d = 2$. Proposition 5.3 in the Appendix provides a decomposition

$$p^{-1}(\omega, \xi) = M^3(A, B, C, D) + M_c^3 \tag{40}$$

with $M_c^3 v = 0$ for $\xi_1 v_1 + \xi_2 v_2 + \xi_3 v_3 = \xi_1 v_4 + \xi_2 v_5 + \xi_3 v_6 = 0$ and

$$A = \frac{1}{i(\omega - \|\xi\|_\varepsilon)}, \quad B = \frac{1}{i(\omega + \|\xi\|_\varepsilon)}, \quad C = \frac{1}{i(\omega - \|\xi\|)}, \quad D = \frac{1}{i(\omega + \|\xi\|)}.$$

Proof of Theorem 1.1, d=3 The estimate

$$\|P(\omega,D)^{-1}\|_{L_0^p(\mathbb{R}^3)\to L_0^q(\mathbb{R}^3)} \lesssim \|((-\Delta)^{\frac{1}{2}}-\omega)^{-1}\|_{L^p\to L^q} + \|((-\Delta)^{\frac{1}{2}}-\omega)^{-1}\|_{L^p\to L^q}$$

for $1 \leq p,q \leq \infty$ with ($1 < p < \infty$ or $1 < q < \infty$) follows from the same argument as in the two-dimensional case: The entries of $M^3(A,B,C,D)$ are linear combinations of A,B,C,D multiplied with components of $\tilde{m}$ and $\tilde{m}^{-1}$, which yield Fourier multipliers by Lemma 2.3.

Below let $\widehat{(\mathcal{R}_i f)}(\xi) = \frac{\xi_i}{\|\xi\|}\hat{f}(\xi)$. To show the lower bound for $1 < p,q < \infty$, we consider the following initial data:

$$J_e = \begin{pmatrix} 0 \\ -\mathcal{R}_3 f \\ \mathcal{R}_2 f \end{pmatrix}, \quad J_m = \underline{0}.$$

Note that $\nabla \cdot J_e = 0$ and again, the initial data is also physically meaningful as the magnetic current vanishes.

Let $\widehat{(e_\pm f)}(\xi) = (\omega \pm \sqrt{b}|\xi|)^{-1}\hat{f}(\xi)$. We compute with m as in (38):

$$(dm^{-1})(\xi)\begin{pmatrix} \hat{J}_e \\ \hat{J}_m \end{pmatrix} = \frac{\sqrt{b}}{2}\begin{pmatrix} 0 \\ 0 \\ \widehat{e_- f} \\ -\widehat{e_+ f} \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} D \\ B \end{pmatrix} = i\begin{pmatrix} 0 \\ -\mathcal{R}_3(e_- f + e_+ f) \\ \mathcal{R}_2(e_- f + e_+ f) \\ -((\mathcal{R}_2^2 + \mathcal{R}_3^2)(e_- f - e_+ f) \\ \mathcal{R}_1\mathcal{R}_2(e_- f - e_+ f) \\ \mathcal{R}_1\mathcal{R}_3(e_- f - e_+ f). \end{pmatrix} \tag{41}$$

We shall see that

$$\|(D,B)\|_{L_0^q} \gtrsim \|e_- f + e_+ f\|_{L^q} + \|e_- f - e_+ f\|_{L^q} \gtrsim \|e_- f\|_{L^q} + \|e_+ f\|_{L^q} \tag{42}$$

either, if f has frequency support in a conic neighbourhood of the ξ_3-axis, or, if f is spherically symmetric.

Assume that $g \in \mathcal{S}(\mathbb{R}^3)$ and

$$\mathrm{supp}(\hat{g}) \subseteq \{\xi \in \mathbb{R}^3 : |\xi/|\xi| - e_3| \leq c \ll 1 \text{ and } \frac{1}{2} \leq |\xi| \leq 2\} =: E.$$

By Theorem 2.1, we have for $1 < p < \infty$

$$\|g\|_{L^p} \lesssim \|\mathcal{R}_3 g\|_{L^p} \text{ and } \|\mathcal{R}_2 g\|_{L^p} \leq C(c)\|g\|_{L^p} \tag{43}$$

with $C(c) \to 0$ as $c \to 0$. If $\text{supp}(\hat{f}) \subseteq E$, then also the Fourier support of $e_- f \pm e_+ f$ is contained in E, and an application of (43) to D_2 and B_1 yields

$$\begin{aligned} \|(D,B)\|_{L_0^q} &\gtrsim \|\mathcal{R}_3(e_- f + e_+ f)\|_{L^q} + \|(\mathcal{R}_2^2 + \mathcal{R}_3^2)(e_- f - e_+ f)\|_{L^q} \\ &\gtrsim \|e_- f + e_+ f\|_{L^q} + \|e_- f - e_+ f\|_{L^q} \\ &\gtrsim \|e_- f\|_{L^q} + \|e_+ f\|_{L^q}, \end{aligned} \tag{44}$$

which is (42).

Next, suppose that $f \in L^p(\mathbb{R}^3)$, $1 < p < \infty$ is spherically symmetric. Since $\mathcal{R}_1^2 + \mathcal{R}_2^2 + \mathcal{R}_3^2 = Id$ and $\|\mathcal{R}_i^2 f\|_{L^p} = \|\mathcal{R}_j^2 f\|_{L^p}$ for $i, j \in \{1, 2, 3\}$ by change of variables and rotation symmetry, we find $\|\mathcal{R}_i^2 f\|_{L^p} \gtrsim \|f\|_{L^p}$. By L^p-boundedness, we have

$$\|f\|_{L^p} \lesssim \|\mathcal{R}_i^2 f\|_{L^p} \lesssim \|\mathcal{R}_i f\|_{L^p} \lesssim \|f\|_{L^p}. \tag{45}$$

Similarly,

$$(\mathcal{R}_1^2 + \mathcal{R}_2^2) + (\mathcal{R}_2^2 + \mathcal{R}_3^2) + (\mathcal{R}_1^2 + \mathcal{R}_3^2) = 2Id,$$

and $\|(\mathcal{R}_i^2 + \mathcal{R}_j^2) f\|_{L^p} = \|(\mathcal{R}_k^2 + \mathcal{R}_l^2) f\|_{L^p}$ again by change of variables and rotation symmetry. Hence, we also find

$$\|(\mathcal{R}_i^2 + \mathcal{R}_j^2) f\|_{L^p} \gtrsim \|f\|_{L^p}. \tag{46}$$

(45) and (46) together allow to argue as well in case of spherical symmetry as in (44). If we can choose f such that the operator norms of $e_\pm$ are approximated, we find

$$\|(D,H)\|_{L_0^q} \gtrsim (\|e_-\|_{L^p \to L^q} + \|e_+\|_{L^p \to L^q}) \|f\|_{L^p}.$$

Lastly, if $\text{supp}(\hat{f}) \subseteq E$, i.e., the frequency support is in a conic neighbourhood of the ξ_3-axis, or is spherically symmetric, we find $\|(J_e, J_m)\|_{L_0^p} \sim \|f\|_{L^p}$. To see that it suffices to consider the frequency support of f as such, we recall the examples from [18, Section 5.2], giving the claimed lower bound for the operator norm of the resolvent of the fractional Laplacian: a Knapp type example, which can be realized with frequency support in a conic neighbourhood of the ξ_3-axis [18, p. 1458], and a spherically symmetric example related with the surface measure on the sphere [18, p. 1459].

