Sign Retrieval in Shift-Invariant Spaces with Totally Positive Generator

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Abstract

We show that a real-valued function f in the shift-invariant space generated by a totally positive function of Gaussian type is uniquely determined, up to a sign, by its absolute values $\{|f(\lambda)| : \lambda \in \Lambda\}$ on any set $\Lambda \subseteq \mathbb{R}$ with lower Beurling density $D^-(\Lambda) > 2$.

We consider a *totally positive function of Gaussian type*, i.e., a function $g \in L^2(\mathbb{R})$ whose Fourier transform factors as

$$\hat{g}(\xi) = \int_{\mathbb{R}} g(x) e^{-2\pi i x \xi} dx = C_0 e^{-\gamma \xi^2} \prod_{v=1}^m (1 + 2\pi i \delta_v \xi)^{-1}, \quad \xi \in \mathbb{R}, \quad (1)$$

with $\delta_1, \dots, \delta_m \in \mathbb{R}$, $C_0, \gamma > 0$, $m \in \mathbb{N} \cup \{0\}$, and the *shift-invariant space*

$$V^\infty(g) = \left\{ f = \sum_{k \in \mathbb{Z}} c_k g(\cdot - k) : c \in \ell^\infty(\mathbb{Z}) \right\},$$

generated by its integer shifts within $L^\infty(\mathbb{R})$. As a consequence of (1), each $f \in V^\infty(g)$ is continuous, the defining series converges unconditionally in the weak* topology of L^∞ , and the coefficients c_k are unique [6, Theorem 3.5].

Keywords Sign retrieval · Phase retrieval · Shift-invariant space · Totally positive function

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The shift-invariant space $V^\infty(g)$ enjoys the following sampling property [4]: every separated set $\Lambda \subseteq \mathbb{R}$ with *lower Beurling density*

$$D^-(\Lambda) := \liminf_{r \rightarrow \infty} \inf_{x \in \mathbb{R}} \frac{\#\left[\Lambda \cap [x-r, x+r]\right]}{2r}$$

strictly larger than 1 provides the norm equivalence

$$\|f\|_{L^\infty(\mathbb{R})} \asymp \|f|_\Lambda\|_{\ell^\infty(\Lambda)}, \quad f \in V^\infty(g). \quad (2)$$

(A similar property holds for all L^p norms, $1 \leq p \leq \infty$; see [4, Theorems 5.2 and 3.1].)

In this article, we show the following uniqueness property for the absolute values of real-valued functions in $V^\infty(g)$.

Theorem 1 (Sign retrieval) *Let g be a totally positive function of Gaussian type, as in (1), and $\Lambda \subseteq \mathbb{R}$ with lower Beurling density*

$$D^-(\Lambda) > 2. \quad (3)$$

Assume that $f_1, f_2 \in V^\infty(g)$ are real-valued and $|f_1| \equiv |f_2|$ on Λ . Then either $f_1 \equiv f_2$ or $f_1 \equiv -f_2$.

See [2] for motivation for sign retrieval in shift-invariant spaces. When g is a Gaussian function—corresponding to $m = 0$ in (1)—Theorem 1 was recently obtained in [3, Theorem 1]. Here, that result is extended to all totally positive functions of Gaussian type.¹

The intuition behind Theorem 1 is as follows. Suppose that $|f_1| \equiv |f_2|$ on Λ , and split Λ into

$$\begin{aligned} \Lambda_1 &= \{\lambda \in \Lambda : f_1(\lambda) = f_2(\lambda)\}, \\ \Lambda_2 &= \{\lambda \in \Lambda : f_1(\lambda) = -f_2(\lambda)\}. \end{aligned} \quad (4)$$

