The Calderón–Zygmund Theorem with an L^1 Mean Hörmander Condition

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Abstract

In 2019, Grafakos and Stockdale introduced an L^q mean Hörmander condition and proved a “limited-range” Calderón–Zygmund theorem. Comparing their theorem with the classical one, it requires weaker assumptions and implies the L^p boundedness for the “limited-range” instead of $1 < p < \infty$. However, in this paper, we show that the L^q mean Hörmander condition is actually enough to obtain the L^p boundedness for all $1 < p < \infty$ even in the worst case $q = 1$. We use a similar method to that used by Fefferman (Acta Math 124:9–36, 1970): form the Calderón–Zygmund decomposition with the bounded overlap property and approximate the bad part. Also we give a criterion of the L^2 boundedness for convolution type singular integral operators under the L^1 mean Hörmander condition.

Keywords Singular integrals · Calderón–Zygmund theory · Fourier analysis

Mathematics Subject Classification 42B20

1 Introduction

The Calderón–Zygmund theorem is a well-known tool to investigate the L^p boundedness of singular integral operators. It was originally developed by Calderón and Zygmund [2] and later improved by Hörmander [6]. Today it is usually stated as follows:

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Theorem A Let T be a singular integral operator with a kernel K . Suppose that T is bounded from $L^{p_0}(\mathbb{R}^d)$ to $L^{p_0,\infty}(\mathbb{R}^d)$ for some $1 < p_0 < \infty$ and its kernel K satisfies the Hörmander condition;

$$[K]_H := \sup_{B \subset \mathbb{R}^d} \sup_{y \in B} \int_{x \in \mathbb{R}^d \setminus 2B} |K(x, y) - K(x, c(B))| dx < \infty, \quad (1.1)$$

where the supremum $\sup_{B \subset \mathbb{R}^d}$ is taken over all balls B in $\mathbb{R}^d$, $c(B)$ is the center of B , $2B$ denotes the ball with the same center as B and whose radius is twice as long. Then T is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$. It follows that T is bounded on $L^p(\mathbb{R}^d)$ for all $1 < p < p_0$.

In 2019, Grafakos and Stockdale [5] introduced an L^q mean Hörmander condition (H_q condition for short);

$$[K]_{H_q} := \sup_{B \subset \mathbb{R}^d} \left(\frac{1}{|B|} \int_{y \in B} \left(\int_{x \in \mathbb{R}^d \setminus 2B} |K(x, y) - K(x, c(B))| dx \right)^q dy \right)^{1/q} < \infty \quad (1.2)$$

and proved the following:

Theorem B [5] Let T be a singular integral with a kernel K . Suppose that T is bounded from $L^{p_0}(\mathbb{R}^d)$ to $L^{p_0,\infty}(\mathbb{R}^d)$ for some $1 < p_0 < \infty$ and its kernel K satisfies the H_q condition for some $1 \leq q < p_0$ where q' denotes the Hölder conjugate of q . Then T is bounded from $L^q(\mathbb{R}^d)$ to $L^{q,\infty}(\mathbb{R}^d)$. It follows that T is bounded on $L^p(\mathbb{R}^d)$ for all $q < p < p_0$.

Note that $[K]_{q_1} \leq [K]_{q_2}$ if $1 \leq q_1 \leq q_2 \leq \infty$ and the H_∞ condition is the same as the classical Hörmander condition (1.1). They named Theorem B ‘limited-range Calderón–Zygmund theorem’ because it implies the L^p boundedness not for all $1 < p < p_0$ but for the ‘limited-range’; $q < p < p_0$. However, as stated in [5], they did not find any operators that satisfy the assumption of Theorem B and not bounded on L^q . In this sense, there is no evidence that it is truly a limited-range theorem. In this paper, we show that it is not actually limited-ranged. In fact, the H_q condition is enough for the $L^1 \rightarrow L^{1,\infty}$ boundedness even in the worst case $q = 1$.