We turn to the proof of (16) for $d = 3$:

$$\begin{aligned} &\|P(\omega, D)^{-1}(J_e, J_m)\|_{L^q} \\ &\quad \lesssim (\|((-\Delta)^{\frac{1}{2}} - \omega)^{-1}\|_{L^p \to L^q} + \|((-\Delta)^{\frac{1}{2}} + \omega)^{-1}\|_{L^p \to L^q}) \|(J_e, J_m)\|_{L^p} \\ &\qquad + \|(-\Delta)^{-\frac{1}{2}} \rho_e\|_{L^q} + \|(-\Delta)^{-\frac{1}{2}} \rho_m\|_{L^q}. \end{aligned} \tag{47}$$

This hinges again on the decomposition

$$(P^{-1}(\omega, D)(J_e, J_m))^\wedge(\xi) = M^3(A, B, C, D)(\hat{J}_e, \hat{J}_m)(\xi) + M_c^3(\hat{J}_e, \hat{J}_m)(\xi).$$

The contribution of $M^3(A, B, C, D)$ is estimated like in the first part of the proof. We compute

$$\begin{aligned} &M_c^3(\hat{J}_e, \hat{J}_m)(\xi) \\ &= -\left(\frac{b\tilde{\xi}_1\hat{\rho}_e(\xi)}{\omega\|\xi\|_\varepsilon}, \frac{a\tilde{\xi}_2\hat{\rho}_e(\xi)}{\omega\|\xi\|_\varepsilon}, \frac{a\tilde{\xi}_3\hat{\rho}_e(\xi)}{\omega\|\xi\|_\varepsilon}, \frac{\xi_1'\hat{\rho}_m(\xi)}{\omega\|\xi\|}, \frac{\xi_2'\hat{\rho}_m(\xi)}{\omega\|\xi\|}, \frac{\xi_3'\hat{\rho}_m(\xi)}{\omega\|\xi\|}\right)^t. \end{aligned}$$

The claim follows by Theorem 2.1 because $\|\xi\|/\|\xi\|_\varepsilon$ and ξ_i' and $\tilde{\xi}_i$ are zero-homogeneous and smooth away from the origin. The proof of Theorem 1.1 is complete. □

3 Local and Global LAP

Let $P(\omega, D)$ be as in the previous section. In the following we want to investigate the limit of

$$P(\omega \pm i\delta, D)^{-1} f \text{ as } \delta \to 0, \quad \omega \in \mathbb{R}\backslash 0,$$

by which we construct solutions to time-harmonic Maxwell's equations. By scaling we see that the following estimates are uniform in ω, provided it varies in a compact set away from the origin. We further suppose that $\omega > 0$; the case $\omega < 0$ can be treated with the obvious modifications.

In the following let $0 < |\delta| < 1/2$. By the above diagonalization, it is equivalent to consider uniform boundedness of

$$e_\pm^{\varepsilon'}(\omega + i\delta) : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d), \quad (e_\pm^{\varepsilon'}(\omega + i\delta) f)\widehat{\ }(\xi) = \frac{\hat{f}(\xi)}{\|\xi\|_{\varepsilon'} \pm (\omega + i\delta)}.$$

Hence, by the results of the previous section, the uniform L_0^p-L_0^q-LAP fails due to the lack of uniform resolvent estimates for the Half-Laplacian in L^p-spaces. This is recorded in Corollary 1.5.

Regarding the local L_0^p-L_0^q-LAP, we observe that the operator

$$(e_+^{\varepsilon'}(\omega \pm i\delta) f)\widehat{\ }(\xi) = \frac{\beta(\xi)\hat{f}(\xi)}{\|\xi\|_{\varepsilon'} + (\omega \pm i\delta)}$$

for $\beta \in C_c^\infty$, $0 < \delta < 1/2$ is bounded from $L^p \to L^q$ for $1 \le p \le q \le \infty$ by Young's inequality, with the obvious limit as $\delta \to 0$. Thus, we focus on

$$(e_\delta f)\widehat{(\xi)} := (e_-(\omega \pm i\delta) f)\widehat{(\xi)} = \frac{\beta(\xi) \hat{f}(\xi)}{\|\xi\|_{\varepsilon'} - (\omega \pm i\delta)} \tag{48}$$

with $0 < \delta < \delta_0 \ll 1$, where $\beta \in C_c^\infty(\mathbb{R}^n)$.

We can be more precise about the limiting operators: For $t \in \mathbb{R}$ recall Sokhotsky's formula, which hold in the sense of distributions:

$$\lim_{\varepsilon \downarrow 0} \frac{1}{t \pm i\varepsilon} = v.p.\frac{1}{t} \mp i\pi \delta_0(t),$$

where δ_0 denotes the delta-distribution at the origin.
Let

$$\mathcal{R}_{\pm}^{loc} f = \lim_{\delta \to \pm 0} e_\delta f.$$

We find

$$\mathcal{R}_{\pm}^{loc} f = v.p. \int \frac{\beta(\xi) e^{ix\xi}}{\|\xi\|_{\varepsilon'} - \omega} \hat{f}(\xi) d\xi \pm i\pi \int e^{ix\xi} \beta(\xi) \delta(\|\xi\|_{\varepsilon'} - \omega) \hat{f}(\xi) d\xi,$$

and by the diagonalization formulae, we find that the limiting operators can be expressed as linear combinations involving possibly generalized Riesz transforms, $\mathcal{R}_{\pm}^{loc}$, and e_+. We recall the L^p-L^q-mapping properties of $\mathcal{R}_{\pm}^{loc}$.

