Duality for Outer $L^p_\mu(\ell^r)$ Spaces and Relation to Tent Spaces

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Abstract

We study the outer L^p spaces introduced by Do and Thiele on sets endowed with a measure and an outer measure. We prove that, in the case of finite sets, for $1 < p \leq \infty$, $1 \leq r < \infty$ or $p = r \in \{1, \infty\}$, the outer $L^p_\mu(\ell^r)$ quasi-norms are equivalent to norms up to multiplicative constants uniformly in the cardinality of the set. This is obtained by showing the expected duality properties between the corresponding outer $L^p_\mu(\ell^r)$ spaces uniformly in the cardinality of the set. Moreover, for $p = 1$, $1 < r \leq \infty$, we exhibit a counterexample to the uniformity in the cardinality of the finite set. We also show that in the upper half space setting the desired properties hold true in the full range $1 \leq p, r \leq \infty$. These results are obtained via greedy decompositions of functions in the outer $L^p_\mu(\ell^r)$ spaces. As a consequence, we establish the equivalence between the classical tent spaces T^p_r and the outer $L^p_\mu(\ell^r)$ spaces in the upper half space. Finally, we give a full classification of weak and strong type estimates for a class of embedding maps to the upper half space with a fractional scale factor for functions on $\mathbb{R}^d$.

Keywords Outer L^p spaces · Tent spaces · Köthe duality · Embedding theorems · Outer measures

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1 Introduction

A classical research topic in harmonic analysis is the study of linear and multilinear operators defined on functions on $\mathbb{R}^d$ and satisfying certain symmetries. It is the case

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of Calderón–Zygmund theory, when the symmetries are given by translations and dilations, and time-frequency analysis, when additional modulation symmetries are included. The symmetries are parametrized by the upper half space $\mathbb{R}^d \times (0, \infty)$ in the first case, and the upper half 3-space $\mathbb{R} \times (0, \infty) \times \mathbb{R}$ in the second. In fact, in both cases we can use a wave packet decomposition to encode the information of a function on $\mathbb{R}^d$ in the space parametrizing the specific symmetries.

In [12], the authors introduced in both the previous settings a new type of function spaces, the so called outer L^p spaces. These spaces were defined via quasi-norms with a structure reminiscent of the iteration of classical Lebesgue norms. The purpose was to formalize a paradigm in proving the boundedness of operators in time-frequency analysis by a two-step program. In particular, the program consisted of a version of Hölder inequality for outer L^p spaces followed by estimates from classical to outer L^p spaces on the embedding maps associated with wave packet decompositions. This is for example the case of the bilinear Hilbert transform in [3, 11, 12], the variational Carleson operator in [8, 21], the variational bilinear iterated Fourier inversion operator in [13], a family of trilinear multiplier forms with singularity over a one-dimensional subspace in [7], and the uniform bilinear Hilbert transform in [23]. Analogous applications of the outer L^p spaces framework in other settings with different geometries can be found in [2, 9, 10, 12, 17, 20].

Moreover, in [12] the authors pointed out that the two-step program outlined above, when applied to the outer L^p spaces on $\mathbb{R}^d \times (0, \infty)$, recovers some results of classical Calderón–Zygmund theory, as detailed for example in [18, 19]. In fact, in this particular setting, the outer L^p spaces are competing with the more classical tent spaces introduced in [5, 6]. The tent spaces are defined by iterated Lebesgue norms, and they have been thoroughly studied and used in the literature. Due to the many analogies in their definition and use, the equivalence between the outer L^p spaces and the tent ones has been conjectured since the publication of [12] but never formally established. We prove the equivalence in Theorem 1.3.

In order to formalize the two-step program described above, in [12], the authors developed the framework of the outer L^p spaces focusing on their real interpolation features, such as Marcinkiewicz interpolation and Hölder's inequality, while other aspects of the theory of these spaces remained untouched. For example, whether the outer L^p quasi-norms are equivalent to norms, or whether they can be recovered as a supremum of a pairing with functions in another appropriate outer $L^{p'}$ space.

Already these simple questions turn out to be difficult. We begin their study in this paper from the case of the outer L^p spaces of functions on $\mathbb{R}^d \times (0, \infty)$ described in [12]. We provide a positive answer to both of the questions in Theorem 1.2. The study of the same questions in the case of the outer L^p spaces on $\mathbb{R} \times (0, \infty) \times \mathbb{R}$ described in [12] is beyond the purpose of the paper, and it will be addressed in future work. We briefly comment on the difference with the previous case. The geometry of the outer measure on the upper half 3-space can be addressed substantially analogously to that on the upper half space. The source of difficulty is the so called size, the object corresponding to the inner Lebesgue norm of the iterated L^p nature of the outer L^p spaces. While on $\mathbb{R}^d \times (0, \infty)$ the size is given by a single Lebesgue norm, on $\mathbb{R} \times (0, \infty) \times \mathbb{R}$ the size is given by the sum of different Lebesgue norms instead. As a consequence, it is more complicated to treat and requires further investigation.

We turn now to a more detailed introduction of the outer L^p spaces. Differently from [12], we specialize the sizes to be themselves Lebesgue norms, so that we can view the L^p theory for outer measure spaces as a generalization of the classical product, or iteration, of L^p quasi-norms. We first focus on the finite setting. This allows us to introduce meaningful outer L^p spaces while at the same time dealing with the least possible amount of technicalities possible. For a more general setting, we refer the interested reader to Appendix A.

We start recalling that on the Cartesian product X of two finite sets equipped with strictly positive weights (Y, μ) , (Z, ν) , we can define the classical product, or iterated, $L^\infty L^r$, $L^p L^r$ spaces for $0 < p, r < \infty$ by the quasi-norms

$$\begin{aligned} \|f\|_{L^\infty((Y, \mu), L^r(Z, \nu))} &= \sup_{y \in Y} \left(\sum_{z \in Z} \nu(z) |f(y, z)|^r \right)^{\frac{1}{r}} \\ &= \sup_{y \in Y} \left(\mu(y)^{-1} \sum_{z \in Z} \omega(y, z) |f(y, z)|^r \right)^{\frac{1}{r}}, \quad (1.1) \\ \|f\|_{L^p((Y, \mu), L^r(Z, \nu))} &= \left(\sum_{y \in Y} \mu(y) \left(\sum_{z \in Z} \nu(z) |f(y, z)|^r \right)^{\frac{p}{r}} \right)^{\frac{1}{p}}, \end{aligned}$$

where we denote by $\omega = \mu \otimes \nu$ the induced weight on X . In both cases, the inner L^r quasi-norm may be replaced by an L^∞ norm as well. For $1 \leq p, r \leq \infty$, the objects defined in the display are in fact norms.

The L^p spaces associated with an outer measure space (X, μ) , or outer L^p spaces, generalize this construction. An *outer measure* μ on X is a monotone, subadditive function from $\mathcal{P}(X)$, the power set of X , to the extended positive half-line, attaining the value 0 on the empty set. In general, an outer measure need not generate an interesting measure by restriction to the Carathéodory measurable sets. For instance, when μ is constantly 1 on every nonempty element of $\mathcal{P}(X)$, the Carathéodory σ -algebra is trivial. A standard way to generate an outer measure is via a pre-measure σ , a function from a collection of subsets $\mathcal{E} \subseteq \mathcal{P}(X)$ to the positive half-line, by means of covering an arbitrary subset of X by elements of $\mathcal{E}$. Namely, for every $A \subseteq X$, we define

$$\mu(A) = \inf \left\{ \sum_{E \in \mathcal{E}'} \sigma(E) : \mathcal{E}' \subseteq \mathcal{E}, A \subseteq \bigcup_{E \in \mathcal{E}'} E \right\}, \quad (1.2)$$

with the understanding that an empty sum is 0 and that if A is not covered by $\mathcal{E}$, then the infimum is ∞ . In fact, this is the way the authors introduced the outer measures in the upper half space and in the upper half 3-space in [12].

For the purpose of defining the outer L^p spaces in the most streamlined fashion, we make the reasonable assumption on μ to be strictly positive and finite on every singleton in $\mathcal{P}(X)$. Next, for a strictly positive weight ω on X , $0 < r < \infty$, let ℓ^∞, ℓ^r be the functions from the set of functions on X to $[0, \infty]^{\mathcal{P}(X)}$ defined by

$$\begin{aligned}\ell^\infty(f)(A) &= \sup_{x \in A} |f(x)|, \\ \ell^r(f)(A) &= \left(\mu(A)^{-1} \sum_{x \in A} \omega(x) |f(x)|^r \right)^{\frac{1}{r}}.\end{aligned}\quad (1.3)$$

The reader familiar with the theory of outer L^p spaces developed in [12] can recognize that ℓ^∞ , ℓ^r are *sizes*.

For $0 < p < \infty$, $0 < r \leq \infty$, we define the outer $L_\mu^\infty(\ell^r)$, $L_\mu^p(\ell^r)$, $L_\mu^{p,\infty}(\ell^r)$ spaces by the quasi-norms

$$\|f\|_{L_\mu^\infty(\ell^r)} = \|f\|_{L_\mu^{\infty,\infty}(\ell^r)} = \sup_{A \subseteq X} \ell^r(f)(A), \quad (1.4)$$

$$\|f\|_{L_\mu^p(\ell^r)} = \left(\int_0^\infty p \lambda^p \inf\{\mu(A) : A \subseteq X, \|f 1_{A^c}\|_{L^\infty(\ell^r)} \leq \lambda\} \frac{d\lambda}{\lambda} \right)^{\frac{1}{p}}, \quad (1.5)$$

$$\|f\|_{L_\mu^{p,\infty}(\ell^r)} = \left(\sup_{\lambda > 0} \lambda^p \inf\{\mu(A) : A \subseteq X, \|f 1_{A^c}\|_{L^\infty(\ell^r)} \leq \lambda\} \right)^{\frac{1}{p}}. \quad (1.6)$$

The integral in (1.5) is reminiscent of the layer-cake representation for the classical L^p norm on a measure space. The novelty and the subtle point of the theory of outer L^p spaces discussed in [12] we want to stress is the different way to evaluate the magnitude of a function to define the level sets. This is done through L^r averages rather than L^∞ norm. As a consequence, due to the L^r averaging interplay between μ and ω , the infima in (1.5) and (1.6) do not stand for outer measures of super level sets $\{f > \lambda\}$ of the function f . In general, this happens only when $r = \infty$, and the L^p quasi-norm becomes a Choquet integral. To shorten the notation, we drop the subscript μ in $L_\mu^p(\ell^r)$ and we refer to the outer L^p spaces with the symbol $L^p(\ell^r)$. Moreover, we denote the infima in (1.5) and (1.6) associated with f , λ by

$$\mu(\ell^r(f) > \lambda), \quad (1.7)$$

and we refer to it as the *super level measure*.

As a side remark, we comment on the definition of the outer L^p quasi-norms in the case of an outer measure μ generated by a pre-measure. Let σ be a pre-measure attaining only strictly positive values on a collection of sets $\mathcal{E}$ covering X , so that μ is strictly positive and finite on every singleton in $\mathcal{P}(X)$. In this case, in (1.4), and hence in (1.5) and (1.6), we can equivalently take the supremum over the elements of $\mathcal{E}$ of the following quantity

$$\begin{aligned}\ell_\sigma^\infty(f)(E) &= \ell^\infty(f)(E), \\ \ell_\sigma^r(f)(E) &= \left(\sigma(E)^{-1} \sum_{x \in E} \omega(x) |f(x)|^r \right)^{\frac{1}{r}},\end{aligned}\quad (1.8)$$

as we will see in Lemma A.3 in Appendix A.

An example of the setting just described is the realisation of the classical iterated $L^\infty L^r$, $L^p L^r$ spaces discussed above as outer L^p spaces. Let X be the set $Y \times Z$, ω

be the strictly positive weight $\mu \otimes \nu$, σ be the pre-measure defined on the collection $\mathcal{E} = \{\{y\} \times Z : y \in Y\}$ of subsets of X by

$$\sigma(\{y\} \times Z) = \mu(y),$$

and consider the outer measure generated by σ as in (1.2). Then, the quasi-norms in (1.1) are the same of those in (1.4) and (1.5) in this setting. In particular, the outer L^p quasi-norms are in fact norms, at least in a certain range of exponents.

In the first part of this paper, we develop the theory of outer L^p spaces addressing the question of the equivalence of the corresponding quasi-norms to norms. The first novelty is to provide a positive answer in the case of the outer $L^p(\ell^r)$ spaces on finite sets. It follows by the sharpness of the Hölder's inequality in the sense of the following inequality,

$$\|f\|_{L^p(\ell^r)} \leq C \sup_{\|g\|_{L^{p'}(\ell^{r'})}=1} \|fg\|_{L^1(X,\omega)}, \quad (1.9)$$

where the constant C is independent of $f \in L^p(\ell^r)$, and $L^1(X, \omega)$ stands for the classical L^1 space on X with the measure associated with the weight ω .

Theorem 1.1 *Let $0 < p, r \leq \infty$. There exists a constant $C = C(p, r)$ such that, for every finite set X , finite outer measure μ strictly positive on every singleton in $\mathcal{P}(X)$, and strictly positive weight ω , the following properties hold true.*

(i) *For $0 < p = r \leq \infty$, for every $f \in L^p(\ell^p)$,*

$$\frac{1}{C} \|f\|_{L^p(X,\omega)} \leq \|f\|_{L^p(\ell^p)} \leq C \|f\|_{L^p(X,\omega)}.$$

(ii) *For $1 < p \leq \infty$, $1 \leq r < \infty$ or $p = r \in \{1, \infty\}$, for every $f \in L^p(\ell^r)$,*

$$\frac{1}{C} \sup_{\|g\|_{L^{p'}(\ell^{r'})}=1} \|fg\|_{L^1(X,\omega)} \leq \|f\|_{L^p(\ell^r)} \leq C \sup_{\|g\|_{L^{p'}(\ell^{r'})}=1} \|fg\|_{L^1(X,\omega)}.$$

(iii) *For $1 < p \leq \infty$, $1 \leq r < \infty$ or $p = r \in \{1, \infty\}$, for every $\{f_n\}_{n \in \mathbb{N}} \subseteq L^p(\ell^r)$,*

$$\left\| \sum_{n \in \mathbb{N}} f_n \right\|_{L^p(\ell^r)} \leq C \sum_{n \in \mathbb{N}} \|f_n\|_{L^p(\ell^r)}.$$

Therefore, for $1 < p \leq \infty$, $1 \leq r < \infty$ or $p = r \in \{1, \infty\}$, the outer $L^p(\ell^r)$ quasi-norm is equivalent to a norm, and the outer $L^p(\ell^r)$ space is the Köthe dual space of the outer $L^{p'}(\ell^{r'})$ space.

The main point of the theorem is the uniformity of the constant in (X, μ, ω) . In fact, for every fixed finite setting, both statements in (ii), (iii) are verified by a certain constant also for $p = 1$, $1 < r \leq \infty$ or $1 < p < \infty$, $r = \infty$, and hence the final considerations of the theorem hold true as well. However, for $p = 1$, $1 < r \leq \infty$, the constant is not uniform in (X, μ, ω) , and we exhibit a counterexample in Lemma 3.4.

For $1 < p < \infty$, $r = \infty$, the question about uniformity remains open. The uniformity of the constant suggests that if an infinite setting is suitably approximated by finite restrictions, the same results could possibly be obtained through a limiting process.

There is a slight abuse in the use of the term Köthe dual space in the statement of Theorem 1.1, since this object is in general defined for Banach function spaces. A *Banach function space*, or *Köthe function space*, $(\mathcal{L}, \|\cdot\|_{\mathcal{L}})$ on a σ -finite measure space $(X, \tilde{\omega})$ is a Banach space of measurable functions containing all the simple functions and such that if f is a measurable function with absolute value bounded $\tilde{\omega}$ -almost everywhere by $g \in \mathcal{L}$, then $f \in \mathcal{L}$ with norm bounded by that of g . The *Köthe dual space*, or *associate space*, of $\mathcal{L}$ is then defined as the space of measurable functions such that the $L^1(X, \tilde{\omega})$ pairing with every element of $\mathcal{L}$ is finite, endowed with the norm of the dual space, see for example [4, 16]. In our setting, we have both a measure associated with the weight ω and an outer measure μ on X . Although it is not clear whether a priori the simple functions with respect to ω belong to the outer $L^p(\ell^r)$ space, it is straight-forward to check that the simple functions with respect to μ belong to $L^p(\ell^r)$. Therefore, with a slight abuse of terminology, we extend the definition of the Köthe duality to the outer $L^p(\ell^r)$ spaces with respect to the $L^1(X, \omega)$ pairing.

