A Note on the L^p Integrability of a Class of Bochner–Riesz Kernels

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Abstract

For a general compact variety Γ of arbitrary codimension, one can consider the L^p mapping properties of the Bochner–Riesz multiplier

$$m_{\Gamma, \alpha}(\zeta) = \text{dist}(\zeta, \Gamma)^\alpha \phi(\zeta)$$

where $\alpha > 0$ and ϕ is an appropriate smooth cutoff function. Even for the sphere $\Gamma = \mathbb{S}^{N-1}$, the exact L^p boundedness range remains a central open problem in Euclidean harmonic analysis. In this paper we consider the L^p integrability of the Bochner–Riesz convolution kernel for a particular class of varieties (of any codimension). For a subclass of these varieties the range of L^p integrability of the kernels differs substantially from the L^p boundedness range of the corresponding Bochner–Riesz multiplier operator.

Keywords Bochner–Riesz conjecture · Multiplier operators · L^p integrability

1 Introduction

When one studies Fourier multiplier operators T_m , with a compactly supported multiplier m , $\widehat{T_m f}(\xi) = m(\xi) \hat{f}(\xi)$, a necessary condition for T_m to be bounded on $L^p(\mathbb{R}^N)$ is that the kernel $K = \check{m} \in L^p$. To see this, one can take a Schwartz function ϕ with $\hat{\phi}(\xi) = 1$ for $\xi \in \text{supp } m$ so that $T_m \phi = \check{m}$.

In a remarkable paper of Heo, Nazarov and Seeger [8], they show that this natural necessary condition is also sufficient for T_m to be bounded on L^p in the range $1 < p <$

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$\frac{2(N-1)}{N+1}$ whenever m is a compactly supported *radial* multiplier. The radial multiplier conjecture states that for radial multipliers $m, \check{m} \in L^p$ implies T_m is bounded on L^p holds in the range $1 < p < \frac{2N}{N+1}$.

The canonical example of a compactly supported radial multiplier is the classical Bochner–Riesz multiplier

$$m_\alpha(\xi) = \left(1 - |\xi|^2\right)_+^\alpha.$$

The corresponding convolution kernel $K_\alpha = \check{m}_\alpha$ is well known to lie in L^p for

$$p > p_{\alpha,N} = \frac{2N}{N+1+2\alpha};$$

see [9]. We have, from the famous work of C. Fefferman on the ball multiplier, [7], the following necessary condition for boundedness of the corresponding operator. For $N \geq 2$ and $\alpha \geq 0$, if the multiplier operator T_{m_α} is bounded on L^p for $p \neq 2$, then $\alpha > 0$. The Bochner–Riesz conjecture states that, for $N \geq 2$, T_{m_α} is bounded on $L^p(\mathbb{R}^N)$ if and only if

$$p_{\alpha,N} < p < p'_{\alpha,N} \text{ and } \alpha > 0.$$

The restrictions on the parameters may be equivalently expressed as

$$\alpha > \max \left\{ N \left| \frac{1}{2} - \frac{1}{p} \right| - \frac{1}{2}, 0 \right\}.$$

The radial multiplier conjecture is a significant generalisation of the Bochner–Riesz conjecture. However, the result in [8] makes no advances on known exponents for the Bochner–Riesz conjecture.

In this paper, we examine a class of compactly supported multipliers arising as Bochner–Riesz multipliers associated to a family of embedded varieties, Γ , in $\mathbb{R}^N$ of arbitrary codimension. Although compactly supported, these multipliers are not radial multipliers. Instances of the corresponding multiplier operators have kernels which are L^p integrable for a *different* range of p than the range of p for which the operators are bounded.

For a compact variety Γ and some $\alpha > 0$,

$$m(\zeta) = d(\zeta, \Gamma)^\alpha \phi(\zeta) \tag{1}$$

is called the Bochner–Riesz multiplier with exponent α . Here ϕ is some appropriate bump function whose support intersects Γ .