We observe that

$$(\mathcal{R}_+^{loc} - \mathcal{R}_-^{loc}) f = 2\pi i \int_{\{\|\xi\|_{\varepsilon'} = 1\}} \beta(\xi) e^{ix\xi} \hat{f}(\xi) d\sigma(\xi).$$

This operator, modulo the bounded operator given by convolution with $\mathcal{F}^{-1}\beta$ and linear change of variables $\xi \to \zeta$ such that $\|\xi\|_{\varepsilon'} = \|\zeta\|$, is known as *restriction-extension operator* (cf. [16, 18]) and is a special case of the Bochner–Riesz operator of negative index:

$$(\mathcal{B}^\alpha f)\widehat{(\xi)} = \frac{1}{\Gamma(1-\alpha)} \frac{\hat{f}(\xi)}{(1 - \|\xi\|^2)_+^\alpha}, \quad 0 < \alpha \le \frac{d+2}{2},$$

$\mathcal{B}_\alpha$ is defined by analytic continuation for $\alpha \ge 1$. Hence, for $\alpha = 1$, it matches the restriction–extension operator. This operator is well-understood due to the works of Börjeson [4], Sogge [29], and Gutiérrez [13, 14]. The most recent results for Bochner–Riesz operators of negative index are due to Kwon–Lee [18]. Gutiérrez showed that $\mathcal{B}^1 : L^p \to L^q$ is bounded if and only if $(1/p, 1/q) \in \mathcal{P}(d)$ with

$$\mathcal{P}(d) = \left\{ (x, y) \in [0,1]^2 \,:\, x - y \ge \frac{2}{d+1},\ x > \frac{d+1}{2d},\ y < \frac{d-1}{2d} \right\}.$$

She used this to show uniform resolvent estimates for

$$(-\Delta - z)^{-1} : L^p \to L^q, \quad z \in \mathbb{S}^1 \backslash \{1\} \text{ for } (1/p, 1/q) \in \mathcal{R}_1(d).$$

We summarize the operator bounds for e_δ and $\mathcal{R}_{\pm}^{\mathrm{loc}}$.

Proposition 3.1 [18, Proposition 4.1] *Let $\omega > 0$, $0 < \delta < 1/2$, $\beta \in C_c^\infty(\mathbb{R}^d)$ and d_δ as in (48). Then, we find the following estimates to hold for $(1/p, 1/q) \in \mathcal{P}(d)$:*

$$\begin{aligned} \|e_\delta\|_{L^p \to L^q} &\leq C(\omega, p, q), \\ \Big\| \int_{\mathbb{R}^d} e^{ix.\xi} \delta(\|\xi\|_{\varepsilon'} - \omega) \hat{f}(\xi) d\xi \Big\|_{L^q} &\leq C(\omega, p, q) \|f\|_{L^p}, \\ \Big\| v.p. \int_{\mathbb{R}^d} e^{ix.\xi} \frac{\beta(\xi)}{\|\xi\|_{\varepsilon'} - \omega} \hat{f}(\xi) d\xi \Big\|_{L^q} &\leq C(\omega, p, q) \|f\|_{L^p}. \end{aligned} \tag{49}$$

We are ready for the proof of the local LAP:

Proposition 3.2 (Local LAP) *We find a local L_0^p-L_0^q-LAP to hold provided that $(1/p, 1/q) \in \mathcal{P}(d)$. This means that for $\omega \in \mathbb{R}\backslash 0$ and $\beta \in C_c^\infty(\mathbb{R}^d)$, we find uniform (in $0 < \delta < 1/2$) resolvent bounds*

$$\|P(\omega \pm i\delta, D)^{-1} \beta(D) f\|_{L_0^q(\mathbb{R}^d)} \lesssim_{p,q,d,\omega} \|f\|_{L_0^p(\mathbb{R}^d)} \tag{50}$$

and there are limiting operators $P_{\pm}^{loc} : L_0^p \to L_0^q$ such that

$$P(\omega \pm i\delta, D)^{-1} \beta(D) f \to P_{\pm}^{loc}(\omega) f \text{ in } (\mathcal{S}'(\mathbb{R}^d))^{m(d)}.$$

Proof We assume that $\omega > 0$ because $\omega < 0$ can be treated *mutatis mutandis*. Recall the bounds for e_δ recorded in Proposition 3.1, easier bounds for $e_{+}^{\varepsilon'}$, and the diagonalization from Sect. 2, which decompose (cf. Lemmas 2.2, 2.3)

$$p(\omega, \xi) = m(\xi) d(\omega, \xi) m^{-1}(\xi).$$

By these, (50) follows for $(1/p, 1/q) \in \mathcal{P}(d)$ provided that $1 < p, q < \infty$ to bound the generalized Riesz transforms. We extend this to all $(1/p, 1/q) \in \mathcal{P}(d)$ by Young's inequality: For $(1/p, 0) \in \mathcal{P}(d)$ we choose $1 < \tilde{q} < \infty$ such that $(1/p, 1/\tilde{q}) \in \mathcal{P}(d)$. By Young's inequality and the previously established bounds for $(1/p, 1/\tilde{q}) \in \mathcal{P}(d)$ follows

$$\|P(\omega \pm i\delta, D)^{-1} \beta(D) f\|_{L_0^\infty} \lesssim \|P(\omega \pm i\delta, D)^{-1} \beta(D) f\|_{L_0^{\tilde{q}}} \lesssim \|f\|_{L_0^p}.$$

The case $(1, 1/q) \in \mathcal{P}(d)$ is treated by the dual argument.

By Sokhotsky's formula and the diagonalization, we can consider the limiting operators

$$P_{\pm}(\omega, D) = \lim_{\delta \to 0} P(\omega \pm i\delta, D)^{-1} \beta(D) : L_0^p \to L_0^q$$

whose mapping properties follow again from Proposition 3.1 and the diagonalization as argued above. We give explicit formulae in Propositions 5.2 and 5.3; however, these are bulky and recorded in the Appendix. □

We are ready for the proof of Theorem 1.6:

Proof of Theorem 1.6 Let $1 \leq p_1, p_2, q \leq \infty$, and $\omega \in \mathbb{R}\backslash 0$. Choose $C = C(\varepsilon, \omega)$ such that $p(\omega, \xi)^{-1}$ is regular for $\|\xi\| \geq C$. Write $J = (J_e, J_m)$ for the sake of brevity. Let $\beta \in C_c^\infty$ with $\beta \equiv 1$ on $\{\|\xi\| \leq C\}$ and decompose

$$J = \beta(D)J + (1-\beta)(D)J =: J_{low} + J_{high}.$$

By Proposition 3.2, we find uniform bounds for $0 < \delta < 1/2$

$$\|P(\omega \pm i\delta, D)^{-1} J_{low}\|_{L_0^q} \lesssim \|J_{low}\|_{L_0^{p_1}}$$

provided that $(\frac{1}{p_1}, \frac{1}{q}) \in \mathcal{P}(d)$. The estimate

$$\|P(\omega \pm i\delta, D)^{-1} J_{high}\|_{L_0^q} \lesssim \|J_{high}\|_{L_0^{p_2}}$$

follows for $0 \leq \frac{1}{p_2} - \frac{1}{q} \leq \frac{1}{d}$ and $(\frac{1}{p_2}, \frac{1}{q}) \notin \{(\frac{1}{d}, 0), (1, \frac{d-1}{d})\}$ by properties of the Bessel kernel. The limiting operators $P_\pm^{loc}(\omega)$ were described in Proposition 3.2: We have

$$P(\omega \pm i\delta, D)^{-1}(J_e, J_m) \to P_\pm^{loc}(\omega)(J_e, J_m) \text{ in } \mathcal{S}'(\mathbb{R}^d)^{m(d)}.$$