The first inequalities of both statements in (i), (ii) were already proved as consequences of more general results obtained in [12, 22], see Proposition A.7 and Proposition A.5 in Appendix A of the present paper. It would be interesting to investigate whether, for example, the outer L^p quasi-norms are equivalent to norms in the generality of sizes discussed in [12] and recalled in Appendix A.

We further develop our research in the case of the outer L^p spaces with size defined by an L^r norm on the infinite setting associated with Calderó–Zygmund theory. We address the question of the equivalence to norms of the outer L^p quasi-norms on functions on the upper half space described in [12]. In particular, let X be $\mathbb{R}^d \times (0, \infty)$ with the topology inherited from $\mathbb{R}^{d+1}$, $\mathcal{D}$ be the collection of the open dyadic cubic boxes with sides parallel to the axes and base on $\mathbb{R}^d$. Let σ be the function on $\mathcal{D}$ given by the classical volume of the base of the box, μ be the outer measure on X generated by σ on $\mathcal{D}$ as in (1.2). Finally, let ω be the measure defined by the density $\omega(y, t) = t^{-1}$ with respect to the Lebesgue measure on $\mathbb{R}^d \times (0, \infty)$, where $y \in \mathbb{R}^d$, $t \in (0, \infty)$. For $0 < r < \infty$, let $\ell_{\sigma}^{\infty}, \ell_{\sigma}^r$ be the functions from $\mathcal{B}(X)$, the set of Borel measurable functions on X , to $[0, \infty]^{\mathcal{D}}$ defined by

$$\begin{aligned}\ell_{\sigma}^{\infty}(f)(D) &= \|f 1_D\|_{L^{\infty}(X, \omega(y, t) \, dy \, dt)}, \\ \ell_{\sigma}^r(f)(D) &= \left(\sigma(D)^{-1} \int_D |f(y, t)|^r \, dy \, \frac{dt}{t} \right)^{\frac{1}{r}} \\ &= \sigma(D)^{-\frac{1}{r}} \|f 1_D\|_{L^r(X, \omega(y, t) \, dy \, dt)}.\end{aligned}\tag{1.10}$$

For $0 < p, r \leq \infty$, let the outer $L^p(\ell_{\sigma}^r), L^{p, \infty}(\ell_{\sigma}^r)$ spaces be defined as in (1.4), (1.5) and (1.6), taking the supremum of the quantity in the previous display over the elements of $\mathcal{D}$ in (1.4). In analogy with the remark concerning the quantities in (1.8), we drop the subscript σ in $L^p(\ell_{\sigma}^r)$.

In this infinite setting, we prove the analogous statement of Theorem 1.1. The properties (ii), (iii) hold true even in the endpoint cases $p = 1$, $1 < r \leq \infty$ and $1 \leq p < \infty$, $r = \infty$.

Theorem 1.2 *Let (X, μ, ω) be the upper half space setting just described, $0 < p, r \leq \infty$. There exists a constant $C = C(p, r)$ such that the analogous properties stated in Theorem 1.1 hold true in the following ranges, property (i) for $0 < p = r \leq \infty$, properties (ii), (iii) for $1 \leq p, r \leq \infty$.*

Therefore, for $1 \leq p, r \leq \infty$, the outer $L^p(\ell^r)$ quasi-norm is equivalent to a norm, and the outer $L^p(\ell^r)$ space is the Köthe dual space of the outer $L^{p'}(\ell^{r'})$ space.

As we recalled in the first part of the introduction, in the upper half space setting there are already classical spaces with a different iterated $L^p L^r$ structure, namely the tent spaces. Let $\Gamma(x)$ be the cone with vertex in $x \in \mathbb{R}^d$, $T(x, s)$ be the tent over the ball in $\mathbb{R}^d$ centred in x with radius s ,

$$\begin{aligned}\Gamma(x) &= \{(y, t) \in \mathbb{R}^d \times (0, \infty) : |x - y| < t\}, \\ T(x, s) &= \{(y, t) \in \mathbb{R}^d \times (0, \infty) : |x - y| < s - t\}.\end{aligned}$$

For $0 < p < \infty$, $0 < r \leq \infty$, let

$$\begin{aligned}A_r(f)(x) &= \|f\|_{L^r(\Gamma(x), dy \frac{dt}{t^{d+1}})}, \\ \|f\|_{T_r^p} &= \|A_r(f)\|_{L^p(\mathbb{R}^d, dx)}.\end{aligned}\tag{1.11}$$

For $p = \infty$, $0 < r \leq \infty$, let

$$\begin{aligned}C_r(f)(x) &= \sup_{s \in (0, \infty)} \|f\|_{L^r(T(x, s), \omega)}, \\ \|f\|_{T_r^\infty} &= \|C_r(f)\|_{L^\infty(\mathbb{R}^d, dx)}.\end{aligned}\tag{1.12}$$

For $0 < p, r \leq \infty$, the tent space T_r^p is defined by the T_r^p quasi-norm. Sometimes in the literature an additional continuity condition is assumed on functions in T_∞^p , see for example [6], but we do not, in order to preserve a uniformity in the definition of the spaces. For $1 \leq p, r \leq \infty$, the quasi-norms defined in the last two displays are in fact norms.

The third result of this paper is to establish the equivalence between the outer $L^p(\ell^r)$ spaces and the tent spaces T_r^p .

Theorem 1.3 *For $0 < p, r \leq \infty$, there exists a constant $C = C(p, r)$ such that, for every $f \in L^p(\ell^r)$,*

$$\frac{1}{C} \|f\|_{T_r^p} \leq \|f\|_{L^p(\ell^r)} \leq C \|f\|_{T_r^p}.$$

Moreover, we have $L^p(\ell^r) = T_r^p$.

It is worth noting that while the tent spaces require to pass from cones to tents in order to define T_r^∞ , the definition of the outer $L^p(\ell^r)$ spaces always relies on the boxes, or equivalently on the tents.

In the second part of the paper, we turn our focus to embedding maps of functions on $\mathbb{R}^d$ to the upper half space $\mathbb{R}^d \times (0, \infty)$. These embeddings are obtained by pairing a function on $\mathbb{R}^d$ with translated and dilated versions of a given test function. More precisely, given a test function ϕ satisfying certain boundedness and decay properties, we define, for every locally integrable function f on $\mathbb{R}^d$, the embedded function $F_\phi(f)$ on $\mathbb{R}^d \times (0, \infty)$ by

$$F_\phi(f)(y, t) = \int_{\mathbb{R}^d} f(x) t^{-d} \phi(t^{-1}(y - x)) \, dx. \quad (1.13)$$

A prominent example of such an embedding is the harmonic extension of a function on $\mathbb{R}^d$ to the upper half space, where ϕ is the Poisson kernel. The interest in embedding maps is part of the aforementioned two-step program to prove the boundedness of operators in Calderón–Zygmund theory.

We study continuous inclusions between outer L^p spaces in the upper half space and continuous embeddings from classical L^p spaces on $\mathbb{R}^d$ to outer L^p spaces in this setting. We start with an improvement over a previous result on Hardy–Littlewood–Sobolev inclusions between tent spaces in [1]. We obtain the boundedness of the map

$$T_{r_1}^p \hookrightarrow T_{r_2}^q, f \mapsto t^{\frac{d}{p} - \frac{d}{q}} f,$$

for $0 < p < q \leq \infty$, $0 < r_2 \leq r_1 \leq \infty$, or equivalently the same statement for outer $L^p(\ell^r)$ spaces. The improvement over the result in [1] consists of allowing for r_1 to be strictly greater than r_2 .

These inclusions allow to recover strong type (p, q) estimates for the embedding maps with a fractional scale factor

$$L^p(\mathbb{R}^d) \hookrightarrow L^q(\ell^r), f \mapsto t^{\frac{d}{p} - \frac{d}{q}} F_\phi(f),$$

for $0 < p < q \leq \infty$, $0 < r \leq \infty$ from the ones for $p = q$, $r = \infty$. The fourth result of the paper is then the full classification of all positive and negative results regarding strong and weak type estimates for a family of embedding maps with a fractional scale factor in Theorem 6.1. More precisely, for $\varepsilon > 0$, $f \in \mathcal{S}(\mathbb{R}^d)$, let the embedded function $F_\varepsilon(f) = F(f)$ be defined by

$$F(f)(y, t) = \sup_{\phi} F_\phi(f)(y, t),$$

where the supremum is taken over the set of functions ϕ such that

$$|\phi(z)| \leq (1 + |z|)^{-d-\varepsilon}. \quad (1.14)$$

With respect to the strong type estimates, we extract the following statement from Theorem 6.1.

Theorem 1.4 *Let*

$$1 \leq p, q \leq \infty, 0 < r \leq \infty. \quad (1.15)$$

Then, for (p, q, r) satisfying one of the following conditions

$$\begin{aligned} 1 < p < q \leq \infty, 0 < r \leq \infty, \\ 1 < p = q \leq \infty, r = \infty, \\ p = 1, q = \infty, 0 < r \leq \infty, \end{aligned} \quad (1.16)$$

there exists a constant $C = C(p, q, r, d, \varepsilon)$ such that, for every $f \in L^p(\mathbb{R}^d)$,

$$\|t^{\frac{d}{p} - \frac{d}{q}} F(f)\|_{L^q(\ell^r)} \leq C \|f\|_{L^p(\mathbb{R}^d)}.$$

For all the triples (p, q, r) satisfying (1.15) but none of the conditions in (1.16), no strong type (p, q) estimate holds true.

It is worth noting that the strong type $(1, \infty)$ estimates hold true for $0 < r \leq \infty$, even if for $r = \infty$ only the weak type $(1, 1)$ estimate holds true. Moreover, in the endpoint $p = q = 1, r = \infty$, we prove in Proposition 6.2 a substitute of the strong type $(1, 1)$ estimate, namely the boundedness of the embedding map

$$H^1(\mathbb{R}^d) \hookrightarrow L^1(\ell^\infty), f \mapsto F_\varphi(f),$$

for $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

We conclude the paper with some applications of these embedding theorems yielding alternative proofs of classical results such as the Hardy–Littlewood–Sobolev inequality, and the Gagliardo–Nirenberg–Sobolev inequality up to the endpoint in the spirit of the aforementioned two-step program.

Guide to the Paper

In Sect. 2 we start with two decomposition results for functions in the outer $L^p(\ell^r)$ spaces in both finite and upper half space settings. We use them to prove Theorem 1.2 and Theorem 1.1 in Sect. 3. Moreover, in Lemma 3.4, we provide a counterexample to the uniformity of the statements in (ii), (iii) in Theorem 1.1 for $p = 1, 1 < r \leq \infty$. In Sect. 4 we prove Theorem 1.3. In Sect. 5, Theorem 5.1, we improve over the result of Amenta on Hardy–Littlewood–Sobolev inclusions between tent spaces. In Sect. 6, Theorem 6.1, we prove a full classification of all positive and negative results regarding strong and weak type estimates for a family of embedding maps with a fractional scale factor from classical L^p spaces on $\mathbb{R}^d$ to outer $L^p(\ell^r)$ spaces on $\mathbb{R}^d \times (0, \infty)$. Moreover, in Proposition 6.2 we prove the boundedness of the embedding map defined by a test function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ from $H^1(\mathbb{R}^d)$ to the outer $L^1(\ell^\infty)$ space. We use the strong type estimates from both results to prove the Hardy–Littlewood–Sobolev

inequality, and the Gagliardo-Nirenberg-Sobolev inequality up to the endpoint in the spirit of the aforementioned two-step program in Sect. 7. Finally, in Appendix A, we review the definitions and recall some results of the theory of outer L^p spaces in the level of generality discussed in [12]. In Appendix B, we prove some properties of the outer measure μ on the upper half space described above.

2 Decompositions for Outer $L^p(\ell^r)$ Spaces

In this section we state and prove two crucial preparatory decomposition results for functions in the outer $L^p(\ell^r)$ spaces in both finite and upper half space settings, used in proving Theorem 1.1 and Theorem 1.2, respectively. Both consist of a recursive greedy selection algorithm that provides a sequence of maximal disjoint subsets of X exhausting the elements of $\mathcal{P}(X)$ where the quantity defined in (1.3) is in the interval $[2^k, 2^{k+1})$, $k \in \mathbb{Z}$. This property guarantees not only an upper bound but also a lower bound on the super level measure in (1.7) at level $\lambda = 2^k$, $k \in \mathbb{Z}$, in terms of the outer measures of the selected subsets, thus providing a concrete substitute for it. Without loss of generality, we can restrict our attention only to these levels. In fact, we can replace the integral in (1.5) with an equivalent discrete version, namely

$$\left(\sum_{k \in \mathbb{Z}} 2^{kp} \mu(\ell^r(f) > 2^k) \right)^{\frac{1}{p}},$$

due to the monotonicity in λ of the super level measure of a fixed function. This quantity is no longer homogeneous in f , hence it is not a quasi-norm, but the discrete levels fit better the recursive process we want to describe.

The decompositions in the two cases are analogous. We could state and prove a unified result in the general setting described in Appendix A, at least in the range of exponents $0 < p, r < \infty$. It would require some adjustments to address the technicalities due to the non-finiteness of the selection process and the generation of the outer measure by a pre-measure. In this exposition, we prefer to focus separately on the two specific settings for the following reasons.

The finite setting offers a full view on the mechanism of the recursive selection algorithm and the proof of the decomposition properties. Moreover, we do not have to worry about our selection process being well-defined, since at each step only finitely many choices are available, and we can choose any subset of X . Again, we stress that the main point in this case is the uniformity of constants in (X, μ, ω) .

The upper half space setting serves two purposes. On one hand, as a privileged case of the general setting described in Appendix A, it provides an example of addressing the technicalities we referred to above. On the other hand, due to the geometry of the outer measure, it allows for an improved version of the decomposition result. First, we can extend it to the case $r = \infty$, which is not included in the finite setting. Second, the decomposition of a function in the outer $L^1(\ell^r)$ space, for $1 < r \leq \infty$, is subtly more efficient for our purpose, as will be clarified in Remark 3.2. We could state sufficient conditions on the geometry of the outer measure to ensure this refined decomposition

in a broader generality, but these considerations are beyond the purpose of the paper, and they will be developed in future work.

We start with the finite setting. Let X be a finite set, μ an outer measure strictly positive and finite on every singleton in $\mathcal{P}(X)$, ω a strictly positive weight. We have the following uniform decomposition result for functions in the outer $L^p(\ell^r)$ spaces defined by (1.5).

Proposition 2.1 *Let $0 < p, r < \infty$. There exists a constant $C = C(p, r)$ such that, for every finite set X , finite outer measure μ strictly positive on every singleton in $\mathcal{P}(X)$, and strictly positive weight ω , the following property holds true. For $f \in L^p(\ell^r)$, there exists a sequence of sets $\{E_k : k \in \mathbb{Z}\} \subseteq \mathcal{P}(X)$ such that if*

$$F_k = \bigcup_{l \geq k} E_l,$$

then, for every $k \in \mathbb{Z}$,

$$\ell^r(f 1_{F_{k+1}^c})(E_k) > 2^k, \quad \text{when } E_k \neq \emptyset, \quad (2.1)$$

$$\|f 1_{F_k^c}\|_{L^\infty(\ell^r)} \leq 2^k, \quad (2.2)$$

$$\mu(\ell^r(f) > 2^k) \leq \sum_{l \geq k} \mu(E_l), \quad (2.3)$$

$$\mu(E_k) \leq C \mu(\ell^r(f) > 2^{k-1}). \quad (2.4)$$

Proof First, we observe qualitatively that by outer Hölder's inequality, Proposition A.5 in Appendix A, we have $L^p(\ell^r) \subseteq L^\infty(\ell^r)$, because $\mu(X)$ is finite.

We define E_k by backward recursion on $k \in \mathbb{Z}$. For k large enough such that

$$\|f\|_{L^\infty(\ell^r)} \leq 2^k,$$

we set E_k to be empty. Now fix k and assume we have selected E_l for $l > k$. In particular, F_{k+1} is already well-defined. If there exists a set $A \subseteq X$ such that

$$\ell^r(f 1_{F_{k+1}^c})(A) > 2^k, \quad (2.5)$$

then we choose such a set A to be E_k , making sure that

$$\|f 1_{(A \cup F_{k+1})^c}\|_{L^\infty(\ell^r)} \leq 2^k. \quad (2.6)$$

In fact, if there exists a set $B \subseteq X$ such that

$$\ell^r(f 1_{(A \cup F_{k+1})^c})(B) > 2^k,$$

then by the subadditivity of the outer measure, we have

$$\ell^r(f 1_{F_{k+1}^c})(A \cup B) > 2^k.$$

Due to the finiteness of X , the condition (2.6) can be achieved in finitely many steps. If no A satisfying (2.5) exists, we set E_k to be empty, and proceed the recursion with $k - 1$.