We may assume that Γ is parameterised as the graph of a smooth function Ψ , with ϕ supported in a small neighbourhood of 0 and Ψ chosen such that $\Psi(0) = 0$ and $\nabla \Psi(0) = 0$. This is achieved by using a partition of unity on the surface and then translating and rotating, with corresponding modifications to the multiplier operator. The mutual boundedness of the initial multiplier operator and the

operators thus obtained is simply verified. As such, we consider the n -dimensional variety

$$\Gamma = \{(\xi, \Psi(\xi)); \xi \in \mathbb{R}^n\} \subset \mathbb{R}^N, \quad (2)$$

with smooth $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^L$ obtained by the above procedure. One can further reduce the study of the L^p mapping properties of the operators T_m to the study of the L^p mapping properties of related multiplier operators $T_{m_{\Gamma,\alpha}}$, whose multipliers are given by

$$m_{\Gamma,\alpha}(\zeta) = m_{\Gamma,\alpha}(\xi, \eta) = \phi(\xi)|\eta - \Psi(\xi)|^\alpha \chi(\eta - \Psi(\xi)). \quad (3)$$

Here $\zeta = (\xi, \eta) \in \mathbb{R}^{n+L} = \mathbb{R}^N$, $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\chi : \mathbb{R}^L \rightarrow \mathbb{R}$ are appropriate radial bump functions with small supports contained in $B^n(0, \delta)$ and $B^L(0, \delta)$, respectively, where $\delta < 1$ is some suitable small parameter. This technique is well known. At its core, it requires an application of the implicit function theorem which shows that the multiplier (1) can be expressed as the product of (3) with a smooth compactly supported function and vice-versa; see [16] for details. Note that the functions ϕ and χ in (3) localise the multiplier about the origin, which is on the surface.

For a particular class of surfaces, we determine the precise $L^p(\mathbb{R}^N)$ integrability range of the corresponding convolution kernel $K_{\Gamma,\alpha} = \check{m}_{\Gamma,\alpha}$.

Let $\Gamma = \{(\xi, \Psi(\xi)) \in \mathbb{R}^N; \xi \in \mathbb{R}^n\}$, where

$$\Psi(x) = (\tilde{\Psi}(\xi), 0) = (|\xi|^{d_1}, |\xi|^{d_2}, \dots, |\xi|^{d_{L_1}}, 0) \in \mathbb{R}^{L_1} \times \mathbb{R}^{L_2} \quad (4)$$

and $2 \leq d_1 < d_2 < \dots < d_{L_1}$ are even. Here $\tilde{\Psi}(\xi) = (|\xi|^{d_1}, |\xi|^{d_2}, \dots, |\xi|^{d_{L_1}})$, which is a smooth radial function. The surface Γ has dimension n and codimension $L = L_1 + L_2$.

Theorem 1.1 *Consider the surface $\Gamma = \{(\xi, \Psi(\xi)) \in \mathbb{R}^N : \xi \in \mathbb{R}^n\}$, where Ψ is given by (4) and $L_1 > 0$. The convolution kernel $K_{\Gamma,\alpha} = \check{m}_{\Gamma,\alpha} \in L^p(\mathbb{R}^N)$ if and only if $p > \frac{L+n}{L+\alpha+\frac{n}{2}}$.*

The proofs of the necessary and sufficient conditions for $K_{\Gamma,\alpha} \in L^p$ in Theorem 1.1 are different. The kernel $K_{\Gamma,\alpha}$ is given as an oscillatory integral with an explicit phase. To prove that

$$K_{\Gamma,\alpha} \in L^p \implies p > \frac{L+n}{L+\alpha+\frac{n}{2}}, \quad (5)$$

we restrict our attention to a region where the critical point of the phase is non-degenerate and use stationary phase techniques to obtain a pointwise estimate on the size of the kernel. To obtain the sufficient condition

$$p > \frac{L+n}{L+\alpha+\frac{n}{2}} \implies K_{\Gamma,\alpha} \in L^p, \quad (6)$$

we adapt a method developed in work by Arkhipov et al. [1], which was used to settle a question arising from Tarry's problem in Number Theory; rather than pointwise control on the size of the kernel, we partition $\mathbb{R}^N$ according to estimates on the kernel's size and then estimate the size of each element of the partition.

Remark 1.2 Theorem 1.1 holds for more general varieties including, for example, the curves of standard type found in [15]; see [16] which obtains the result by a more delicate application of the methods contained herein.

