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Time–Frequency Analysis in $\mathbb{R}^n$

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Abstract

A time–frequency transform is a sesquilinear mapping from a suitable family of test functions to functions on the time–frequency plane. The goal is to quantify the energy present in the signal at any given time and frequency. The transform is further specified by imposing conditions such as those stemming from basic transformations of signals and those which an energy density should satisfy. We present a systematic study on how properties of a time–frequency transform are reflected in the associated evaluation at time–frequency origin, integral kernel and quantization and discuss some examples of time–frequency transforms.

Keywords Cohen class · Quadratic time–frequency transforms · Time–frequency analysis

Mathematics Subject Classification 42B10 · 47G30

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1 Introduction

Representing a signal $u : \mathbb{R}^n \to \mathbb{C}$ in terms of simpler components is the basic problem in signal analysis. We consider quadratic time–frequency transforms $(u, v) \mapsto Q(u, v)$, where $Q(u, u)$ acts as an energy density of the signal u in the time–frequency plane $\mathbb{R}^n \times \widehat{\mathbb{R}^n}$. For an ideal energy density $(x, \eta) \mapsto Q(u, u)(x, \eta)$ the value at the point (x, η) could be interpreted as the energy content of the component at frequency η at time x. However, such a pointwise density cannot exist since several of the properties of energy densities are mutually incompatible for time–frequency transforms. For instance, if we insist on pointwise positive energy densities, we lose the correct marginal distributions both in time and in frequency.

We study characterizations of quadratic time–frequency transforms in the space $\mathbb{R}^n$. In general, a time–frequency transform is a function

$$Q(u, v) : \mathbb{R}^n \times \widehat{\mathbb{R}^n} \to \mathbb{C}$$

where the form $(u, v) \mapsto Q(u, v)$ is sesquilinear and $\widehat{\mathbb{R}^n}$ denotes the unitary dual of $\mathbb{R}^n$. We will call the quadratic form $u \mapsto Q[u] := Q(u, u)$ the corresponding time–frequency distribution.

The most well-known and widely-applied time–frequency transform is, largely due to its simplicity and positivity, the *spectrogram*. It is defined using the short-time Fourier transform[1]

$$V_g u(y, \xi) = \int_{\mathbb{R}^n} e^{-i2\pi x \cdot \xi} u(x) \overline{g(x - y)} \, dx \tag{1}$$

as

$$u \mapsto |V_g u|^2, \tag{2}$$

where the function $g : \mathbb{R}^n \to \mathbb{C}$ is referred to as the *window function*.

Another example of a time–frequency transforms is the Wigner transform

$$W(u, v)(x, \eta) = \int_{\mathbb{R}^n} e^{-i2\pi y \cdot \eta} u(x + y/2) \overline{v(x - y/2)} \, dy.$$

The Wigner distribution $u \mapsto W[u]$ can also be interpreted as an energy distribution of the signal u. However, this interpretation is more difficult due to the Wigner distribution attaining also negative pointwise values for most signals.

Time–frequency transforms $(x, \eta) \mapsto Q(u, v)(x, \eta)$ may be characterized by properties related to basic transformations of signals. If they are considered as functions on the Heisenberg group, the correspondence between automorphisms of the Heisenberg group and certain unitary operators on signals $u \in L^2(\mathbb{R}^n)$ yields these so-called covariance properties. Further characterizations are given by properties defining an energy distribution. For instance, the time–frequency distribution

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[1] We use the variables (y, ξ) for the short-time Fourier transform, since it satisfies $V_g u(y, \xi) = FR(u, g)(y, \xi)$, where $R(u, g)(x, \eta)$ is the Rihaczek transform defined in 26 and F is the symplectic Fourier transform, see Definition 10.

$(x, \eta) \mapsto Q[u](x, \eta)$ should yield the total energy of the signal if integrated over the whole time–frequency plane.

2 Fourier Analysis

We recall first some basic Fourier analysis. The basic theory can be found in many introductory texts, for example, in [22]. For any $u \in L^1(\mathbb{R}^n)$ the Fourier transform $\widehat{u} : \widehat{\mathbb{R}^n} \to \mathbb{C}$ is defined by the integral

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i2\pi x \cdot \xi} u(x) \, dx.$$

We use the operator notation $\mathcal{F}u := \widehat{u}$ as well. Next we consider some simple operations on signals and their interplay with the Fourier transform.

Definition 1 For any signal $u : \mathbb{R}^n \to \mathbb{C}$ the *modulation* of u by $\xi \in \widehat{\mathbb{R}^n}$ is given by

$$M_\xi u(x) = e^{i2\pi x \cdot \xi} u(x).$$

The *translation* of u by $y \in \mathbb{R}^n$ is given by

$$T_y u(x) = u(x - y),$$

and the energy preserving *dilations* D_s are defined by

$$D_s u(x) = |s|^{-n/2} u(x/s), \quad s \in \mathbb{R} \setminus \{0\}.$$

It follows directly from the definitions that modulations, translations and dilations satisfy

$$\mathcal{F} \circ M_\xi = T_\xi \circ \mathcal{F},$$
$$\mathcal{F} \circ T_y = M_{-y} \circ \mathcal{F},$$
$$\mathcal{F} \circ D_s = D_{1/s} \circ \mathcal{F}$$

as operators in $L^1(\mathbb{R}^n)$. In addition to the previous formulas, we have for the complex conjugation, $z \mapsto \bar{z}$,

$$\mathcal{F}(\iota(\overline{u})) = \overline{\mathcal{F}u}$$

where $\iota u(x) = u(-x)$. The notation ι is used throughout for the reflection operator. For functions of two variables $x, y \in \mathbb{R}^n$ the separate reflections in each variable are denoted by

$$\iota_1 u(x, y) = u(-x, y),$$
$$\iota_2 u(x, y) = u(x, -y).$$

We define the convolution $u * v : \mathbb{R}^n \to \mathbb{C}$ of two absolutely integrable functions $u, v : \mathbb{R}^n \to \mathbb{C}$ by the formula

$$(u * v)(x) = \int_{\mathbb{R}^n} u(x - y)v(y) \, dy.$$

Convolution is commutative and associative as an algebraic operation and its Fourier transform is the point-wise product of the individual transforms:

$$\widehat{u * v}(\xi) = \widehat{u}(\xi)\widehat{v}(\xi).$$

For functions $u(x, y)$ and $v(x, y)$ of two variables $x, y \in \mathbb{R}^n$ we use the notation $u *_1 v$ for the partial convolution with respect to the first variable defined by

$$(u *_1 v)(x, y) = \int_{\mathbb{R}^n} u(x - t, y)v(t, y) \, dt.$$

The partial convolution $u *_2 v$ is defined similarly.

The Fourier transform satisfies the following self-dual property which is used to define it for distributions.

Lemma 1 *If u and v are in $L^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} \widehat{u}(x)v(x) \, dx = \int_{\mathbb{R}^n} u(x)\widehat{v}(x) \, dx.$$

The Fourier inversion formula is defined by the integral

$$u(x) = \mathcal{F}^{-1}\widehat{u}(x) := \int_{\mathbb{R}^n} e^{i2\pi x \cdot \xi}\widehat{u}(\xi) \, d\xi$$

in the case of both u and $\widehat{u}$ being absolutely integrable. This may be extended to other spaces e.g. by the Plancherel theorem as follows:

Theorem 1 (Plancherel Theorem) *If $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then the equality*

$$\|u\|_{L^2(\mathbb{R}^n)} = \|\widehat{u}\|_{L^2(\mathbb{R}^n)}$$

holds. This allows the definition of the Fourier transform on $L^2(\mathbb{R}^n)$ as the bounded extension of the transform on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. The transform thus defined is a unitary operator with the inverse

$$u(x) \mapsto \widehat{u}(-x).$$

We will mainly consider the space of Schwartz rapidly decreasing functions and the corresponding distributions. As references for the theory of distributions we use the books [13,20,23]. The definitions and theorems below as well as their proofs can be found in these monographs.

We will use the standard multi-index notation $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and define

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$
$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n},$$

and $|\alpha| = \sum_{i=1}^n \alpha_i$.

The space of smooth functions is denoted by $\mathscr{C}^\infty(\mathbb{R}^n)$. It is a topological vector space with the topology given by the countable family of seminorms $\varphi \mapsto |\varphi|_{m,K}$, where

$$|\varphi|_{m,K} = \sup_{|\alpha| \leq m} \left(\sup_{x \in K} \left| \partial_x^\alpha \varphi(x) \right| \right)$$

and K is a compact subset of $\mathbb{R}^n$. The space of compactly supported smooth functions is denoted by $\mathscr{C}_c^\infty(\mathbb{R}^n)$.

Definition 2 Consider the vector space of functions $\varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$ satisfying

$$\sup_{x \in \mathbb{R}^n} |x^\beta \partial_x^\alpha \varphi(x)| < \infty \tag{3}$$

for all multi-indices α and β. This space equipped with the topology given by the family of seminorms defined by the left-hand side of (3) is called the *Schwartz space* $\mathscr{S}(\mathbb{R}^n)$ of rapidly decreasing smooth functions, also known as the space of *Schwartz test functions*.

Definition 3 The space $\mathscr{S}'(\mathbb{R}^n)$ of *temperate distributions* is the space of continuous linear functionals on $\mathscr{S}(\mathbb{R}^n)$. The notation

$$u(\varphi) =: \langle u, \varphi \rangle_{\mathscr{S}', \mathscr{S}}$$

means the evaluation of the temperate distribution u at φ.

Any function $u \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ has an interpretation as a temperate distribution Λ_u by defining

$$\Lambda_u(\varphi) := \int_{\mathbb{R}^n} u(x)\varphi(x) \, dx. \tag{4}$$

With this interpretation we have

$$\mathscr{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n),$$

and furthermore:

Lemma 2 *The space $\mathscr{S}(\mathbb{R}^n)$ is dense in the space of temperate distributions.*

We will use subscripts to differentiate the bilinear distribution pairing $\langle u, \varphi \rangle_{\mathscr{S}',\mathscr{S}}$ from the usual sesquilinear Hilbert space inner product $\langle u, v \rangle$. We define the complex conjugate of a temperate distribution u by

$$\langle \overline{u}, \varphi \rangle_{\mathscr{S}',\mathscr{S}} := \overline{\langle u, \overline{\varphi} \rangle}_{\mathscr{S}',\mathscr{S}}.$$

Given a polynomial $p(x) = \sum_\alpha a_\alpha x^\alpha$ we define $p(\partial_x)$ to be the constant-coefficient differential operator formed from the polynomial $p(x)$ by the formal substitution $x^\alpha \mapsto \partial_x^\alpha$. The operator $p(\partial_x)$ is thus given by

$$p(\partial_x) = \sum_\alpha a_\alpha \partial_x^\alpha.$$

For the Schwartz test functions we may now formulate the following duality between differentiation and multiplication operators.

Theorem 2 *If $\varphi \in \mathscr{S}(\mathbb{R}^n)$ and p is a polynomial then the identities*

$$\mathcal{F}(p(\partial_x)\varphi)(\xi) = p(\mathrm{i}2\pi\xi) \cdot \mathcal{F}\varphi(\xi)$$
$$\mathcal{F}(p \cdot \varphi)(\xi) = p((-\mathrm{i}2\pi)^{-1}\partial_\xi)\mathcal{F}\varphi(\xi)$$

hold.

Finally, motivated by Lemma (1), we define the Fourier transform for the space of temperate distributions:

Definition 4 (*Fourier transform for temperate distributions*) For any $\varphi \in \mathscr{S}(\mathbb{R}^n)$ we define the Fourier transform $\widehat{u} \in \mathscr{S}'(\mathbb{R}^n)$ of $u \in \mathscr{S}'(\mathbb{R}^n)$ as a temperate distribution by

$$\langle \widehat{u}, \varphi \rangle_{\mathscr{S}',\mathscr{S}} = \langle u, \widehat{\varphi} \rangle_{\mathscr{S}',\mathscr{S}}.$$

The Fourier transform thus defined is an automorphism of $\mathscr{S}'$ and it may be restricted to automorphisms of L^2 and $\mathscr{S}$.