Theorem 1 Let T be a singular integral operator with a kernel K . Suppose that T is bounded from $L^{p_0}(\mathbb{R}^d)$ to $L^{p_0,\infty}(\mathbb{R}^d)$ for some $1 < p_0 < \infty$ and its kernel K satisfies the H_1 condition;

$$[K]_{H_1} = \sup_{B \subset \mathbb{R}^d} \frac{1}{|B|} \int_{y \in B} \int_{x \in \mathbb{R}^d \setminus 2B} |K(x, y) - K(x, c(B))| dx dy < \infty. \quad (1.3)$$

Then T is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ with a constant proportional to $\|T\|_{L^{p_0} \rightarrow L^{p_0,\infty}} + [K]_{H_1}$.

Our proof is motivated by Fefferman's proof of the $L^1 \rightarrow L^{1,\infty}$ boundedness of strongly singular integral operators (see [4, Theorem 2']). In the proof, we form the Calderón–Zygmund decomposition of f ; $f = g + b$, and approximate the bad part b by a certain function $\tilde{b}$.

Also we will give a criterion of the L^2 boundedness for convolution type singular integral operators under the H_1 condition.

Theorem 2 *Let $K \in \mathcal{S}'(\mathbb{R}^d) \cap L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ be such that*

$$A := \sup_{0 < a < b < \infty} \left| \int_{a < |x| < b} K(x) dx \right| < \infty, \quad (1.4)$$

$$B := \sup_{a > 0} \frac{1}{a} \int_{|x| < a} |x| |K(x)| dx < \infty, \quad (1.5)$$

$$[K]_{H_1} := \sup_{r > 0} \frac{1}{V_d r^d} \int_{|y| \leq r} \int_{|x| \geq 2r} |K(x-y) - K(x)| dx dy < \infty, \quad (1.6)$$

where V_d denotes the volume of the d dimensional unit ball, and define $K_{\varepsilon,R} := K \chi_{\{\varepsilon < |x| < R\}}$ for any $0 < \varepsilon < R < \infty$. Then $K_{\varepsilon,R}$ satisfies

$$\sup_{0 < \varepsilon < R < \infty} \sup_{\xi \in \mathbb{R}^d} |\widehat{K_{\varepsilon,R}}(\xi)| < \infty$$

with a constant proportional to $A + B + [K]_{H_1}$.

This is a natural generalization of the classical result stated by using the H_∞ condition (see [1, Theorem 3], [3, Proposition 5.5]).

Note that it remains an open question: is the H_1 condition actually weaker than the classical one? As of this writing, we have no examples of K such that $[K]_{H_\infty} = \infty$ but $[K]_{H_1} < \infty$.

This paper is organized as follows. We prove Theorem 1 in Sect. 2 and Theorem 2 in Sect. 3. In Sect. 4, we will remark on the $H^1 \rightarrow L^1$ boundedness under the assumption of Theorem 1.

2 Proof of Theorem 1

We use the following lemma:

Lemma A [4, Decomposition Lemma] *Let $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$. Then there exists a family of disjoint dyadic cubes $\{Q_j\}_j$ such that*

$$\begin{aligned} |f(x)| &\leq \lambda \text{ a.e. } x \in \mathbb{R}^d \setminus \Omega, \\ \frac{1}{|B_j|} \int_{B_j} |f(x)| dx &\leq 9^d \lambda, \\ |\Omega^*| &\leq d^{d/2} V_d |\Omega| \leq C_d d^{d/2} V_d \frac{\|f\|_1}{\lambda}, \end{aligned} \quad (2.1)$$

$$\sum_j \chi_{2B_j} \leq (33\sqrt{d}/2)^d V_d, \quad (2.2)$$

where

- C_d denotes a constant which depends only on the dimension d ,
- B_j denotes the smallest ball circumscribing Q_j ,
- $\Omega := \bigcup_j Q_j$,
- $\Omega^* := \bigcup_j 2B_j$.