We now focus on the $L^p(\mathbb{R}^N)$ boundedness of the multiplier operator $T_{m_{\Gamma,\alpha}}$. After recalling some known results on the relation between the Bochner–Riesz conjecture and the restriction conjecture, we are able to state a sharp result, Proposition 1.6.

For general varieties Γ , it is well known that the Bochner–Riesz conjecture is connected to another fundamental part of Euclidean harmonic analysis, the Fourier restriction conjecture. In this context, the conjecture concerns the precise $L^p(\mathbb{R}^N) \rightarrow L^q(\Gamma, \mu)$ mapping properties of the restriction operator $Rf = \hat{f}|_{\Gamma}$, where μ denotes the surface measure on Γ . Even in the original setting of $\Gamma = S^{N-1}$, as proposed by Stein in the mid 1960's, the Fourier restriction conjecture is unsolved for $N \geq 3$.

Progress with the Bochner–Riesz conjecture has historically paralleled progress with the restriction conjecture. T. Tao established that the Bochner–Riesz conjecture implies the restriction conjecture on the sphere, [14]. The table contained therein also outlines some of the parallel progress in these two areas.

It is well known that sharp L^p estimates for Bochner–Riesz multiplier operators, $T_{m_{\Gamma,\alpha}}$, follow from L^2 Fourier restriction estimates for Γ ; the relevant restriction estimates are written as

$$\left(\int \left| \hat{f}(\xi, \Psi(\xi)) \right|^2 \phi(\xi) d\xi \right)^{\frac{1}{2}} \leq C \|f\|_{L^q(\mathbb{R}^N)}. \quad (7)$$

The implication to obtain sharp Bochner–Riesz estimates and surrounding ideas date back to C. Fefferman's thesis, [6], where the analysis is for the sphere $\Gamma = S^{N-1}$.

For varieties Γ of arbitrary codimension we have the following result of G. Mockenhaupt, [11].

Theorem 1.3 *Let*

$$\Gamma = \{(\xi, \Psi(\xi)) ; \xi \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}^L.$$

Suppose that the $L^q \rightarrow L^2$ restriction inequality (7) holds. Then, for $1 \leq p \leq q$ or $q' \leq p < \infty$, the Bochner–Riesz multiplier $m_{\Gamma,\alpha}$, (3) with $\alpha > 0$, defines a multiplier operator, $T_{m_{\Gamma,\alpha}}$, which is bounded on L^p for

$$\alpha > (n+L) \left| \frac{1}{p} - \frac{1}{2} \right| - \frac{L}{2}.$$

In particular, when $1 \leq p \leq q < 2$, we see that $T_{m_{\Gamma,\alpha}}$ is bounded on $L^p(\mathbb{R}^N)$ if $p > \frac{L+n}{L+\alpha+\frac{n}{2}}$, which is precisely the L^p integrability range for the Bochner–Riesz kernels

$K_{\Gamma, \alpha}$ considered in Theorem 1.1. Hence, for the varieties considered in Theorem 1.1, we obtain sharp L^p estimates for $T_{m_{\Gamma, \alpha}}$, provided the Fourier restriction inequality (7) holds.

In this paper, we observe that there are surfaces for which the range of p such that the Bochner–Riesz kernel $K_{m_{\Gamma, \alpha}} = \check{m}_{\Gamma, \alpha} \in L^p$ differs from the L^p boundedness range for $T_{m_{\Gamma, \alpha}}$. These examples are instances of varieties

$$\Gamma = \{(\xi, \tilde{\Psi}(\xi)) ; \xi \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}^{L_1} \times \mathbb{R}^{L_2},$$

with $\tilde{\Psi} : \mathbb{R}^n \mapsto \mathbb{R}^{L_1}$ smooth. These lie in a hyperplane in $\mathbb{R}^N$ if $L_1 < L$. In this case it is well known that the Fourier restriction inequality (7) can not hold for any $1 < q$. Nevertheless, a restriction inequality may hold within the ambient hyperplane $\mathbb{R}^{n+L_1}$. Indeed, let us suppose that

$$\left(\int_{\mathbb{R}^n} |\hat{g}(\xi, \tilde{\Psi}(\xi))|^2 \phi(\xi) d\xi \right)^{\frac{1}{2}} \leq C \|g\|_{L^q(\mathbb{R}^{n+L_1})} \quad (8)$$

holds for some $1 < q < 2$ and some constant C . Then we have the following generalisation of Theorem 1.3.