The high frequency is limit is easier to analyze because the multiplier remains regular by construction. Let $M^d \in \mathbb{C}^{m(d)\times m(d)}$ be as in Propositions 5.2 and 5.3. For $d = 2$, let

$$A = \frac{1}{i(\omega - \|\xi\|_{\varepsilon'})}, \quad B = \frac{1}{i(\omega + \|\xi\|_{\varepsilon'})},$$

and we have

$$\begin{aligned} P(\omega \pm i\delta, D)^{-1} J_{high} \to \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix.\xi} M^2(A, B)(1-\beta(\xi))\hat{J}(\xi) d\xi \text{ in } (\mathcal{S}'(\mathbb{R}^2))^3 \\ =: P^{high}(\omega) J. \end{aligned} \tag{51}$$

For $d = 3$, let

$$A = \frac{1}{i(\omega - \sqrt{b}\|\xi\|)}, \quad B = \frac{1}{i(\omega + \sqrt{b}\|\xi\|)}, \quad C = \frac{1}{i(\omega - \|\xi\|_\varepsilon)}, \quad D = \frac{1}{i(\omega + \|\xi\|_\varepsilon)},$$

and we have with convergence in $(\mathcal{S}'(\mathbb{R}^3))^6$

$$P(\omega \pm i\delta, D)^{-1}(1-\beta(D))J \to \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{ix.\xi} M^3(A,B,C,D)(1-\beta(\xi))\hat{J}(\xi)d\xi \\ =: P^{high}(\omega)J. \tag{52}$$

Let $P_\pm(\omega) = P_\pm^{loc}(\omega) + P^{high}(\omega)$. By Proposition 3.2, and (51), (52), we have

$$P(\omega \pm i\delta, D)^{-1}J \to P_\pm^{loc}(\omega)J + P^{high}(\omega)J \text{ in } (\mathcal{S}'(\mathbb{R}^d))^{m(d)}.$$

Let $(D,B)_\delta^\pm = P(\omega \pm i\delta, D)^{-1}J$ and $(D,B)^\pm = P_\pm(\omega)J$. At last, we show that

$$P(\omega, D)(D,B)^\pm = J. \tag{53}$$

For this purpose, we show that for $\delta \to 0$ we have

$$P(\omega, D)(D,B)_\delta^\pm \to J \text{ in } \mathcal{S}'(\mathbb{R}^d)^{m(d)}. \tag{54}$$

As $(D,B)_\delta^\pm \to (D,B)^\pm$ in $\mathcal{S}'(\mathbb{R}^d)^{m(d)}$, (54) concludes the proof.
To show (54), we return to the diagonalizations (cf. Lemmas 2.2 and 2.3):

$$p(\tilde{\omega},\xi) = im(\xi)d(\tilde{\omega},\xi)m^{-1}(\xi) \text{ for } \tilde{\omega} \in \mathbb{C}.$$

We find for $\omega \in \mathbb{R}$:

$$\begin{aligned} p(\omega,\xi)p^{-1}(\omega \pm i\delta,\xi) &= m(\xi)d(\omega,\xi)d(\omega \pm i\delta,\xi)^{-1}m^{-1}(\xi) \\ &= m(\xi)(1_{m(d)\times m(d)} \pm \delta d(\omega \pm i\delta,\xi)^{-1})m^{-1}(\xi) \\ &= 1_{m(d)\times m(d)} \pm \delta p(\omega \pm i\delta,\xi)^{-1}. \end{aligned}$$

Hence,

$$\begin{aligned} P(\omega, D)(D,B)_\delta^\pm &= J \pm \delta P(\omega \pm i\delta, D)^{-1}J, \\ \|P(\omega,D)(D,B)_\delta^\pm - J\|_{L_0^q(\mathbb{R}^d)} &\lesssim \delta\|J\|_{L_0^{p_1}\cap L_0^{p_2}} \to 0. \end{aligned}$$

In particular, (54) holds true in $\mathcal{S}'(\mathbb{R}^d)^{m(d)}$.

Next, we suppose additionally that $J \in (W^{s-1,q}(\mathbb{R}^d))^{m(d)}$ for $s \geq 1$. By Young's inequality, we have

$$\|P(\omega \pm i\delta, D)^{-1}\beta(D)J\|_{(W^{s,q}(\mathbb{R}^d))^{m(d)}} \lesssim \|P(\omega \pm i\delta, D)^{-1}\beta(D)J\|_{L_0^q(\mathbb{R}^d)}.$$

Hence, the low frequencies can be estimated like before. For the high frequencies, we recall that the multipliers M^2 and M^3 yield smoothing of one derivative and by

Theorem 2.1, we find

$$\begin{aligned}
&\|P(\omega \pm i\delta, D)^{-1}(1-\beta(D))J\|_{(W^{s,q}(\mathbb{R}^d))^{m(d)}} \\
&\lesssim \|\frac{(1-\Delta)^{s/2}}{(1-\Delta)^{1/2}}(1-\beta(D))J\|_{(L^q(\mathbb{R}^d))^{m(d)}} \\
&\lesssim \|(1-\Delta)^{(s-1)/2}(1-\beta(D))J\|_{(L^q(\mathbb{R}^d))^{m(d)}} \\
&= \|(1-\beta(D))J\|_{(W^{s-1,q}(\mathbb{R}^d))^{m(d)}}.
\end{aligned}$$

The proof of Theorem 1.6 is complete. □

4 Localization of Eigenvalues

At last, we use the ω-dependent resolvent estimates to localize eigenvalues for operators $P(\omega, D) + V$ acting in L^q. For this purpose, we consider for $\ell > 0$ and $(1/p, 1/q) \in \tilde{\mathcal{R}}_0^{\frac{1}{2}}$ the region, where uniform resolvent estimates are possible:

$$\begin{aligned}
\mathcal{Z}_{p,q}(\ell) &= \{\omega \in \mathbb{C}\backslash\mathbb{R} \; : \; \kappa_{p,q}(\omega) \leq \ell\} \\
&= \{\omega \in \mathbb{C}\backslash\mathbb{R} \; : \; |\omega|^{-\alpha_{p,q}} |\omega|^{\gamma_{p,q}} |\Im\omega|^{-\gamma_{p,q}} \leq \ell\}, \quad \alpha_{p,q} = 1 - d\big(\frac{1}{p} - \frac{1}{q}\big).
\end{aligned} \tag{55}$$