Furthermore, if we define

$$g := f \chi_{\mathbb{R}^d \setminus \Omega}, \quad (2.3)$$

$$b_j := f \chi_{Q_j}, \quad b := f \chi_{\Omega} = \sum_j b_j, \quad (2.4)$$

then immediately it follows that

$$\|g\|_{p_0}^{p_0} \leq \lambda^{p_0-1} \|f\|_1, \quad (2.5)$$

$$\frac{1}{|B_j|} \|b_j\|_1 \leq 9^d \lambda, \quad (2.6)$$

$$\|b\|_1 = \sum_j \|b_j\|_1 \leq \|f\|_1. \quad (2.7)$$

Lemma A is essentially the Whitney decomposition of $\{x \in \mathbb{R}^d : Mf(x) > \lambda\}$, where M is the Hardy–Littlewood maximal function with uncentered balls.¹

Note that our good part g (2.3) and bad part b (2.4) are different from usual ones. Ordinarily, they are defined by

$$g := f \chi_{\mathbb{R}^d \setminus \Omega} + \sum_j \left(\frac{1}{|Q_j|} \int_{Q_j} f \right) \chi_{Q_j},$$

$$b_j := \left(f - \frac{1}{|Q_j|} \int_{Q_j} f \right) \chi_{Q_j}, \quad b := \sum_j b_j$$

to guarantee the zero mean condition $\int b_j = 0$. However, our proof does not require it, hence we use our simpler definition.

Now we are going to give the proof of Theorem 1.

Proof of Theorem 1 Fix $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $t, \lambda > 0$ and form the Calderón–Zygmund decomposition of f at height $t\lambda$ (where t is given later to set appropriate estimates). In addition, fix $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\text{supp } \varphi \subset B(0, 1), \quad \int \varphi = 1, \quad \varphi \geq 0 \quad (2.8)$$

¹ The constant C_d in Lemma A is the $L^1(\mathbb{R}^d) \rightarrow L^{1,\infty}(\mathbb{R}^d)$ bound of M , hence it can be taken to be 3^d .

and write $\varphi_j(x) := s_j^{-d} \varphi(s_j^{-1}x)$ where r_j is the radius of B_j and $s_j := r_j/2$. We approximate b_j by $\tilde{b}_j := b_j * \varphi_j$ and b by $\tilde{b} := \sum_j \tilde{b}_j$.

Now we have $f = g - (\tilde{b} - b) + \tilde{b}$ and it suffices to show the following inequalities.

$$\lambda |\{x \in \mathbb{R}^d : |Tg(x)| > \lambda\}| \lesssim \|f\|_1, \quad (2.9)$$

$$\lambda |\{x \in \mathbb{R}^d : |T(\tilde{b} - b)(x)| > \lambda\}| \lesssim \|f\|_1, \quad (2.10)$$

$$\lambda |\{x \in \mathbb{R}^d : |T\tilde{b}(x)| > \lambda\}| \lesssim \|f\|_1. \quad (2.11)$$

Proof of (2.9) Since T is bounded from $L^{p_0}(\mathbb{R}^d)$ to $L^{p_0, \infty}(\mathbb{R}^d)$, it follows that

$$\begin{aligned} \lambda |\{x \in \mathbb{R}^d : |Tg(x)| > \lambda\}| &= \frac{1}{\lambda^{p_0-1}} \lambda^{p_0} |\{x \in \mathbb{R}^d : |Tg(x)| > \lambda\}| \\ &\leq \frac{1}{\lambda^{p_0-1}} \|Tg\|_{p_0, \infty}^{p_0} \\ &\leq \frac{1}{\lambda^{p_0-1}} \|T\|_{L^{p_0} \rightarrow L^{p_0, \infty}}^{p_0} \|g\|_{p_0}^{p_0} \\ &\stackrel{(2.5)}{\leq} \frac{1}{\lambda^{p_0-1}} \|T\|_{L^{p_0} \rightarrow L^{p_0, \infty}}^{p_0} (t\lambda)^{p_0-1} \|f\|_1 \\ &= t^{p_0-1} \|T\|_{L^{p_0} \rightarrow L^{p_0, \infty}}^{p_0} \|f\|_1. \end{aligned}$$