Definition 5 (*Convolution of distributions*) The convolution of a Schwartz function φ and a temperate distribution u is the function given by

$$(\varphi * u)(x) := \langle \iota u, T_{-x}\varphi \rangle_{\mathscr{S}',\mathscr{S}}. \tag{5}$$

We will consider time–frequency transforms which are based on convolutions of particular temperate distributions. In this context we need the spaces of multipliers and convolutors. For the following definition we refer to [23, p. 275]:

Definition 6 The space of *multipliers*, $\mathscr{O}_M(\mathbb{R}^n)$, is the space of functions $\varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$ such that for every multi-index α there is a polynomial P_α in $\mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$,

$$|\partial_\alpha \varphi(x)| \leq |P_\alpha(x)|.$$

The space of convolutors is defined in [23, p. 315] as follows:

Definition 7 The space of *convolutors*, $\mathscr{O}'_C(\mathbb{R}^n)$, is the space of distributions Λ satisfying the following property: Given any integer $h \geq 0$, there is a finite family of continuous functions in $\mathbb{R}^n$, f_α, with the multi-index α satisfying $|\alpha| \leq m(h)$, such that

$$\Lambda = \sum_{|\alpha| \leq m(h)} \partial_\alpha f_\alpha$$

and such that for all multi-indices α, $|\alpha| \leq m(h)$,

$$\lim_{|x| \to \infty} (1 + |x|)^h |f_\alpha(x)| = 0.$$

Referring to [23, Chap. 30], we point out that the space of convolutors is precisely the subspace of temperate distributions such that Schwartz functions are mapped to Schwartz functions in convolution, that is, $f \in \mathscr{O}'_C(\mathbb{R}^n)$ if and only if

$$f * \varphi \in \mathscr{S}(\mathbb{R}^n)$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$. The Fourier transform is a bijective linear map from $\mathscr{O}_M(\mathbb{R}^n)$ to $\mathscr{O}'_C(\mathbb{R}^n)$ and from $\mathscr{O}'_C(\mathbb{R}^n)$ to $\mathscr{O}_M(\mathbb{R}^n)$.

3 Time–Frequency Plane

In this section, we introduce the structure of the time–frequency plane and present some basic properties of functions defined on it using [17] and [10] as basic references. In applications, the time–frequency plane is often called the phase space or the position-momentum space. Time–frequency transforms are certain covariant functions on the time–frequency plane.

Definition 8 Time–frequency plane is defined to be the direct sum of vector spaces

$$\mathbb{R}^n \oplus \widehat{\mathbb{R}^n},$$

where $\widehat{\mathbb{R}^n}$ is the unitary dual of $\mathbb{R}^n$ consisting of the functions $e_\xi(x) = e^{i2\pi x \cdot \xi}$ with $\xi \in \mathbb{R}^n$. As the mapping $\xi \mapsto e_\xi$ is an isomorphism $\mathbb{R}^n \simeq \widehat{\mathbb{R}^n}$ of groups we will regard the time–frequency plane as the space $\mathbb{R}^n \oplus \mathbb{R}^n \simeq \mathbb{R}^{2n}$. This version of the time–frequency plane is endowed with the symplectic form

$$B((x, \eta), (y, \xi)) = y \cdot \eta - x \cdot \xi. \tag{6}$$

Of particular importance in the time–frequency plane are those coordinate transforms which preserve the symplectic form. The set of these transforms forms the symplectic group.

Definition 9 Symplectic group $\mathrm{Sp}(2n)$ is the group of linear transformations $\mathcal{A}$ on $\mathbb{R}^{2n}$ satisfying

$$B(\mathcal{A}(x, \eta), \mathcal{A}(y, \xi)) = B((x, \eta), (y, \xi)),$$

where B is the symplectic form defined in (6).

The symplectic group is generated by the mappings

$$(x, \eta) \mapsto (Tx, (T^{-1})^* \eta), \tag{7}$$
$$(x_k, \eta_k) \mapsto (\eta_k, -x_k), \quad k = 1, \ldots, n, \text{ other coordinates remain fixed}, \tag{8}$$
$$(x, \eta) \mapsto (x, \eta + Sx), \tag{9}$$

where T is an automorphism of $\mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric linear mapping, see [17, Sect. 4.4.1].

The standard symplectic form gives rise to a version of the Fourier transform which we call the symplectic Fourier transform.

Definition 10 (*Symplectic Fourier Transform*) We define the symplectic Fourier transform $Fa \in \mathscr{S}(\mathbb{R}^{2n})$ for functions $a \in \mathscr{S}(\mathbb{R}^{2n})$ in the time–frequency plane as

$$
\begin{aligned}
(Fa)(y, \xi) &:= \iint_{\mathbb{R}^{2n}} e^{\mathrm{i}2\pi \, B((x,\eta),(y,\xi))} a(x, \eta) \, \mathrm{d}x \, \mathrm{d}\eta \\
&= \iint_{\mathbb{R}^{2n}} e^{-\mathrm{i}2\pi (x \cdot \xi - y \cdot \eta)} a(x, \eta) \, \mathrm{d}x \, \mathrm{d}\eta.
\end{aligned}
$$

The space of Fourier variables $(y, \xi) \in \mathbb{R}^{2n}$ is called the *ambiguity plane*. The symplectic Fourier transform is also given by

$$F = S \circ \mathcal{F}_1 \circ \mathcal{F}_2^{-1}, \tag{10}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the partial Fourier transforms defined by

$$(\mathcal{F}_1 a)(\xi, y) = \int_{\mathbb{R}^n} e^{-\mathrm{i}2\pi x \cdot \xi} a(x, y) \, \mathrm{d}x,$$
$$(\mathcal{F}_2 a)(x, \eta) = \int_{\mathbb{R}^n} e^{-\mathrm{i}2\pi y \cdot \eta} a(x, y) \, \mathrm{d}y,$$

and $S : a(\xi, y) \mapsto a(y, \xi)$.

A direct consequence of the definition is that the symplectic Fourier transform is its own inverse $F^{-1} = F$ and that it commutes with symplectic coordinate transforms, that is,

$$F \circ T_{\mathcal{A}} = T_{\mathcal{A}} \circ F, \tag{11}$$

where $T_{\mathcal{A}}a(x,\eta) = a(\mathcal{A}(x,\eta))$ and $\mathcal{A} \in \mathrm{Sp}(2n)$.

We state some basic mapping properties of the symplectic Fourier transform which, in view of (10) are analogous to the properties for the standard Fourier transform in $\mathbb{R}^{2n}$.

Lemma 3 *The symplectic Fourier transform is an automorphism of $L^2(\mathbb{R}^{2n})$, $\mathscr{S}(\mathbb{R}^{2n})$ and $\mathscr{S}'(\mathbb{R}^{2n})$. It is also a bijective map $\mathscr{O}_M(\mathbb{R}^{2n}) \to \mathscr{O}'_C(\mathbb{R}^{2n})$ and satisfies the Plancherel formula*

$$\langle Fa, Fb \rangle_{L^2(\mathbb{R}^{2n})} = \langle a, b \rangle_{L^2(\mathbb{R}^{2n})}$$

and the identities

$$F \circ T_{(z,\zeta)} = M_{(z,\zeta)} \circ F,$$
$$\overline{Fa} = \iota F(\overline{a}),$$
$$F(a * b) = Fa \cdot Fb,$$

where $a, b \in \mathscr{S}(\mathbb{R}^{2n})$, translations and modulations in time and frequency are

$$T_{(z,\zeta)}a(x,\eta) := a(x - z, \eta - \zeta),$$
$$M_{(z,\zeta)}a(x,\eta) := \mathrm{e}^{\mathrm{i}2\pi(x\cdot\zeta - z\cdot\eta)}a(x,\eta) = \mathrm{e}^{\mathrm{i}2\pi B((z,\zeta),(x,\eta))}a(x,\eta),$$

and the phase space convolution is an ordinary convolution with respect to both time and frequency.

Particularly interesting in terms of time–frequency analysis are those unitary operators on $L^2(\mathbb{R}^n)$ which are connected to symplectic transforms in the time–frequency plane. This group of operators is the metaplectic group. We present a few basic facts referring to Sects. 2.1.4 and 4.4.2 of [17]. The following definition for the metaplectic group is presented in [17, p. 60]:

Definition 11 The metaplectic group $\mathrm{Mp}(n)$ is the subgroup of unitary operators on $L^2(\mathbb{R}^n)$ generated by

$$Pu(x) = \mathrm{e}^{\mathrm{i}\theta}u(x), \tag{12}$$
$$Au(x) = |\det T|^{-1/2}u(T^{-1}x), \tag{13}$$
$$Bu(\check{x}_k) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi x_k \xi_k}u(x)\,\mathrm{d}x_k, \quad k = 1, \ldots, n, \tag{14}$$
$$Cu(x) = \mathrm{e}^{\mathrm{i}\pi\langle Sx, x\rangle}u(x), \tag{15}$$

where $\theta \in \mathbb{R}$, T is an automorphism of $\mathbb{R}^n$, $\check{x}_k = (x_1, \ldots, x_{k-1}, \xi_k, x_{k+1}, \ldots, x_n)$ and S is a symmetric matrix.

Remark 1 The Metaplectic group is often defined as the connected double cover of the symplectic group. This definition differs from the one presented in [17], see [17, Remark 4.4.20].

There is a surjective homomorphism

$$\Pi : \mathrm{Mp}(n) \to \mathrm{Sp}(2n) \tag{16}$$

with kernel $\mathbb{C}^* \, \mathrm{id}_{L^2(\mathbb{R}^n)}$ such that if $\mathcal{A} = \Pi(M)$, the Wigner transform W satisfies the Segal formula

$$W(Mu, Mv)(x, \eta) = (W(u, v) \circ \mathcal{A}^{-1})(x, \eta), \tag{17}$$

see [17, p. 61]. The mapping Π maps the phase transformations P in (12) to the identity, $\Pi(P) = \mathrm{id}_{\mathrm{Sp}(2n)}$ and the operators A, B and C of Definition 11 to the symplectic mappings (7), (8) and (9), respectively. We will take the formula (17) as a source of transformation properties that a time–frequency transform should ideally have.

4 Time–Frequency Transforms and Quantizations

We follow the common tradition of representing time–frequency transform in terms of the Wigner transform. Since the Wigner transform satisfies many desirable properties, such as (17), it is reasonable to have it as the starting point for defining other transforms. We will consider transforms which can be defined as convolution smoothings of the Wigner transform. In the usual terminology these transforms are referred to as the *Cohen class* transforms named after Leon Cohen who first studied such transform systematically in [6].

Definition 12 The Wigner transform $W(u, v)$ is defined for $u, v \in \mathscr{S}(\mathbb{R}^n)$ by the formula

$$W(u, v)(x, \eta) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}2\pi y \cdot \eta} u(x + y/2) \overline{v(x - y/2)} \, \mathrm{d}y. \tag{18}$$

The Wigner transform is a particular example $\tau = 1/2$ of the τ-Wigner transforms defined by

$$W_\tau(u, v)(x, \eta) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}2\pi y \cdot \eta} u(x + \tau y) \overline{v(x - (1 - \tau)y)} \, \mathrm{d}y, \quad \tau \in \mathbb{R}, \tag{19}$$

which will be addressed in more in detail in Example 5 of Sect. 6.

The symplectic Fourier transform of the Wigner transform is the (narrowband) *ambiguity transform*

$$\chi(u, v)(y, \xi) := FW(u, v)(y, \xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}2\pi x \cdot \xi} u(x + y/2) \overline{v(x - y/2)} \, \mathrm{d}x. \tag{20}$$

The ambiguity transform is also given by the matrix coefficients

$$\chi(u, v)(y, \xi) = \langle u, \rho(y, \xi)v \rangle, \tag{21}$$

where

$$\rho(y, \xi) f(x) = M_{\xi/2} T_y M_{\xi/2} f(x) = \mathrm{e}^{-\mathrm{i}\pi\, y\cdot\xi + \mathrm{i}2\pi x\cdot\xi} f(x - y) \tag{22}$$

is the symmetric version of the *Schrödinger representation* ρ of the Heisenberg group.