Then $f_1 - f_2$ vanishes on Λ_1 while $f_1 + f_2$ vanishes on Λ_2 . Under (3), one may expect one of the two subsets Λ_j to have lower Beurling density larger than 1. The sampling inequalities (2) would then imply that either $f_1 - f_2$ or $f_1 + f_2$ are identically zero. This argument breaks down, however, because Beurling's lower density is not subadditive. For example, $\Lambda_1 := \Lambda \cap (-\infty, 0]$ and $\Lambda_2 := \Lambda \cap (0, \infty)$ have always zero lower Beurling density. The proof of the sign retrieval theorem for Gaussian generators in [3] resorts instead to a special property of the Gaussian function, namely that $V^\infty(g) \cdot V^\infty(g)$ is contained in a dilation of $V^\infty(\tilde{g})$ by a factor of 2, where $\tilde{g}$ is another Gaussian function. Thus, in the Gaussian case, the sampling theorem can be applied after rescaling to the set Λ to conclude that $(f_1 - f_2) \cdot (f_1 + f_2) \equiv 0$,

¹ In contrast to [3, Theorem 1], in Theorem 1 it is not assumed that Λ is separated, or even relatively separated. See [3] for an example of a separated set with $D^-(\Lambda) < 2$ for which sign retrieval fails in a shift-invariant space with Gaussian generator.

and, by analyticity, that either $f_1 \equiv f_2$ or $f_1 \equiv -f_2$. A similar argument applies to Paley–Wiener spaces [1, Theorem 2.5]. We are unaware of an analogous dilation property for general totally positive generators.

To prove Theorem 1 for all totally positive generators of Gaussian type we take a different route. We define the *upper average circular density* of a set $\Lambda \subseteq \mathbb{R}$ as

$$D_{\text{circ}}^+(\Lambda) = \limsup_{r \rightarrow \infty} \frac{4}{\pi r^2} \int_0^r \sum_{\lambda \in \Lambda \cap [-t, t]} \sqrt{t^2 - \lambda^2} \frac{dt}{t}, \quad (5)$$

with the convention that $D_{\text{circ}}^+(\Lambda) = \infty$, if Λ is uncountably infinite. The density is named circular because, in (5), each point $\lambda \in [-t, t]$ is weighted with the measure of the largest vertical segment $\{\lambda\} \times (-a, a)$ contained in the two-dimensional open disk $B_t(0) \subseteq \mathbb{R}^2$.

The upper average circular density can be alternatively described as follows: as shown in Lemma 1 below, for any lattice $\alpha\mathbb{Z}$, $\alpha > 0$,

$$D_{\text{circ}}^+(\Lambda) = \limsup_{r \rightarrow \infty} \frac{2\alpha}{\pi r^2} \int_0^r \#[(\Lambda \setminus \{0\}) \times \alpha\mathbb{Z}] \cap B_t(0) \frac{dt}{t}. \quad (6)$$

From here, it follows easily that D_{circ}^+ dominates Beurling's lower density:

$$D_{\text{circ}}^+(\Lambda) \geq D^-(\Lambda); \quad (7)$$

see Lemma 1 below. We call D_{circ}^+ an upper density because, due to the sublinearity of $\limsup$,

$$D_{\text{circ}}^+(\Lambda_1 \cup \Lambda_2) \leq D_{\text{circ}}^+(\Lambda_1) + D_{\text{circ}}^+(\Lambda_2), \quad (8)$$

for any two sets $\Lambda_1, \Lambda_2 \subseteq \mathbb{R}$.

Below we prove the following uniqueness result formulated in terms of the upper average circular density of the zero set $\{f = 0\}$ of a function f (counted without multiplicities).

Theorem 2 (Uniqueness theorem) *Let g be a totally positive function of Gaussian type, as in (1). Let $f \in V^\infty(g)$ be non-zero. Then $D_{\text{circ}}^+(\{f = 0\}) \leq 1$.*

The uniqueness theorem (which applies also to non-real-valued functions) allows one to carry out the following natural proof of the sign retrieval theorem.

Proof (Proof of Theorem 1, assuming Theorem 2) Assume that $D^-(\Lambda) > 2$ and write $\Lambda = \Lambda_1 \cup \Lambda_2$ as in (4). Then, by (7) and (8), either $D_{\text{circ}}^+(\Lambda_1) > 1$ or $D_{\text{circ}}^+(\Lambda_2) > 1$ (possibly both). In the first case, Theorem 2 shows that $f_1 \equiv f_2$, while in the second, $f_1 \equiv -f_2$. $\square$

As the proof shows, Theorem 1 remains valid if (3) is relaxed to $D_{\text{circ}}^+(\Lambda) > 2$. Sign retrieval also holds under the same density condition for the shift-invariant spaces $V^p(g)$ defined with respect to L^p norms, $1 \leq p \leq \infty$, as these are contained in $V^\infty(g)$.