By construction, we have (2.1) for every nonempty selected set E_k , (2.2) and (2.3) for every $k \in \mathbb{Z}$.

We observe that for every k such that 2^k is greater than the $L^\infty(\ell^r)$ quasi-norm of f , the statement (2.4) is true. To prove (2.4) for any other k , let A_{k-1} be a set witnessing the super level measure at level 2^{k-1} . In particular,

$$\begin{aligned} \|f 1_{A_{k-1}^c}\|_{L^\infty(\ell^r)} &\leq 2^{k-1}, \\ \mu(\ell^r(f) > 2^{k-1}) &= \mu(A_{k-1}). \end{aligned}$$

By (2.2) for $k + 1$, we have

$$\mu(A_{k-1}) \geq 2^{-r(k+1)} \sum_{x \in A_{k-1} \setminus F_{k+1}} \omega(x) |f(x)|^r. \quad (2.7)$$

By the definition of A_{k-1} and E_k , we have

$$\begin{aligned} \sum_{x \in E_k \setminus A_{k-1}} \omega(x) |f(x)|^r &\leq 2^{r(k-1)} \mu(E_k), \\ \sum_{x \in E_k \setminus F_{k+1}} \omega(x) |f(x)|^r &> 2^{rk} \mu(E_k), \end{aligned}$$

hence

$$\sum_{x \in (A_{k-1} \cap E_k) \setminus F_{k+1}} \omega(x) |f(x)|^r > C 2^{r(k-1)} \mu(E_k).$$

Combining this with (2.7) gives

$$\mu(\ell^r(f) > 2^{k-1}) \geq C \mu(E_k),$$

concluding the proof of (2.4) for the given k . $\square$

Now we move to the upper half space setting. Let X be the upper half space and μ the outer measure generated by the pre-measure σ on $\mathcal{D}$, the collection of the open dyadic cubic boxes in the upper half space, as in (1.2). In particular,

$$\begin{aligned} X &= \mathbb{R}^d \times (0, \infty), \\ \mathcal{D} &= \{(x, 0) + (0, 2^j)^{d+1} : x \in 2^j \mathbb{Z}^d, j \in \mathbb{Z}\}, \\ \sigma(E) &= |B(E)|, \quad \text{for every } E \in \mathcal{D}, \\ \omega(y, t) &= t^{-1}, \end{aligned} \quad (2.8)$$

where $B(E)$ is the base in $\mathbb{R}^d$ of the dyadic box $E \in \mathcal{D}$, and $|B(E)|$ its volume. Moreover, for every dyadic box $E = (x, 0) + (0, s)^{d+1} \in \mathcal{D}$, we define E^+ by

$$E^+ = (x, 0) + ((0, s)^d \times (s/2, s)).$$

Finally, let ω be the measure defined by the density $\omega(y, t)$ with respect to the Lebesgue measure on $\mathbb{R}^d \times (0, \infty)$, where $y \in \mathbb{R}^d$, $t \in (0, \infty)$, and for every $0 < r \leq \infty$ let ℓ^r be the size defined in (1.10).

We make the following observations involving the geometry of the elements of $\mathcal{D}$ and the values of σ, μ on them. We postpone the proofs to Appendix B.

Lemma 2.2 *For every two dyadic boxes $E_1, E_2 \in \mathcal{D}$ with nonempty intersection, we have either $E_1 \subseteq E_2$ or $E_2 \subseteq E_1$.*

Lemma 2.3 *Let $\{E_n: n \in \mathbb{N}\}$ be a collection of pairwise disjoint dyadic boxes in $\mathcal{D}$, and let $\{D_n: n \in \mathbb{N}\}$ be a collection of subsets of X such that, for every $n \in \mathbb{N}$, we have $D_n \subseteq E_n$ and $D_n \cap E_n^+ \neq \emptyset$. Then we have*

$$\mu\left(\bigcup_{n \in \mathbb{N}} D_n\right) = \sum_{n \in \mathbb{N}} \sigma(E_n).$$

In the following statement, the elements of a double sequence are parametrized by a pair (k, n) , for $k \in \mathbb{Z}, n \in \mathbb{N}_k$, where $\mathbb{N}_k$ is either the set of positive natural numbers or a possibly empty finite initial string of positive natural numbers. We consider the lexicographic order of such pairs as follows: $(l, m) < (k, n)$ if either $l > k$, or $l = k$ and $m < n$.

We have the following decomposition result for functions in the intersection between the outer $L^p(\ell^r)$ and $L^\infty(\ell^r)$ spaces defined by (1.5) and (1.4), respectively.

Proposition 2.4 *Let $0 < p < \infty, 0 < r \leq \infty$. There exists a constant $C = C(p, r)$ such that the following property holds true. For $f \in L^p(\ell^r) \cap L^\infty(\ell^r)$, there exists a double sequence of dyadic boxes $\{E_{k,n}: k \in \mathbb{Z}, n \in \mathbb{N}_k\} \subseteq \mathcal{D}$ such that if*

$$\begin{aligned} F_k &= \bigcup_{n \in \mathbb{N}_k} F_{k,n}, \\ F_{k,n} &= F_{k,n-1} \cup E_{k,n}, \\ F_{k,0} &= \bigcup_{i \in I_k} Q_i, \end{aligned}$$

where $\{Q_i: i \in I_k\} \subseteq \mathcal{D}$ is the collection of maximal dyadic boxes such that

$$|B(Q_i)| \leq 2|B(Q_i) \cap \bigcup_{(l,m): l > k} B(E_{l,m})|, \quad (2.9)$$

then, for every $k \in \mathbb{Z}$, $n \in \mathbb{N}_k$,

$$\ell^r(f 1_{F_{k,n-1}^c})(E_{k,n}) > 2^k, \quad \text{when } E_{k,n} \neq \emptyset, \quad (2.10)$$

$$\|f 1_{F_k^c}\|_{L^\infty(\ell^r)} \leq 2^k, \quad (2.11)$$

$$\mu(\ell^r(f) > 2^k) \leq C \sum_{(l,m): l \geq k} \sigma(E_{l,m}), \quad (2.12)$$

$$\sum_{n \in \mathbb{N}_k} \sigma(E_{k,n}) \leq C \mu(\ell^r(f) > 2^{k-1}). \quad (2.13)$$

Moreover, the collection $\{B(E_{k,n}): k \in \mathbb{Z}, n \in \mathbb{N}_k\}$ of the bases of the chosen boxes is 2-Carleson, i.e. for every dyadic box $E \in \mathcal{D}$

$$\sum_{(k,n): E_{k,n} \subseteq E} \sigma(E_{k,n}) \leq 2\sigma(E). \quad (2.14)$$

For the definition of the Λ -Carleson condition and, later in the proof, of the η -sparse condition for collections of cubes, as well as for their equivalence, we refer for example to [15].

Before starting the proof, we briefly comment that a dyadic box satisfies the condition in (2.9) for a certain $k \in \mathbb{Z}$ when at least half of its base is covered by the bases of the elements of the double sequence selected up to the level $k + 1$.

Proof Case I: $0 < r < \infty$. The selection algorithm is analogous to that described in the previous proof. We define $E_{k,n}$ by a double recursion, backward on $k \in \mathbb{Z}$, and, for every fixed k , forward on $n \in \mathbb{N}_k$. In parallel, we prove the properties in (2.10)–(2.13) by backward induction on $k \in \mathbb{Z}$.

For k large enough such that

$$\|f\|_{L^\infty(\ell^r)} \leq 2^k,$$

we set $\mathbb{N}_k$ empty. The properties in (2.10)–(2.13) are trivially satisfied.

Now fix (k, n) and assume we have selected $E_{l,m}$ for $(l, m) < (k, n)$, and that the properties in (2.10)–(2.13) are satisfied for every $l > k$. In particular, F_{k+1} is already well-defined and satisfies (2.11), and $F_{k,n-1}$ is already well-defined. If there exists a dyadic box $A \in \mathcal{D}$ such that

$$\ell^r(f 1_{F_{k,n-1}^c})(A) > 2^k, \quad (2.15)$$

then we choose such a dyadic box A to be $E_{k,n}$, making sure that $\sigma(A)$ is maximal. The maximality of $\sigma(A)$ is achieved because the set of values of σ is discrete and doubling, namely it is $\{2^{id}: i \in \mathbb{Z}\}$, and we have an upper bound on $\sigma(A)$ when A satisfies the condition (2.15). In fact, we have

$$\sigma(A) \leq C \mu(\ell^r(f) > 2^{k-1}) \leq C 2^{-kp} \|f\|_{L^p(\ell^r)}^p < \infty. \quad (2.16)$$

To prove the first inequality, we use an argument analogous to that used to prove (2.4) above. For every $\varepsilon > 0$, let $A_{k-1}(\varepsilon)$ be an optimal set witnessing the super level measure at level 2^{k-1} up to the multiplicative constant $(1 + \varepsilon)$. Next, let $\mathcal{E}_{k-1}(\varepsilon)$ be an optimal covering of $A_{k-1}(\varepsilon)$ witnessing its outer measure up to the multiplicative constant $(1 + \varepsilon)$. In particular,

$$\begin{aligned} \|f 1_{A_{k-1}(\varepsilon)^c}\|_{L^\infty(\ell^r)} &\leq 2^{k-1}, \\ A_{k-1}(\varepsilon) &\subseteq \bigcup_{E \in \mathcal{E}_{k-1}(\varepsilon)} E, \\ (1 + \varepsilon)^2 \mu(\ell^r(f) > 2^{k-1}) &\geq (1 + \varepsilon) \mu(A_{k-1}(\varepsilon)) \geq \sum_{E \in \mathcal{E}_{k-1}(\varepsilon)} \sigma(E). \end{aligned}$$

By (2.11) for $k + 1$, we have, for every $E \in \mathcal{E}_{k-1}(\varepsilon)$,

$$\sigma(E) \geq 2^{-r(k+1)} \|f 1_{E \setminus F_{k+1}}\|_{L^r(X, \omega)}^r,$$

which yields, together with the covering of $A_{k-1}(\varepsilon)$ by the elements of $\mathcal{E}_{k-1}(\varepsilon)$,

$$\begin{aligned} \sum_{E \in \mathcal{E}_{k-1}(\varepsilon)} \sigma(E) &\geq \sum_{E \in \mathcal{E}_{k-1}(\varepsilon)} 2^{-r(k+1)} \|f 1_{E \setminus F_{k+1}}\|_{L^r(X, \omega)}^r \\ &\geq 2^{-r(k+1)} \|f 1_{(\bigcup_{E \in \mathcal{E}_{k-1}(\varepsilon)} E) \setminus F_{k+1}}\|_{L^r(X, \omega)}^r \\ &\geq 2^{-r(k+1)} \|f 1_{A_{k-1}(\varepsilon) \setminus F_{k+1}}\|_{L^r(X, \omega)}^r. \end{aligned} \quad (2.17)$$

By the definition of $A_{k-1}(\varepsilon)$ and A , we have

$$\begin{aligned} \|f 1_{A \setminus A_{k-1}(\varepsilon)}\|_{L^r(X, \omega)}^r &\leq 2^{r(k-1)} \sigma(A), \\ \|f 1_{A \setminus F_{k+1}}\|_{L^r(X, \omega)}^r &\geq \|f 1_{A \setminus F_{k, n-1}}\|_{L^r(X, \omega)}^r > 2^{rk} \sigma(A), \end{aligned}$$

hence

$$\|f 1_{(A_{k-1}(\varepsilon) \cap A) \setminus F_{k+1}}\|_{L^r(X, \omega)}^r > C 2^{r(k-1)} \sigma(A).$$

Combining this with (2.17) and taking ε arbitrarily small give the desired inequality

$$\mu(\ell^r(f) > 2^{k-1}) \geq C \sigma(A).$$

If no A satisfying (2.15) exists, we set $\mathbb{N}_k = \{1, \dots, n-1\}$, $\mathbb{N}_k$ empty if $n = 1$. If we are able to choose $E_{k, n}$ for all $n \in \mathbb{N}$, we fix such $E_{k, n}$. Before proceeding the recursion with $(k-1, 1)$, we prove the properties in (2.10)–(2.13) for k .

By construction, we have (2.10) for every nonempty selected dyadic box $E_{k, n}$.

The proof of (2.13) for k assuming (2.11) for $k+1$, which we have by the induction hypothesis, is analogous to that of the first inequality in (2.16). In fact, we have

$$\begin{aligned}\|f 1_{\bigcup_{n \in \mathbb{N}_k} E_{k,n} \setminus A_{k-1}(\varepsilon)}\|_{L^r(X, \omega)}^r &\leq \sum_{n \in \mathbb{N}_k} \|f 1_{E_{k,n} \setminus A_{k-1}(\varepsilon)}\|_{L^r(X, \omega)}^r \\ &\leq 2^{r(k-1)} \sum_{n \in \mathbb{N}_k} \sigma(E_{k,n}), \\ \|f 1_{\bigcup_{n \in \mathbb{N}_k} E_{k,n} \setminus F_{k+1}}\|_{L^r(X, \omega)}^r &\geq \sum_{n \in \mathbb{N}_k} \|f 1_{E_{k,n} \setminus F_{k,n-1}}\|_{L^r(X, \omega)}^r \\ &> 2^{rk} \sum_{n \in \mathbb{N}_k} \sigma(E_{k,n}),\end{aligned}$$

hence

$$\|f 1_{(A_{k-1}(\varepsilon) \cap \bigcup_{n \in \mathbb{N}_k} E_{k,n}) \setminus F_{k+1}}\|_{L^r(X, \omega)}^r > C 2^{r(k-1)} \sum_{n \in \mathbb{N}_k} \sigma(E_{k,n}),$$

where $A_{k-1}(\varepsilon)$ is defined as above. We conclude as above.

Now we prove (2.11) for k . If $\mathbb{N}_k$ is finite, then by construction there is no dyadic box $A \in \mathcal{D}$ such that

$$\ell^r(f 1_{F_k^c})(A) > 2^k.$$

If $\mathbb{N}_k$ is infinite, we observe by (2.13) for this k , that

$$\sum_{n \in \mathbb{N}_k} \sigma(E_{k,n}) < \infty,$$

since $f \in L^p(\ell^r)$. Therefore, $\sigma(E_{k,n})$ tends to zero as n tends to ∞ . Since each $E_{k,n}$ is chosen to maximize $\sigma(E_{k,n})$, there exists no dyadic box $A \in \mathcal{D}$ which can violate (2.11) as such A would contradict the choice of $E_{k,n}$ for sufficiently large n . This concludes the proof of (2.11) for the given k .

With (2.11), we also have (2.12). In fact, we have

$$\begin{aligned}\mu(F_k) &\leq \mu(F_{k-1,0}) \\ &\leq \sum_{i \in I_{k-1}} |B(Q_i)| \\ &\leq 2 \left| \bigcup_{i \in I_{k-1}} B(Q_i) \cap \bigcup_{(l,m): l \geq k} B(E_{l,m}) \right| \\ &\leq C \sum_{(l,m): l \geq k} \sigma(E_{l,m}),\end{aligned}$$

where we used (2.9) and the disjointness of the elements of $\{Q_i: i \in I_{k-1}\}$ in the third inequality.

Case II: $r = \infty$. The only difference is in the selection of $E_{k,n}$. Fix (k, n) and assume we have selected $E_{l,m}$ for $(l, m) < (k, n)$, and that the properties in (2.10)–

(2.13) are satisfied for every $l > k$. If there exists a dyadic box $A \in \mathcal{D}$ such that

$$\ell^\infty(f 1_{F_{k,n-1}^c} 1_{A^+})(A) > 2^k, \quad (2.18)$$

then we choose such a dyadic box A to be $E_{k,n}$, making sure that $\sigma(A)$ is maximal.

As in the previous case, the maximality of $\sigma(A)$ is achieved because the set of values of σ is discrete and doubling, and we have an upper bound on $\sigma(A)$ when A satisfies the condition (2.18). In fact, we have

$$\sigma(A) \leq \mu(\ell^\infty(f) > 2^{k-1}) \leq C 2^{-kp} \|f\|_{L^p(\ell^\infty)}^p < \infty. \quad (2.19)$$

To prove the first inequality, we observe that for $E = A^+ \cap \{|f| \geq 2^k\}$, we have $\omega(E) > 0$, hence

$$\mu(E) \leq \mu(\ell^\infty(f) > 2^{k-1}).$$

We conclude by Lemma 2.3.