□

Proof of (2.10) Since

$$\{x \in \mathbb{R}^d : |T(\tilde{b} - b)(x)| > \lambda\} \subset \Omega^* \cup \{x \in \mathbb{R}^d \setminus \Omega^* : |T(\tilde{b} - b)(x)| > \lambda\},$$

it follows that

$$\begin{aligned} \lambda |\{x \in \mathbb{R}^d : |T(\tilde{b} - b)(x)| > \lambda\}| &\leq \lambda |\Omega^*| + \lambda |\{x \in \mathbb{R}^d \setminus \Omega^* : |T(\tilde{b} - b)(x)| > \lambda\}| \\ &\stackrel{(2.1)}{\leq} t^{-1} C_d d^{d/2} V_d \|f\|_1 + \|T(\tilde{b} - b)\|_{L^1(\mathbb{R}^d \setminus \Omega^*)}. \end{aligned}$$

We will estimate the second term by the H_1 condition (1.3). For each j and $x \in \mathbb{R}^d \setminus 2B_j$, we write

$$\begin{aligned} Tb_j(x) &= \int_{y \in \mathbb{R}^d} K(x, y) b_j(y) dy \\ &= \int_{z \in B(0, s_j)} \left(\int_{y \in B_j} K(x, y) b_j(y) dy \right) \varphi_j(z) dz \end{aligned}$$

and

$$\begin{aligned}
T\tilde{b}_j(x) &= \int_{y \in \mathbb{R}^d} K(x, y) \tilde{b}_j(y) dy \\
&= \int_{y \in \mathbb{R}^d} K(x, y) \int_{z \in B(0, s_j)} b_j(y - z) \varphi_j(z) dz dy \\
&= \int_{z \in B(0, s_j)} \int_{y \in \mathbb{R}^d} K(x, y) b_j(y - z) dy \varphi_j(z) dz \\
&= \int_{z \in B(0, s_j)} \left(\int_{y \in B_j} K(x, y + z) b_j(y) dy \right) \varphi_j(z) dz
\end{aligned}$$

since T is a singular integral operator with a kernel K . Therefore, for each j , we have

$$\begin{aligned}
&\|T(\tilde{b}_j - b_j)\|_{L^1(\mathbb{R}^d \setminus \Omega^*)} \\
&\leq \int_{x \in \mathbb{R}^d \setminus 2B_j} |T\tilde{b}_j(x) - Tb_j(x)| dx \\
&= \int_{x \in \mathbb{R}^d \setminus 2B_j} \left| \int_{z \in B(0, s_j)} \left(\int_{y \in B_j} (K(x, y + z) - K(x, y)) b_j(y) dy \right) \varphi_j(z) dz \right| dx \\
&\leq \int_{y \in B_j} \left(\int_{z \in B(0, s_j)} \int_{x \in \mathbb{R}^d \setminus 2B_j} |K(x, y + z) - K(x, y)| dx \varphi_j(z) dz \right) |b_j(y)| dy \\
&\leq \|\varphi_j\|_\infty \int_{y \in B_j} \left(\int_{z \in B(0, s_j)} \int_{x \in \mathbb{R}^d \setminus B(y, r_j)} |K(x, y + z) - K(x, y)| dx dz \right) |b_j(y)| dy \\
&= V_d \|\varphi\|_\infty \int_{y \in B_j} \left(\frac{1}{V_d s_j^d} \int_{z \in B(y, s_j)} \int_{x \in \mathbb{R}^d \setminus B(y, 2s_j)} |K(x, z) - K(x, y)| dx dz \right) |b_j(y)| dy \\
&\leq V_d \|\varphi\|_\infty \int_{y \in B_j} [K]_{H_1} |b_j(y)| dy \\
&= V_d \|\varphi\|_\infty [K]_{H_1} \|b_j\|_1.
\end{aligned}$$

It follows that

$$\|T(\tilde{b} - b)\|_{L^1(\mathbb{R}^d \setminus \Omega^*)} \leq \sum_j \|T(\tilde{b}_j - b_j)\|_{L^1(\mathbb{R}^d \setminus \Omega^*)} \leq V_d \|\varphi\|_\infty [K]_{H_1} \|f\|_1.$$