Theorem 1.4 *Let*

$$\Gamma = \{(\xi, \tilde{\Psi}(\xi), 0) ; \xi \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}^{L_1} \times \mathbb{R}^{L_2}$$

and $\tilde{\Gamma}$ be the corresponding projection to $\mathbb{R}^n \times \mathbb{R}^{L_1}$,

$$\tilde{\Gamma} = \{(\xi, \tilde{\Psi}(\xi)) ; \xi \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}^{L_1}.$$

Suppose that the $L^q \rightarrow L^2$ restriction inequality (8) holds. Then, for $1 \leq p \leq q$ or $q' \leq p < \infty$, the Bochner–Riesz multiplier $m_{\Gamma, \alpha}$ given by (3) with $\alpha > 0$ defines a multiplier operator, $T_{m_{\Gamma, \alpha}}$, which is bounded on L^p if

$$\alpha > (n + L_1) \left| \frac{1}{p} - \frac{1}{2} \right| - \frac{L_1}{2}.$$

The proof of Theorem 1.4 follows along known lines of argument, those which establish Theorem 1.3; see [16]. It works via an $L^2(\tilde{\Gamma}) \rightarrow L^{q'}(\mathbb{R}^N)$ extension estimate. Where $L_2 > 0$, no non-trivial extension estimate from the surface Γ holds so that Theorem 1.4 is not a direct corollary of Theorem 1.3.

Let $\Gamma = \{(\xi, \Psi(\xi)) ; \xi \in \mathbb{R}^n\}$, with Ψ given by (4) such that $L_1 < L$. Then a necessary condition for the restriction inequality (8) to hold is

$$\frac{q'}{2} \geq 1 + \frac{D}{n}, \quad (9)$$

where $D = \sum_{j=1}^{L_1} d_j$. This follows from a standard Knapp example. The following result has been established by Wright [17].

Proposition 1.5 *Let $\Gamma = \{(\xi, \Psi(\xi)); \xi \in \mathbb{R}^n\}$, with Ψ given by (4). If $d_1 \geq n(L_1 + 1)$, then (8) holds if $\frac{q'}{2} \geq 1 + \frac{D}{n}$.*

We sketch a proof of Proposition 1.5 in Appendix A.

Proposition 1.5 gives us a class of varieties of arbitrary codimension where the optimal L^2 Fourier restriction estimate holds. Combining this with Theorem 1.4, de Leeuw's multiplier theorem, duality, and Theorem 1.1 we establish the following sharp result; see Sect. 4.

Proposition 1.6 *Let*

$$\Gamma = \{(\xi, \Psi(\xi)); \xi \in \mathbb{R}^n\},$$

with Ψ defined at (4) chosen such that $L_1 < L$ and $d_1 \geq n(L_1 + 1)$. Consider the Bochner–Riesz multiplier $m_{\Gamma, \alpha}$, (3) with $\alpha > 0$. Suppose that $1 \leq p \leq q$ or $q' \leq p < \infty$ and $\frac{q'}{2} = 1 + \frac{D}{n}$. The operator $T_{m_{\Gamma, \alpha}}$ is bounded on L^p if and only if $\frac{L_1+n}{L_1+\alpha+\frac{n}{2}} < p < \frac{L_1+n}{L_1-\alpha+\frac{n}{2}}$.

One expects these propositions to hold without the extra condition $d_1 \geq n(L_1 + 1)$. For the curves described by the Eq. (4) when $n = 1$ and $L_1 = L$ this is indeed the case; see [5].

In distinction to Proposition 1.6, Theorem 1.1 implies that the convolution kernel $K_{\Gamma, \alpha} = \check{m}_{\Gamma, \alpha}$ lies in $L^p(\mathbb{R}^N)$ if and only if $p > \frac{L+n}{L+\alpha+\frac{n}{2}}$. It is easily verified, for $\alpha < \frac{n}{2}$, that $\frac{L_1+n}{L_1+\alpha+\frac{n}{2}} \geq \frac{L+n}{L+\alpha+\frac{n}{2}}$, with strict inequality if $L_1 \neq L$. Hence, for the varieties in Theorem 1.1 with $L_1 < L$, the L^p boundedness range for $T_{m_{\Gamma, \alpha}}$ differs from the L^p integrability range of $K_{\Gamma, \alpha}$, as previously suggested.