The proof of (2.10)–(2.13) for k then follows in a straight-forward way. As in the previous case, the proof of (2.13) is analogous to that of the first inequality in (2.19). In fact, we observe that for $D_{k,n} = E_{k,n}^+ \cap \{|f| \geq 2^k\}$, we have $\omega(D_{k,n}) > 0$, hence

$$\mu\left(\bigcup_{n \in \mathbb{N}_k} D_{k,n}\right) \leq \mu(\ell^\infty(f) > 2^{k-1}).$$

We conclude by Lemma 2.3 upon observing that for fixed k , the selected dyadic boxes $E_{k,n}$ are pairwise disjoint, by Lemma 2.2 and the definition of $E_{k,n}$.

To conclude, for every $0 < r \leq \infty$, we observe that the collection $\{B(E_{k,n}) : k \in \mathbb{Z}, n \in \mathbb{N}_k\}$ is $1/2$ -sparse, i.e. one can choose pairwise disjoint measurable sets $\tilde{B}_{k,n} \subseteq B(E_{k,n})$ with $|\tilde{B}_{k,n}| \geq |B(E_{k,n})|/2$. This follows by (2.9) and the maximality in the choice of $E_{k,n}$. Therefore, the collection is 2-Carleson. $\square$

3 Equivalence with Norms

In this section we prove Theorem 1.2 and Theorem 1.1. We start with the upper half space setting. First, we prove property (i). After that, for every $f \in L^p(\ell^r) \cap L^\infty(\ell^r)$, for $1 \leq p, r \leq \infty$, we provide a candidate function g to realize (1.9), up to normalization of its outer $L^{p'}(\ell^{r'})$ quasi-norm. Upon showing an upper bound on the outer $L^{p'}(\ell^{r'})$ quasi-norm of g and a lower bound on the $L^1(X, \omega)$ norm of fg , properties (ii), (iii) follow. Then we turn to the finite setting and when possible we follow analogous arguments to prove properties (i), (ii), and (iii). In almost all the definitions and proofs we make use of the decompositions provided by Proposition 2.4 and Proposition 2.1. Finally, in Lemma 3.4 we exhibit a counterexample to the uniformity in every finite setting (X, μ, ω) of both statements in (ii), (iii) for $p = 1$, $1 < r \leq \infty$.

We start with the upper half space setting, where (X, μ, ω) is the setting described in (2.8).

Proof of Theorem 1.2, property (i) The case $p = \infty$ follows by definition.

Therefore, we can assume without loss of generality $p = 1$, since

$$\|f\|_{L^p(\ell^p)}^p = \|f^p\|_{L^1(\ell^1)}.$$

For $f \in L^1(\ell^1) \cap L^\infty(\ell^1)$, let $\{E_{k,n}\}$ be the collection of the dyadic boxes from Proposition 2.4. We have

$$\begin{aligned} \|f\|_{L^1(\ell^1)} &\leq C \sum_{k \in \mathbb{Z}} 2^k \mu(\ell^1(f) > 2^k) \\ &\leq C \sum_{k \in \mathbb{Z}} 2^k \sum_{(l,m): l \geq k} \sigma(E_{l,m}) \\ &\leq C \sum_{l \in \mathbb{Z}} \sum_{m \in \mathbb{N}_l} 2^l \sigma(E_{l,m}) \\ &\leq C \sum_{l \in \mathbb{Z}} \sum_{m \in \mathbb{N}_l} \|f\|_{L^1(E_{l,m} \setminus F_{l,m-1}, \omega)} \\ &\leq C \|f\|_{L^1(X, \omega)}, \end{aligned}$$

where we used (2.12) in the second inequality, Fubini and the bounds on the geometric series in the third, (2.10) in the fourth, and disjointness of the sets in the fifth.

We note that f vanishes ω -almost everywhere outside the union of all the selected dyadic boxes $\{E_{k,n}\}$, since $\mathcal{D}$ covers all of X . We have

$$\begin{aligned} \|f\|_{L^1(X, \omega)} &= \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}_k} \|f 1_{E_{k,n} \setminus F_{k,n-1}}\|_{L^1(X, \omega)} + \sum_{k \in \mathbb{Z}} \|f 1_{F_{k,0} \setminus F_{k+1}}\|_{L^1(X, \omega)} \\ &\leq \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}_k} \|f 1_{E_{k,n} \setminus F_{k+1}}\|_{L^1(X, \omega)} + \sum_{k \in \mathbb{Z}} \sum_{i \in I_k} \|f 1_{Q_i \setminus F_{k+1}}\|_{L^1(X, \omega)} \\ &\leq \sum_{k \in \mathbb{Z}} 2^{k+1} \sum_{n \in \mathbb{N}_k} \sigma(E_{k,n}) + \sum_{k \in \mathbb{Z}} 2^{k+1} \sum_{i \in I_k} \sigma(Q_i) \\ &\leq \sum_{k \in \mathbb{Z}} 2^{k+1} \sum_{n \in \mathbb{N}_k} \sigma(E_{k,n}) + \sum_{k \in \mathbb{Z}} 2^{k+1} \sum_{(l,m): l > k} \sigma(E_{l,m}) \\ &\leq \sum_{k \in \mathbb{Z}} 2^{k+1} \sum_{n \in \mathbb{N}_k} \sigma(E_{k,n}) + C \sum_{l \in \mathbb{Z}} 2^{l+1} \sum_{m \in \mathbb{N}_l} \sigma(E_{l,m}) \\ &\leq C \sum_{k \in \mathbb{Z}} 2^{k-1} \mu(\ell^1(f) > 2^{k-1}) \\ &\leq C \|f\|_{L^1(\ell^1)}, \end{aligned}$$

where we used (2.11) in the second inequality, (2.9) and the disjointness of the dyadic boxes $\{Q_i\}$ in the third, Fubini and the bounds on the geometric series in the fourth, and (2.13) in the fifth.

A standard approximation argument yields the result for arbitrary $f \in L^1(\ell^1)$. $\square$

Now we provide the candidate function g for $f \in L^p(\ell^r) \cap L^\infty(\ell^r)$, for $1 \leq p, r \leq \infty$. We separate the definition into four cases depending on p and r .

Case 1: $1 \leq p, r < \infty$. For $f \in L^p(\ell^r) \cap L^\infty(\ell^r)$, let $\{E_{k,n}\}$ be the collection from Proposition 2.4, and define

$$g(x, s) = \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}_k} 2^{k(p-r)} 1_{E_{k,n} \setminus F_{k,n-1}}(x, s) |f(x, s)|^{r-1}.$$

Case 2: $1 \leq p < \infty$ and $r = \infty$. For $f \in L^p(\ell^\infty) \cap L^\infty(\ell^\infty)$, let $\{E_{k,n}\}$ be the collection from Proposition 2.4, and define

$$g(x, s) = \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}_k} 2^{k(p-1)} 1_{\tilde{E}_{k,n}}(x, s) (\ell^1(1_{\tilde{E}_{k,n}})(E_{k,n}))^{-1},$$

where

$$\tilde{E}_{k,n} = E_{k,n}^+ \cap \{|f| > 2^k\},$$

and $E_{k,n}^+$ is the upper half of $E_{k,n}$.

Case 3: $p = \infty$ and $1 \leq r < \infty$. For $f \in L^\infty(\ell^r)$, let the dyadic box $E \in \mathcal{D}$ witness the outer $L^\infty(\ell^r)$ quasi-norm of f up to a factor 2, and define

$$g(x, s) = 1_E(x, s) |f(x, s)|^{r-1}.$$

Case 4: $p = r = \infty$. For $f \in L^\infty(\ell^\infty)$, let the dyadic box $E \in \mathcal{D}$ witness the outer $L^\infty(\ell^\infty)$ quasi-norm of f up to a factor 2 in a subset of strictly positive measure in E^+ , and define

$$g(x, s) = 1_{\tilde{E}}(x, s) (\ell^1(1_{\tilde{E}})(E))^{-1},$$

where

$$\tilde{E} = E^+ \cap \{|f| > \|f\|_{L^\infty(\ell^\infty)}/2\}.$$

We have the following upper bounds on the outer $L^{p'}(\ell^{r'})$ quasi-norm of g , where g is defined according to the four (p, r) -dependent cases.

Lemma 3.1 *Case I:* $p = 1$ and $1 \leq r \leq \infty$. We have

$$\|g\|_{L^\infty(\ell^{r'})} \leq C.$$

Case II: $1 < p < \infty$ and $1 \leq r \leq \infty$. We have

$$\|g\|_{L^{p'}(\ell^{r'})}^{p'} \leq C \|f\|_{L^p(\ell^r)}^p.$$

Case III: $p = \infty$ and $1 \leq r < \infty$. We have

$$\|g\|_{L^1(\ell^{r'})} \leq \|f\|_{L^\infty(\ell^r)}^{r-1} \sigma(E).$$

Case IV: $p = r = \infty$. We have

$$\|g\|_{L^1(\ell^1)} \leq \sigma(E).$$

Proof Case I: $p = 1$ and $1 \leq r \leq \infty$. Let $1 < r < \infty$. For every dyadic box $A \in \mathcal{D}$, we have

$$\begin{aligned} (\ell^{r'}(g)(A))^{r'} &= \frac{1}{\sigma(A)} \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}_k} 2^{-kr} \int_{A \cap (E_{k,n} \setminus F_{k,n-1})} |f(y, t)|^r \omega(y, t) \, dy \, dt \\ &\leq \frac{1}{\sigma(A)} \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}_k} 2^{-kr} \int_{A \cap (E_{k,n} \setminus F_{k+1})} |f(y, t)|^r \omega(y, t) \, dy \, dt \\ &\leq C \frac{1}{\sigma(A)} \left(\sigma(A) + \sum_{(k,n): E_{k,n} \subseteq A} \sigma(E_{k,n}) \right) \\ &\leq C, \end{aligned} \quad (3.1)$$

where we used (2.11) and the nested structure of $\mathcal{D}$, namely the fact that for $A, B \in \mathcal{D}$, $A \cap B \neq \emptyset$, then either $A \subseteq B$ or $B \subseteq A$, in the second inequality, and (2.14) in the third.

In an analogous way, for every dyadic box $A \in \mathcal{D}$, for $r = \infty$, we have

$$\ell^1(g)(A) \leq C,$$

and it is easy to see that, for $r = 1$, we have

$$\ell^\infty(g)(A) \leq 1.$$

Therefore, for $1 \leq r \leq \infty$, we have

$$\|g\|_{L^\infty(\ell^{r'})} \leq C.$$

Case II: $1 < p < \infty$ and $1 \leq r \leq \infty$. Let $1 < r < \infty$. For a fixed k and every dyadic box $A \in \mathcal{D}$, we have

$$\begin{aligned} (\ell^{r'}(g1_{F_k^c})(A))^{r'} &= \frac{1}{\sigma(A)} \sum_{(l,m): l < k} 2^{l(p-r)r'} \int_{A \cap (E_{l,m} \setminus F_{l,m-1})} |f(y, t)|^r \omega(y, t) \, dy \, dt \\ &\leq \sum_{l < k} 2^{l(p-r)r'} \frac{1}{\sigma(A)} \int_{A \setminus F_{l+1}} |f(y, t)|^r \omega(y, t) \, dy \, dt \\ &\leq c \sum_{l < k} 2^{l(p-r+r-1)r'} \\ &\leq c 2^{k(p-1)r'}, \end{aligned} \quad (3.2)$$

where we used (2.11) in the second inequality, and the bounds on the geometric series in the third.

In an analogous way, for every dyadic box $A \in \mathcal{D}$, for $r = \infty$, we have

$$\begin{aligned} \ell^1(g1_{F_k^c})(A) &= \frac{1}{\sigma(A)} \sum_{l < k} \sum_{m \in \mathbb{N}_l} 2^{l(p-1)} \int_{A \cap \tilde{E}_{l,m}} (\ell^1(1_{\tilde{E}_{l,m}})(E_{l,m}))^{-1} \omega(y, t) \, dy \, dt \\ &\leq \sum_{l < k} 2^{l(p-1)} \frac{1}{\sigma(A)} \sum_{m: E_{l,m} \subseteq A} \sigma(E_{l,m}) \\ &\leq c2^{k(p-1)}, \end{aligned} \quad (3.3)$$

where we used the disjointness of the elements of $\{E_{l,m}: m \in \mathbb{N}_l\}$ due to the maximality in their choice, and the bounds on the geometric series in the second inequality.

It is easy to see that, for every dyadic box $A \in \mathcal{D}$, for $r = 1$, we have

$$\ell^\infty(g1_{F_k^c})(A) \leq 2^{k(p-1)}.$$

As a consequence, for $1 \leq r \leq \infty$, for every dyadic box $A \in \mathcal{D}$, we have

$$\ell^{r'}(g1_{F_k^c})(A) \leq c2^{k(p-1)},$$

hence

$$\mu(\ell^{r'}(g) > c2^{k(p-1)}) \leq \mu(F_k) \leq C \sum_{(l,m): l \geq k} \sigma(E_{l,m}). \quad (3.4)$$

Therefore, we have

$$\begin{aligned} \|g\|_{L^{p'}(\ell^{r'})}^{p'} &\leq C \sum_{k \in \mathbb{Z}} 2^{kp} \mu(\ell^{r'}(g) > c2^{k(p-1)}) \\ &\leq C \sum_{k \in \mathbb{Z}} 2^{kp} \sum_{(l,m): l \geq k} \sigma(E_{l,m}) \\ &\leq C \sum_{l \in \mathbb{Z}} 2^{lp} \sum_{m \in \mathbb{N}_l} \sigma(E_{l,m}) \\ &\leq C \sum_{l \in \mathbb{Z}} 2^{lp} \mu(\ell^r(f) > 2^{l-1}) \\ &\leq C \|f\|_{L^p(\ell^r)}^p, \end{aligned}$$

where we used (3.4) in the second inequality, Fubini and the bounds on the geometric series in the third, and (2.13) in the fourth.

Case III: $p = \infty$ and $1 \leq r < \infty$. By construction we have

$$\|g\|_{L^\infty(\ell^{r'})} \leq \|f\|_{L^\infty(\ell^r)}^{r-1},$$

therefore, by outer Hölder's inequality, Proposition A.5, we have

$$\|g\|_{L^1(\ell^{r'})} \leq \|g\|_{L^\infty(\ell^{r'})} \|1_E\|_{L^1(\ell^\infty)} \leq \|f\|_{L^\infty(\ell^r)}^{r-1} \sigma(E).$$

Case IV: $p = r = \infty$. In an analogous way, we have

$$\|g\|_{L^1(\ell^1)} \leq \sigma(E),$$

since by construction $\|g\|_{L^\infty(\ell^1)} = 1$.

Remark 3.2 Without the crucial property of the decomposition established by (2.14), the argument in (3.1) above produces the empty upper bound

$$(\ell^{r'}(g)(E))^{r'} \leq \sum_{k \in \mathbb{Z}} 1.$$

Nevertheless, when $1 < p < \infty$, in (3.2) and in (3.3) we can already get a summable decay in $l < k$ for the upper bound on the $\ell^{r'}$ size of g over the sets $A \cap (F_l \setminus F_{l+1})$, and it is not necessary to invoke (2.14).

We have the following lower bounds on the $L^1(X, \omega)$ norm of fg , where as above g is defined according to the four (p, r) -dependent cases.

Lemma 3.3 *Case I:* $1 \leq p < \infty$ and $1 \leq r \leq \infty$. We have

$$\|fg\|_{L^1(X, \omega)} \geq C \|f\|_{L^p(\ell^r)}^p.$$

Case II: $p = \infty$ and $1 \leq r < \infty$. We have

$$\|fg\|_{L^1(X, \omega)} \geq C \|f\|_{L^\infty(\ell^r)}^r \sigma(E).$$

Case III: $p = r = \infty$. We have

$$\|fg\|_{L^1(X, \omega)} \geq C \|f\|_{L^\infty(\ell^r)} \sigma(E).$$

Proof *Case I:* $1 \leq p < \infty$ and $1 \leq r \leq \infty$. Let $1 \leq r < \infty$. For every fixed (k, n) such that $E_{k,n}$ is not empty, we have

$$\ell^1(fg 1_{F_{k,n-1}^c})(E_{k,n}) = 2^{k(p-r)} (\ell^r(f 1_{F_{k,n-1}^c})(E_{k,n}))^r > 2^{kp}, \quad (3.5)$$

where we used (2.10) in the inequality.