Describing the regions, we start with observing the symmetry in the real and imaginary part. For $\alpha_{p,q} = 0$, $\ell < 1$, we find $\mathcal{Z}_{p,q}(\ell) = \emptyset$. For $\ell \geq 1$, $\mathcal{Z}_{p,q}(\ell)$ describes a cone around the y-axis with aperture getting larger. For $\alpha_{p,q} > 0$ the boundaries become slightly curved. Pictorial representations for $\Re\omega > 0$ were provided in [18, Figures 9(a)–(c)]. The region in the left half plane is obtained by reflection along the imaginary axis. We shall see that eigenvalues of $P(\omega, D) + V$ must lie in $\mathbb{C}\backslash\mathcal{Z}_{p,q}(\ell)$. Previously in [11], for non-self-adjoint Schrödinger operators analogous arguments were used to show that in a range of (p, q), a sequence of eigenvalues λ_j with $\Re\lambda_j \to \infty$ has to satisfy $\Im\lambda_j \to 0$ as a consequence of the shape of $\mathcal{Z}_{p,q}(\ell)$. This is not the case presently and the shape of $\mathcal{Z}_{p,q}(\ell)$ only yields a bound for the asymptotic growth of $|\Im\lambda_j|$ as $|\Re\lambda_j| \to \infty$. This also raises the question for counterexamples, where the behavior $\Re\lambda_j \to \infty$ and $\Im\lambda_j \to 0$ fails. We also refer to Cuenin [6] for resolvent estimates for the fractional Laplacian in this context.

Let C be the constant such that

$$\|P(\omega, D)^{-1}\|_{L_0^p(\mathbb{R}^d) \to L_0^q(\mathbb{R}^d)} \leq C\kappa_{p,q}(\omega). \tag{56}$$

Corollary 4.1 *Let $d \in \{2, 3\}$, $\ell > 0$, and $1 < p, q < \infty$ such that $(1/p, 1/q) \in \tilde{\mathcal{R}}_0^{1/2}$. Suppose that there is $t \in (0, 1)$ such that*

$$\|V\|_{\frac{pq}{q-p}} \leq t(C\ell)^{-1}.$$

If $E \in \mathbb{C}\backslash\mathbb{R}$ is an eigenvalue of $P+V$ acting in L^q_0, then E must lie in $\mathbb{C}\backslash\mathcal{Z}_{p,q}(\ell)$.

Proof The short argument is standard by now (cf. [18, 19]), but contained for the sake of completeness. Let $u \in L^q_0(\mathbb{R}^d)$ be an eigenfunction of $P+V$ with eigenvalue $E \in \mathbb{C}\backslash\mathbb{R}$ and suppose that $E \in \mathcal{Z}_{p,q}(\ell)$. By Hölder's inequality, we find $-(P-E)u = (V-(P-E+V))u = Vu \in L^p$. By definition of $\mathcal{Z}_{p,q}(\ell)$, we find

$$\|(P-E)^{-1}\|_{p\to q} \le C\kappa_{p,q}(E) \le C\ell.$$

By the triangle and Hölder's inequality, we find

$$\begin{aligned}\|(P-E)^{-1}(P-E)u\|_q &\le C\ell(\|(P-E+V)u\|_p + \|Vu\|_p) \\ &\le C\ell\|V\|_{\frac{pq}{q-p}}\|u\|_q \le t\|u\|_q,\end{aligned}$$

which implies $u=0$ as $t<1$. Hence, $E \notin \mathcal{Z}_{p,q}(\ell)$. □

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5 Appendix

Lemma 5.1 *With the notations from Section 2.2, let $m(\xi)$ be as in (38) and $\alpha(\xi)$ as in (39). Then, we have*

$$|\det m(\xi)| \sim \alpha^4(\xi).$$

Proof We compute the determinant by taking linear combinations of the third and fourth column and fifth and sixth column, and aligning the columns as block matrices:

$$\det m(\xi) = \begin{vmatrix} 0 & \tilde\xi_1/a & 0 & 0 & \tilde\xi_2^2+\tilde\xi_3^2 & -(\tilde\xi_2^2+\tilde\xi_3^2) \\ 0 & \tilde\xi_2/b & -\xi_3'/\sqrt{b} & \xi_3'/\sqrt{b} & -\tilde\xi_1\tilde\xi_2 & \tilde\xi_1\tilde\xi_2 \\ 0 & \tilde\xi_3/b & \xi_2'/\sqrt{b} & -\xi_2'/\sqrt{b} & -\tilde\xi_1\tilde\xi_3 & \tilde\xi_1\tilde\xi_3 \\ \xi_1' & 0 & -((\xi_2')^2+(\xi_3')^2) & -((\xi_2')^2+(\xi_3')^2) & 0 & 0 \\ \xi_2' & 0 & \xi_1'\xi_2' & \xi_1'\xi_2' & -\tilde\xi_3 & -\tilde\xi_3 \\ \xi_3' & 0 & \xi_1'\xi_3' & \xi_1'\xi_3' & \tilde\xi_2 & \tilde\xi_3 \end{vmatrix}$$
$$\sim \begin{vmatrix} 0 & \tilde\xi_1/a & 0 & 0 & \tilde\xi_2^2+\tilde\xi_3^2 & 0 \\ 0 & \tilde\xi_2/b & 0 & -\xi_3'/\sqrt{b} & -\tilde\xi_1\tilde\xi_2 & 0 \\ 0 & \tilde\xi_3/b & 0 & -\xi_2'/\sqrt{b} & -\tilde\xi_1\tilde\xi_3 & 0 \\ \xi_1' & 0 & (\xi_2')^2+(\xi_3')^2 & 0 & 0 & 0 \\ \xi_2' & 0 & -\xi_1'\xi_2' & 0 & 0 & -\tilde\xi_3 \\ \xi_3' & 0 & -\xi_1'\xi_3' & 0 & 0 & \tilde\xi_3 \end{vmatrix}$$
$$\sim \begin{vmatrix} \tilde\xi_1/a & 0 & \tilde\xi_2^2+\tilde\xi_3^2 & 0 & 0 & 0 \\ \tilde\xi_2/b & -\xi_3'/\sqrt{b} & -\tilde\xi_1\tilde\xi_2 & 0 & 0 & 0 \\ \tilde\xi_3/b & \xi_2'/\sqrt{b} & -\tilde\xi_1\tilde\xi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_1' & (\xi_2')^2+(\xi_3')^2 & 0 \\ 0 & 0 & 0 & \xi_2' & -\xi_1'\xi_2' & -\tilde\xi_3 \\ 0 & 0 & 0 & \xi_3' & -\xi_1'\xi_3' & \tilde\xi_2 \end{vmatrix} =: A_2 \cdot A_1.$$