Since the Wigner transform is taken to be the basic time–frequency transform, its properties are of interest when defining other transforms. The lists of properties for the Wigner transform and several other transforms can be found, for instance, in [1]. The Wigner transform is a member of the Cohen class (Definition 14) having several desirable properties such as full symplectic covariance, correct marginal distributions, Moyal property and invertibility. The Wigner distribution is not positive, however, as can be readily verified by considering $W[u]$ for odd u in $\mathbb{R}$. Positivity is characterized by Hudson's theorem which states that the Wigner distribution is positive only for generalized Gaussians [11, p. 70].

As mapping properties Wigner transform satisfies, for instance,

$$W : \mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^{2n}), \tag{23}$$

$$W : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n}). \tag{24}$$

When defining other time–frequency transforms Q we will mainly consider those that have mapping properties like (23) and (24). Not all well-known transform satisfy (23) and (24) as can be seen by examining the spectrogram $|V_g u|^2(x, \eta)$. For the short-time Fourier transform we have by [11, Chap. 11] that assuming $g \in \mathscr{S}(\mathbb{R}^n)$ nonzero and $u \in \mathscr{S}'(\mathbb{R}^n)$ then $V_g u \in \mathscr{S}(\mathbb{R}^{2n})$ if and only if $u \in \mathscr{S}(\mathbb{R}^n)$. Hence by using the identity

$$V_g u(y, \xi) = \mathrm{e}^{-\mathrm{i}2\pi\, y\cdot\xi} \overline{V_u g(-y, -\xi)},$$

most spectrograms are not mappings from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^{2n})$.

Translations move the signal in time and modulations move the signal in frequency. The corresponding property jointly in time and frequency is time–frequency covariance.

Definition 13 A sesquilinear map $(u, v) \mapsto Q(u, v)(x, \eta)$ is *time–frequency covariant* if

$$Q(T_{x_0} M_{\eta_0} u, T_{x_0} M_{\eta_0} v)(x, \eta) = Q(u, v)(x - x_0, \eta - \eta_0) \tag{25}$$

for all (x, η), $(x_0, \eta_0) \in \mathbb{R}^{2n}$ and $u, v \in \mathscr{S}(\mathbb{R}^n)$.

Based on the mapping properties (23), (24) and time–frequency covariance we define the Cohen class time–frequency transforms considered in this article as follows.

Definition 14 A sesquilinear map $Q : \mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^{2n})$ is a time–frequency transform if it satisfies

$$F Q(u, v)(y, \xi) = \phi_Q(y, \xi) \chi(u, v)(y, \xi),$$

where χ is the ambiguity transform defined in (20) and $\phi_Q \in \mathscr{O}_M(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$.

A time–frequency transform Q can be specified by different kernels which are Fourier transforms of one another. We will use the following notation and terminology:

$$\text{ambiguity kernel,} \quad \phi_Q(y, \xi),$$
$$\text{time–frequency kernel,} \quad \psi_Q(x, \eta) := F\phi_Q(x, \eta),$$
$$\text{time-delay kernel,} \quad \varphi_Q(x, y) := \mathcal{F}_2^{-1}\psi_Q(x, y).$$

Definition 14 implies that a time–frequency transform Q is a convolution of the Wigner transform,

$$Q(u, v) = \psi_Q * W(u, v),$$

for some distribution $\psi_Q \in \mathscr{O}'_C(\mathbb{R}^{2n})$. Time–frequency covariance of the Wigner transform implies that the transforms defined in 14 are also covariant.

The boundedness requirement of the ambiguity kernel $\phi_Q \in L^\infty(\mathbb{R}^{2n})$ makes the transform square-integrable for finite-energy signals by the Plancherel theorem and the estimate

$$\|Q(u, v)\|_{L^2(\mathbb{R}^{2n})} \le \|\phi_Q\|_{L^\infty(\mathbb{R}^{2n})} \|u\|_{L^2(\mathbb{R}^n)} \|u\|_{L^2(\mathbb{R}^n)}.$$

Remark 2 Time–frequency transforms can also be based on the Rihaczek transform $R := W_0$ given by

$$R(u, v)(x, \eta) = u(x)e^{-i2\pi x \cdot \eta}\overline{\widehat{v}(\eta)}, \tag{26}$$

which can be generalized easily to groups without dilation. The difficulty with the Wigner transform is $y \mapsto y/2$ which does not work well on groups in general. Rihaczek transform is the starting point for defining Cohen class time–frequency transforms on groups in the article by Turunen [24].

In view of (25), the value of a covariant time–frequency transform at any point (x, η) is determined by the sesquilinear form

$$(u, v) \mapsto Q(u, v)(0, 0),$$

where $u, v \in \mathscr{S}(\mathbb{R}^n)$. We use this sesquilinear form to define the *evaluation at origin operator* $Q_0 : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ by

$$\langle \overline{Q_0 v}, u \rangle_{\mathscr{S}', \mathscr{S}} := Q(u, v)(0, 0). \tag{27}$$

This operator admits a representation by a distribution kernel $\delta^Q \in \mathscr{S}'(\mathbb{R}^{2n})$ by the Schwartz kernel theorem as

$$\langle Q_0 v, u \rangle_{\mathscr{S}', \mathscr{S}} = \langle \delta^Q, u \otimes v \rangle_{\mathscr{S}', \mathscr{S}}. \tag{28}$$

We collect these observations into a lemma which is a version of [11, Theorem 4.5.1]. The original lemma excludes transforms which are not in general point-wise bounded for signals of finite energy. An example of such a transform is the Rihaczek transform (26).

Lemma 4 *Let Q be a time–frequency transform according to Definition 14. The evaluation at the time–frequency origin defines the distribution kernel $\delta^Q \in \mathscr{S}'(\mathbb{R}^{2n})$ and*

$$Q(u, v)(x, \eta) = \left\langle \overline{\delta^Q}, \rho(x, \eta)^{-1} u \otimes \overline{\rho(x, \eta)^{-1} v} \right\rangle_{\mathscr{S}', \mathscr{S}}, \tag{29}$$

where $\rho(x, \eta) = M_{\eta/2} T_x M_{\eta/2}$ is the symmetric time–frequency shift defining the Schrödinger representation and δ_Q the distribution kernel defined in (28). The corresponding time–frequency distribution is expressed in terms of the τ-Wigner transform as

$$Q(u, v)(x, \eta) = (W_\tau(u, v) *_{\iota_1} \mathcal{F}_2 P_\tau \delta^Q)(x, \eta), \tag{30}$$

where the operator P_τ is defined by $P_\tau a(x, y) = a(x + \tau y, x + (\tau - 1)y)$ with $\tau \in \mathbb{R}$ and $a \in \mathscr{S}(\mathbb{R}^{2n})$.

Proof The evaluation at the origin defines the operator Q_0 by

$$
\begin{aligned}
Q(u, v)(0, 0) &= \langle \overline{Q_0 v}, u \rangle_{\mathscr{S}', \mathscr{S}} \\
&= \overline{\langle Q_0 v, \overline{u} \rangle_{\mathscr{S}', \mathscr{S}}} \\
&= \overline{\langle \delta^Q, \overline{u} \otimes v \rangle_{\mathscr{S}', \mathscr{S}}} \\
&= \langle \overline{\delta^Q}, u \otimes \overline{v} \rangle_{\mathscr{S}', \mathscr{S}}.
\end{aligned}
$$

By time–frequency covariance

$$
\begin{aligned}
Q(u, v)(x, \eta) &= \left\langle \overline{\delta^Q}, (T_{-x} M_{-\eta} u) \otimes \overline{T_{-x} M_{-\eta} v} \right\rangle_{\mathscr{S}', \mathscr{S}} \\
&= \left\langle \overline{\delta^Q}, \rho(x, \eta)^{-1} u \otimes \overline{\rho(x, \eta)^{-1} v} \right\rangle_{\mathscr{S}', \mathscr{S}},
\end{aligned}
$$

where the phase introduced by the Schrödinger representation is cancelled by the complex conjugation.

The τ-Wigner transform is given by

$$W_\tau(u, v) = \mathcal{F}_2 P_\tau(u \otimes \overline{v}).$$

We express the evaluation of Q at the origin in terms of the τ-Wigner transform

$$
\begin{aligned}
Q(u, v)(0, 0) &= \langle \overline{\delta^Q}, P_\tau^{-1} \mathcal{F}_2^{-1} W_\tau(u, v) \rangle_{\mathscr{S}', \mathscr{S}} \\
&= \langle \mathcal{F}_2^{-1} P_\tau \overline{\delta^Q}, W_\tau(u, v) \rangle_{\mathscr{S}', \mathscr{S}},
\end{aligned}
$$

where $P_\tau^{-1} a(x, y) = a((1 - \tau)x + \tau y, x - y)$. For any point in the time–frequency plane we find by covariance of the τ-Wigner transform

$$
\begin{aligned}
Q(u, v)(x, \eta) &= \langle \mathcal{F}_2^{-1} P_\tau \overline{\delta^Q}, T_{(-x, -\eta)} W_\tau(u, v) \rangle_{\mathscr{S}', \mathscr{S}} \\
&= (W_\tau(u, v) * \iota \mathcal{F}_2^{-1} P_\tau \overline{\delta^Q})(x, \eta) \\
&= (W_\tau(u, v) * \iota_1 \mathcal{F}_2 P_\tau \overline{\delta^Q})(x, \eta).
\end{aligned}
$$

$\square$

When defining a time–frequency transform in terms of the τ-Wigner transform, we use the notation

$$
Q(u, v) = \psi_Q^\tau * W_\tau(u, v),
$$

where ψ_Q^τ is the τ-*Wigner time–frequency kernel*. The corresponding ambiguity kernel and time-delay kernel are denoted by ϕ_Q^τ and φ_Q^τ, respectively.

Proposition 1 *The kernel of the evaluation at origin Q_0 is related to the τ-Wigner time-delay kernel φ_Q^τ of the transform Q by*

$$
\overline{\delta^Q} = P_\tau^{-1} \iota_1 \varphi_Q^\tau. \tag{31}
$$

Proof The transform Q is defined with the τ-Wigner time-delay kernel as

$$
Q(u, v)(x, \eta) = \mathcal{F}_2(\varphi_Q^\tau *_1 P_\tau(u \otimes \overline{v}))
$$

and at the time–frequency origin we find

$$
\begin{aligned}
Q(u, v)(0, 0) &= \mathcal{F}_2(P_\tau(u \otimes \overline{v}) *_1 \varphi_Q^\tau)(0, 0) \\
&= \langle \iota_1 \varphi_Q^\tau, P_\tau(u \otimes \overline{v}) \rangle_{\mathscr{S}', \mathscr{S}} \\
&= \langle P_\tau^{-1} \iota_1 \varphi_Q^\tau, u \otimes \overline{v} \rangle_{\mathscr{S}', \mathscr{S}},
\end{aligned}
$$

which proves the claim.

$\square$

We define next the the Q-quantization related to a time–frequency transform Q.

Definition 15 Let $(u, v) \mapsto Q(u, v)$ be a Cohen class time–frequency transform. The Q-quantization is the map $a \mapsto \mathrm{Op}_Q a$ taking temperate distributions $a \in \mathscr{S}'(\mathbb{R}^{2n})$ to linear operators from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$ defined by

$$
\langle \overline{a}, Q(u, v) \rangle_{\mathscr{S}'(\mathbb{R}^{2n}), \mathscr{S}(\mathbb{R}^{2n})} = \overline{\langle (\mathrm{Op}_Q a)v, u \rangle_{\mathscr{S}'(\mathbb{R}^n), \mathscr{S}(\mathbb{R}^n)}}. \tag{32}
$$

The distribution a is called a symbol of the operator $\mathrm{Op}_Q a$.

As an example of quantization we have previously set $Q_0 := \mathrm{Op}_Q(\delta_{(0,0)})$. If the equality 32 makes sense in L^2 we write it using the usual inner product as

$$\langle Q(u, v), a \rangle = \langle u, (\mathrm{Op}_Q a)v \rangle. \tag{33}$$

Example 1 (Weyl–Wigner quantization) Choosing the time–frequency transform in the above formula as the Wigner transform we obtain the Weyl–Wigner quantization. The operator corresponding to the symbol $a \in \mathscr{S}(\mathbb{R}^{2n})$ is given by

$$(\mathrm{Op}_W a)u(x) = \iint_{\mathbb{R}^{2n}} \mathrm{e}^{\mathrm{i}2\pi(x-y)\cdot\eta} a([x+y]/2, \eta)u(y)\,\mathrm{d}y\,\mathrm{d}\eta, \tag{34}$$

where $u \in \mathscr{S}(\mathbb{R}^n)$.