Regarding the Fourier restriction problem for the moment curve,

$$(t, t^2, t^3, \dots, t^d),$$

there is the result of Drury, [4], which tells us that the restriction inequality (7) holds for $q' \geq d(d+1)$. This is sharp, by the example in [1]. As mentioned previously, it is a method from [1] that we adapt to show the sufficient condition for $K_{\Gamma, \alpha} \in L^p$, the implication (6), of Theorem 1.1.

2 Outline of Paper

Section 3 introduces notation and derives the Bochner–Riesz kernels. We also observe some basic properties of the kernels.

In Sect. 4 we show how Proposition 1.6 follows from Theorem 1.4 and Theorem 1.1 applied with reference to de Leeuw's theorem.

In Sects. 5 and 6 we prove Theorem 1.1. In Sect. 5 we prove the sufficient condition for $K_{\Gamma, \alpha} \in L^p$, (6). In Sect. 6 the reverse implication, (5), is proved.

In Appendix A we sketch the proof of Theorem 1.5.

3 Notation

Throughout this paper C will be used to denote a constant, its value may change from line to line. We use the notation $X \lesssim Y$ or $Y \gtrsim X$ if there exists some constant C such that $X \leq CY$. Where $X \lesssim Y$ and $X \gtrsim Y$, we write $X \sim Y$. When we wish to highlight the dependence of the implied constant C on some other parameter, say $C = C(M)$, we will use the notation $X \lesssim_M Y$. We use the notation $X \ll Y$ or $Y \gg X$ if there exists some suitable large constant D such that $DX \leq Y$.

We denote by $B^m(x, r)$ the Euclidean ball of radius r centred at x in $\mathbb{R}^m$, or simply $B(x, r)$ if the dimension is clear.

We use the notation

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int e^{-2\pi i \xi \cdot x} f(x) dx,$$

for the Fourier transform and likewise for the inverse Fourier transform we write

$$\mathcal{F}^{-1}g(x) = \check{g}(x) = \int e^{2\pi i x \cdot \xi} g(\xi) d\xi.$$

We also denote by $\hat{\sigma}$ the Fourier transform of the surface measure of the sphere S^{m-1} ,

$$\hat{\sigma}(x) = \int_{S^{m-1}} e^{2\pi i x \cdot \omega} d\sigma(\omega).$$

For $p \in (1, \infty)$ we denote by $\mathcal{M}_p$ the space of Borel measurable functions m for which the multiplier operator defined a priori by

$$f \mapsto \left(m \cdot \hat{f} \right)^\vee$$

is bounded from L^p to L^p .

Recall that we write $L = L_1 + L_2$. Typically, in what follows x , y , and z will denote points in $\mathbb{R}^n$, $\mathbb{R}^{L_1}$, and $\mathbb{R}^{L_2}$, respectively. Since we may have that $L_2 = 0$ but integrate with respect to dz we follow the convention that, when $L_2 = 0$, $\mathbb{R}^{L_2} = \mathbb{R}^0 = \{0\}$ and dz is the counting measure. Similarly for S^0 , which we regard as the set $\{-1, 1\}$ equipped with the counting measure.

All multiplier operators are expressible as convolution operators. In this paper, we consider the Bochner–Riesz multipliers corresponding to the surfaces

$$\Gamma = \{(\xi, \Psi(\xi)); \xi \in \mathbb{R}^n\} = \{(\xi, \tilde{\Psi}(\xi), 0); \xi \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}^{L_1} \times \mathbb{R}^{L_2}$$

with Ψ as given in (4). We turn our sights to the convolution kernel of T_m , where $m = m_{\Gamma, \alpha}$ is given at (3). The kernel, $K_{\Gamma, \alpha} = K = \check{m}$, is given by