Towards the proof of Theorem 2, we first prove (6) and (7).

Lemma 1 Let $\Lambda \subseteq \mathbb{R}$ and $\alpha > 0$. Then (6) and (7) hold true.

Proof We assume that Λ has no accumulation points, since, otherwise, both sides of (6) are infinite. Denote provisionally the right hand side of (6) by $\tilde{D}_{\text{circ}}^+(\Lambda)$, and set $\Lambda' := \Lambda \setminus \{0\}$.

Step 1. Let $\varepsilon \in (0, 1)$, and $C = C_{\alpha, \varepsilon} > 0$ a constant to be specified. We claim:

$$D_{\text{circ}}^+(\Lambda) = \limsup_{r \rightarrow \infty} \frac{4}{\pi r^2} \int_C^r \sum_{\lambda \in \Lambda' \cap [-t, t]} \sqrt{t^2 - \lambda^2} \frac{dt}{t}, \quad (9)$$

$$(1 + \delta)^2 \tilde{D}_{\text{circ}}^+(\Lambda) = \limsup_{r \rightarrow \infty} \frac{2\alpha}{\pi r^2} \int_C^r \#[(\Lambda' \times \alpha\mathbb{Z}) \cap B_{(1+\delta)t}(0)] \frac{dt}{t}, \quad \delta = \pm\varepsilon. \quad (10)$$

To prove the first claim, we first note that $D_{\text{circ}}^+(\Lambda) = D_{\text{circ}}^+(\Lambda')$, since, if $0 \in \Lambda$, the contribution of the point $\lambda = 0$ is

$$\frac{4}{\pi r^2} \int_0^r dt = \frac{4}{\pi r},$$

and does not affect the limit on r . Similarly, since $\sqrt{t^2 - \lambda^2} \leq t$,

$$\frac{4}{\pi r^2} \int_0^C \sum_{\lambda \in \Lambda' \cap [-t, t]} \sqrt{t^2 - \lambda^2} \frac{dt}{t} \leq \#(\Lambda' \cap [-C, C]) \cdot \frac{4C}{\pi r^2}.$$

The last quantity is finite because Λ has no accumulation points. This proves (9), because it shows that limiting the integral to $[C, r]$ does not affect the limit on r . Second, a change of variables shows that (10) holds with C replaced by 0. In addition, since Λ has no accumulation points, there exists $\eta > 0$ such that $\Lambda' \cap (-\eta, \eta) = \emptyset$. Therefore, we can estimate the part of the integral excluded in (10) as

$$\frac{2\alpha}{\pi r^2} \int_0^C \#[(\Lambda' \times \alpha\mathbb{Z}) \cap B_{(1+\delta)t}(0)] \frac{dt}{t} \leq \#[(\Lambda' \times \alpha\mathbb{Z}) \cap B_{2C}(0)] \cdot \frac{2\alpha}{\pi r^2} \int_{\frac{\eta}{1+\varepsilon}}^C \frac{dt}{t},$$

which is finite because $\eta > 0$. This proves (10).

Step 2. For $t > 0$, we note that

$$\begin{aligned} \#(\Lambda' \times \alpha\mathbb{Z}) \cap B_t(0) &= \sum_{\lambda \in \Lambda' \cap [-t, t]} \# \left\{ k \in \mathbb{Z} : \lambda^2 + \alpha^2 k^2 < t^2 \right\} \\ &= \sum_{\lambda \in \Lambda' \cap [-t, t]} \# \left\{ k \in \mathbb{Z} : |k| < \alpha^{-1} \sqrt{t^2 - \lambda^2} \right\}. \end{aligned}$$