For $r = \infty$, by the definition of g , we have the same inequality.

Therefore, for $1 \leq r \leq \infty$, we have

$$\begin{aligned}
 \|fg\|_{L^1(X,\omega)} &\geq \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}_k} \|fg\|_{L^1(E_{k,n} \setminus F_{k,n-1}, \omega)} \\
 &\geq \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}_k} 2^{kp} \sigma(E_{k,n}) \\
 &\geq C \sum_{l \in \mathbb{Z}} 2^{lp} \sum_{(k,n): l \leq k} \sigma(E_{k,n}) \\
 &\geq C \sum_{l \in \mathbb{Z}} 2^{lp} \mu(\ell^r(f) > 2^l) \\
 &\geq C \|f\|_{L^p(\ell^r)}^p,
 \end{aligned}$$

where we used (3.5) in the second inequality, the bounds on the geometric series and Fubini in the third, and (2.12) in the fourth.

Case II: $p = \infty$ and $1 \leq r < \infty$. Let $E \in \mathcal{D}$ be the dyadic box associated with g , in particular

$$\ell^r(f)(E) \geq C \|f\|_{L^\infty(\ell^r)}.$$

Therefore, we have

$$\begin{aligned}
 \|fg\|_{L^1(X,\omega)} &= \|f 1_E\|_{L^r(X,\omega)}^r \\
 &= \ell^r(f)(E)^r \sigma(E) \\
 &\geq C \|f\|_{L^\infty(\ell^r)}^r \sigma(E).
 \end{aligned}$$

Case III: $p = r = \infty$. In an analogous way, we have

$$\|fg\|_{L^1(X,\omega)} \geq C \|f\|_{L^\infty(\ell^r)} \sigma(E).$$

□

Proof of Theorem 1.2, properties (ii),(iii) The first inequality in (ii) is given by outer Hölder's inequality, Proposition A.5.

The second inequality in (ii) is a corollary of the previous Lemmata for $f \in L^p(\ell^r) \cap L^\infty(\ell^r)$. A standard approximation argument yields the case of an arbitrary $f \in L^p(\ell^r)$.

The statement in (iii) is a corollary of the triangle inequality for the $L^1(X, \omega)$ norm and property (ii). □

We conclude the part of the section about the upper half space with the following observation.

Let X be the upper half space and ν the outer measure generated by the pre-measure σ on $\mathcal{E}$, the collection of all the open cubic boxes in the upper half space, as in (1.2).

In particular,

$$\begin{aligned} X &= \mathbb{R}^d \times (0, \infty), \\ \mathcal{E} &= \{(x, 0) + (0, s)^{d+1} : x \in \mathbb{R}^d, s \in (0, \infty)\}, \\ \sigma(E) &= |B(E)|, \quad \text{for every } E \in \mathcal{E}, \\ \omega(y, t) &= t^{-1}, \end{aligned} \quad (3.6)$$

where $B(E)$ is the base in $\mathbb{R}^d$ of the box E , and $|B(E)|$ its volume. We observe that $\mathcal{D} \subseteq \mathcal{E}$, and every box in $\mathcal{E}$ can be covered up to a set of measure zero by finitely many dyadic boxes in $\mathcal{D}$ of comparable pre-measure. Therefore, the outer $L^p(\ell^r)$ space quasi-norms in the settings (2.8) and (3.6) are equivalent by Proposition A.4. As a consequence, all the previous results obtained in the setting (2.8) extend to the setting (3.6). An analogous argument applies to the outer measure structure generated by triangular tents in place of cubic boxes.

We turn now to the finite setting.

Proof of Theorem 1.1 The proof of property (i) and, for $1 < p \leq \infty$, $1 \leq r < \infty$, of property (ii) follows by arguments analogous to those in the previous proofs, using the decomposition in Proposition 2.1.

For $p = r \in \{1, \infty\}$, the statement in (ii) follows by the equivalence between $L^p(\ell^p)$ and $L^p(X, \omega)$ by property (i).

The statement in (iii) is again a corollary of the triangle inequality for the $L^1(X, \omega)$ norm and property (ii). $\square$

Lemma 3.4 *Let $1 < r \leq \infty$. For every $M > 0$, there exist a finite set X , a finite outer measure μ strictly positive on every singleton in $\mathcal{P}(X)$, a strictly positive weight ω , functions $f, f_n \in L^1(\ell^r)$ such that*

$$\begin{aligned} \|f\|_{L^1(\ell^r)} &\geq M \sup_{\|g\|_{L^\infty(\ell^{r'})}=1} \|fg\|_{L^1(\ell^1)}, \\ \left\| \sum_{n \in \mathbb{N}} f_n \right\|_{L^1(\ell^r)} &\geq M \sum_{n \in \mathbb{N}} \|f_n\|_{L^1(\ell^r)}. \end{aligned}$$

Proof Let $\mathcal{D}$ be the set of dyadic intervals. For every $m \in \mathbb{N}$, let

$$\begin{aligned} X_m &= \{I \in \mathcal{D} : I \subseteq [0, 1], |I| \geq 2^{-m}\}, \\ \mathcal{E}_m &= \{E_I = \{J \in \mathcal{D} : I \subseteq J \subseteq [0, 1]\} : I \in X_m, |I| = 2^{-m}\}, \\ \sigma_m(E_I) &= 1, \quad \text{for every } I \in X_m, |I| = 2^{-m}, \\ \omega_m(J) &= 1, \quad \text{for every } J \in X_m, \\ f_m(J) &= 2^m |J|, \\ f_I(J) &= 1_{E_I}(J), \quad \text{for every } I \in X_m, |I| = 2^{-m}. \end{aligned}$$

We have

$$\left\| \sum_{I \in X_m, |I|=2^{-m}} f_I \right\|_{L^1(\ell^r)} = \|f_m\|_{L^1(\ell^r)} \geq 2^m \frac{m+1}{2},$$

$$\sum_{I \in X_m, |I|=2^{-m}} \|f_I\|_{L^1(\ell^r)} = \sum_{I \in X_m, |I|=2^{-m}} (m+1)^{\frac{1}{r}} = 2^m (m+1)^{\frac{1}{r}}.$$

For m big enough, we get the second statement. In particular, this yields a counterexample to the uniformity of the constant in the statement of Theorem 1.1, property (iii). Therefore, also the uniformity of the constant in the statement of Theorem 1.1, property (ii) does not hold true. $\square$

4 Equivalence with Tent Spaces

In this section we prove the equivalence between the outer $L^p(\ell^r)$ spaces in the upper half space setting (3.6) and the tent spaces T_r^p stated in Theorem 1.3. First, in Lemma 4.1 we prove the equivalence for certain exponents p, r . After that, we extend it to the full range $0 < p, r \leq \infty$ via the Köthe duality result for the outer $L^p(\ell^r)$ spaces, equivalent to that stated in Theorem 1.2, property (ii), and the analogous result for tent spaces T_r^p , stated in Proposition 4.2.

Lemma 4.1 *For $p = \infty, 0 < r < \infty$ or $0 < p < \infty, r = \infty$, there exists a constant $C = C(p, r)$ such that, for every $f \in L^p(\ell^r)$,*

$$\frac{1}{C} \|f\|_{T_r^p} \leq \|f\|_{L^p(\ell^r)} \leq C \|f\|_{T_r^p}.$$

Proof Without loss of generality, it is enough to consider the cases

$$\begin{aligned} p &= \infty, r = 1, \\ p &= 1, r = \infty. \end{aligned} \tag{4.1}$$

In fact, let $q < \infty$ be the minimum of p and r . We have

$$\begin{aligned} \|f\|_{T_r^p}^q &= \|f^q\|_{T_{r/q}^{p/q}}, \\ \|f\|_{L^p(\ell^r)}^q &= \|f^q\|_{L^{p/q}(\ell^{r/q})}, \end{aligned}$$

where $\infty/q = \infty$, thus recovering one of the cases in (4.1).

Case I: $p = \infty, r = 1$. The quantities associated with the spaces $L^\infty(\ell^1)$, T_1^∞ are equivalent by definition, up to a constant determined by a simple covering argument between boxes and tents.

Case II: $p = 1, r = \infty$. Let $f \in L^1(\ell^\infty)$. For every $\lambda > 0$, let $\mathcal{E}_\lambda \subseteq \mathcal{E}$ be a covering witnessing the super level measure at level λ up to a factor 2. In particular, we have

$$2\mu(\ell^\infty(f) > \lambda) \geq \sum_{E \in \mathcal{E}_\lambda} \sigma(E).$$

For

$$B_\lambda = \bigcup_{E \in \mathcal{E}_\lambda} 10B(E) \subseteq \mathbb{R}^d,$$

where $10B$ is the cube in $\mathbb{R}^d$ with the same centre of B and 10 times its side length, we have

$$|B_\lambda| \leq C \sum_{E \in \mathcal{E}_\lambda} \sigma(E) \leq C\mu(\ell^\infty(f) > \lambda).$$

Moreover, for every $x \in B_\lambda^c$, we have

$$A_\infty(f)(x) \leq \lambda,$$

otherwise we get a contradiction with the definition of $\mathcal{E}_\lambda$. Therefore, we have

$$|\{x \in \mathbb{R}^d : A_\infty(f)(x) > \lambda\}| \leq C\mu(\ell^\infty(f) > \lambda). \quad (4.2)$$

Now let $f \in T_\infty^p$. For every $\lambda > 0$, let D_λ be

$$D_\lambda = \{x \in \mathbb{R}^d : A_\infty(f)(x) > \lambda\},$$

and define

$$E_\lambda = \bigcup_{i \in I_\lambda} 10Q_i \subseteq X,$$

where $\{B(Q_i)\}$ is a Whitney decomposition of D_λ , and $10Q$ is the box whose base $B(10Q)$ has the same centre of $B(Q)$ and 10 times its side length. In particular, we have

$$\mu(E_\lambda) \leq C|D_\lambda|.$$

Moreover, for every $E \in \mathcal{E}$, we have

$$\ell^\infty(f1_{E_\lambda^c})(E) \leq \lambda,$$

otherwise we get a contradiction with the definition of D_λ . Therefore, we have

$$\mu(\ell^\infty(f) > \lambda) \leq C|\{x \in \mathbb{R}^d : A_\infty(f)(x) > \lambda\}|. \quad (4.3)$$

The desired equivalence follows by integrating the inequalities (4.2), (4.3) over all levels $\lambda > 0$. $\square$

For the tent spaces T_r^p we have the following Köthe duality result, see for example Theorem 5.2 in [14].

Proposition 4.2 For $1 \leq p, r \leq \infty$, for every $f \in T_r^p$,

$$\sup_{\|g\|_{T_{r'}^{p'}=1}} \|fg\|_{L^1(X,\omega)} \leq \|f\|_{T_r^p} \leq \sup_{\|g\|_{T_{r'}^{p'}=1}} \|fg\|_{L^1(X,\omega)}.$$

Proof of Theorem 1.3 Without loss of generality, it is enough to consider the cases

$$\begin{aligned} p &= r = \infty, \\ 1 < p \leq \infty, r &= 1, \\ p &= 1, 1 \leq r \leq \infty, \end{aligned}$$

due to an argument analogous to that in the previous proof.

Case I: $p = r = \infty$. The equivalence between $L^\infty(\ell^\infty)$, T_∞^∞ follows by definition.

Case II: $1 < p \leq \infty, r = 1$. For $p = \infty$ the quantities associated with the spaces $L^\infty(\ell^1)$, T_1^∞ are equivalent by Lemma 4.1.

For $1 < p < \infty$, let $f \in L^p(\ell^1)$. By Theorem 1.2, property (ii), we have

$$\frac{1}{C} \sup_{\|g\|_{L^{p'}(\ell^\infty)} \leq 1} \|fg\|_{L^1(X,\omega)} \leq \|f\|_{L^p(\ell^1)} \leq C \sup_{\|g\|_{L^{p'}(\ell^\infty)} \leq 1} \|fg\|_{L^1(X,\omega)}.$$

Applying Lemma 4.1 to g , we have

$$\frac{1}{C} \sup_{\|g\|_{T_\infty^{p'} \leq 1}} \|fg\|_{L^1(X,\omega)} \leq \|f\|_{L^p(\ell^1)} \leq C \sup_{\|g\|_{T_\infty^{p'} \leq 1}} \|fg\|_{L^1(X,\omega)}.$$

Finally, by Proposition 4.2, we conclude

$$\frac{1}{C} \|f\|_{T_1^p} \leq \|f\|_{L^p(\ell^1)} \leq C \|f\|_{T_1^p}.$$

Case III: $p = 1, 1 \leq r \leq \infty$. For $p = 1, r = \infty$, the quantities associated with the spaces $L^1(\ell^\infty)$, T_∞^1 are equivalent by Lemma 4.1.

For $p = 1, 1 \leq r < \infty$, an argument analogous to that used to prove Case II yields the desired equivalence. If $p = r = 1$, we use Case I in place of Lemma 4.1.

To conclude, we observe that the set of bounded functions with compact support in X is dense in T_r^p for $1 \leq p < \infty, r = 1$ and $p = 1, 1 \leq r < \infty$. However, these functions are also in $L^p(\ell^r)$. Therefore, the two spaces coincide. $\square$

5 Hardy–Littlewood–Sobolev Inclusions for Tent Spaces

In this section we improve over a result of Amenta on continuous inclusions between tent spaces T_r^p , see Theorem 2.19 and Lemma 2.20 in [1]. In his notation, we have the

weighted tent spaces $T_s^{p,r}$ defined, for $0 < p, r \leq \infty, s \in \mathbb{R}$, by

$$T_s^{p,r} = \{f: t^{-ds} f \in T_r^p\}, \quad \|f\|_{T_s^{p,r}} = \|t^{-ds} f\|_{T_r^p},$$

where T_r^p is defined in (1.11) and (1.12), and the continuous inclusions

$$T_0^{p,r} \hookrightarrow T_{\frac{1}{q}-\frac{1}{p}}^{q,r}, \quad f \mapsto f,$$

for $0 < p < q \leq \infty, 0 < r \leq \infty$. The improvement consists of allowing for two different values of r , under certain conditions, in each of the two spaces in the last display.

Due to the equivalence proved in the previous section, we get an analogous result for the outer $L^p(\ell^r)$ spaces in the upper half space setting (3.6). This result is auxiliary in proving strong type estimates in the following section.

Theorem 5.1 *For $0 < p < q \leq \infty, 0 < r_2 \leq r_1 \leq \infty$, there exists a constant $C = C(p, q, r_1, r_2)$ such that, for every $f \in T_{r_1}^p$,*

$$\|t^{\frac{d}{p}-\frac{d}{q}} f\|_{T_{r_2}^q} \leq C \|f\|_{T_{r_1}^p}.$$

Equivalently, for every $f \in L^p(\ell^{r_1})$,

$$\|t^{\frac{d}{p}-\frac{d}{q}} f\|_{L^q(\ell^{r_2})} \leq C \|f\|_{L^p(\ell^{r_1})}.$$

The main ingredient is the following. We define a function a to be a T_r^p -atom associated with the ball $B \subseteq \mathbb{R}^d$ if a is essentially supported in $T(B)$ and

$$\|a\|_{T_r^p} \leq |B|^{\frac{1}{r}-\frac{1}{p}}. \quad (5.1)$$

Lemma 5.2 *Let $1 < q \leq r_2 \leq r_1 \leq \infty$. Suppose that a is a $T_{r_1}^1$ -atom. Then a is in $T_{r_2}^q$ with norm smaller than 1.*

Proof For $q < \infty$, let $0 < r, s \leq \infty$ be such that

$$\frac{1}{r} + \frac{1}{r_1} = \frac{1}{r_2}, \quad \frac{1}{s} + \frac{1}{r_1} = \frac{1}{q}.$$

We have

$$\begin{aligned}
 \|t^{d-\frac{d}{q}}a\|_{T_2^q} &= \|A_{r_2}(t^{d-\frac{d}{q}}a)\|_{L^q(B)} \\
 &\leq \|A_r(t^{d-\frac{d}{q}}1_{T(B)})A_{r_1}(a)\|_{L^q(B)} \\
 &\leq \|A_r(t^{d-\frac{d}{q}}1_{T(B)})\|_{L^s(B)}\|A_{r_1}(a)\|_{L^{r_1}(B)} \\
 &\leq |B|^{1-\frac{1}{r_1}}\|a\|_{T_1^{r_1}} \\
 &\leq 1,
 \end{aligned}$$

where we used Hölder's inequality in the first and in the second inequality, and (5.1) in the fourth.