Symbols in other quantizations have unique representations in terms of Weyl–Wigner symbols.

Lemma 5 *Given a Q-symbol $a \in \mathscr{S}(\mathbb{R}^{2n})$ and denoting $A = \mathrm{Op}_Q a : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$, there is a unique Weyl-Wigner symbol a_W such that $A = \mathrm{Op}_W a_W$. This symbol is given by*

$$a_W = F^{-1}(\overline{\phi_Q}Fa). \tag{35}$$

Proof Writing out the quantization formula for any $u, v \in \mathscr{S}(\mathbb{R}^n)$, we find

$$
\begin{aligned}
\langle \overline{Av}, u \rangle_{\mathscr{S}',\mathscr{S}} &= \langle Q(u, v), a \rangle \\
&= \langle FQ(u, v), Fa \rangle \\
&= \langle \phi_Q FW(u, v), Fa \rangle \\
&= \langle FW(u, v), \overline{\phi_Q}Fa \rangle \\
&= \langle W(u, v), F^{-1}(\overline{\phi_Q}Fa) \rangle,
\end{aligned}
$$

so the Weyl-Wigner symbol of the operator A is $a_W = F^{-1}(\overline{\phi_Q}Fa)$. $\qquad\square$

We note that if the symbol satisfies $\mathrm{supp}\, Fa \subset \{(y, \xi) : \phi_Q(y, \xi) = 0\}$, then $\mathrm{Op}_Q a = 0$. The quantization map may thus be non-injective.

So far we have seen that a time–frequency transform can be characterized by the sesquilinear form at the time–frequency origin and the time–frequency kernel ψ_Q. In addition to these, it suffices to give the quantization of symbols in $\mathscr{S}(\mathbb{R}^{2n})$ to specify the corresponding time–frequency transform.

Lemma 6 *If a quantization Op_Q is the quantization related to some time–frequency transform Q, this transform is uniquely specified by considering the quantization of symbols in $\mathscr{S}(\mathbb{R}^{2n})$.*

This lemma is just a consequence of the continuity of the quantization map and the Lemma 2. We recover the operator Q_0 of the transform given by the distribution

symbol $\delta_{(0,0)}$. Furthermore, the distribution kernel δ^Q of the operator Q_0 has the representation in terms of the Wigner time-delay kernel as

$$\overline{\delta^Q} = P_{1/2}^{-1} \iota_1 \varphi_Q.$$

5 Characterization of the Properties of Cohen-Class Time–Frequency Transforms

We show next how properties of a time–frequency transform are reflected in the associated kernel and quantization. We study also other ways of characterizing these properties. The conditions are usually formulated with respect to some of the following objects.

- The transform Q itself.
- The operator Q_0.
- Some of the kernels e.g. ϕ_Q or φ_Q.
- The quantization Op_Q.

There are extensive treatments of the various desirable properties of time–frequency transforms. Many such properties with the corresponding conditions for kernels have been studied first in relation to quantum physics in [19] and [16]. Further works in time–frequency analysis include [1,5,9,14]. An overview of quadratic time–frequency distributions with an extensive list of references can be found in [12].

We study the properties elaborating on the tables found in [1, pp. 146–147] and [9, p. 131].

5.1 Covariance Properties

The term covariant is commonly used when a time–frequency transform behaves regularly under a symplectic coordinate transform of the time–frequency plane. These are also the center-fixing automorphisms of the Heisenberg group, see [11, Sect. 9.4].

Definition 16 A time–frequency transform Q is covariant under a symplectic coordinate transform $\mathcal{A}$ of the time–frequency plane if

$$Q(u, v)(\mathcal{A}^{-1}(x, \eta)) = Q(\mu(\mathcal{A})u, \mu(\mathcal{A})v)(x, \eta) \tag{36}$$

holds for any $u, v \in \mathscr{S}(\mathbb{R}^n)$, where $\mu(\mathcal{A}) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a *symplectic operator* given by the *metaplectic representation* μ such that $\Pi \circ \mu(\mathcal{A}) = \mathcal{A}$ with the projection Π as in (16).

The symplectic operator $\mu(\mathcal{A})$ is defined up to a complex phase factor by applying Schur's lemma to the Schrödinger representation ρ. It is the unitary operator satisfying

$$\rho(\mathcal{A}(x, \eta)) = \mu(\mathcal{A})\rho(x, \eta)\mu(\mathcal{A})^{-1}, \tag{37}$$

see [11, Sect. 9.4]. By Schur's lemma, the metaplectic representation $\mathcal{A} \mapsto \mu(\mathcal{A})$ also satisfies

$$\mu(\mathcal{A}\mathcal{B}) = c\mu(\mathcal{A})\mu(\mathcal{B}), \tag{38}$$

where $|c| = 1$ is a complex phase factor.

Definition 16 is motivated by the full symplectic covariance of the Wigner transform as expressed in the following lemma:

Lemma 7 ([10], p. 180) *The Wigner transform satisfies*

$$W(u, v)(\mathcal{A}^{-1}(x, \eta)) = W(\mu(\mathcal{A})u, \mu(\mathcal{A})v)(x, \eta) \tag{39}$$

for any $u, v \in \mathscr{S}'(\mathbb{R}^n)$ and for any $\mathcal{A} \in \mathrm{Sp}(2n)$.

We will denote the symplectic transform acting on functions in the time–frequency or ambiguity plane also by $T_{\mathcal{A}}a(x, \eta) = a(\mathcal{A}(x, \eta))$.

Theorem 3 *Let $\mu(\mathcal{A})$ be a metaplectic operator such that $\Pi \circ \mu(\mathcal{A}) = \mathcal{A}$. The following conditions are equivalent for a time–frequency transform Q:*

1. *Time–frequency transform Q is covariant under $\mathcal{A}$.*
2. *The evaluation at origin Q_0 commutes with $\mu(\mathcal{A})$.*
3. *The ambiguity kernel ϕ_Q satisfies $\phi_Q = T_{\mathcal{A}^{-1}}\phi_Q$.*
4. *The quantization Op_Q satisfies $\mathrm{Op}_Q(T_{\mathcal{A}}a) = \mu(\mathcal{A})^{-1} \circ \mathrm{Op}_Q\, a \circ \mu(\mathcal{A})$.*

Proof *(a) $\Leftrightarrow$ (b)*: We have on one hand

$$
\begin{aligned}
Q(u, v)(\mathcal{A}^{-1}(x, \eta)) &= \langle \rho[\mathcal{A}^{-1}(x, \eta)]^{-1}u, Q_0\rho[\mathcal{A}^{-1}(x, \eta)]^{-1}v\rangle \\
&= \langle \mu(\mathcal{A})^{-1}\rho(x, \eta)^{-1}\mu(\mathcal{A})u, Q_0\mu(\mathcal{A})^{-1}\rho(x, \eta)^{-1}\mu(\mathcal{A})v\rangle \\
&= \langle \rho(x, \eta)^{-1}\mu(\mathcal{A})u, \mu(\mathcal{A})Q_0\mu(\mathcal{A})^{-1}\rho(x, \eta)^{-1}\mu(\mathcal{A})v\rangle \tag{40}
\end{aligned}
$$

and on the other hand

$$Q(\mu(\mathcal{A})u, \mu(\mathcal{A})v)(x, \eta) = \langle \rho(x, \eta)^{-1}\mu(\mathcal{A})u, Q_0\rho(x, \eta)^{-1}\mu(\mathcal{A})v\rangle \tag{41}$$

for all $u, v \in \mathscr{S}(\mathbb{R}^n)$. Assuming covariance under $\mathcal{A}$, we combine (40) and (41) into the equality

$$
\begin{aligned}
&\langle \rho(x, \eta)^{-1}\mu(\mathcal{A})u, \mu(\mathcal{A})Q_0\mu(\mathcal{A})^{-1}\rho(x, \eta)^{-1}\mu(\mathcal{A})v\rangle \\
&= \langle \rho(x, \eta)^{-1}\mu(\mathcal{A})u, Q_0\rho(x, \eta)^{-1}\mu(\mathcal{A})v\rangle
\end{aligned}
$$

which shows

$$\mu(\mathcal{A})Q_0\mu(\mathcal{A})^{-1} = Q_0$$

since $\mu(\mathcal{A})$ and $\rho(x, \eta)$ are isomorphisms mapping $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^n)$. Assuming *(b)* also implies *(a)* by the previous calculations.

(a) $\Leftrightarrow$ *(c)*: Taking the symplectic Fourier transform of $T_{\mathcal{A}^{-1}} Q$ we find

$$
\begin{aligned}
F T_{\mathcal{A}^{-1}} Q(u, v)(y, \xi) &= T_{\mathcal{A}^{-1}} F Q(u, v)(y, \xi) \\
&= T_{\mathcal{A}^{-1}} (\phi_Q \chi(u, v))(y, \xi) \\
&= T_{\mathcal{A}^{-1}} \phi_Q(y, \xi) T_{\mathcal{A}^{-1}} \chi(u, v))(y, \xi) \\
&= T_{\mathcal{A}^{-1}} \phi_Q(y, \xi) \langle u, \rho \circ \mathcal{A}^{-1}(y, \xi) v \rangle \\
&= T_{\mathcal{A}^{-1}} \phi_Q(y, \xi) \langle u, \mu(\mathcal{A})^{-1} \rho(y, \xi) \mu(\mathcal{A}) v \rangle \\
&= T_{\mathcal{A}^{-1}} \phi_Q(y, \xi) \langle \mu(\mathcal{A}) u, \rho(y, \xi) \mu(\mathcal{A}) v \rangle \\
&= T_{\mathcal{A}^{-1}} \phi_Q(y, \xi) \chi(\mu(\mathcal{A}) u, \mu(\mathcal{A}) v)(y, \xi).
\end{aligned}
$$

In addition, we have

$$
F Q(\mu(\mathcal{A}) u, \mu(\mathcal{A}) v)(y, \xi) = \phi_Q(y, \xi) \chi(\mu(\mathcal{A}) u, \mu(\mathcal{A}) v)(y, \xi)
$$

so by the Fourier inversion the condition $\phi_Q(y, \xi) = T_{\mathcal{A}^{-1}} \phi_Q(y, \xi)$ is equivalent to covariance under $\mathcal{A}$.

(a) $\Rightarrow$ *(d)*: Assuming the covariance we have on one hand

$$
\begin{aligned}
\langle Q(\mu(\mathcal{A}) u, \mu(\mathcal{A}) v), a \rangle &= \langle \mu(\mathcal{A}) u, (\mathrm{Op}_Q a) \mu(\mathcal{A}) v \rangle \\
&= \langle u, \mu(\mathcal{A})^{-1} (\mathrm{Op}_Q a) \mu(\mathcal{A}) v \rangle
\end{aligned}
$$

and on the other hand

$$
\begin{aligned}
\langle Q(\mu(\mathcal{A}) u, \mu(\mathcal{A}) v), a \rangle &= \langle T_{\mathcal{A}^{-1}} Q(u, v), a \rangle \\
&= \langle Q(u, v), T_{\mathcal{A}} a \rangle \\
&= \langle u, \mathrm{Op}_Q(T_{\mathcal{A}} a) v \rangle.
\end{aligned}
$$

Hence, if the transform is $\mathcal{A}$-covariant we have

$$
\langle u, \mu(\mathcal{A})^{-1} (\mathrm{Op}_Q a) \mu(\mathcal{A}) v \rangle = \langle u, \mathrm{Op}_Q(T_{\mathcal{A}} a) v \rangle
$$

for all $u, v \in \mathscr{S}(\mathbb{R}^n)$ which implies

$$
\mathrm{Op}_Q(T_{\mathcal{A}} a) = \mu(\mathcal{A})^{-1} \circ \mathrm{Op}_Q a \circ \mu(\mathcal{A}).
$$

(d) $\Rightarrow$ *(a)*: Assuming the quantization condition we find from the previous calculations

$$
\langle Q(\mu(\mathcal{A}) u, \mu(\mathcal{A}) v), a \rangle = \langle T_{\mathcal{A}^{-1}} Q(u, v), a \rangle
$$

for any $u, v \in \mathscr{S}(\mathbb{R}^n)$ and $a \in \mathscr{S}(\mathbb{R}^{2n})$. This implies the equality

$$Q(\mu(\mathcal{A})u, \mu(\mathcal{A})v) = T_{\mathcal{A}^{-1}} Q(u, v).$$

□

The full symplectic covariance is restrictive enough to single out the Weyl–Wigner quantization, which is proven, for instance, in Chap. 30 of [26]. This uniqueness result has also been further studied in [8]. To admit other transforms one has to consider covariance under certain subgroups of the symplectic group. The usual particular cases of covariance in the space $\mathbb{R}^n$ are given by rotations scalings and shears.