Next, we choose the constant $C_{\alpha,\varepsilon}$ so that the following estimates hold for $t \geq C_{\alpha,\varepsilon}$:

$$\begin{aligned} \#[(\Lambda' \times \alpha\mathbb{Z}) \cap B_{(1+\varepsilon)t}(0)] &\geq \sum_{\lambda \in \Lambda' \cap [-t, t]} \left[\frac{2}{\alpha} \sqrt{(1+\varepsilon)^2 t^2 - \lambda^2} - 1 \right] \\ &= \sum_{\lambda \in \Lambda' \cap [-t, t]} \frac{2}{\alpha} \sqrt{(1+\varepsilon)^2 t^2 - \lambda^2 - \alpha \sqrt{(1+\varepsilon)^2 t^2 - \lambda^2} + \alpha^2/4} \\ &\geq \sum_{\lambda \in \Lambda' \cap [-t, t]} \frac{2}{\alpha} \sqrt{(1+\varepsilon)^2 t^2 - \lambda^2 - \alpha(1+\varepsilon)t + \alpha^2/4} \\ &\geq \sum_{\lambda \in \Lambda' \cap [-t, t]} \frac{2}{\alpha} \sqrt{t^2 - \lambda^2}, \end{aligned}$$

where the last estimate requires choosing $C_{\alpha,\varepsilon}$ large. Similarly,

$$\begin{aligned} \#[(\Lambda' \times \alpha\mathbb{Z}) \cap B_{(1-\varepsilon)t}(0)] &\leq \sum_{\lambda \in \Lambda' \cap [-(1-\varepsilon)t, (1-\varepsilon)t]} \left[\frac{2}{\alpha} \sqrt{(1-\varepsilon)^2 t^2 - \lambda^2} + 1 \right] \\ &= \sum_{\lambda \in \Lambda' \cap [-(1-\varepsilon)t, (1-\varepsilon)t]} \frac{2}{\alpha} \sqrt{(1-\varepsilon)^2 t^2 - \lambda^2 + \alpha \sqrt{(1-\varepsilon)^2 t^2 - \lambda^2} + \alpha^2/4} \\ &\leq \sum_{\lambda \in \Lambda' \cap [-(1-\varepsilon)t, (1-\varepsilon)t]} \frac{2}{\alpha} \sqrt{(1-\varepsilon)^2 t^2 - \lambda^2 + \alpha(1-\varepsilon)t + \alpha^2/4} \\ &\leq \sum_{\lambda \in \Lambda' \cap [-t, t]} \frac{2}{\alpha} \sqrt{t^2 - \lambda^2}, \end{aligned}$$

where the last estimate requires choosing $C_{\alpha,\varepsilon}$ large. Combining the last two estimates with (9) and (10), and letting $\varepsilon \rightarrow 0$, we deduce (6).

Step 3. To prove (7), let $r > s > 0$ and use (6) with $\alpha = 1$ to estimate

$$\begin{aligned} D_{\text{circ}}^+(\Lambda) &\geq \liminf_{r \rightarrow \infty} \frac{2}{\pi r^2} \int_s^r \inf_{z \in \mathbb{R}^2} \#[(\Lambda' \times \mathbb{Z}) \cap B_t(z)] \frac{dt}{t} \\ &\geq \inf_{t \geq s} \left[\frac{1}{|B_t(0)|} \inf_{z \in \mathbb{R}^2} \#[(\Lambda' \times \mathbb{Z}) \cap B_t(z)] \right] \liminf_{r \rightarrow \infty} \frac{2}{\pi r^2} \int_s^r |B_t(0)| \frac{dt}{t} \\ &= \inf_{t \geq s} \left[\inf_{z \in \mathbb{R}^2} \frac{1}{|B_t(z)|} \#[(\Lambda' \times \mathbb{Z}) \cap B_t(z)] \right]. \end{aligned}$$

Letting $s \rightarrow \infty$, we see that

$$D_{\text{circ}}^+(\Lambda) \geq \liminf_{t \rightarrow \infty} \inf_{z \in \mathbb{R}^2} \frac{1}{|B_t(z)|} \#[(\Lambda' \times \mathbb{Z}) \cap B_t(z)] = D^-(\Lambda') = D^-(\Lambda),$$

where we used that the two-dimensional lower Beurling density of $\Lambda' \times \mathbb{Z}$ is $D^-(\Lambda)$. $\square$

We can now prove the uniqueness Theorem. The proof builds on [4, Lemma 5.1] and [5, Theorems 4.3 and 4.9].