□

Proof of (2.11) By the same argument as in the proof of (2.9), we have

$$\begin{aligned}
\lambda |\{x \in \mathbb{R}^d : |T\tilde{b}(x)| > \lambda\}| &\leq \frac{1}{\lambda^{p_0-1}} \|T\|_{L^{p_0} \rightarrow L^{p_0, \infty}}^{p_0} \|\tilde{b}\|_{p_0}^{p_0} \\
&\leq \frac{1}{\lambda^{p_0-1}} \|T\|_{L^{p_0} \rightarrow L^{p_0, \infty}}^{p_0} \|\tilde{b}\|_\infty^{p_0-1} \|\tilde{b}\|_1.
\end{aligned}$$

Since it is obvious that

$$\|\tilde{b}\|_1 \leq \sum_j \|b_j * \varphi_j\|_1 \leq \sum_j \|b_j\|_1 \stackrel{(2.7)}{\leq} \|f\|_1,$$

it is enough to show that $\|\tilde{b}\|_\infty \lesssim \lambda$. For each j , we have

$$\|b_j * \varphi_j\|_\infty \leq \|b_j\|_1 \|\varphi_j\|_\infty = \|\varphi\|_\infty \frac{\|b_j\|_1}{s_j^d} \stackrel{(2.6)}{\leq} 18^d V_d \|\varphi\|_\infty t \lambda.$$

Therefore, it follows from the bounded overlap property (2.2),

$$\|\tilde{b}\|_\infty \leq 297^d d^{d/2} V_d^2 \|\varphi\|_\infty t \lambda.$$

Hence we conclude that

$$\lambda |\{x \in \mathbb{R}^d : |T\tilde{b}(x)| > \lambda\}| \leq t^{p_0-1} (297^d d^{d/2} V_d^2 \|\varphi\|_\infty)^{p_0-1} \|T\|_{L^{p_0} \rightarrow L^{p_0,\infty}}^{p_0} \|f\|_1.$$

□

Combining estimates above, we obtain

$$\begin{aligned} \|Tf\|_{1,\infty} &\leq 3\{t^{p_0-1} \|T\|_{L^{p_0} \rightarrow L^{p_0,\infty}}^{p_0} + (t^{-1} C_d d^{d/2} V_d + V_d \|\varphi\|_\infty [K]_{H_1}) \\ &\quad + t^{p_0-1} (297^d d^{d/2} V_d^2 \|\varphi\|_\infty)^{p_0-1} \|T\|_{L^{p_0} \rightarrow L^{p_0,\infty}}^{p_0}\} \|f\|_1. \end{aligned}$$

Finally, remember that t and φ are arbitrary. Since $\inf_{\varphi \text{ satisfies (2.8)}} \|\varphi\|_\infty = V_d^{-1}$, we conclude that

$$\begin{aligned} &\|T\|_{L^1 \rightarrow L^{1,\infty}} \\ &\leq 3 \inf_{t>0} (t^{-1} C_d d^{d/2} V_d + [K]_{H_1} + t^{p_0-1} (1 + (297^d d^{d/2} V_d)^{p_0-1}) \|T\|_{L^{p_0} \rightarrow L^{p_0,\infty}}^{p_0}) \\ &= 3 \left(p_0 (1 + (297^d d^{d/2} V_d)^{p_0-1})^{1/p_0} \left(\frac{C_d d^{d/2} V_d}{p_0 - 1} \right)^{1-1/p_0} \|T\|_{L^{p_0} \rightarrow L^{p_0,\infty}} + [K]_{H_1} \right). \end{aligned}$$

□

3 The Proof of Theorem 2

We use the following lemma:

Lemma 1 *If $K \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ satisfies (1.5) and (1.6), then*

$$\sup_{0 < \varepsilon < R < \infty} [K_{\varepsilon,R}]_{H_1} \leq [K]_{H_1} + 7B. \quad (3.1)$$