Example 2 Let S be a symmetric matrix in $\mathbb{R}^{n \times n}$. A time–frequency transform is covariant under time–frequency shears $\mathcal{A}(x, \eta) = (x, \eta + Sx)$ given by the chirp transforms $u(x) \mapsto e^{i\pi \langle Sx, x \rangle} u(x)$ if its ambiguity kernel satisfies

$$\phi_Q(y, \xi) = \phi_Q(y, \xi - Sy)$$

for all $(y, \xi) \in \mathbb{R}^{2n}$.

Next we consider the covariance under complex conjugation of the signals defined in [14, Eq. (2.39)].

Definition 17 Time–frequency transform Q is covariant under complex conjugation if $Q(\overline{u}, \overline{v})(x, \eta) = \overline{Q(u, v)(x, -\eta)}$.

Theorem 4 *The following conditions are equivalent for a time–frequency transform Q:*

1. *Time–frequency transform Q is covariant under complex conjugation.*
2. *The evaluation at origin Q_0 commutes with complex conjugation.*
3. *The ambiguity kernel ϕ_Q satisfies $\phi_Q(y, \xi) = \overline{\phi_Q(y, -\xi)}$.*
4. *The quantization Op_Q satisfies $\mathrm{Op}_Q(\iota_2(\overline{a}))u = \overline{\mathrm{Op}_Q(a)(\overline{u})}$.*

Proof *(a)* ⇔ *(c)*: We find the equivalence of covariance and the kernel condition using the symplectic Fourier transform by

$$FQ(\overline{u}, \overline{v})(y, \xi) = \phi_Q(y, \xi)\chi(\overline{u}, \overline{v})(y, \xi)$$
$$= \phi_Q(y, \xi)\overline{\chi(u, v)(y, -\xi)}$$

and

$$F\overline{\iota_2 Q(u, v)}(y, \xi) = \overline{\iota_2 FQ(u, v)(y, \xi)}$$
$$= \overline{\phi_Q(y, -\xi)\chi(u, v)(y, -\xi)}.$$

(b) ⇔ *(c)*: The Schrödinger representation satisfies

$$\overline{\rho(y, \xi)u(x)} = \overline{e^{-i\pi y \cdot \xi + i2\pi x \cdot \xi}u(x - y)}$$
$$= \rho(y, -\xi)\overline{u(x)} \tag{42}$$

and the evaluation at origin may be expressed by

$$Q(u, v)(0, 0) = \iint_{\mathbb{R}^{2n}} FQ(u, v)(y, \xi) \, dy \, d\xi$$

$$= \iint_{\mathbb{R}^{2n}} \phi_Q(y, \xi) \chi(u, v)(y, \xi) \, dy \, d\xi. \tag{43}$$

Applying (21), (42) and (43), we find that the condition for the evaluation Q_0 and the kernel condition are equivalent by

$$Q_0 \overline{u}(x) = \overline{\iint_{\mathbb{R}^{2n}} \overline{\phi_Q(y, \xi)} \rho(y, \xi) \overline{u(x)} \, dy \, d\xi}$$

$$= \overline{\iint_{\mathbb{R}^{2n}} \phi_Q(y, \xi) \rho(y, -\xi) u(x) \, dy \, d\xi}$$

$$= \iint_{\mathbb{R}^{2n}} \phi_Q(y, -\xi) \rho(y, \xi) u(x) \, dy \, d\xi$$

and

$$\overline{Q_0 u(x)} = \overline{\iint_{\mathbb{R}^{2n}} \overline{\phi_Q(y, \xi)} \rho(y, \xi) u(x) \, dy \, d\xi}.$$

$(a) \Leftrightarrow (d)$: The quantization condition is equivalent to the covariance by

$$\langle u, \mathrm{Op}_Q(\iota_2(\overline{a})) v \rangle = \langle Q(u, v), \iota_2(\overline{a}) \rangle$$

$$= \langle \iota_2 Q(u, v), \overline{a} \rangle$$

$$= \overline{\langle \iota_2 Q(u, v), a \rangle}$$

and

$$\langle u, \overline{\mathrm{Op}_Q(a)(\overline{v})} \rangle = \overline{\langle \overline{u}, \mathrm{Op}_Q(a)(\overline{v}) \rangle}$$

$$= \overline{\langle Q(\overline{u}, \overline{v}), a \rangle}.$$

$\square$

Example 3 (Particular examples of kernel conditions) The kernel condition for the time–frequency kernel ψ_Q is similar to the condition for the ambiguity kernel as can be seen by

$$F\psi_Q = \phi_Q = T_{\mathcal{A}}\phi_Q = T_{\mathcal{A}}F\psi_Q = FT_{\mathcal{A}}\psi_Q.$$

The Rihaczek ambiguity kernel ϕ_Q^0 (τ-Wigner with $\tau = 0$) is given in terms of the (Wigner) ambiguity kernel ϕ_Q by $\phi_Q^0(y, \xi) = e^{i\pi y \cdot \xi} \phi_Q(y, \xi)$. As an example, the

kernel condition for the covariance under complex conjugation becomes

$$
\begin{aligned}
\phi_Q^0(y, \xi) &= e^{i\pi y \cdot \xi} \phi_Q(y, \xi) \\
&= e^{i\pi y \cdot \xi} \overline{\phi_Q(y, -\xi)} \\
&= \overline{e^{i\pi y \cdot (-\xi)} \phi_Q(y, -\xi)} \\
&= \overline{\phi_Q^0(y, -\xi)}.
\end{aligned}
$$

5.2 Energy Distribution Properties

An ideal time–frequency distribution would measure the energy content of a signal at a given time and frequency. However, such an ideal density cannot exist. For instance, positivity and correct marginal densities of a time–frequency distribution are mutually exclusive as was first proven in [25]. In addition to this, the Heisenberg uncertainty principle shows that a single point in the time–frequency plane cannot be interpreted as a single frequency component located precisely at the given time.

It is possible to try to mimic an energy distribution by requiring the time–frequency distribution to satisfy different properties characteristic to an energy distribution. We will discuss several of these properties starting with the requirement of the distribution being real-valued.

Symmetry:

Definition 18 A time–frequency transform Q is symmetric if it satisfies

$$
Q(u, v) = \overline{Q(v, u)}
$$

for any $u, v \in \mathscr{S}(\mathbb{R}^n)$.

If a time–frequency transform Q is symmetric, then the time–frequency distributions $Q[u]$ are real-valued for any $u \in \mathscr{S}(\mathbb{R}^n)$.

Theorem 5 *The following conditions are equivalent for a time–frequency transform* Q.

1. *Time–frequency transform Q is symmetric.*
2. *The evaluation at origin Q_0 is a symmetric operator.*
3. *The ambiguity kernel ϕ_Q satisfies $\overline{\phi_Q(y, \xi)} = \phi_Q(-y, -\xi)$.*
4. *The quantization Op_Q satisfies $\mathrm{Op}_Q(\overline{a}) = (\mathrm{Op}_Q a)^*$.*

Proof *(a)* $\Rightarrow$ *(b)*: Assume first that the transform is symmetric. Then the operator Q_0 is symmetric by

$$
\langle u, Q_0 v \rangle = Q(u, v)(0, 0) = \overline{Q(v, u)(0, 0)} = \overline{\langle v, Q_0 u \rangle} = \langle Q_0 u, v \rangle
$$

which holds for all $u, v \in \mathscr{S}(\mathbb{R}^n)$.

(b) $\Rightarrow$ *(a)*: Conversely, we have by shift-covariance

$$
\begin{aligned}
Q(u, v)(x, \eta) &= \langle T_{-x} M_{-\eta} u, \, Q_0 T_{-x} M_{-\eta} v \rangle \\
&= \langle Q_0 T_{-x} M_{-\eta} u, \, T_{-x} M_{-\eta} v \rangle \\
&= \overline{\langle T_{-x} M_{-\eta} v, \, Q_0 T_{-x} M_{-\eta} u \rangle} \\
&= \overline{Q(v, u)(x, \eta)}
\end{aligned}
$$

for all $u, v \in \mathscr{S}(\mathbb{R}^n)$.

(a) $\Leftrightarrow$ *(c)*: Equivalence of the kernel condition and the symmetry of the transform follows by taking symplectic Fourier transforms. The ambiguity kernel is defined by

$$
F Q(u, v)(y, \xi) = \phi_Q(y, \xi) \chi(u, v)(y, \xi).
$$

Taking the symplectic Fourier transform of the function $\overline{Q(v, u)}$ we find

$$
\begin{aligned}
F(\overline{Q(v, u)})(y, \xi) &= \overline{\phi_Q(-y, -\xi) \chi(v, u)(-y, -\xi)} \\
&= \overline{\phi_Q(-y, -\xi)} \chi(u, v)(y, \xi),
\end{aligned}
$$

where the last equality follows by the basic properties of the ambiguity transform. Hence, if the transform is symmetric we have the kernel condition

$$
\phi_Q(y, \xi) = \overline{\phi_Q(-y, -\xi)}
$$

and also the converse by Fourier inversion.

(a) $\Rightarrow$ *(d)*: Assuming symmetry of the transform we find

$$
\begin{aligned}
\langle u, (\mathrm{Op}_Q a) v \rangle &= \langle Q(u, v), a \rangle \\
&= \langle \overline{Q(v, u)}, a \rangle \\
&= \overline{\langle Q(v, u), \overline{a} \rangle} \\
&= \overline{\langle v, (\mathrm{Op}_Q \overline{a}) u \rangle} \\
&= \langle (\mathrm{Op}_Q \overline{a}) u, v \rangle
\end{aligned}
$$

which shows that the quantization is symmetric.

(d) $\Rightarrow$ *(b)*: If the quantization is symmetric we have for any real-valued symbol $a \in \mathscr{S}(\mathbb{R}^{2n})$ the equality

$$
\langle u, (\mathrm{Op}_Q a) v \rangle = \langle (\mathrm{Op}_Q a) u, v \rangle.
$$

In particular, taking the sequence of real-valued symbols converging to the Dirac delta distribution $\delta_{(0,0)}$ we find

$$
\begin{aligned}
\langle u, Q_0 v \rangle &= \lim_\alpha \langle Q(u, v), a_\alpha \rangle \\
&= \lim_\alpha \langle u, (\mathrm{Op}_Q a_\alpha) v \rangle \\
&= \lim_\alpha \overline{\langle v, (\mathrm{Op}_Q a_\alpha) u \rangle} \\
&= \lim_\alpha \overline{\langle Q(v, u), a_\alpha \rangle} \\
&= \overline{\langle v, Q_0 u \rangle} \\
&= \langle Q_0 u, v \rangle,
\end{aligned}
$$

for any $u, v \in \mathscr{S}(\mathbb{R}^n)$ which proves the claim. $\qquad\square$

Remark 3 The Rihaczek ambiguity kernel ϕ_Q^0 yields a symmetric transform Q if it satisfies

$$
\phi_Q^0(y, \xi) = e^{i2\pi y \cdot \xi} \overline{\phi_Q^0(-y, -\xi)}. \tag{44}
$$

This equation has the expression in terms of the Rihaczek time-delay kernel φ_Q^0 as

$$
\begin{aligned}
\varphi_Q^0(x, y) &= \int_{\mathbb{R}^n} e^{i2\pi x \cdot \xi} \phi_Q^0(y, \xi) \, d\xi \\
&= \int_{\mathbb{R}^n} e^{i2\pi (y+x) \cdot \xi} \overline{\phi_Q^0(-y, -\xi)} \, d\xi \\
&= \overline{\varphi_Q^0(x + y, -y)}.
\end{aligned}
$$

Positivity: As a function of time and frequency, the transform should be positive if it represents an energy density of a given signal. Hence, we say that a time–frequency transform Q is positive if $Q[u](x, \eta) \geq 0$ for all $u \in \mathscr{S}(\mathbb{R}^n)$ and $(x, \eta) \in \mathbb{R}^{2n}$. There are several articles and results about positivity and many references can be found in the survey [14].