For $q = r_2 = r_1 = \infty$, the statement follows directly from (5.1).

Proof of Theorem 5.1 The proof of the first statement follows along the lines of that of Theorem 2.19 in [1], using Lemma 5.2 above in place of Lemma 2.20.

The second statement then follows by Theorem 1.3. $\square$

6 Embedding into Outer $L^p(\ell^r)$ Spaces with a Fractional Scale Factor

In this section we state and prove a full classification of all positive and negative results regarding strong and weak type estimates for a family of embedding maps with a fractional scale factor from classical L^p spaces on $\mathbb{R}^d$ to outer $L^p(\ell^r)$ spaces in the upper half space setting.

The positive results for $d = 1$, $1 \leq p = q \leq \infty$, $r = \infty$ were already proved in [12], see Theorem 4.1. Although there ϕ was assumed to be smooth and compactly supported, the same argument can be extended with minor adjustments to the test functions satisfying the boundedness and decay condition (1.14) and to all dimensions.

We conclude the section by stating and proving an embedding theorem with a fractional scale factor for functions in the Hardy space $H^1(\mathbb{R}^d)$ into the outer $L^1(\ell^\infty)$ space. The embedded function in this case is that defined in (1.13) for a smooth test function $\phi \in \mathcal{S}(\mathbb{R}^d)$.

Theorem 6.1 *Let*

$$1 \leq p, q \leq \infty, 0 < r \leq \infty. \quad (6.1)$$

Then, for (p, q, r) satisfying one of the following conditions, which are also displayed in Fig. 1 below,

$$\begin{aligned}
 1 &< p < q \leq \infty, 0 < r \leq \infty, \\
 1 &< p = q \leq \infty, r = \infty, \\
 p &= 1, q = \infty, 0 < r \leq \infty,
 \end{aligned} \quad (6.2)$$

there exists a constant $C = C(p, q, r, d, \varepsilon)$ such that, for every $f \in L^p(\mathbb{R}^d)$,

$$\|t^{\frac{d}{p}-\frac{d}{q}}F(f)\|_{L^q(\ell^r)} \leq C\|f\|_{L^p(\mathbb{R}^d)}.$$

Fig. 1 Range of exponents p, q, r and weak/strong type estimates

For all the triples (p, q, r) satisfying (6.1) but none of the conditions in (6.2), no strong type (p, q) estimate holds true.

Moreover, for (p, q, r) satisfying one of the following conditions, which are also displayed in Fig. 1 below,

$$\begin{aligned} 1 = p < q < \infty, 0 < r \leq \infty, \\ p = q = 1, r = \infty, \end{aligned} \quad (6.3)$$

there exists a constant $C = C(q, r, d, \varepsilon)$ such that, for every $f \in L^1(\mathbb{R}^d)$,

$$\|F(f)\|_{L^{q,\infty}(\ell^r)} \leq C \|f\|_{L^1(\mathbb{R}^d)}.$$

For all the triples (p, q, r) satisfying (6.1) but none of the conditions in (6.2), (6.3), no weak type (p, q) estimate holds true.

In the next proof, the constants c, C are allowed to depend on d, ε, p, q, r but not on f .

Proof of Theorem 6.1 Without loss of generality, we can assume f to be nonnegative. In fact, by definition (1.13), we have the pointwise bound

$$|F_\phi(f)(y, t)| \leq F_{|\phi|}(|f|)(y, t) \leq F(|f|)(y, t).$$

In particular, we have

$$F(f)(y, t) = \int_{\mathbb{R}^d} f(z) t^{-d} (1 + t^{-1}|y - z|)^{-d-\varepsilon} dz.$$

This expression can be bounded either by means of the centred maximal function

$$F(f)(y, t) \leq CMf(y), \quad (6.4)$$

or by Young's convolution inequality

$$F(f)(y, t) \leq Ct^{-\frac{d}{p}} \|f\|_{L^p(\mathbb{R}^d)}. \quad (6.5)$$

6.1 Strong Type (p, q) Estimates for $0 < r \leq \infty$ in the Range for $p \neq 1, q$ Displayed in Fig. 1

The strong type (p, q) estimates in the range $1 < p < q \leq \infty, 0 < r \leq \infty$ follow by the already known strong type (p, p) estimate for $1 < p \leq \infty, r = \infty$ and Theorem 5.1.

6.2 Strong Type $(1, \infty)$ Estimates for $0 < r \leq \infty$

We aim to prove that, for every $E \in \mathcal{E}$,

$$\ell^r(t^d F(f))(E) \leq C \|f\|_{L^1(\mathbb{R}^d)}. \quad (6.6)$$

If $r = \infty$, the claim follows by (6.5).

Now let $0 < r < \infty$. By Theorem 1.2, property (iii), the decay property of ϕ , and the translation invariance of the $L^\infty(\ell^r)$ quasi-norm, it is enough to prove the inequality assuming that f is supported in $(-1, 1)^d$ and $\phi = 1_{(-1, 1)^d}$. In this case, we have

$$F_\phi(f)(y, t) \leq Ct^{-d} \|f 1_{y+(-t, t)^d}\|_{L^1(\mathbb{R}^d)} 1_{\{(-1-s, 1+s)^d \times \{s\}, s>0\}}(y, t),$$

and it is enough to prove (6.6) for the elements of $\mathcal{E}$ of the form

$$E_{x,u} = (x + (-u, u)^d) \times (0, 2u) \in \mathcal{E},$$

for every $u > 0, x \in (-1-u, 1+u)^d$. We distinguish two cases, $r \geq 1$ and $0 < r < 1$.

Case I: $r \geq 1$. Let $r = 1$. We have

$$\begin{aligned} \ell^1(t^d F_\phi(f))(E_{x,u}) &\leq \frac{C}{u^d} \int_0^{2u} \int_{x+(-u,u)^d} \int_{(-1,1)^d} f(z) 1_{y+(-t,t)^d}(z) \, dz \, dy \, \frac{dt}{t} \\ &\leq \frac{C}{u^d} \int_{(-1,1)^d} f(z) \int_0^{2u} \int_{x+(-u,u)^d} 1_{z+(-t,t)^d}(y) \, dy \, \frac{dt}{t} \, dz \\ &\leq \frac{C}{u^d} \|f\|_{L^1(\mathbb{R}^d)} \int_0^{2u} t^d \frac{dt}{t} \\ &\leq C \|f\|_{L^1(\mathbb{R}^d)}, \end{aligned}$$

where we used Fubini in the second inequality.

If $1 < r < \infty$, Proposition A.8 implies the strong type $(1, \infty)$ estimate for $L^\infty(\ell^r)$ from those for $L^\infty(\ell^1)$, $L^\infty(\ell^\infty)$.

Case II: $0 < r < 1$. We have

$$\begin{aligned} \ell^r(t^d F_\phi(f))(E_{x,u}) &\leq \ell^1(t^{d-\frac{1}{2}} F_\phi(f))(E_{x,u}) \ell^{\frac{r}{1-r}}(t^{\frac{1}{2}})(E_{x,u}) \\ &\leq C \left(\frac{1}{u^d} \int_0^{2u} \int_{x+(-u,u)^d} t^{-\frac{1}{2}} \int_{(-1,1)^d} f(z) 1_{y+(-t,t)^d}(z) \, dz \, dy \, \frac{dt}{t} \right) \times \\ &\quad \times \left(\frac{1}{u^d} \int_0^{2u} \int_{x+(-u,u)^d} t^{\frac{r}{2(1-r)}} \, dy \, \frac{dt}{t} \right)^{\frac{1-r}{r}} \\ &\leq C \|f\|_{L^1(\mathbb{R}^d)} \left(\frac{1}{u^d} \int_0^{2u} t^{d-\frac{1}{2}} \frac{dt}{t} \right) \left(\int_0^{2u} t^{\frac{r}{2(1-r)}} \frac{dt}{t} \right)^{\frac{1-r}{r}} \\ &\leq C \|f\|_{L^1(\mathbb{R}^d)}, \end{aligned}$$

where we used Hölder's inequality with exponents $(1, \frac{r}{1-r})$ in the first inequality, and then we proceeded as in the previous case.

6.3 Weak Type $(1, q)$ Estimates for $0 < r \leq \infty$ in the Range for $q \neq \infty$ Displayed in Fig. 1

We aim to prove that, for every $\lambda > 0$,

$$\lambda^q \mu(\ell^r(t^{d-\frac{d}{q}} F(f)) > \lambda) \leq C \|f\|_{L^1(\mathbb{R}^d)}^q.$$

This requires to construct, for every $\lambda > 0$, a set with appropriate outer measure approximating the super level measure at level λ .

For fixed f and $\lambda > 0$, let D_λ be the set

$$D_\lambda = \{x \in \mathbb{R}^d : Mf(x) > \lambda^q \|f\|_{L^1(\mathbb{R}^d)}^{1-q}\}.$$

We have

$$|D_\lambda| \leq C \lambda^{-q} \|f\|_{L^1(\mathbb{R}^d)}^q,$$

because of the weak type $(1, 1)$ estimate for the maximal function operator on $\mathbb{R}^d$.

Let $\{B_i : i \in I_\lambda\}$ be a Whitney covering of D_λ up to a set of measure 0 by pairwise disjoint open dyadic cubes in $\mathbb{R}^d$, and denote by x_i and s_i the centre and the side length of B_i , respectively. Let $Q(B_i) = Q_i \in \mathcal{D}$ be the open dyadic box over the cube B_i , and define

$$E_\lambda = \bigcup_{i \in I_\lambda} Q_i \subseteq X.$$

In particular, we have

$$\mu(E_\lambda) \leq |D_\lambda| \leq C\lambda^{-q} \|f\|_{L^1(\mathbb{R}^d)}^q.$$

We are left with proving that for every $E \in \mathcal{E}$,

$$\ell^r(t^{d-\frac{d}{q}} F(f) 1_{E_\lambda^c})(E) \leq C\lambda.$$

If $(x, s) \in E_\lambda^c$, $x \in D_\lambda$, then $x \in Q_i$ for some $i \in I_\lambda$, $s > s_i$, and there exists $u \in \mathbb{S}^{d-1}$ such that $x + s'u \in D_\lambda^c$, for $cs_i \leq s' \leq Cs_i$. As a consequence, for $t \geq s$, we have

$$t^{d-\frac{d}{q}} F(f)(x, t) \leq C(t + s')^{d-\frac{d}{q}} F(f)(x + s'u, t + s').$$

Therefore, we have

$$\ell^r(t^{d-\frac{d}{q}} F(f) 1_{E_\lambda^c})(E) \leq C \sup_{x \in D_\lambda^c} \|t^{d-\frac{d}{q}} F(f)\|_{L^r(\{x\} \times (0, \infty), \frac{dt}{t})},$$

and it is enough to show that for every $x \in D_\lambda^c$, we have

$$\|t^{d-\frac{d}{q}} F(f)\|_{L^r(\{x\} \times (0, \infty), \frac{dt}{t})} \leq C\lambda.$$

We split the norm on the left hand side at height $0 < R(x) < \infty$ soon to be fixed

$$\|t^{d-\frac{d}{q}} F(f)\|_{L^r(\{x\} \times (0, R(x)), \frac{dt}{t})} + \|t^{d-\frac{d}{q}} F(f)\|_{L^r(\{x\} \times (R(x), \infty), \frac{dt}{t})}. \quad (6.7)$$

We bound $F(f)$ by (6.4) in the first summand obtaining

$$CMf(x)R(x)^{d-\frac{d}{q}},$$

and by (6.5) in the second summand obtaining

$$C\|f\|_{L^1(\mathbb{R}^d)} R(x)^{-\frac{d}{q}}.$$

If $0 < r < \infty$, we require the additional hypothesis $q > 1$ to guarantee the L^r -integrability at 0 of the estimate for the first summand.

Optimizing the choice of $R(x)$ with

$$R(x) = CMf(x)^{-\frac{1}{d}} \|f\|_{L^1(\mathbb{R}^d)}^{\frac{1}{d}},$$

we get the bound for (6.7)

$$CMf(x)^{\frac{1}{q}} \|f\|_{L^1(\mathbb{R}^d)}^{1-\frac{1}{q}}.$$

We conclude by the estimate for every $x \in D_\lambda^c$,

$$Mf(x) \leq \lambda^q \|f\|_{L^1(\mathbb{R}^d)}^{1-\frac{1}{q}}.$$

6.4 Counterexample to the Strong Type $(1, q)$ Estimates for $1 \leq q < \infty$, $0 < r \leq \infty$

In the following counterexamples we are going to use test functions ϕ satisfying the condition (1.14) with a multiplicative factor different from 1. While it does not effect the nature of the counterexamples, it spares us the definition of other appropriate constants.

For $f = 1_{(-1,1)^d}$, $\phi = 1_{(-1,1)^d}$, we have

$$F_\phi(f)(y, t) \geq t^{-d} 1_{\{(-s,s)^d \times \{s\}, s \geq 1\}}(y, t).$$

For every $u \geq 1$, let

$$E_u = (0, u)^{d+1} \in \mathcal{E}.$$

Then, for $0 < r < \infty$, we have

$$\ell^r(t^{d-\frac{d}{q}} F(f) 1_{(\mathbb{R}^d \times (0,u))^c})(E_{2u}) \geq \left(\frac{1}{(2u)^d} \int_u^{2u} \int_{(0,u)^d} t^{-\frac{dr}{q}} dy \frac{dt}{t} \right)^{\frac{1}{r}} \geq Cu^{-\frac{d}{q}},$$

and it is easy to see that, for $r = \infty$, we have

$$\ell^\infty(t^{d-\frac{d}{q}} F(f) 1_{(\mathbb{R}^d \times (0,u))^c})(E_{2u}) = u^{-\frac{d}{q}}.$$

Therefore, for every fixed $u \geq 1$, if $A \subseteq X$ is such that

$$\ell^r(t^{d-\frac{d}{q}} F(f) 1_{A^c})(E_{2u}) \leq Cu^{-\frac{d}{q}},$$

then $A \setminus (\mathbb{R}^d \times (0, u)) \neq \emptyset$, hence we have

$$\mu(\ell^r(t^{d-\frac{d}{q}} F(f))) > Cu^{-\frac{d}{q}} \geq u^d.$$

As a consequence, we have

$$\|t^{d-\frac{d}{q}} F_\phi(f)\|_{L^q(\ell^r)}^q \geq C \int_0^C u^{-d} \mu(\ell^r(t^{d-\frac{d}{q}} F(f))) > C u^{-\frac{d}{q}} \frac{du}{u} = \infty.$$

6.5 Counterexample to the Weak Type (p, q) Estimates for $1 \leq q \leq p \leq \infty$, $0 < r < \infty$ and $1 \leq q < p \leq \infty$, $r = \infty$

For f, ϕ as above, we have

$$F_\phi(f)(y, t) \geq 1_{\{(-1+s, 1-s)^d \times \{s\}, s \leq 1\}}(y, t).$$

For every $x \in (0, \frac{1}{4})^d$, $u \leq \frac{1}{4}$, let

$$E_{x,u} = (x + (-u, u)^d) \times (0, 2u) \in \mathcal{E}.$$

Then, for $1 \leq q \leq p \leq \infty$, $0 < r < \infty$, we have

$$\ell^r(t^{\frac{d}{p}-\frac{d}{q}} F_\phi(f))(E_{x,u}) \geq \left(\frac{1}{(2u)^d} \int_0^{2u} \int_{x+(-u,u)^d} t^{\frac{dr}{p}-\frac{dr}{q}} dy \frac{dt}{t} \right)^{\frac{1}{r}} = \infty,$$

thus exhibiting a counterexample in the case $p = q = \infty$. Moreover, it is easy to see that, for $1 \leq q < p \leq \infty$, $r = \infty$, we have

$$\ell^\infty(t^{\frac{d}{p}-\frac{d}{q}} F_\phi(f))(E_{x,u}) = \infty.$$

Let $A \subseteq (-1, 1)^d \times (0, \infty)$ be such that, for every $x \in (0, \frac{1}{4})^d$, $u \leq \frac{1}{4}$,

$$\ell^r(t^{\frac{d}{p}-\frac{d}{q}} F(f) 1_{A^c})(E_{x,u}) < \infty. \quad (6.8)$$

For every finite collection $\mathcal{E}' \subseteq \mathcal{E}$ covering A , let

$$A_{\mathcal{E}'} = \bigcup_{E \in \mathcal{E}'} \overline{B(E)},$$

where $B(E)$ is the base in $\mathbb{R}^d$ of E , and $\overline{B}$ is the closure of B in $\mathbb{R}^d$. If $A_{\mathcal{E}'} \cap [0, \frac{1}{4}]^d \neq \emptyset$, there would exist x, u such that $E_{x,u} \cap A = \emptyset$, hence contradicting (6.8). Therefore, for every $\lambda > 0$, we have

$$\mu(\ell^r(t^{\frac{d}{p}-\frac{d}{q}} F_\phi(f)) > \lambda) \geq C,$$

where C does not depend on λ .