Proof of Lemma 1 It is obvious that

$$\begin{aligned}
 & \frac{1}{V_d r^d} \int_{|y| \leq r} \int_{|x| \geq 2r} |K_{\varepsilon, R}(x-y) - K_{\varepsilon, R}(x)| dx dy \\
 &= \frac{1}{V_d r^d} \int_{|y| \leq r} \int_{|x| \geq 2r, \varepsilon < |x-y| < R, \varepsilon < |x| < R} |K(x-y) - K(x)| dx dy \\
 &+ \frac{1}{V_d r^d} \int_{|y| \leq r} \int_{|x| \geq 2r, \varepsilon < |x-y| < R, |x| \leq \varepsilon} |K(x-y)| dx dy \\
 &+ \frac{1}{V_d r^d} \int_{|y| \leq r} \int_{|x| \geq 2r, \varepsilon < |x-y| < R, |x| \geq R} |K(x-y)| dx dy \\
 &+ \frac{1}{V_d r^d} \int_{|y| \leq r} \int_{|x| \geq 2r, |x-y| \leq \varepsilon, \varepsilon < |x| < R} |K(x)| dx dy \\
 &+ \frac{1}{V_d r^d} \int_{|y| \leq r} \int_{|x| \geq 2r, |x-y| \geq R, \varepsilon < |x| < R} |K(x)| dx dy
 \end{aligned}$$

and the first term is bounded by $[K]_{H_1}$. To estimate other terms, note that (1.5) implies

$$\sup_{a>0} \int_{a<|x|<ca} |K(x)| dx \leq \sup_{a>0} \int_{a<|x|<ca} \frac{|x|}{a} |K(x)| dx \leq cB$$

for any $c > 1$. Since we have

$$\begin{aligned}
 \varepsilon < |x-y| < R, |x| \leq \varepsilon &\Rightarrow |x-y| \leq |x| + |y| \leq 3|x|/2 \leq 3\varepsilon/2, \\
 \varepsilon < |x-y| < R, |x| \geq R &\Rightarrow |x-y| \geq |x| - |y| \geq |x|/2 \geq R/2, \\
 |x-y| \leq \varepsilon, \varepsilon < |x| < R &\Rightarrow |x| \leq 2(|x-y| + |y|) - |x| \leq 2|x-y| < 2\varepsilon, \\
 |x-y| \geq R, \varepsilon < |x| < R &\Rightarrow |x| \geq 2(|x-y| - |y|)/3 + |x|/3 \geq 2|x-y|/3 \geq 2R/3
 \end{aligned}$$

under the condition $2|y| \leq 2r \leq |x|$, the second and fifth terms are bounded by $3B/2$, the third and fourth terms are bounded by $2B$. $\square$

Proof of Theorem 2 Fix $0 < \varepsilon < R < \infty$ and $\xi \in \mathbb{R}^d$. Since it is obvious that

$$|\widehat{K_{\varepsilon, R}}(0)| = \left| \int_{\varepsilon < |x| < R} K(x) dx \right| \stackrel{(1.4)}{\leq} A,$$

we assume $\xi \neq 0$ and write $s := |\xi|^{-1}$. If we decompose $\widehat{K_{\varepsilon, R}}(\xi)$ as

$$\begin{aligned}
 & \widehat{K_{\varepsilon, R}}(\xi) \\
 &= \int_{x \in \mathbb{R}^d} K_{\varepsilon, R}(x) e^{-2\pi i x \cdot \xi} dx \\
 &= \int_{|x| < 2s} K_{\varepsilon, R}(x) (e^{-2\pi i x \cdot \xi} - 1) dx + \int_{|x| < 2s} K_{\varepsilon, R}(x) dx + \int_{2s \leq |x|} K_{\varepsilon, R}(x) e^{-2\pi i x \cdot \xi} dx \\
 &=: I_1 + I_2 + I_3,
 \end{aligned}$$

then we easily get

$$|I_1| \leq \int_{|x| < 2s} |K_{\varepsilon, R}(x)| |e^{-2\pi i x \cdot \xi} - 1| dx \leq 4\pi \frac{1}{2s} \int_{|x| < 2s} |x| |K_{\varepsilon, R}(x)| dx \stackrel{(1.5)}{\leq} 4\pi B,$$

$$|I_2| = \left| \int_{\varepsilon < |x| < 2s} K_{\varepsilon, R}(x) dx \right| \stackrel{(1.4)}{\leq} A.$$