As was the case in Theorem 5, the pointwise values of a Cohen-class time–frequency distribution Q are related to the numerical range of the operator Q_0. In particular, if $Q_0 : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ we may write

$$
Q[u](0, 0) = \langle u, Q_0 u \rangle.
$$

An example of a positive time–frequency transform is obtained by setting $Q_0 = P$ for some projection in $L^2(\mathbb{R}^n)$ as is seen by

$$
Q[u](0, 0) = \langle u, Pu \rangle = \langle u, P^2 u \rangle = \langle Pu, Pu \rangle = \|Pu\|^2 \geq 0.
$$

Any spectrogram is such a transform since for the w-windowed spectrogram the operator Q_0 is the projection to the one-dimensional subspace spanned by w. Another

example is given by the projection onto odd functions defined by $Pu(x) = [u(x) - u(-x)]/2$. This results in

$$Q[u](x, \eta) = \frac{\|u\|^2 \mathbf{1}(x, \eta)}{2} - \frac{W[u](x, \eta)}{4}.$$

Related to positive time–frequency transforms we define the positivity of a quantization.

Definition 19 The quantization Op_Q is said to be positive if every non-negative symbol $a(x, \eta)$ yields a positive operator $\mathrm{Op}_Q a$.

Referring to this definition we formulate the simple observations about positivity into a theorem.

Theorem 6 *The following conditions are equivalent for a time–frequency transform Q:*

1. *Time–frequency transform Q is positive.*
2. *The evaluation at origin Q_0 is a positive operator.*
3. *The quantization Op_Q is positive.*

Proof *(a) $\Leftrightarrow$ (b)*: The equivalence of the first conditions is immediate by the shift-covariance of the transform.

   (a) $\Rightarrow$ (c): By the definition of Op_Q we obtain

$$\langle u, (\mathrm{Op}_Q a)u \rangle = \langle Q[u], a \rangle \geq 0$$

for all test functions u.

   (c) $\Rightarrow$ (b): The delta distribution $\delta_{(0,0)}$ can be approximated, for instance, by Gaussian functions. Let $\lim_\alpha a_\alpha = \delta_{(0,0)}$. Then

$$\langle u, Q_0 u \rangle = \langle Q[u], \delta_{(0,0)} \rangle = \lim_\alpha \langle Q[u], a_\alpha \rangle = \lim_\alpha \langle u, \mathrm{Op}_Q a_\alpha u \rangle \geq 0$$

by the positivity of the quantization. □

Concerning the form of the ambiguity kernel, we find a simple characterization for the case of trace-class Q_0. This proposition is inspired by [9, p. 118].

Proposition 2 *Let Q be a positive time–frequency transform on $L^2(\mathbb{R}^n)$. If the corresponding operator Q_0 is a trace class operator, then the ambiguity kernel $\phi_Q(y, \xi)$ is a weighted sum of ambiguity transforms. Hence, the transform itself is a weighted sum of spectrograms.*

Proof If Q_0 is a trace class operator there is a total orthonormal set $\{f_k\}$ such that the spectral decomposition formula

$$Q_0 u = \sum_{k \geq 0} q_k \langle u, f_k \rangle f_k$$

holds with $(q_k)_{k=0}^{\infty} \in \ell^1$. The positive square root A of Q_0 given by $a_k = \sqrt{q_k}$ is then a Hilbert-Schmidt operator and

$$Q_0 u(x) = A^* A u(x) = \sum_{k \geq 0} q_k \langle u, f_k \rangle f_k = \int_{\mathbb{R}^n} K(x, y) u(y) \, dy,$$

where $K(x, y) \in L^2(\mathbb{R}^{2n})$ and

$$K(x, y) = \sum_{k \geq 0} q_k f_k(x) \overline{f_k(y)}.$$

The kernel of the operator Q_0 can also be given by the time-delay kernel as

$$K(x, y) = P_{1/2}^{-1} \iota_1 \varphi_Q(x, y).$$

Taking the coordinate transform $P_{1/2}$ on both sides we find

$$\varphi_Q(-x, y) = P_{1/2} K(x, y) = \sum_{k \geq 0} q_k f_k(x + y/2) \overline{f_k(x - y/2)} \tag{45}$$

and by the inverse Fourier transform in the x-variable we have

$$\phi_Q(y, \xi) = \sum_{k \geq 0} q_k \int_{\mathbb{R}^n} e^{-i2\pi x \cdot \xi} f_k(x + y/2) \overline{f_k(x - y/2)} \, dx$$
$$= \sum_{k \geq 0} q_k \chi[f_k](y, \xi),$$

which completes the proof.   □

In addition to real-valuedness and positivity, pointwise boundedness of a time–frequency transform can be directly seen from Q_0. If the operator Q_0 is bounded in $L^2(\mathbb{R}^n)$, then the time–frequency transform Q satisfies the bound

$$\|Q(u, v)\|_{L^\infty(\mathbb{R}^{2n})} \leq \|Q_0\| \|u\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)}$$

and is thus bounded for finite-energy signals.

Distribution integrates to the total energy: The integral of an energy distribution over the whole space yields the total energy. This should also be satisfied by an ideal time–frequency distribution. We say that a time–frequency distribution is normalized if it satisfies

$$\iint_{\mathbb{R}^{2n}} Q[u](x, \eta) \, dx \, d\eta = \|u\|^2 \tag{46}$$

for any $u \in L^2(\mathbb{R}^n)$. This is connected to the calculation of traces of operators under the Q-quantization.

The distribution kernel of the operator $A = \text{Op}_{W_\tau} a$ given by the τ-Wigner symbol a is [21, p. 202]

$$K_A(x, y) = \int_{\mathbb{R}^n} e^{i2\pi(x-y)\cdot\eta} a((1 - \tau)x + \tau y, \eta) \, d\eta.$$

If the operator is trace-class its trace is thus given by

$$\text{Tr}(A) = \int_{\mathbb{R}^n} K_A(x, x) \, dx = \iint_{\mathbb{R}^{2n}} a(x, \eta) \, dx \, d\eta.$$

We say that a quantization Q has correct traces if it satisfies the previous formula for every symbol $a \in \mathscr{S}(\mathbb{R}^{2n})$.

Theorem 7 *The following conditions are equivalent for a time–frequency transform Q:*

1. *Time–frequency transform Q is normalized.*
2. *The ambiguity kernel ϕ_Q satisfies $\phi_Q(0, 0) = 1$.*
3. *The quantization Op_Q has correct traces.*

Proof *(a) $\Leftrightarrow$ (b)*: The equivalence follows by noting that the integral is just the evaluation of the symplectic Fourier transform at the origin. Hence, we have

$$\iint_{\mathbb{R}^{2n}} Q(u, v) \, d\eta \, dx = FQ(u, v)(0, 0) = \phi_Q(0, 0)\chi(u, v)(0, 0) = \phi_Q(0, 0)\langle u, v \rangle.$$

(b) $\Leftrightarrow$ (c): We find using the formula for the Weyl-Wigner symbol

$$\begin{aligned}
\text{Tr}(\text{Op}_W a_W) &= \iint_{\mathbb{R}^{2n}} a_W(x, \eta) \, d\eta \, dx \\
&= \iint_{\mathbb{R}^{2n}} F^{-1}(\overline{\phi_Q} Fa)(x, \eta) \, d\eta \, dx \\
&= \overline{\phi_Q(0, 0)} Fa(0, 0) \\
&= \overline{\phi_Q(0, 0)} \iint_{\mathbb{R}^{2n}} a(x, \eta) \, d\eta \, dx
\end{aligned}$$

for any $a \in \mathscr{S}(\mathbb{R}^{2n})$. Hence, if the kernel condition is satisfied we have *(c)*. Conversely, assume that the trace condition holds. Then

$$\text{Tr}(A) = \iint_{\mathbb{R}^{2n}} a(x, \eta) \, d\eta \, dx = \overline{\phi_Q(0, 0)} \iint_{\mathbb{R}^{2n}} a(x, \eta) \, d\eta \, dx = \overline{\phi_Q(0, 0)} \text{Tr}(A)$$

for any operator A given by Q-quantization by the Schwartz symbol a. Hence, we have $\phi_Q(0, 0) = 1$. $\qquad\square$

Remark 4 If the evaluation at origin operator Q_0 is trace class, then we may write using the same notation as in (45),

$$\mathrm{Tr}(Q_0) = \sum_{k \geq 0} q_k = \int_{\mathbb{R}^n} \varphi_Q(-x, 0)\, \mathrm{d}x = \phi_Q(0, 0).$$

Thus, such transform Q is normalized if and only if $\mathrm{Tr}(Q_0) = 1$.

Marginal distributions: Integrating a time–frequency energy distribution either over time or frequency should yield the corresponding marginal distribution in frequency or time, respectively.

Definition 20 A time–frequency transform has the *correct frequency marginal distribution* if it satisfies

$$\int_{\mathbb{R}^n} Q[u](x, \eta)\, \mathrm{d}x = |\widehat{u}(\eta)|^2 \tag{47}$$

for all $u \in \mathscr{S}(\mathbb{R}^n)$, and the *correct time marginal distribution* if

$$\int_{\mathbb{R}^n} Q[u](x, \eta)\, \mathrm{d}\eta = |u(x)|^2 \tag{48}$$

We characterize next the time–frequency distributions which have the correct marginal distributions in time and frequency.

Theorem 8 *The following conditions are equivalent for a time–frequency transform Q:*

1. *Time–frequency transform Q has the correct frequency marginal distribution.*
2. *The ambiguity kernel ϕ_Q satisfies $\phi_Q(y, 0) = 1$.*
3. *Symbols a of the form $a(x, \eta) = \widehat{f}(\eta)$, where $f \in \mathscr{S}(\mathbb{R}^n)$, map to convolution operators*

$$(\mathrm{Op}_Q a)v = f * v.$$

Proof *(b)* $\Rightarrow$ *(a)*: The ambiguity kernel condition gives

$$\int_{\mathbb{R}^n} Q(u, v)(x, \eta)\, \mathrm{d}x = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}2\pi y \cdot \eta} \phi_Q(y, 0) \chi(u, v)(y, 0)\, \mathrm{d}y$$

$$= \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}2\pi y \cdot \eta} \chi(u, v)(y, 0)\, \mathrm{d}y$$

$$= \widehat{u}(\eta)\overline{\widehat{v}(\eta)}$$

for any Schwartz test functions u and v.

(a) $\Rightarrow$ *(b)*: Conversely, by the previous calculation we find

$$\int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}2\pi y \cdot \eta} \phi_Q(y, 0) \chi[u](y, 0)\, \mathrm{d}y = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}2\pi y \cdot \eta} \chi[u](y, 0)\, \mathrm{d}y$$

and by the Fourier inversion

$$(1 - \phi_Q(y, 0))\chi[u](y, 0) = 0$$

for every $y \in \mathbb{R}^n$ and every Schwartz test function u which implies the claim.

(b) $\Rightarrow$ (c): Assuming the ambiguity kernel condition we find for the quantization

$$
\begin{aligned}
\langle Q(u, v), \mathbf{1} \otimes \widehat{f} \rangle &= \langle \phi_Q \chi(u, v), f \otimes \delta_0 \rangle \\
&= \int_{\mathbb{R}^n} \phi_Q(y, 0)\overline{f(y)} \int_{\mathbb{R}^n} u(x + y/2)\overline{v(x - y/2)} \, dx \, dy \\
&= \int_{\mathbb{R}^n} \phi_Q(x - y, 0)\overline{f(x - y)} \int_{\mathbb{R}^n} u(x)\overline{v(y)} \, dx \, dy \\
&= \int_{\mathbb{R}^n} u(x) \int_{\mathbb{R}^n} \overline{v(y)} \underbrace{\phi_Q(x - y, 0)\, \overline{f(x - y)}}_{=1} \, dy \, dx \\
&= \langle u, f * v \rangle.
\end{aligned}
$$

(c) $\Rightarrow$ (b): Conversely, if frequency-dependent symbols map to convolution operators the previous calculation leads to

$$
\begin{aligned}
&\int_{\mathbb{R}^n} u(x) \int_{\mathbb{R}^n} \phi_Q(x - y, 0)\overline{f(x - y)v(y)} \, dy \, dx \\
&= \langle Q(u, v), \mathbf{1} \otimes \widehat{f} \rangle \\
&= \langle u, f * v \rangle \\
&= \int_{\mathbb{R}^n} u(x) \int_{\mathbb{R}^n} \overline{f(x - y)v(y)} \, dy \, dx
\end{aligned}
$$

for any $u, v \in \mathscr{S}(\mathbb{R}^n)$ and convolution kernel $f \in \mathscr{S}(\mathbb{R}^n)$, which implies the claim. $\quad\square$

The characterization of correct time marginal distribution is similar.