As a consequence, for $q \neq \infty$, we have

$$\|t^{\frac{d}{p}-\frac{d}{q}} F_\phi(f)\|_{L^{q,\infty}(\ell^r)}^q \geq C \sup_{\lambda>0} \lambda^q = \infty.$$

□

Before stating and proving the embedding result for functions in $H^1(\mathbb{R}^d)$, we recall the definition of H^1 -atom. A function f is a H^1 -atom associated with the cube $B \subseteq \mathbb{R}^d$ if f is essentially supported in B and

$$\int_B f(x) \, dx = 0, \quad \|f\|_{L^\infty(\mathbb{R}^d)} \leq |B|^{-1}.$$

Proposition 6.2 *Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then there exists a constant $C = C(d, \varphi)$ such that, for every $f \in H^1(\mathbb{R}^d)$,*

$$\|F_\varphi(f)\|_{L^1(\ell^\infty)} \leq C \|f\|_{H^1(\mathbb{R}^d)}.$$

Proof By Theorem 1.2, property (iii), the decay properties of φ and its derivatives, and the definition of the Hardy space $(H^1(\mathbb{R}^d), \|\cdot\|_{H^1(\mathbb{R}^d)})$, it is enough to prove the inequality assuming that φ is a smooth function compactly supported in a cube of side length 2 and f is a H^1 -atom associated with a cube B . Moreover, due to the translation invariance of the $L^1(\ell^\infty)$ quasi-norm, we can assume that both f, φ are supported in cubes centred in the origin. Therefore it is enough to show that

$$\|F_\varphi(f)\|_{L^1(\ell^\infty)} \leq C.$$

Let $2B$ be the cube with the same centre of B and double the side length. For $0 < t < |B|^{\frac{1}{d}}$, $y \in 2B$, we have

$$|F_\varphi(f)(y, t)| \leq C|B|^{-1},$$

where we used the L^∞ bounds for f .

For $t \geq |B|^{\frac{1}{d}}$, $y \in (-|B|^{\frac{1}{d}} - t, |B|^{\frac{1}{d}} + t)^d$, we have

$$\begin{aligned} |F_\varphi(f)(y, t)| &= Ct^{-d} \left| \int_B f(z) \varphi(t^{-1}(y - z)) \, dz \right| \\ &= Ct^{-d} \left| \int_B f(z) (\varphi(t^{-1}(y - z)) - \varphi(t^{-1}y)) \, dz \right| \\ &\leq Ct^{-d} \int_B |f(z)| t^{-1} |z| \, dz \\ &\leq C|B|^{\frac{1}{d}} t^{-(d+1)}, \end{aligned}$$

where we used the L^∞ bounds, the localized support and the cancellation property of f together with the smoothness of φ .

For all the others (y, t) , we have $F_\varphi(f)$ is 0, since the supports of f and the dilated version of φ are disjoint.

As a consequence, for $\lambda > C|B|^{-1}$, we have

$$\mu(\ell^\infty(F_\varphi(f)) > \lambda) = 0,$$

and for $0 < \lambda \leq C|B|^{-1}$, we have

$$\mu(\ell^\infty(F_\varphi(f)) > \lambda) \leq C|B|^{\frac{1}{d+1}} \lambda^{-\frac{d}{d+1}}.$$

Therefore, we have

$$\|F_\varphi(f)\|_{L^1(\ell^\infty)} \leq C \int_0^{C|B|^{-1}} \mu(\ell^\infty(F_\varphi(f)) > \lambda) d\lambda \leq C.$$

□

7 Applications

In this section we show some applications of the strong type estimates in Theorem 6.1 and Proposition 6.2. We use them to give alternative proofs of the Hardy–Littlewood–Sobolev inequality, and the Gagliardo–Nirenberg–Sobolev inequality up to the endpoint in the spirit of the two-step program outlined in the introduction.

Theorem 7.1 (HLS inequality) *For $1 < p, q < \infty$, $0 < \alpha < d$ such that*

$$\frac{1}{p} + \frac{1}{q} + \frac{\alpha}{d} = 2,$$

there exists a constant $C = C(p, q, d)$ such that, for every $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$,

$$\left| \int_{\mathbb{R}^{2d}} \frac{f(x)g(y)}{|x-y|^\alpha} dx dy \right| \leq C \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$

Proof Let $\psi \in \mathcal{S}(\mathbb{R})$ be such that $\text{supp } \hat{\psi} \subseteq [\frac{1}{2}, 2]$, $\int_0^\infty \hat{\psi}^2(t) \frac{dt}{t} = 1$, and define $\Psi, \Phi \in \mathcal{S}(\mathbb{R}^d)$ by

$$\hat{\Psi}(\xi) = \hat{\psi}(|\xi|), \quad \hat{\Phi}(\xi) = |\xi|^{\alpha-d} \hat{\psi}(|\xi|).$$

Let $f, g \in \mathcal{S}(\mathbb{R}^d)$. By a frequency localization argument, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^{2d}} \frac{f(x)g(y)}{|x-y|^\alpha} dx dy \right| &\leq C \left| \int_{\mathbb{R}^{2d}} \hat{f}(\xi) \hat{g}(\eta) |\xi - \eta|^{\alpha-d} \delta(\xi + \eta) d\xi d\eta \right| \\ &\leq C \left| \int_{\mathbb{R}^{2d} \times (0, \infty)} \hat{f}(\xi) \hat{g}(\eta) |\xi - \eta|^{\alpha-d} \delta(\xi + \eta) \hat{\psi}^2(t) d\xi d\eta \frac{dt}{t} \right| \\ &\leq C \left| \int_{\mathbb{R}^d \times (0, \infty)} t^{d-\alpha} \hat{f}(\xi) \hat{\psi}(|\xi|t) \hat{g}(-\xi) (t|\xi|)^{\alpha-d} \hat{\psi}(|\xi|t) d\xi \frac{dt}{t} \right| \\ &\leq C \left| \int_{\mathbb{R}^d \times (0, \infty)} t^{d-\alpha} F_\Psi(f)(y, t) G_\Phi(g)(y, t) dy \frac{dt}{t} \right|. \end{aligned}$$

By Theorem 1.2, property (i), the integral in the last display is bounded by

$$\|t^{d-\alpha} F_\Psi(f) G_\Phi(g)\|_{L^1(\ell^1)}.$$

Applying outer Hölder's inequality, Proposition A.5, we estimate it in terms of

$$\|t^{d-\alpha} F_\Psi(f)\|_{L^{q'}(\ell^1)} \|G_\Phi(g)\|_{L^q(\ell^\infty)},$$

which by the strong type estimates in Theorem 6.1 is bounded by

$$\|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$

A standard approximation argument yields the result for arbitrary $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$. $\square$

Theorem 7.2 (GNS inequality) *For $1 \leq p < d$, there exists a constant $C = C(p, d)$ such that, for every $f \in W^{1,p}(\mathbb{R}^d)$,*

$$\|f\|_{L^{p_*}(\mathbb{R}^d)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^d)},$$

where $p_* = \frac{dp}{d-p}$.

Moreover, there exists a constant $C = C(d)$ such that, for every $f \in W^{1,d}(\mathbb{R}^d)$,

$$\|f\|_{BMO(\mathbb{R}^d)} \leq C \|\nabla f\|_{L^d(\mathbb{R}^d)}.$$

Proof Let $\{\varphi_i\}_{i=1}^d$ be a smooth partition of the unity on the set $\{\frac{1}{2} \leq |\xi| \leq 2\}$ such that $\text{supp } \varphi_i \subseteq \{|\xi_i| > \frac{1}{4d}\} \cap \{\frac{1}{4} \leq |\xi| \leq 4\}$.

For $\psi \in \mathcal{S}(\mathbb{R})$ as above, let $\Psi_i \in \mathcal{S}(\mathbb{R}^d)$ be defined by

$$\hat{\Psi}_i(\xi) = \frac{\hat{\psi}(|\xi|)}{\xi_i} \varphi_i(\xi).$$

For $1 < p < d$, let $f, g \in \mathcal{S}(\mathbb{R}^d)$. By a frequency localization argument, we have

$$\begin{aligned} |\langle f, g \rangle| &\leq C \left| \int_{\mathbb{R}^{2d}} \hat{f}(\xi) \hat{g}(\eta) \delta(\xi + \eta) d\xi d\eta \right| \\ &\leq C \left| \int_{\mathbb{R}^{2d} \times (0, \infty)} \hat{f}(\xi) \hat{g}(\eta) \delta(\xi + \eta) \hat{\psi}^2(t) d\xi d\eta \frac{dt}{t} \right| \\ &\leq C \sum_{i=1}^d \left| \int_{\mathbb{R}^d \times (0, \infty)} t \xi_i \hat{f}(\xi) \frac{\hat{\psi}(|\xi|t)}{t \xi_i} \varphi_i(\xi) \hat{g}(-\xi) \hat{\psi}(|\xi|t) d\xi \frac{dt}{t} \right| \\ &\leq C \sum_{i=1}^d \left| \int_{\mathbb{R}^d \times (0, \infty)} t F_{\Psi_i}(\partial_i f)(y, t) G_{\Psi}(g)(y, t) dy \frac{dt}{t} \right|. \end{aligned}$$

By Theorem 1.2, property (i), the integral in the last display is bounded by

$$\sum_{i=1}^d \|t F_{\Psi_i}(\partial_i f) G_{\Psi}(g)\|_{L^1(\ell^1)}.$$

Applying outer Hölder's inequality, Proposition A.5, we estimate it in terms of

$$\sum_{i=1}^d \|t F_{\Psi_i}(\partial_i f)\|_{L^{p*}(\ell^1)} \|G_{\Psi}(g)\|_{L^{p*'}(\ell^\infty)},$$

which by the strong type estimates in Theorem 6.1 is bounded by

$$\sum_{i=1}^d \|\partial_i f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^{p*'}(\mathbb{R}^d)}.$$

The duality between $L^p(\mathbb{R}^d)$ spaces and the density of Schwartz functions in $L^p(\mathbb{R}^d)$ yield the desired inequality. A standard approximation argument yields the result for arbitrary $f \in W^{1,p}(\mathbb{R}^d)$.

For $p = d$, we proceed in the same way with $f \in \mathcal{S}(\mathbb{R}^d)$ and $g \in H^1(\mathbb{R}^d) \cap \mathcal{S}(\mathbb{R}^d)$, getting

$$|\langle f, g \rangle| \leq \sum_{i=1}^d \|t F_{\Psi_i}(\partial_i f)\|_{L^\infty(\ell^1)} \|G_{\Psi}(g)\|_{L^1(\ell^\infty)},$$

which by the strong type estimates in Theorem 6.1 and by Proposition 6.2 is bounded by

$$\sum_{i=1}^d \|\partial_i f\|_{L^d(\mathbb{R}^d)} \|g\|_{H^1(\mathbb{R}^d)}.$$

The duality between the spaces $BMO(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$ and the density of Schwartz functions in $H^1(\mathbb{R}^d)$ yield the desired inequality. A standard approximation argument yields the result for arbitrary $f \in W^{1,d}(\mathbb{R}^d)$.

For $p = 1, d > 1$, the statement can be classically proved by the Loomis-Whitney inequality. $\square$

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Appendix A: Outer L^p Spaces Theory

In this Appendix we review the theory of outer L^p spaces in the level of generality discussed in [12].

Definition A.1 (Outer measure, pre-measure) Let X be a set. An outer measure μ on X is a function from $\mathcal{P}(X)$, the power set of X , to $[0, \infty]$ that satisfies the following properties.

- (1) $\mu(\emptyset) = 0$.
- (2) If $E \subseteq F$ for two subsets of X , then $\mu(E) \leq \mu(F)$.
- (3) If $\{E_i\}$ is a countable collection of sets in X , then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

A pre-measure $(\sigma, \mathcal{E})$ on X is defined by a collection $\mathcal{E}$ of subsets of X and a function σ from $\mathcal{E}$ to $[0, \infty)$.

Since $\mathcal{E}$ is implicit in the definition of σ , we drop it in the notation $(\sigma, \mathcal{E})$, and we refer to the pre-measure with the symbol σ .

Definition A.2 (Size) Let (X, Σ) be a measurable space. A size $(S, \mathcal{A})$ on X is defined by a collection $\mathcal{A}$ of measurable subsets of X and a function S from $\mathcal{M}(X)$, the set of measurable functions on X , to $[0, \infty]^{\mathcal{A}}$, that satisfies, for every $f, g \in \mathcal{M}(X)$, for every $A \in \mathcal{A}$, the following properties.

- (1) If $|f| \leq |g|$, then $S(f)(A) \leq S(g)(A)$.
- (2) $S(\lambda f)(A) = |\lambda| S(f)(A)$ for every $\lambda \in \mathbb{C}$.

(3) There exists a constant C depending only on S but not on f, g, A such that

$$S(f + g)(A) \leq C[S(f)(A) + S(g)(A)].$$

Since $\mathcal{A}$ is implicit in the definition of S , we drop it in the notation $(S, \mathcal{A})$, and we refer to the size with the symbol S .

Now, let (X, Σ) be a measurable space, and let $\mathcal{E}$ be a countable collection of measurable subsets of X such that

$$X = \bigcup_{E \in \mathcal{E}} E.$$

Let σ be a pre-measure defined on $\mathcal{E}$ attaining only strictly positive values, and let μ be the outer measure generated by σ as in (1.2). Let $(S, \mathcal{A})$ be a size on X .

In particular, let ω be a measure on (X, Σ) , and assume that for all $A \in \Sigma$

$$\mu(A) = 0 \Rightarrow \omega(A) = 0.$$

For $0 < r < \infty$, we can define the following sizes. First, let $\ell_\sigma^\infty, \ell_\sigma^r$ be the sizes defined by, for every function $f \in \mathcal{M}(X)$, for every $E \in \mathcal{E}$,

$$\begin{aligned} \ell_\sigma^\infty(f)(E) &= \|f 1_E\|_{L^\infty(X, \omega)}, \\ \ell_\sigma^r(f)(E) &= \sigma(E)^{-\frac{1}{r}} \|f 1_E\|_{L^r(X, \omega)}. \end{aligned} \quad (\text{A.1})$$

Next, for $\tilde{\Sigma}$ defined by

$$\tilde{\Sigma} = \{A \in \Sigma : \mu(A) \in (0, \infty)\},$$

let ℓ^∞, ℓ^r be the sizes defined by, for every function $f \in \mathcal{M}(X)$, for every $A \in \tilde{\Sigma}$,

$$\begin{aligned} \ell^\infty(f)(A) &= \|f 1_A\|_{L^\infty(X, \omega)}, \\ \ell^r(f)(A) &= \mu(A)^{-\frac{1}{r}} \|f 1_A\|_{L^r(X, \omega)}. \end{aligned} \quad (\text{A.2})$$

For every function $f \in \mathcal{M}(X)$, we define

$$\|f\|_{L^\infty(S)} = \sup_{A \in \mathcal{A}} S(f)(A),$$

and the outer $L^\infty(S)$ space to be the set of functions in $\mathcal{M}(X)$ for which this quantity is finite.

For $\lambda > 0$, we define the super level measure

$$\mu(S(f) > \lambda) = \inf\{\mu(A) : A \in \Sigma, \|f 1_{A^c}\|_{L^\infty(S)} \leq \lambda\}.$$

For $0 < p < \infty$, for every function $f \in \mathcal{M}(X)$, we define

$$\|f\|_{L^p(S)} = \left(\int_0^\infty p\lambda^p \mu(S(f) > \lambda) \frac{d\lambda}{\lambda} \right)^{\frac{1}{p}},$$

$$\|f\|_{L^{p,\infty}(S)} = \left(\sup_{\lambda>0} \lambda^p \mu(S(f) > \lambda) \right)^{\frac{1}{p}},$$

and the outer $L^p(S)$, $L^{p,\infty}(S)$ spaces to be the sets of functions in $\mathcal{M}(X)$ for which these quantities are finite, respectively.