We find by noting that $(\xi_1')^2 + (\xi_2')^2 + (\xi_3')^2 = 1$

$$A_1 \sim \begin{vmatrix} (\xi_2')^2+(\xi_3')^2 & -\xi_1'\xi_2' & -\xi_1'\xi_3' \\ \xi_1' & \xi_2' & \xi_3' \\ 0 & -\tilde\xi_3 & \tilde\xi_2 \end{vmatrix} = \begin{vmatrix} 1-(\xi_1')^2 & -\xi_1'\xi_2' & -\xi_1'\xi_3' \\ \xi_1' & \xi_2' & \xi_3' \\ 0 & -\tilde\xi_3 & \tilde\xi_2 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 0 & 0 \\ \xi_1' & \xi_2' & \xi_3' \\ 0 & -\tilde\xi_3 & \tilde\xi_2 \end{vmatrix} - \xi_1' \begin{vmatrix} \xi_1' & \xi_2' & \xi_3' \\ \xi_1' & \xi_2' & \xi_3' \\ 0 & -\tilde\xi_3 & \tilde\xi_2 \end{vmatrix} = \xi_2'\tilde\xi_2 + \xi_3'\tilde\xi_3.$$

Next, by a similar argument,

$$A_2 \sim \begin{vmatrix} \tilde\xi_1/a & \tilde\xi_2/b & \tilde\xi_3/b \\ 0 & -\xi_3'/\sqrt{b} & \xi_2'/\sqrt{b} \\ \tilde\xi_2^2+\tilde\xi_3^2 & -\tilde\xi_1\tilde\xi_2 & -\tilde\xi_1 \quad \tilde\xi_3 \end{vmatrix} = \frac{1}{a} \begin{vmatrix} \tilde\xi_1/a & \tilde\xi_2/b & \tilde\xi_3/b \\ 0 & -\xi_3'/\sqrt{b} & \xi_2'/\sqrt{b} \\ a(\tilde\xi_2^2+\tilde\xi_3^2) & -a\tilde\xi_1\tilde\xi_2 & -a\tilde\xi_1\tilde\xi_3 \end{vmatrix}$$
$$= \frac{1}{a} \begin{vmatrix} \tilde\xi_1/a & \tilde\xi_2/b & \tilde\xi_3/b \\ 0 & -\xi_3'/\sqrt{b} & \xi_2'/\sqrt{b} \\ b\tilde\xi_1^2 + a(\tilde\xi_2^2+\tilde\xi_3^2) - b\tilde\xi_1^2 & -a\tilde\xi_1\tilde\xi_2 & -a\tilde\xi_1\tilde\xi_3 \end{vmatrix}.$$

We use multilinearity to write

$$A_2 \sim \frac{1}{a}\left(\begin{vmatrix} \tilde{\xi}_1/a & \tilde{\xi}_2/b & \tilde{\xi}_3/b \\ 0 & -\xi_3'/\sqrt{b} & \xi_2'/\sqrt{b} \\ 1 & 0 & 0 \end{vmatrix} - \begin{vmatrix} \tilde{\xi}_1/a & \tilde{\xi}_2/b & \tilde{\xi}_3/b \\ 0 & -\xi_3'/\sqrt{b} & \xi_2'/\sqrt{b} \\ -b\tilde{\xi}_1^2 & -a\tilde{\xi}_1\tilde{\xi}_2 & -a\tilde{\xi}_1\tilde{\xi}_3 \end{vmatrix}\right)$$

$$\sim \begin{vmatrix} 1 & 0 & 0 \\ 0 & -\xi_3'/\sqrt{b} & \xi_2'/\sqrt{b} \\ \tilde{\xi}_1/a & \tilde{\xi}_2/b & \tilde{\xi}_3/b \end{vmatrix} \sim (\xi_3'\tilde{\xi}_3 + \xi_2'\tilde{\xi}_2).$$

□

In the following we give explicit formulae for the resolvents and for limiting operators in two dimensions:

Proposition 5.2 *Let $d = 2$ and*

$$M^2(A,B) = \begin{pmatrix} \frac{A+B}{2\mu}((\xi_2')^2\varepsilon_{11} - (\xi_1'\xi_2')\varepsilon_{12}) & \frac{A+B}{2\mu}((\xi_2')^2\varepsilon_{21} - \xi_1'\xi_2'\varepsilon_{22}) & \frac{\xi_2'}{2\mu}(A-B) \\ \frac{A+B}{2\mu}((\xi_1')^2\varepsilon_{21} - \xi_1'\xi_2'\varepsilon_{11}) & \frac{A+B}{2\mu}((\xi_1')^2\varepsilon_{22} - \varepsilon_{12}(\xi_1')(\xi_2')) & \frac{\xi_1'}{2\mu}(B-A) \\ \frac{A-B}{2}(\xi_2'\varepsilon_{11} - \xi_1'\varepsilon_{21}) & \frac{B-A}{2}(\xi_1'\varepsilon_{22} - \xi_2'\varepsilon_{21}) & \frac{A+B}{2} \end{pmatrix},$$

furthermore,

$$M_c^2 = \frac{1}{i\omega\mu}\begin{pmatrix} \varepsilon_{22}(\xi_1')^2 - \varepsilon_{12}\xi_1'\xi_2' & \varepsilon_{22}\xi_1'\xi_2' - \varepsilon_{12}(\xi_2')^2 & 0 \\ \varepsilon_{11}\xi_1'\xi_2' - \varepsilon_{12}(\xi_1')^2 & \varepsilon_{11}(\xi_2')^2 - \varepsilon_{12}\xi_1'\xi_2' & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, we have for $\omega \in \mathbb{C}\backslash\mathbb{R}$ and almost all $\xi \in \mathbb{R}^2$:

$$(P(\omega,D)^{-1}u)\widehat{(\xi)} = (M^2(A,B) + M_c^2)\hat{u}(\xi)$$

with

$$A = \frac{1}{i(\omega - \|\xi\|_{\varepsilon'})}, \quad B = \frac{1}{i(\omega + \|\xi\|_{\varepsilon'})}.$$