To estimate I_3 , fix a radial function $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\text{supp } \varphi \subset B(0, 1), \quad \int \varphi = 1, \quad \varphi \geq 0, \quad |\widehat{\varphi}(1)| < 1$$

and define $\varphi_s(x) := s^{-d} \varphi(s^{-1}x)$. Moreover, rewrite

$$I_3 = \int_{|x| \geq 2s} \int_{|y| \leq s} K_{\varepsilon, R}(x) \varphi_s(y) dy e^{-2\pi i x \cdot \xi} dx \quad (3.2)$$

and introduce

$$I_4 := \int_{|x| \geq 2s} \int_{|y| \leq s} K_{\varepsilon, R}(x - y) \varphi_s(y) dy e^{-2\pi i x \cdot \xi} dx, \quad (3.3)$$

$$I_5 := \int_{|x| < 2s} \int_{|y| \leq s} K_{\varepsilon, R}(x - y) \varphi_s(y) dy e^{-2\pi i x \cdot \xi} dx \quad (3.4)$$

$$\begin{aligned} &= \int_{|x| < 2s} \int_{|x-y| \leq s} K_{\varepsilon, R}(y) \varphi_s(x - y) dy e^{-2\pi i x \cdot \xi} dx \\ &= \int_{|x| < 2s} \int_{|y| \leq 3s} K_{\varepsilon, R}(y) \varphi_s(x - y) dy e^{-2\pi i x \cdot \xi} dx, \end{aligned} \quad (3.5)$$

$$I_6 := \int_{|x| < 2s} \int_{|y| \leq 3s} K_{\varepsilon, R}(y) \varphi_s(x) dy e^{-2\pi i x \cdot \xi} dx \quad (3.6)$$

$$= \widehat{\varphi}_s(\xi) \int_{|y| \leq 3s} K_{\varepsilon, R}(y) dy. \quad (3.7)$$

We decompose I_3 into $(I_3 - I_4) + (I_4 + I_5) - (I_5 - I_6) - I_6$. By (3.2), (3.3) and Lemma 1, we get

$$\begin{aligned}
& |I_4 - I_3| \\
& \stackrel{(3.2), (3.3)}{=} \left| \int_{|x| \geq 2s} \int_{|y| \leq s} (K_{\varepsilon, R}(x - y) - K_{\varepsilon, R}(x)) \varphi_s(y) e^{-2\pi i x \cdot \xi} dy dx \right| \\
& \leq \int_{|y| \leq s} \int_{|x| \geq 2s} |K_{\varepsilon, R}(x - y) - K_{\varepsilon, R}(x)| \varphi_s(y) dx dy \\
& \leq V_d \|\varphi\|_{\infty} \frac{1}{V_d s^d} \int_{|y| \leq s} \int_{|x| < 2s} |K_{\varepsilon, R}(x - y) - K_{\varepsilon, R}(x)| dx dy \\
& \leq V_d \|\varphi\|_{\infty} [K_{\varepsilon, R}]_{H_1} \\
& \stackrel{(3.1)}{\leq} V_d \|\varphi\|_{\infty} ([K]_{H_1} + 7B).
\end{aligned}$$