Proof of Theorem 2 We proceed by induction on m in (1).

In the case $m = 0$, the function g is a Gaussian $g(x) = C'_0 e^{-ax^2}$, with $a = \frac{\pi^2}{\gamma} > 0$. Let $f = \sum_k c_k g(\cdot - k) \in V^\infty(g)$ with c bounded be non-zero, and denote by $\Lambda \subseteq \mathbb{R}$ its zero set.

As shown in [5, Lemma 4.1], f possesses an extension to an entire function on $\mathbb{C}$, satisfying the growth estimate $|f(x + iy)| \lesssim e^{ay^2}$ for $x, y \in \mathbb{R}$, where the implied constant depends on f . Let $n \geq 0$ be the order of f at $z = 0$, and consider the analytic function

$$F(z) := C_1 z^{-n} f(z) e^{\frac{a}{2} z^2},$$

where $C_1 \in \mathbb{C}$ is chosen so that $F(0) = 1$. Then F satisfies

$$|F(x + iy)| \leq C e^{ay^2} e^{\frac{a}{2}(x^2 - y^2)} = C e^{\frac{a}{2}(x^2 + y^2)}, \quad x, y \in \mathbb{R}, \quad (11)$$

for some constant $C > 0$. Moreover, the zero set of f is invariant under addition of $i \frac{\pi}{a} \mathbb{Z}$ [5, Lemma 4.2]. Therefore, the set of complex zeros of F contains $(\Lambda \setminus \{0\}) + i \frac{\pi}{a} \mathbb{Z}$:

$$F(\lambda + i \frac{\pi}{a} k) = 0, \quad \lambda \in \Lambda \setminus \{0\}, k \in \mathbb{Z}.$$

By (6) with $\alpha = \frac{\pi}{a}$, the zero-counting function of F ,

$$n_F(t) := \#\{z \in \mathbb{C} : F(z) = 0, |z| \leq t\},$$

satisfies

$$\limsup_{r \rightarrow \infty} \frac{1}{r^2} \int_0^r \frac{n_F(t)}{t} dt \geq \frac{a}{2} D_{\text{circ}}^+(\Lambda). \quad (12)$$

On the other hand, by Jensen's formula combined with (11), for all $r > 0$,

$$\frac{1}{r^2} \int_0^r \frac{n_F(t)}{t} dt = \frac{1}{2\pi r^2} \int_0^{2\pi} \log |F(re^{i\theta})| d\theta \leq \frac{\log(C)}{r^2} + \frac{a}{2}. \quad (13)$$

By combining (12) and (13) we conclude that $D_{\text{circ}}^+(\Lambda) \leq 1$, as claimed. Note, in particular, that Λ has no accumulation points.

Inductive step. Let g be as in (1) and $f = \sum_k c_k g(\cdot - k) \in V^\infty(g)$ non-zero. Without loss of generality we assume that, in (1), $\delta_1, \dots, \delta_m$ are non-zero. Since g is real-valued, by replacing c_k with $\Re(c_k)$ or $\Im(c_k)$ if necessary, we may assume that f is real-valued. Let g_1 be given by (1), except that the product runs only up to $m - 1$. Then

$$f_1 := f + \delta_m f' = \sum_{k \in \mathbb{Z}} c_k g_1(\cdot - k)$$

Fig. 1 Each positive zero λ_k of f is assigned to a zero γ_k of f_1 in such a way that the corresponding vertical segments contained in the disk $B_t(0)$ become longer

belongs to $V^\infty(g_1)$ and is real-valued and non-zero, because the coefficients c_k are real and at least one of them is non-zero. We assume by inductive hypothesis that $D_{\text{circ}}^+(\{f_1 = 0\}) \leq 1$.

Let $\Lambda = \{f = 0\}$ and $\Gamma = \{f_1 = 0\}$. Suppose that Λ has no accumulation points. Assume for the moment that, in addition, f has a non-negative zero, and order the set of non-negative zeros of f increasingly:

$$\Lambda \cap [0, \infty) = \{\lambda_k : k = 0, \dots, N\},$$

where $N \in \mathbb{N} \cup \{0, \infty\}$, and we do not count multiplicities. Hence, $\lambda_k > 0$, if $k > 0$.