For $\omega > 0$, $\beta \in C_c^\infty(\mathbb{R}^2)$, and $u \in \mathcal{S}(\mathbb{R}^2)^3$, we find

$$P(\omega \pm i\delta, D)^{-1}\beta(D)u \to P_\pm^{loc}(\omega)\beta(D)u$$

with

$$P_\pm^{loc}(\omega)\beta(D)u(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{ix.\xi}(M^2(A,B) + M_c^2)\beta(\xi)\hat{u}(\xi),$$

where

$$A = \frac{1}{i}\left\{v.p.\frac{1}{\omega - \|\xi\|_{\varepsilon'}} \mp i\pi\delta(\omega - \|\xi\|_{\varepsilon'})\right\}, \quad B = \frac{1}{i(\omega + \|\xi\|_{\varepsilon'})}.$$

Proof The first claim follows from computing $p^{-1}(\omega, \xi)$ (cf. Lemma 2.2). We decompose

$$\begin{aligned} m^{-1}(\xi) &= m_1(\xi) + m_2(\xi) \\ &= \begin{pmatrix} 0 & 0 & 0 \\ \frac{\xi_1' \varepsilon_{21} - \xi_2' \varepsilon_{11}}{2} & \frac{\varepsilon_{22} \xi_1' - \varepsilon_{21} \xi_2'}{2} & -\frac{1}{2} \\ \frac{\xi_2' \varepsilon_{11} - \xi_1' \varepsilon_{12}}{2} & \frac{\xi_2' \varepsilon_{12} - \xi_1' \varepsilon_{22}}{2} & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} \mu^{-1} \xi_1' & \mu^{-1} \xi_2' & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

based on the observation that $m_2(\xi) v(\xi) = 0$ for $\xi_1 v_1(\xi) + \xi_2 v_2(\xi) = 0$. We compute for A and B as in the first claim:

$$M^2(A, B) = m(\xi) d(\omega, \xi)^{-1} m_1(\xi).$$

The computation is simplified by noting that:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ \frac{\xi_1' \varepsilon_{21} - \xi_2' \varepsilon_{11}}{2} & \frac{\varepsilon_{22} \xi_1' - \varepsilon_{21} \xi_2'}{2} & -\frac{1}{2} \\ \frac{\xi_2' \varepsilon_{11} - \varepsilon_{12} \xi_1'}{2} & \frac{\xi_2' \varepsilon_{12} - \xi_1' \varepsilon_{22}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{A(\xi_1' \varepsilon_{21} - \xi_2' \varepsilon_{11})}{2} & \frac{A(\varepsilon_{22} \xi_1' - \varepsilon_{21} \xi_2'}{2} & -\frac{A}{2} \\ \frac{B(\xi_2' \varepsilon_{11} - \varepsilon_{12} \xi_1')}{2} & \frac{B(\xi_2' \varepsilon_{12} - \xi_1' \varepsilon_{22})}{2} & -\frac{B}{2} \end{pmatrix}.$$

We find for $m(\xi) d(\omega, \xi)^{-1} m_1(\xi)$:

$$\begin{pmatrix} \frac{A+B}{2\mu}((\xi_2')^2 \varepsilon_{11} - (\xi_1' \xi_2') \varepsilon_{12}) & \frac{A+B}{2\mu}((\xi_2')^2 \varepsilon_{21} - \xi_1' \xi_2' \varepsilon_{22}) & \frac{\xi_2'}{2\mu}(A - B) \\ \frac{A+B}{2\mu}((\xi_1')^2 \varepsilon_{21} - \xi_1' \xi_2' \varepsilon_{11}) & \frac{A+B}{2\mu}((\xi_1')^2 \varepsilon_{22} - \varepsilon_{12}(\xi_1')(\xi_2')) & \frac{\xi_1'}{2\mu}(B - A) \\ \frac{A-B}{2}(\xi_2' \varepsilon_{11} - \xi_1' \varepsilon_{21}) & \frac{B-A}{2}(\xi_1' \varepsilon_{22} - \xi_2' \varepsilon_{21}) & \frac{A+B}{2} \end{pmatrix}$$

In $M_c^2 = m(\xi) d(\omega, \xi)^{-1} m_2(\xi)$ we have separated the contribution of non-trivial charges. The second claim follows with the same computation from Sokhotsky's formula. □

For $d = 3$, we define $M^3(A, B, C, D) \in \mathbb{C}^{6 \times 6}$:

$$M_{11}^3 = \frac{a(C + D)(\tilde{\xi}_2^2 + \tilde{\xi}_3^2)}{2}, \quad M_{12}^3 = -\frac{b(C + D)\tilde{\xi}_1 \tilde{\xi}_2}{2}, M_{13}^3 = -\frac{b(C + D)\tilde{\xi}_1 \tilde{\xi}_3}{2},$$

$$M_{14}^3 = 0, \quad M_{15}^3 = \frac{(D - C)\tilde{\xi}_3}{2}, \quad M_{16}^3 = \frac{(C - D)\tilde{\xi}_2}{2}.$$

Furthermore,

$$M_{21}^3 = -\frac{a(C + D)\tilde{\xi}_1 \tilde{\xi}_2}{2}, \quad M_{22}^3 = \frac{(A + B)\xi_3^2}{2(\xi_2^2 + \xi_3^2)} + \frac{b(C + D)\tilde{\xi}_1^2 \xi_2^2}{2(\xi_2^2 + \xi_3^2)},$$

$$M_{23}^3 = -\frac{(A + B)\xi_2 \xi_3}{2(\xi_2^2 + \xi_3^2)} + \frac{b(C + D)\tilde{\xi}_1^2 \xi_2 \xi_3}{2(\xi_2^2 + \xi_3^2)}, \quad M_{24}^3 = \frac{\xi_3'(A + B)}{2\sqrt{b}},$$

$$M_{25}^3=\frac{(B-A)\xi_1'\xi_2\xi_3}{2\sqrt{b}(\xi_2^2+\xi_3^2)}+\frac{(C-D)\tilde{\xi}_1\xi_2\xi_3}{2(\xi_2^2+\xi_3^2)},$$
$$M_{26}^3=\frac{(B-A)\xi_1'\xi_3^2}{2\sqrt{b}(\xi_2^2+\xi_3^2)}+\frac{(D-C)\tilde{\xi}_1\xi_2^2}{2(\xi_2^2+\xi_3^2)}.$$