For $I_5 - I_6$, use (3.5), (3.6) and the mean value theorem to obtain

$$\begin{aligned}
& |I_5 - I_6| \\
& \stackrel{(3.5), (3.6)}{=} \left| \int_{|x| < 2s} \int_{|y| \leq 3s} K_{\varepsilon, R}(y) (\varphi_s(x - y) - \varphi_s(x)) e^{-2\pi i x \cdot \xi} dy dx \right| \\
& \leq \int_{|x| < 2s} \int_{|y| \leq 3s} |K_{\varepsilon, R}(y)| |\varphi_s(x - y) - \varphi_s(x)| dy dx \\
& \leq \int_{|x| < 2s} \int_{|y| \leq 3s} |K_{\varepsilon, R}(y)| s^{-d-1} |y| \|\nabla \varphi\|_{\infty} dy dx \\
& = 3s^{-d} \|\nabla \varphi\|_{\infty} \int_{|x| < 2s} \left(\frac{1}{3s} \int_{|y| \leq 3s} |y| |K_{\varepsilon, R}(y)| dy \right) dx \\
& \stackrel{(1.5)}{\leq} 3s^{-d} \|\nabla \varphi\|_{\infty} \int_{|x| \leq 2s} B dx \\
& = 3 \cdot 2^d V_d \|\nabla \varphi\|_{\infty} B.
\end{aligned}$$

For $I_4 + I_5$ and I_6 , remark that $\widehat{\varphi_s}(\xi) = \widehat{\varphi}(s\xi) = \widehat{\varphi}(1)$ because φ is radial and $s = |\xi|^{-1}$. Then it follows immediately that

$$\begin{aligned}
I_4 + I_5 & \stackrel{(3.3), (3.4)}{=} \widehat{K_{\varepsilon, R} * \varphi_s}(\xi) = \widehat{\varphi}(1) \widehat{K_{\varepsilon, R}}(\xi), \\
|I_6| & \stackrel{(3.7)}{=} \left| \widehat{\varphi_s}(\xi) \int_{|x| \leq 3s} K_{\varepsilon, R}(y) dy \right| \stackrel{(1.4)}{\leq} |\widehat{\varphi}(1)| A.
\end{aligned}$$

Now we have

$$\begin{aligned}
& |\widehat{K_{\varepsilon, R}}(\xi)| \\
& \leq |I_1| + |I_2| + |I_3 - I_4| + |I_4 + I_5| + |I_5 - I_6| + |I_6| \\
& \leq 4\pi B + A + V_d \|\varphi\|_{\infty} ([K]_{H_1} + 7B) + |\widehat{\varphi}(1)| |\widehat{K_{\varepsilon, R}}(\xi)| \\
& \quad + 3 \cdot 2^d V_d \|\nabla \varphi\|_{\infty} B + |\widehat{\varphi}(1)| A
\end{aligned}$$

for any $\xi \in \mathbb{R}^d$ (it is still valid in the case $\xi = 0$). Finally, remember $|\widehat{\varphi}(1)| < 1$ to conclude that

$$|\widehat{K_{\varepsilon,R}}(\xi)| \leq \frac{(1 + |\widehat{\varphi}(1)|)A + (4\pi + V_d(7\|\varphi\|_{\infty} + 3 \cdot 2^d \|\nabla \varphi\|_{\infty}))B + V_d \|\varphi\|_{\infty} [K]_{H_1}}{1 - |\widehat{\varphi}(1)|}.$$

□

4 Remark

We can also obtain the $H^1 \rightarrow L^1$ boundedness under the assumption of Theorem 1.

Theorem 3 *Let T be a singular integral operator with a kernel K . Suppose that T is bounded from $L^{p_0}(\mathbb{R}^d)$ to $L^{p_0,\infty}(\mathbb{R}^d)$ for some $1 < p_0 < \infty$ and its kernel K satisfies the H_1 condition. Then T is bounded from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$.*

To see this, note that Theorem 1 implies that T is bounded on $L^p(\mathbb{R}^d)$ for any $1 < p < p_0$. Hence we assume that T is bounded on $L^{p_0}(\mathbb{R}^d)$ for some $1 < p_0 < \infty$ without loss of generality. Now we can show the $H^1(\mathbb{R}^d) \rightarrow L^1(\mathbb{R}^d)$ boundedness. We do not prove it here because its proof is the almost same as that of the classical theorem (see [3, Proposition 6.2, Corollary 6.3]).

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