$$\begin{aligned} K(x, y, z) &= \int_{\mathbb{R}^L} \int_{\mathbb{R}^n} e^{2\pi i(x \cdot \xi + (y, z) \cdot \eta)} m(\xi, \eta) d\xi d\eta \\ &= \int_{\mathbb{R}^L} \int_{\mathbb{R}^n} e^{2\pi i(x \cdot \xi + (y, z) \cdot \eta)} \phi(\xi) |\eta - \Psi(\xi)|^\alpha \chi(\eta - \Psi(\xi)) d\xi d\eta \\ &= \int_{\mathbb{R}^L} \int_{\mathbb{R}^n} e^{2\pi i(x \cdot \xi + (y, z) \cdot (\eta + \Psi(\xi)))} \phi(\xi) |\eta|^\alpha \chi(\eta) d\xi d\eta \\ &= A_\alpha(y, z) \left(\int_{\mathbb{R}^n} e^{2\pi i(x \cdot \xi + (y, z) \cdot \Psi(\xi))} \phi(\xi) d\xi \right) \\ &= A_\alpha(y, z) k(x, y), \end{aligned} \quad (10)$$

where

$$k(x, y) = \int_{\mathbb{R}^n} e^{2\pi i(x \cdot \xi + y \cdot \tilde{\Psi}(\xi))} \phi(\xi) d\xi \quad (11)$$

and

$$A_\alpha(y, z) = \int_{\mathbb{R}^L} e^{2\pi i(y, z) \cdot \eta} |\eta|^\alpha \chi(\eta) d\eta. \quad (12)$$

Unlike for A_α , obtaining pointwise control k is a delicate matter. Such control will only be required when we prove the necessary condition (5) in Sect. 6.

We have the following bounds on the size of the function A_α . Their proof, which is routine, is omitted; see [16].

Lemma 3.1 *For large $|(y, z)|$, there exist constants $0 < c_\alpha \leq C_\alpha$ such that*

$$c_\alpha |(y, z)|^{-L-\alpha} \leq |A_\alpha((y, z))| \leq C_\alpha |(y, z)|^{-L-\alpha}. \quad (13)$$

4 Proof of Proposition 1.6

Proof By Proposition 1.5, we see that the $L^q \rightarrow L^2$ restriction inequality (8) also holds as an $L^p \rightarrow L^2$ inequality for $1 \leq p \leq q$. Therefore, we can apply Theorem 1.4 to see that, where $1 \leq p \leq q$ or $q' \leq p < \infty$, T_m is bounded on L^p for $\frac{L_1+n}{L_1+\alpha+\frac{n}{2}} < p < \frac{L_1+n}{L_1-\alpha+\frac{n}{2}}$. Instead of considering the multiplier

$$m(\xi, \eta, \lambda) = |(\eta, \lambda) - \Psi(\xi)|^\alpha \chi((\eta, \lambda) - \Psi(\xi)) \phi(\xi),$$

we restrict this continuous multiplier to $V = \mathbb{R}^{n+L_1}$, corresponding to $\lambda = 0$. de Leeuw's theorem, [3], tells us that if $m \in \mathcal{M}_p(\mathbb{R}^N)$, then $m|_V \in \mathcal{M}_p(\mathbb{R}^{n+L_1})$. This is the Bochner–Riesz multiplier on $\mathbb{R}^{n+L_1}$ corresponding to the surface $\tilde{\Gamma} = \{(\xi, \tilde{\Psi}(\xi)); \xi \in \mathbb{R}^n\}$. We apply Theorem 1.1 to see that the corresponding kernel is

in L^p only for $p > \frac{L_1+n}{L_1+\alpha+\frac{n}{2}}$. By duality, we must also have $p < \frac{L_1+n}{L_1-\alpha+\frac{n}{2}}$ for T_m to be bounded on L^p . $\square$

5 A Sufficient Condition for $K \in L^p$; Proof of (6)

In this section we prove one part of Theorem 1.1, the implication (6). That is the sufficient condition for the kernel K , (10), to be in L^p .

We use the following lemma, this van der Corput estimate is essential to the proof of our main result. It is proven analogously to the corresponding classical estimate, as a simple corollary of the van der Corput estimate proved in [2].