Theorem 9 *The following conditions are equivalent for a time–frequency transform Q:*

1. *Time–frequency transform Q has the correct time marginal distribution.*
2. *The ambiguity kernel ϕ_Q satisfies $\phi_Q(0, \xi) = 1$.*
3. *Symbols a of the form $a(x, \eta) = g(x)$, where $g \in \mathscr{S}(\mathbb{R}^n)$, map to multiplication operators*

$$(\mathrm{Op}_Q a)v = gv.$$

Proof *(b)* ⇔ *(a)*: We assume first the ambiguity kernel condition and find

$$\int_{\mathbb{R}^n} Q(u,v)(x,\eta)\,\mathrm{d}\eta = \int_{\mathbb{R}^n} e^{\mathrm{i}2\pi x\cdot\xi}\phi_Q(0,\xi)\chi(u,v)(0,\xi)\,\mathrm{d}\xi$$

$$= \int_{\mathbb{R}^n} e^{\mathrm{i}2\pi x\cdot\xi}\chi(u,v)(0,\xi)\,\mathrm{d}\xi$$

$$= u(x)\overline{v(x)}.$$

The converse follows by a similar argument as in the case of the frequency marginal distribution.

(b) ⇒ *(c)*: The quantization property follows from the ambiguity kernel condition by

$$\langle Q(u,v), g\otimes\mathbf{1}\rangle = \langle \phi_Q\cdot\chi(u,v), \delta_0\otimes\widehat{g}\rangle$$

$$= \int_{\mathbb{R}^n}\underbrace{\phi_Q(0,\xi)}_{=1}\overline{\widehat{g}(\xi)}\int_{\mathbb{R}^n} e^{-\mathrm{i}2\pi x\cdot\xi}u(x)\overline{v(x)}\,\mathrm{d}x\,\mathrm{d}\xi$$

$$= \int_{\mathbb{R}^n} u(x)\overline{g(x)v(x)}\,\mathrm{d}x$$

$$= \langle u, gv\rangle.$$

(c) ⇒ *(b)*: Conversely we find

$$\int_{\mathbb{R}^n} u(x)\overline{g(x)v(x)}\,\mathrm{d}x = \langle u, gv\rangle$$

$$= \langle Q(u,v), g\otimes\mathbf{1}\rangle$$

$$= \int_{\mathbb{R}^n} u(x)\int_{\mathbb{R}^n} e^{-\mathrm{i}2\pi x\cdot\xi}\phi_Q(0,\xi)\overline{\widehat{g}(\xi)}\,\mathrm{d}\xi\,\overline{v(x)}\,\mathrm{d}x$$

for any $u, v \in \mathscr{S}(\mathbb{R}^n)$ and $g \in \mathscr{S}'(\mathbb{R}^n)$. Hence,

$$g(x) = \int_{\mathbb{R}^n} e^{\mathrm{i}2\pi x\cdot\xi}\overline{\phi_Q(0,\xi)}\widehat{g}(\xi)\,\mathrm{d}\xi$$

and by the Fourier inversion $\phi_Q(0,\xi) = 1$ for any $\xi \in \mathbb{R}^n$.   □

Remark 5 The correct marginal distributions can be characterized by the time-delay kernel φ_Q. For correct frequency marginal distribution it has to satisfy

$$\int_{\mathbb{R}^n} \varphi_Q(x,y)\,\mathrm{d}x = 1$$

for any $y \in \mathbb{R}^n$. The time-delay condition for the correct time marginal distribution is

$$\varphi_Q(x,0) = \delta_0(x).$$

Higher moments: Some of the moments have been used in applications previously. In particular, calculations for the second moments can be found in [7] and [18]. These are usually related to modeling the signal using the instantaneous frequency and instantaneous amplitude, see [1, Chap. 2]. For a time–frequency transform in $\mathbb{R}$ the frequency moments are

$$
\int_{\mathbb{R}} \eta^k Q(u, v)(x, \eta) \, d\eta = \int_{\mathbb{R}} \eta^k F^{-1}(\phi_Q \chi (u, v))(x, \eta) \, d\eta
$$
$$
= \int_{\mathbb{R}} e^{-i2\pi x \cdot \xi} \frac{1}{(i2\pi)^k} \partial_y^k \left[\phi_Q(y, \xi) \chi(u, v)(y, \xi) \right]_{y=0} \, d\xi
$$

for any $u, v \in \mathscr{S}(\mathbb{R})$. The transform has thus the same frequency moments as the Wigner transform up to kth order if

$$
\phi_Q(0, \xi) = 1 \quad \text{and} \quad (\partial_y^l \phi_Q)(0, \xi) = 0
$$

for all $1 \leq l \leq k$ and $\xi \in \mathbb{R}$.

Given a single-component signal $u(x) = a(x)e^{i\theta(x)}$, the first moment of the Wigner transform is

$$
\int_{\mathbb{R}} \eta W[u](x, \eta) \, d\eta = \frac{1}{2\pi} \theta'(x) |a(x)|^2
$$

which corresponds to the mean frequency at x as can be seen by the remark in [9, p. 30]. We may thus formulate that the transform Q conserves instantaneous frequency if

$$
\int_{\mathbb{R}} \eta Q(u, v)(x, \eta) \, d\eta = \int_{\mathbb{R}} \eta W(u, v)(x, \eta) \, d\eta,
$$

which leads to the conditions

$$
\phi_Q(0, \xi) = 1 \quad \text{and} \quad (\partial_y \phi_Q)(0, \xi) = 0
$$

for all $\xi \in \mathbb{R}$. The conditions are similar for the time moment.

The second moments would ideally be positive and describe the spread of the signal in time or frequency. The higher moments of the Wigner transform do not satisfy this requirement so different definitions for the correct second moments have been proposed. For a discussion about these we refer to [18] and the references therein.

5.3 Support Properties

A time–frequency transform should be able to detect the times and frequencies where the signal has energy. The Wigner transform conserves supports in the following sense [10, Proposition (1.97)].

Proposition 3 *Let $\pi_1(x, \eta) = x$ and $\pi_2(x, \eta) = \eta$ be the projections to the axes in the time–frequency plane. Let $H(E)$ denote the closed convex hull of $E \subset \mathbb{R}^n$. Then*

$$\pi_1(\operatorname{supp} W[u]) \subset H(\operatorname{supp} u),$$
$$\pi_2(\operatorname{supp} W[u]) \subset H(\operatorname{supp} \widehat{u}).$$

Since in the one-dimensional case the convex hull of the support is the smallest closed interval containing it, we call $H(\operatorname{supp} u)$ the *lifetime* of the signal. A time–frequency transform is said to conserve the lifetime of a signal if it satisfies

$$\pi_1(\operatorname{supp} Q[u]) \subset H(\operatorname{supp} u).$$

The transforms satisfying this property are characterized by their time-delay kernels.

Lemma 8 *A time–frequency transform Q conserves the lifetime of the signal if and only if its time-delay kernel φ_Q satisfies*

$$\operatorname{supp} \varphi_Q \subset \{(x, y) \in \mathbb{R}^{2n} : x = (\tau - 1/2)y, \ \tau \in [0, 1]\}.$$

Proof The distribution $Q[u]$ can be given by its time-delay kernel as

$$Q[u](x, \eta) = \int_{\mathbb{R}^n} e^{-i2\pi y \cdot \eta} \int_{\mathbb{R}^n} \varphi_Q(t - x, y) u(t + y/2) \overline{u(t - y/2)} \, dt \, dy.$$

Denoting $U = H(\operatorname{supp} u)$ we find that the lifetime of the function $t \mapsto u(t + y/2)\overline{u(t - y/2)}$ is the set $K_y = (U - y/2) \cap (U + y/2)$. Let L_y be the support of the distribution $t \mapsto \varphi_Q(-t, y)$. The support of the function

$$x \mapsto \int_{\mathbb{R}^n} \varphi_Q(t - x, y) u(t + y/2) \overline{u(t - y/2)} \, dt \tag{49}$$

is then given by $K_y + L_y$. The condition for the conservation of the lifetime of u becomes now

$$K_y + L_y \subset U$$

for all $y \in \mathbb{R}^n$. For this inclusion to hold, the map (49) may increase the support only in the direction of y by no more than $y/2$. Otherwise there are signals for any y for which the lifetime increases. This leads to

$$\operatorname{supp} \varphi_Q \subset \{(x, y) \in \mathbb{R}^{2n} : x = \tau y, \ \tau \in [-1/2, 1/2]\}$$

or using the convention compatible with the τ-Wigner transform

$$\operatorname{supp} \varphi_Q \subset \{(x, y) \in \mathbb{R}^{2n} : x = (\tau - 1/2)y, \ \tau \in [0, 1]\},$$

completing the proof. $\qquad\square$

Example 4 Let f be a finite Borel measure on $\mathbb{R}$ considered as a distribution and $g \in \mathscr{O}_M(\mathbb{R}^n)$. Define a family of time–frequency transforms by $Q(u, v) = \mathcal{F}_2 r(u, v)$, where, abusing the notation of the distribution f slightly, we set formally

$$r(u, v)(x, y) = g(y) \int_{\mathbb{R}} f(\tau) u(x + \tau y) \overline{v(x + (\tau - 1)y)} \, d\tau.$$

Such transforms conserve the lifetime of the signal if and only if supp $f \subset [0, 1]$. The time-delay kernel is $\varphi_Q(x, y) = g(y)\delta_{(\tau - 1/2)y}(x) f$ and the distribution $Q[u]$ may be written for any $u \in \mathscr{S}(\mathbb{R}^n)$ as

$$
\begin{aligned}
Q[u](x, \eta) &= \int_{\mathbb{R}^n} e^{-i2\pi y \cdot \eta} \int_{\mathbb{R}} f(\tau) g(y) u(x + \tau y) \overline{u(x + (\tau - 1)y)} \, d\tau \, dy \\
&= \int_{\mathbb{R}} f(\tau) \int_{\mathbb{R}^n} e^{-i2\pi y \cdot \eta} g(y) u(x + \tau y) \overline{u(x + (\tau - 1)y)} \, dy \, d\tau \\
&= f(\widehat{g} *_2 W_\tau[u]).
\end{aligned}
$$

Setting $g = 1$ and $f = \delta_{1/2}$ one obtains the Wigner distribution. Other examples are presented in Sect. 6.

Remark 6 The strict conservation of supports is more restrictive and only frequency-averaged sums of W_0 and W_1 satisfy it. This result can be found in [9, p.125].

5.4 Unitarity

Definition 21 The time–frequency transform Q is *unitary* or satisfies the *Moyal property* if

$$\langle Q(u, v), Q(f, g) \rangle = \langle u, f \rangle \langle g, v \rangle$$

for all $u, v, f, g \in L^2(\mathbb{R}^n)$.