Next,

$$M_{31}^3=-\frac{a(C+D)\tilde{\xi}_1\tilde{\xi}_3}{2},\quad M_{32}^3=-\frac{(A+B)\xi_2\xi_3}{2(\xi_2^2+\xi_3^2)}+\frac{b(C+D)\tilde{\xi}_1^2\xi_2\xi_3}{2(\xi_2^2+\xi_3^2)},$$
$$M_{33}^3=\frac{(A+B)\xi_2^2}{2(\xi_2^2+\xi_3^2)}+\frac{b(C+D)\tilde{\xi}_1^2\xi_3^2}{2(\xi_2^2+\xi_3^2)},\quad M_{34}^3=\frac{(B-A)\xi_2'}{2\sqrt{b}},$$
$$M_{35}^3=\frac{(A-B)\xi_1'\xi_2^2}{2\sqrt{b}(\xi_2^2+\xi_3^2)}+\frac{(C-D)\tilde{\xi}_1\xi_3^2}{2(\xi_2^2+\xi_3^2)},\quad M_{36}^3=\frac{(A-B)\xi_1'\xi_2\xi_3}{2\sqrt{b}(\xi_2^2+\xi_3^2)}+\frac{(D-C)\tilde{\xi}_1\xi_2\xi_3}{2(\xi_2^2+\xi_3^2)}.$$
$$M_{41}^3=0,\quad M_{42}^3=\frac{\sqrt{b}(A-B)\xi_3'}{2},$$
$$M_{43}^3=\frac{\sqrt{b}(B-A)\xi_2'}{2},\quad M_{44}^3=\frac{(A+B)(\xi_2'^2+\xi_3'^2)}{2},$$
$$M_{45}^3=-\frac{(A+B)\xi_1'\xi_2'}{2},\quad M_{46}^3=-\frac{(A+B)\xi_1'\xi_3'}{2}.$$
$$M_{51}^3=\frac{a(D-C)}{2}\tilde{\xi}_3,\quad M_{52}^3=\frac{\sqrt{b}(B-A)\xi_1'\xi_2\xi_3}{2(\xi_2^2+\xi_3^2)}+\frac{b(C-D)\tilde{\xi}_1\xi_2\xi_3}{2(\xi_2^2+\xi_3^2)},$$
$$M_{53}^3=\frac{\sqrt{b}(A-B)\xi_1'\xi_2^2}{2(\xi_2^2+\xi_3^2)}+\frac{b(C-D)\tilde{\xi}_1\xi_3^2}{2(\xi_2^2+\xi_3^2)},\quad M_{54}^3=-\frac{(A+B)\xi_1'\xi_2'}{2},$$
$$M_{55}^3=\frac{(A+B)\xi_1'^2\xi_2^2}{2(\xi_2^2+\xi_3^2)}+\frac{(C+D)\xi_3^2}{2(\xi_2^2+\xi_3^2)},\quad M_{56}^3=\frac{(A+B)\xi_1'^2\xi_2\xi_3}{2(\xi_2^2+\xi_3^2)}-\frac{(C+D)\xi_2\xi_3}{2(\xi_2^2+\xi_3^2)}.$$

Lastly,

$$M_{61}^3=\frac{a(C-D)\tilde{\xi}_2}{2},\quad M_{62}^3=\frac{\sqrt{b}(B-A)\xi_1'\xi_3^2}{2(\xi_2^2+\xi_3^2)}+\frac{b(D-C)\tilde{\xi}_1\xi_2^2}{2(\xi_2^2+\xi_3^2)},$$
$$M_{63}^3=\frac{\sqrt{b}(A-B)\xi_1'\xi_2\xi_3}{2(\xi_2^2+\xi_3^2)}+\frac{b(D-C)\tilde{\xi}_1\xi_2\xi_3}{2(\xi_2^2+\xi_3^2)},\quad M_{64}^3=-\frac{(A+B)\xi_1'\xi_3'}{2},$$
$$M_{65}^3=\frac{(A+B)\xi_1'^2\xi_2\xi_3}{2(\xi_2^2+\xi_3^2)}-\frac{(C+D)\xi_2\xi_3}{2(\xi_2^2+\xi_3^2)},\quad M_{66}^3=\frac{(A+B)\xi_1'^2\xi_3^2}{2(\xi_2^2+\xi_3^2)}+\frac{(C+D)\xi_2^2}{2(\xi_2^2+\xi_3^2)}.$$

We let moreover

$$M_c^3 = \frac{1}{i\omega}\begin{pmatrix} b\tilde{\xi}_1^2 & b\tilde{\xi}_1\tilde{\xi}_2 & b\tilde{\xi}_1\tilde{\xi}_3 & 0 & 0 & 0 \\ a\tilde{\xi}_1\tilde{\xi}_2 & a\tilde{\xi}_2^2 & a\tilde{\xi}_2\tilde{\xi}_3 & 0 & 0 & 0 \\ a\tilde{\xi}_1\tilde{\xi}_3 & a\tilde{\xi}_2\tilde{\xi}_3 & a\tilde{\xi}_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_1'^2 & \xi_1'\xi_2' & \xi_1'\xi_3' \\ 0 & 0 & 0 & \xi_1'\xi_2' & \xi_2'^2 & \xi_2'\xi_3' \\ 0 & 0 & 0 & \xi_1'\xi_3' & \xi_2'\xi_3' & \xi_3'^2 \end{pmatrix}.$$

We have the following analog of Proposition 5.2:

Proposition 5.3 *Let $d = 3$. We find for $\omega \in \mathbb{C}\backslash\mathbb{R}$ and almost all $\xi \in \mathbb{R}^3$*

$$(P(\omega, D)^{-1}u)\widehat{}(\xi) = (M^3(A, B, C, D) + M_c^3)\hat{u}(\xi)$$

with

$$A = \frac{1}{i(\omega - \sqrt{b}\|\xi\|)}, \; B = \frac{1}{i(\omega + \sqrt{b}\|\xi\|)}, \; C = \frac{1}{i(\omega - \|\xi\|_\varepsilon)}, \; D = \frac{1}{i(\omega + \|\xi\|_\varepsilon)}.$$

For $\omega > 0$, $\beta \in C_c^\infty(\mathbb{R}^3)$, and $u \in \mathcal{S}(\mathbb{R}^3)^6$, we find

$$P(\omega \pm i\delta, D)^{-1}\beta(D)u \to P_\pm^{loc}(\omega)\beta(D)u \text{ in } (\mathcal{S}'(\mathbb{R}^3))^6$$

with

$$P_\pm^{loc}(\omega)\beta(D)u(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{ix.\xi}(M^3(A, B, C, D) + M_c^3)\beta(\xi)\hat{u}(\xi),$$

where

$$A = \frac{1}{i}\left\{v.p.\frac{1}{\omega - \sqrt{b}\|\xi\|} \mp i\pi\delta(\omega - \sqrt{b}\|\xi\|)\right\}, \; B = \frac{1}{i(\omega + \sqrt{b}\|\xi\|)},$$
$$C = \frac{1}{i}\left\{v.p.\frac{1}{\omega - \|\xi\|_\varepsilon} \mp i\pi\delta(\omega - \|\xi\|_\varepsilon)\right\}, \qquad D = \frac{1}{i(\omega + \|\xi\|_\varepsilon)}.$$

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