By Rolle's Theorem applied to the differential operator $I + \delta_m \partial_x$, there is a sequence of zeros of f_1 that interlaces $\Lambda \cap [0, \infty)$. (Indeed, we note that $\frac{d}{dx}[e^{x/\delta_m} f(x)] = \frac{1}{\delta_m} e^{x/\delta_m} f_1(x)$ and apply the standard version of Rolle's theorem; see [4, Lemma 5.1] or [5, Lemma 4.8] for details.) We parameterize that sequence as $\Gamma^+ = \{\gamma_k : k = 1, \dots, N\}$, with

$$\lambda_{k-1} < \gamma_k < \lambda_k, \quad k > 0.$$

The set Γ^+ is empty if $N = 0$. For $t > 0$, this ordering associates with each $\lambda_k \in (0, t]$, $k \neq 0$, a distinct zero of f_1 , $\gamma_k \in (0, t]$, such that the vertical segment through γ_k contained in the disk $B_t(0)$ is longer than the corresponding segment through λ_k :

$$\sum_{k>0, \lambda_k \in (0, t]} \sqrt{t^2 - \lambda_k^2} \leq \sum_{k>0, \gamma_k \in (0, t]} \sqrt{t^2 - \gamma_k^2} \leq \sum_{\gamma \in \Gamma \cap (0, t]} \sqrt{t^2 - \gamma^2};$$

see Fig. 1.

We now add the term corresponding to λ_0 , and bound $\sqrt{t^2 - \lambda_0^2} \leq t$ to obtain

$$\sum_{\lambda \in \Lambda \cap [0, t]} \sqrt{t^2 - \lambda^2} \leq t + \sum_{\gamma \in \Gamma \cap (0, t]} \sqrt{t^2 - \gamma^2}.$$

The previous estimate is also trivially true if f has no non-negative zeros. Arguing similarly with the non-positive zeros of f we conclude that

$$\sum_{\lambda \in \Lambda \cap [-t, t]} \sqrt{t^2 - \lambda^2} \leq 2t + \sum_{\gamma \in \Gamma \cap [-t, t]} \sqrt{t^2 - \gamma^2}.$$

This shows that $D_{\text{circ}}^+(\{f = 0\}) \leq D_{\text{circ}}^+(\{f_1 = 0\})$. The latter quantity is bounded by 1 by inductive hypothesis. Finally, if $\Lambda = \{f = 0\}$ has an accumulation point, Rolle's theorem applied as before shows that so does $\Gamma = \{f_1 = 0\}$, and, therefore, $D_{\text{circ}}^+(\{f_1 = 0\}) = \infty$ contradicting the inductive hypothesis. $\square$

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References

1. Alaifari, R., Grohs, P.: Phase retrieval in the general setting of continuous frames for Banach spaces. *SIAM J. Math. Anal.* **49**(3), 1895–1911 (2017)
2. Cheng, C., Jiang, J., Sun, Q.: Phaseless sampling and reconstruction of real-valued signals in shift-invariant spaces. *J. Fourier Anal. Appl.* **25**(4), 1361–1394 (2019)
3. Gröchenig, K.: Phase-retrieval in shift-invariant spaces with Gaussian generator. *J. Fourier Anal. Appl.* **26**(3), 15 (2020). (Paper No. 52)
4. Gröchenig, K., Romero, J.L., Stöckler, J.: Sampling theorems for shift-invariant spaces, Gabor frames, and totally positive functions. *Invent. Math.* **211**(3), 1119–1148 (2018)
5. Gröchenig, K., Romero, J.L., Stöckler, J.: Sharp results on sampling with derivatives in shift-invariant spaces and multi-window Gabor frames. *Constr. Approx.* **51**(1), 1–25 (2020)
6. Jia, R.Q., Micchelli, C.A.: Using the refinement equations for the construction of pre-wavelets. II. Powers of two. In: *Curves and Surfaces (Chamonix-Mont-Blanc, 1990)*, pp 209–246. Academic, Boston (1991)

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