Interpreting this property in terms of the Q-quantization with the assumption $\|u\| = \|v\| = 1$ we have

$$\langle Q[u], Q[v] \rangle = \langle u, v \rangle \langle v, u \rangle = \langle u, \langle u, v \rangle v \rangle = \langle u, P_v u \rangle.$$

Hence, the symbol $Q[v]$ in the Q-quantization yields the projection operator P_v. An interesting example is given by the identity operator in a Hilbert space with a total orthonormal set h_α. The identity operator is given by the sum

$$I = \sum_\alpha P_{h_\alpha}$$

so we find

$$\langle u, Iv \rangle = \sum_\alpha \langle u, P_{h_\alpha} v \rangle = \sum_\alpha \langle Q(u, v), Q[h_\alpha] \rangle = \langle Q(u, v), a \rangle,$$

with the Q-symbol of the identity as $a = \sum_\alpha Q[v_\alpha]$. If Q is normalized then the identity operator is given by the constant symbol $\mathbf{1}$, so we conclude

$$\sum_\alpha Q[v_\alpha] = \mathbf{1}.$$

We call the quantization Op_Q unitary if it satisfies

$$\langle a_1, a_2 \rangle_{L^2(\mathbb{R}^{2n})} = \langle \mathrm{Op}_Q a_1, \mathrm{Op}_Q a_2 \rangle_{HS} := \mathrm{Tr}[\mathrm{Op}_Q a_1 \circ (\mathrm{Op}_Q a_2)^*]$$

for all $a_1, a_2 \in L^2(\mathbb{R}^{2n})$ and where the subscript HS refers to the Hilbert–Schmidt inner product of operators. If the distribution associated with a unitary quantization is interpreted as a probability distribution, we find the expectation $\langle a \rangle_Q$ of a phase space function a as

$$\langle a \rangle_{Q[u]} = \langle Q[u], a \rangle = \mathrm{Tr}[P_u (\mathrm{Op}_Q a)^*],$$

which corresponds to the usual formula for the expectation of an hermitian observable A in the state u

$$\langle A \rangle_u = \mathrm{Tr}(P_u A).$$

Theorem 10 (Moyal Property) *The following conditions are equivalent for a time–frequency transform Q:*

1. *Time–frequency transform Q is unitary.*
2. *The ambiguity kernel ϕ_Q satisfies $|\phi_Q(\xi, y)| = 1$ for all ξ, y.*
3. *The quantization Op_Q is unitary.*

Proof $(b) \Rightarrow (a)$: Assuming the kernel condition we calculate

$$\begin{aligned}
\langle Q(u, v), Q(f, g) \rangle &= \langle F Q(u, v), F Q(f, g) \rangle \\
&= \langle \phi_Q \chi(u, v), \phi_Q \chi(f, g) \rangle \\
&= \langle |\phi_Q|^2 \chi(u, v), \chi(f, g) \rangle \\
&= \langle \chi(u, v), \chi(f, g) \rangle \\
&= \langle W(u, v), W(f, g) \rangle \\
&= \langle u, f \rangle \langle g, v \rangle
\end{aligned}$$

for all $u, v, f, g \in L^2(\mathbb{R}^n)$.

$(a) \Rightarrow (b)$: Using the previous calculation we find

$$\langle (1 - |\phi_Q|^2) \chi(u, v), \chi(f, g) \rangle = 0$$

which implies $|\phi_Q(y, \xi)| = 1$ for all $(y, \xi) \in \mathbb{R}^{2n}$.

$(b) \Leftrightarrow (c)$: Assuming the kernel condition we find the Hilbert–Schmidt inner product of the two operators by

$$
\begin{aligned}
&\langle \text{Op}_Q a_1, \text{Op}_Q a_2 \rangle_{HS} \\
&= \text{Tr}(\text{Op}_Q a_1 \circ (\text{Op}_Q a_2)^*) \\
&= \iint_{\mathbb{R}^{2n}} K_{a_1}(y, x) \overline{K_{a_2}(y, x)} \, dx \, dy \\
&= \iint_{\mathbb{R}^{2n}} \int_{\mathbb{R}^n} e^{i2\pi\xi \cdot x} \overline{\phi_Q(y, \xi)} F a_1(y, \xi) \, d\xi \int_{\mathbb{R}^n} e^{-i2\pi\xi' \cdot x} \phi_Q(y, \xi') \overline{F a_2(y, \xi')} \, d\xi' \, dx \, dy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \overline{\phi_Q(y, \xi)} F a_1(y, \xi) \phi_Q(y, \xi) \overline{F a_2(y, \xi')} \, d\xi \, dy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\phi_Q(y, \xi)|^2 F a_1(y, \xi) \overline{F a_2(y, \xi')} \, d\xi \, dy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F a_1(y, \xi) \overline{F a_2(y, \xi')} \, d\xi \, dy \\
&= \langle F a_1, F a_2 \rangle \\
&= \langle a_1, a_2 \rangle.
\end{aligned}
$$

The converse follows since by unitarity of the quantization we have

$$\| \phi_Q F a \| = \| \text{Op}_Q a \|_{HS} = \| a \| = \| F a \|,$$

which yields

$$\langle (1 - |\phi_Q|^2) a, a \rangle = 0$$

for any symbol $a \in \mathscr{S}(\mathbb{R}^{2n})$ and, consequently, $|\phi_Q|^2 = 1$. $\square$

The form of the ambiguity kernel has been previously deduced, for instance, in [9, p. 126] and [11, p. 80]. The unitarity of the Weyl and Kohn–Nirenberg quantization has been proven in [11, Theorem 14.6.1].

6 Examples of Time–Frequency Transforms

In addition to the usual Wigner transform, other symmetric time–frequency transforms may be defined in terms of the τ-Wigner transform. This transform has been studied notably in [21] and more recently by Boggiatto, Oliaro and their collaborators in [2,3].

Example 5 (τ-Wigner Transform) The transform W_τ, $\tau \in \mathbb{R}$, was defined in (19). It can be presented with the ambiguity kernel ϕ_{W_τ} given by

$$\phi_{W_\tau}(y, \xi) = e^{i2\pi(\tau - 1/2)y \cdot \xi}.$$

Notable special cases in the family of τ-Wigner transforms are the Rihaczek transform if $R = W_0$ and the Wigner transform $W = W_{1/2}$.

The τ-Wigner transforms can be used to present Cohen-class transforms which are covariant under the transforms

$$u(x) \mapsto |\det T|^{-1/2} u(T^{-1}x), \tag{50}$$

$$u(x) \mapsto \int_{\mathbb{R}} e^{-i2\pi x_k \xi_k} u(x)\, dx_k, \quad k = 1, \dots, n. \tag{51}$$

Let $\phi_Q(y, \xi) = k(y \cdot \xi)$ and $f(\tau) = \widehat{k}(\tau)$. Then a class of time–frequency transforms can be given by

$$FQ(u, v)(y, \xi) = \int_{\mathbb{R}} f(\tau)e^{i2\pi\tau y \cdot \xi} \chi(u, v)(y, \xi)\, d\tau, \tag{52}$$

which yields

$$Q(u, v)(x, \eta) = \int_{\mathbb{R}} f(\tau - 1/2) W_\tau(u, v)(x, \eta)\, d\tau.$$

These transforms are covariant under (50) and (51) by the kernel condition of Theorem 3.

In one dimension the situation is a bit simpler since the only transformations (50) are scalings and in that case the transforms 52 are known as transforms with a product-type kernel [1, p. 158]. They are characterized by the following proposition:

Proposition 4 *Cohen-class time–frequency transforms in $\mathbb{R}$ which are scale-covariant and have correct marginal distributions can be represented by weighted τ-Wigner transforms as*

$$Q(u, v)(x, \eta) = \int_{\mathbb{R}} f(\tau - 1/2) W_\tau(u, v)(x, \eta)\, d\tau. \tag{53}$$

Proof By the marginal distribution condition we have $\phi_Q(0, \xi) = \phi_Q(y, 0) = 1$ for all $y, \xi \in \mathbb{R}$. Scale-covariance yields

$$\phi_Q(y, \xi) = \phi_Q(y/a, a\xi)$$

and, in particular, if $y, \xi \neq 0$ we may take $a = y$ to obtain

$$\phi_Q(y, \xi) = \phi_Q(1, y\xi) =: k(y\xi).$$

Defining further $k(0) = 1$ we have $\phi_Q(y, \xi) = k(y\xi)$ for any $y, \xi \in \mathbb{R}$. Hence, we find

$$FQ(u, v)(y, \xi) = \int_{\mathbb{R}} f(\tau) e^{i2\pi \tau y \xi} \chi(u, v)(y, \xi) \, d\tau,$$

where $f(\tau) = \widehat{k}(\tau)$ and the result follows by Fourier inversion. □

Particular examples of these transforms are the Born–Jordan distribution with

$$f(\tau) = \mathbf{1}_{[-1/2, 1/2]}(\tau),$$

and the Choi–Williams transform with

$$f(\tau) = e^{-\tau^2/2\sigma^2} / \sqrt{2\pi\sigma^2}.$$

We note that by the form of the function f the Born–Jordan transform conserves the lifetime of the signal while the Choi–Williams distribution does not. Historically, Born–Jordan quantization was presented first in [4] and the corresponding time–frequency transform was introduced by Cohen in [6]. We present its explicit definition with some further observations.

Example 6 (Born–Jordan transform in $\mathbb{R}$) The Born–Jordan time–frequency transform, given by $f(\tau) = \mathbf{1}_{[-1/2, 1/2]}(\tau)$, has the integral expression

$$Q(u, v)(x, \eta) = \int_{\mathbb{R}} e^{-i2\pi y \cdot \eta} \frac{1}{y} \int_{x-y/2}^{x+y/2} u(t + y/2) \overline{v(t - y/2)} \, dt \, dy.$$

The time-delay kernel here is thus

$$\varphi_Q(x, y) = \begin{cases} \frac{1}{|y|} \mathbf{1}_{[-|y|/2, |y|/2]}(x), & \text{if } y \neq 0, \\ \delta_0(x), & \text{if } y = 0, \end{cases}$$

and the ambiguity kernel is $\varphi_Q(y, \xi) = \operatorname{sinc}(\xi y) \in \mathcal{O}_M(\mathbb{R}^2)$.

This transform can be motivated physically by considering the quantization of Poisson brackets. Poisson correspondence is the formula

$$\left[\operatorname{Op}_Q a, \operatorname{Op}_Q b \right] = \operatorname{Op}_Q \{a, b\}, \tag{54}$$

where the brackets denote the commutator, and

$$\{a, b\} = \sum_{i=i}^{n} \frac{\partial a}{\partial x_i} \frac{\partial b}{\partial \eta_i} - \frac{\partial a}{\partial \eta_i} \frac{\partial b}{\partial x_i}$$

is the Poisson bracket of symbols a and b. Replacing Poisson brackets by operator commutators would be the most direct analogy between classical mechanics and quantum mechanics. However, no quantization satisfies the full Poisson correspondence.

The Born–Jordan quantization is partly motivated by a reduced version of the Poisson correspondence it satisfies, see [15].

Proposition 5 *The reduced Poisson correspondence*

$$\mathrm{Op}_Q \{a_1 + b_1, a_2 + b_2\} = \left[\mathrm{Op}_Q a_1 + \mathrm{Op}_Q b_1, \mathrm{Op}_Q a_2 + \mathrm{Op}_Q b_2 \right]$$

with a_1, a_2 being functions of x and b_1, b_2 being functions of η characterizes the Born–Jordan quantization. In particular, the quantization satisfies

$$\mathrm{Op}_Q(\partial_x a \cdot \partial_\eta b) = [\mathrm{Op}_Q a, \mathrm{Op}_Q b].$$

The Weyl–Wigner quantization, on the other hand, satisfies (54) only for polynomial symbols with degree at most two.

As an example of a zero-localization operator Q_0 we consider the one related to the Born–Jordan transform. The value of the Born–Jordan transform at the origin is given after a change of variables by

$$\langle u, Q_0 v \rangle = \int_{\mathbb{R}} u(x) \int_{\mathbb{R}} \frac{\theta(x) - \theta(y)}{x - y} \overline{v(y)} \, dy \, dx,$$

where θ is the unit step function and the inner integral should be interpreted as a principal value. The operator Q_0 can be written in terms of the Hilbert transform as the commutator

$$\pi[P, H]u(x) = \theta(x)\, \mathrm{p.\,v.} \int_{\mathbb{R}} \frac{u(y)}{x - y} \, dy - \mathrm{p.\,v.} \int_{\mathbb{R}} \frac{\theta(y)u(y)}{x - y} \, dy,$$

where H is the Hilbert transform and $Pu(x) = \theta(x)u(x)$. This form yields after a brief calculation the L^2 bound

$$\|Q(u, v)\|_{L^\infty(\mathbb{R}^2)} \leq \pi \|u\|_{L^2(\mathbb{R})} \|v\|_{L^2(\mathbb{R})}$$

by the L^2-boundedness of the Hilbert transform.

Declarations

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