We have the following equality between the outer L^p spaces associated with the ℓ_σ^r sizes and the ℓ^r ones under some reasonable assumptions.

Lemma A.3 *Let (X, Σ) be a measurable space, and let $\mathcal{E}$ be a countable collection of measurable subsets of X such that*

$$X = \bigcup_{E \in \mathcal{E}} E.$$

Let σ be a pre-measure defined on $\mathcal{E}$ attaining only strictly positive values, and let μ be the outer measure generated by σ as in (1.2). Let ω be a measure on (X, Σ) , and assume that for all $A \in \Sigma$

$$\mu(A) = 0 \Rightarrow \omega(A) = 0.$$

Let ℓ_σ^r, ℓ^r be the sizes defined in (A.1), (A.2). Then, for $0 < p, r \leq \infty$, for every function $f \in \mathcal{M}(X)$, we have

$$\|f\|_{L^p(\ell_\sigma^r)} = \|f\|_{L^p(\ell^r)}.$$

Proof It is enough to prove the equality for $p = \infty$. The case $0 < p < \infty$ follows by this particular case and the definition of the outer L^p quasi-norm.

Case I: $r = \infty$. For every $E \in \mathcal{E}$, we have

$$\mu(E) \leq \sigma(E) < \infty. \quad (\text{A.3})$$

Now, if $\mu(E) = 0$, then $\omega(E) = 0$, hence

$$\ell_\sigma^\infty(f)(E) = 0 \leq \|f\|_{L^\infty(\ell^\infty)}.$$

If $\mu(E) \neq 0$, then $E \in \widetilde{\Sigma}$, hence

$$\ell_\sigma^\infty(f)(E) = \ell^\infty(f)(E) \leq \|f\|_{L^\infty(\ell^\infty)}.$$

Next, for every $A \in \widetilde{\Sigma}$, there exists a countable collection $\mathcal{E}_A \subseteq \mathcal{E}$ such that

$$A \subseteq \bigcup_{E \in \mathcal{E}_A} E,$$

hence

$$\ell^\infty(f)(A) \leq \sup_{E \in \mathcal{E}_A} \ell^\infty_\sigma(f)(E) \leq \|f\|_{L^\infty(\ell^\infty_\sigma)}.$$

Therefore, we have

$$\|f\|_{L^\infty(\ell^\infty_\sigma)} = \sup_{E \in \mathcal{E}} \ell^\infty_\sigma(f)(E) \leq \|f\|_{L^\infty(\ell^\infty)} = \sup_{A \in \tilde{\Sigma}} \ell^\infty(f)(A) \leq \|f\|_{L^\infty(\ell^\infty_\sigma)}.$$

Case II: $0 < r < \infty$. Let $E \in \mathcal{E}$. If $\mu(E) = 0$, then $\omega(E) = 0$, hence

$$\ell^r_\sigma(f)(E) = 0 \leq \|f\|_{L^\infty(\ell^r)}.$$

If $\mu(E) \neq 0$, then $E \in \tilde{\Sigma}$, hence we have, by (A.3),

$$\begin{aligned} \ell^r_\sigma(f)(E) &= \left(\sigma(E)^{-1} \sum_{x \in E} \omega(x) |f(x)|^r \right)^{\frac{1}{r}} \leq \left(\mu(E)^{-1} \sum_{x \in E} \omega(x) |f(x)|^r \right)^{\frac{1}{r}} \\ &\leq \ell^r(f)(E). \end{aligned}$$

Next, let $A \in \tilde{\Sigma}$. For every $\varepsilon > 0$, there exists a countable collection $\mathcal{E}_A(\varepsilon) \subseteq \mathcal{E}$ such that

$$\begin{aligned} A &\subseteq \bigcup_{E \in \mathcal{E}_A(\varepsilon)} E, \\ \mu(A) &\leq \sum_{E \in \mathcal{E}_A(\varepsilon)} \sigma(E) \leq (1 + \varepsilon)\mu(A), \end{aligned}$$

hence

$$\begin{aligned} \ell^r(f)(A) &\leq \mu(A)^{-\frac{1}{r}} \left(\sum_{E \in \mathcal{E}_A} \|f 1_E\|^r \right)^{\frac{1}{r}} \\ &\leq \mu(A)^{-\frac{1}{r}} \sup_{E \in \mathcal{E}_A} \ell^r_\sigma(f)(E) \left(\sum_{E \in \mathcal{E}_A} \sigma(E) \right)^{\frac{1}{r}} \\ &\leq (1 + \varepsilon) \|f\|_{L^\infty(\ell^r_\sigma)}, \end{aligned}$$

By taking ε arbitrarily small, we obtain the desired inequality.

Therefore, we have

$$\begin{aligned} \|f\|_{L^\infty(\ell^r_\sigma)} &= \sup_{E \in \mathcal{E}} \ell^r_\sigma(f)(E) \leq \|f\|_{L^\infty(\ell^r)} \\ &= \sup_{A \in \tilde{\Sigma}} \ell^r(f)(A) \leq \|f\|_{L^\infty(\ell^r_\sigma)}. \end{aligned}$$

□

Finally, we recall some important results in a setting satisfying the properties stated above with the additional assumption $\mathcal{E} = \mathcal{A}$.

Proposition A.4 (Pull back, Proposition 3.2 in [12]) *For $i = 1, 2$, let (X_i, Σ_i) be a measurable space, $(\sigma_i, \mathcal{A}_i)$ be a pre-measure satisfying the properties stated above, and $(S_i, \mathcal{A}_i)$ be a size. Let $\Phi: X_1 \rightarrow X_2$ be a measurable map. Assume that for every $E_2 \in \mathcal{A}_2$ we have*

$$\mu_1(\Phi^{-1}E_2) \leq A\mu_2(E_2).$$

Further assume that for each $E_1 \in \mathcal{A}_1$, there exists $E_2 \in \mathcal{A}_2$ such that for every $f \in \mathcal{M}(X_2)$ we have

$$S_1(f \circ \Phi)(E_1) \leq BS_2(f)(E_2).$$

Then we have for every $f \in \mathcal{M}(X_2)$ and $0 < p \leq \infty$ and some universal constant C

$$\begin{aligned} \|f \circ \Phi\|_{L^p(S_1)} &\leq A^{1/p} BC \|f\|_{L^p(S_2)}, \\ \|f \circ \Phi\|_{L^{p,\infty}(S_1)} &\leq A^{1/p} BC \|f\|_{L^{p,\infty}(S_2)}. \end{aligned}$$

Proposition A.5 (outer Hölder's inequality, Proposition 3.4 in [12]) *Let (X, Σ) be a measurable space. Let $(\sigma, \mathcal{A})$, $(\sigma_1, \mathcal{A}_1)$, $(\sigma_2, \mathcal{A}_2)$ be three pre-measures on X satisfying the properties stated above and such that the generated outer measures satisfy $\mu \leq \mu_i$, for $i = 1, 2$. Let $(S, \mathcal{A})$, $(S_1, \mathcal{A}_1)$, $(S_2, \mathcal{A}_2)$ be three sizes on X such that for any $A \in \mathcal{A}$, there exist $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$ such that for all $f_1, f_2 \in \mathcal{M}(X)$ we have*

$$S(f_1 f_2)(A) \leq S_1(f_1)(A_1) S_2(f_2)(A_2).$$

Let $p, p_1, p_2 \in (0, \infty]$ such that $1/p = 1/p_1 + 1/p_2$. Then, for every $f_1, f_2 \in \mathcal{M}(X)$,

$$\|f_1 f_2\|_{L^p(S)} \leq 2 \|f_1\|_{L^{p_1}(S_1)} \|f_2\|_{L^{p_2}(S_2)}.$$

Proposition A.6 (Marcinkiewicz interpolation) *Let (X, Σ) be a measurable space, $(\sigma, \mathcal{A})$ be a pre-measure satisfying the properties stated above, and $(S, \mathcal{A})$ be a size. Let (Y, ν) be a measure space. Let $1 \leq p_1 < p_2 \leq \infty$, $1 \leq q_1 \neq q_2 \leq \infty$ such that $p_i \leq q_i$, for $i = 1, 2$. Let T be a homogeneous quasi-subadditive operator that maps $L^{p_1}(Y, \nu)$ and $L^{p_2}(Y, \nu)$ to the space $\mathcal{M}(X)$ such that*

$$\begin{aligned} \|T(f)\|_{L^{q_1,\infty}(S)} &\leq A_1 \|f\|_{L^{p_1}(Y,\nu)}, \\ \|T(f)\|_{L^{q_2,\infty}(S)} &\leq A_2 \|f\|_{L^{p_2}(Y,\nu)}. \end{aligned}$$

Then we also have

$$\|T(f)\|_{L^q(S)} \leq A_1^\theta A_2^{1-\theta} C_{p,p_1,p_2} \|f\|_{L^p(Y,\nu)},$$

where $0 < \theta < 1$ is such that

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{p} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Proof See Appendix B in [18]. It is enough, for a function h on X , to replace the quantity $\mu(\{h > \lambda\})$ with the super level measure at level λ in the definition of the non-increasing rearrangement h^* . In particular, for a function $h: X \rightarrow \mathbb{R}$, the function $h^*: (0, \infty) \rightarrow (0, \infty)$ is defined by

$$h^*(t) = \inf\{\lambda: \mu(S(h) > \lambda) \leq t\}.$$

Proposition A.7 (Radon-Nikodym measure differentiation, Proposition 3.6 in [12], Proposition 1.9 in [22]) *Let (X, ω) be a measure space, $(\sigma, \mathcal{A})$ be a pre-measure satisfying the properties stated above, and $(S, \mathcal{A})$ be a size. Then, if either for all $A \in \mathcal{A}$*

$$\mu(A) = 0 \Rightarrow \omega(A) = 0,$$

or for all $A \in \mathcal{A}$

$$\mu(A)^{-1} \int_A |f(x)| d\omega(x) \leq C \|f\|_{L^\infty(S)},$$

we have, for every $f \in L^\infty(S)$,

$$\left| \int_X f(x) d\omega(x) \right| \leq C \|f\|_{L^1(S)},$$

where the implicit constant C is independent of $\|f\|_{L^\infty(S)}$.

Proposition A.8 *Let (X, ω) be a measure space, and $(\sigma, \mathcal{A})$ be a pre-measure satisfying the properties stated above. For $0 < r_1 < r_2 \leq \infty$, let $\ell_\sigma^{r_1}, \ell_\sigma^{r_2}$ be the sizes defined in (A.1). Then, for every $0 < p \leq \infty, r_1 < r < r_2$, there exists a constant $C = C(p, r, r_1, r_2)$ such that, for every $f \in \mathcal{M}(X)$,*

$$\begin{aligned} \|f\|_{L^p(\ell_\sigma^r)} &\leq C(\|f\|_{L^p(\ell_\sigma^{r_1})} + \|f\|_{L^p(\ell_\sigma^{r_2})}), \\ \|f\|_{L^{p,\infty}(\ell_\sigma^r)} &\leq C(\|f\|_{L^{p,\infty}(\ell_\sigma^{r_1})} + \|f\|_{L^{p,\infty}(\ell_\sigma^{r_2})}). \end{aligned}$$

Proof It is enough to prove that there exists a constant $c = c(r, r_1, r_2)$ such that, for every $\lambda > 0$,

$$\mu(\ell_\sigma^r(f) > c\lambda) \leq C(\mu(\ell_\sigma^{r_1}(f) > \lambda) + \mu(\ell_\sigma^{r_2}(f) > \lambda)).$$

The desired inequalities follow by multiplying the last display by λ^p and either integrating or taking the supremum over all levels $\lambda > 0$.

Let $E_1, E_2 \subseteq X$ be two sets witnessing the super level measure at λ up to a factor 2 with respect to the sizes $\ell_\sigma^{r_1}$ and $\ell_\sigma^{r_2}$, respectively. In particular, we have

$$2\mu(\ell_\sigma^{r_1}(f) > \lambda) \geq \mu(E_1), \quad 2\mu(\ell_\sigma^{r_2}(f) > \lambda) \geq \mu(E_2).$$

Now let $E' = E_1 \cup E_2$. Then, for every $A \subseteq X$, we have

$$\ell_\sigma^r(f 1_{(E')^c})(A) \leq c(\ell_\sigma^{r_1}(f 1_{(E_1)^c})(A))^\theta (\ell_\sigma^{r_1}(f 1_{(E_2)^c})(A))^{1-\theta} \leq c\lambda,$$

by logarithmic convexity of the L^r spaces, where $0 < \theta < 1$ satisfies

$$\frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}.$$

To conclude, we observe that $\mu(E') \leq \mu(E_1) + \mu(E_2)$. $\square$

Appendix B: Geometry of the Dyadic Upper Half Space

In this Appendix we prove Lemma 2.2 and Lemma 2.3.

Proof of Lemma 2.2 For every dyadic box $E \in \mathcal{D}$, we have

$$E = B(E) \times (0, |B(E)|),$$

where $B(E)$ is its base in $\mathbb{R}^d$, and $|B(E)|$ the volume of the base.

Therefore, the desired property follows from the analogous one for the dyadic cubes $B(E_1)$, $B(E_2)$. $\square$

We state and prove an auxiliary result.

Lemma B.1 *For every dyadic box $E \in \mathcal{D}$ and for every collection of pairwise disjoint dyadic boxes $\{E_n : n \in \mathbb{N}\}$ such that $E_n \subseteq E$ for every $n \in \mathbb{N}$, we have*

$$\sum_{n \in \mathbb{N}} \sigma(E_n) \leq \sigma(E).$$

Proof The dyadic cubes in the collection $\{B(E_n) : n \in \mathbb{N}\}$ are pairwise disjoint and such that $B(E_n) \subseteq B(E)$ for every $n \in \mathbb{N}$. Therefore, we have

$$\sum_{n \in \mathbb{N}} \sigma(E_n) = \sum_{n \in \mathbb{N}} |B(E_n)| = \left| \bigcup_{n \in \mathbb{N}} B(E_n) \right| \leq |B(E)| = \sigma(E).$$

Proof of Lemma 2.3 The inequality

$$\mu\left(\bigcup_{n \in \mathbb{N}} D_n\right) \leq \sum_{n \in \mathbb{N}} \sigma(E_n)$$

is trivially satisfied by the definition of μ .

If the left hand side is infinite, there is nothing else to prove. If it is finite, for every $\varepsilon > 0$, let $\mathcal{E}(\varepsilon) \subseteq \mathcal{D}$ be an optimal covering of the union of the elements of $\{D_n : n \in \mathbb{N}\}$ witnessing its outer measure up to the multiplicative constant $(1 + \varepsilon)$. In particular,

$$\begin{aligned} \bigcup_{n \in \mathbb{N}} D_n &\subseteq \bigcup_{E \in \mathcal{E}(\varepsilon)} E, \\ \mu\left(\bigcup_{n \in \mathbb{N}} D_n\right) &\leq \sum_{E \in \mathcal{E}(\varepsilon)} \sigma(E) \leq (1 + \varepsilon) \mu\left(\bigcup_{n \in \mathbb{N}} D_n\right) < \infty. \end{aligned} \tag{B.1}$$

Without loss of generality, we can assume the elements of $\mathcal{E}(\varepsilon)$ to be pairwise disjoint. In fact, given two elements of $\mathcal{E}(\varepsilon)$ with nonempty intersection, by Lemma 2.2 one is contained in the other, and we can discard the smaller from the collection. The upper bound on $\sigma(E)$ for every $E \in \mathcal{E}(\varepsilon)$ by (B.1) guarantees that we still end up with a collection.

Next, we observe that for every E_n , there exists an element of $\mathcal{E}(\varepsilon)$ such that $E_n^+ \cap E \neq \emptyset$, hence $E_n \subseteq E$ by Lemma 2.2. Since the elements of $\mathcal{E}(\varepsilon)$ are pairwise disjoint, the element E is unique, hence we can split the collection $\{E_n : n \in \mathbb{N}\}$ into pairwise disjoint subcollections $\mathcal{D}(E) = \{E_n : n \in \mathbb{N}, E_n \subseteq E\}$, one for each $E \in \mathcal{E}(\varepsilon)$.

By Lemma B.1, we have

$$\sum_{n \in \mathbb{N}} \sigma(E_n) = \sum_{E \in \mathcal{E}(\varepsilon)} \sum_{E_n \in \mathcal{D}(E)} \sigma(E_n) \leq \sum_{E \in \mathcal{E}(\varepsilon)} \sigma(E).$$

Combining this with (B.1) and taking ε arbitrarily small give the desired inequality. $\square$

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