Lemma 5.1 *Let $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial of degree d and ϕ be a smooth function with $\text{supp } \phi \subset (a, b)$. Given the bound $\inf_{t \in [a, b]} \sum_{j=1}^d |\Phi^{(j)}(t)|^{\frac{1}{j}} \geq \kappa$, we have that*

$$\left| \int_a^b e^{2\pi i \Phi(t)} \phi(t) dt \right| \lesssim_d \min \left\{ (b-a), \kappa^{-1} \right\} \|\phi'\|_{L^1}.$$

Henceforth, until stated otherwise, we denote by $\Phi(t)$ the phase appearing in certain one dimensional oscillatory integrals $I_{\pm}(x, y) = \int e^{2\pi i \Phi(t)} \phi_0(t) dt$. These phases are

$$\Phi(t) = \Phi_{x,y,\pm}(t) = \pm|x|t + \sum_{j=1}^{L_1} y_j \psi_j(t),$$

with $\psi_j(t) = |t|^{d_j}$. Recall δ , given at (3), is a small parameter bounding the radius of the support of ϕ . For $t \in [-\delta, \delta]$, we denote by $H_t(x, y)$ and $H(x, y)$ the quantities

$$H_t(x, y) = \sum_{j=1}^{d_{L_1}} \left| \Phi_{x,y}^{(j)}(t) \right|^{\frac{1}{j}} \text{ and } H(x, y) = \inf_{t \in [-\delta, \delta]} H_t(x, y).$$

We will make use the following lemma.

Lemma 5.2 *For $|y| \gtrsim 1$, $H(x, y) \gtrsim |y|^{\frac{1}{d_{L_1}}}$.*

Proof We relate H to a homogeneous $\widetilde{H}^1$, where

$$\widetilde{H}^1(x, y) = \max_{j=1,2,\dots,d_{L_1}} \inf_{t \in [-\delta, \delta]} \left| \Phi_{x,y}^{(j)}(t) \right|.$$

We show that $\widetilde{H}^1$, which is clearly homogeneous of degree 1, is bounded positively away from 0 on the sphere. We first show that it is non-vanishing on the sphere.

Suppose that $\widetilde{H}^1(x, y) = 0$. Then, in particular, $|y_{L_1}| \sim \inf_{t \in [a, b]} \left| \Phi_{x,y}^{(d_{L_1})}(t) \right| = 0$ so that $y_{L_1} = 0$. Considering the d_{L_1-1} th derivative in turn, we find that $|y_{L_1-1}| \sim$

$\inf_{t \in [-\delta, \delta]} |\Phi^{(d_{L_1-1})}| = 0$ so that $y_{L_1-1} = 0$. Continuing in this fashion shows that if $\widetilde{H}^1(x, y) = 0$, then $(x, y) = 0$.

To show that $\inf_{(x,y) \in S^{N-1}} \widetilde{H}^1(x, y) > 0$, we suppose by way of contradiction that there exists a sequence $z_n \in S^{N-1}$ with $\widetilde{H}^1(z_n) \rightarrow 0$. Passing to a subsequence if necessary, we can assume that $z_n \rightarrow z \in S^{N-1}$. We see that

$$\begin{aligned} \widetilde{H}^1(z_n) &\geq \inf_{t \in I} |\Phi_{z_n}^{(j)}(t)| = |\Phi_{z_n}^{(j)}(t_n)| \geq |\Phi_z^{(j)}(t_n)| - |\Phi_z^{(j)}(t_n) - \Phi_{z_n}^{(j)}(t_n)| \\ &\geq \inf_{t \in I} |\Phi_z^{(j)}(t)| - C|z_n - z|. \end{aligned}$$

Taking the maximum over j shows that

$$\widetilde{H}^1(z_n) \geq \widetilde{H}^1(z) - C|z_n - z|.$$

We know that $\widetilde{H}^1(z) \neq 0$ and $|z_n - z| \rightarrow 0$ so taking $n \rightarrow \infty$ we obtain a contradiction, $0 > 0$. Therefore we can conclude $\inf_{(x,y) \in S^{N-1}} \widetilde{H}^1(x, y) > 0$. As a consequence of this and homogeneity, we have that $\widetilde{H}^1(x, y) \gtrsim |(x, y)| \geq |y|$.

Now, by relating H to the homogeneous $\widetilde{H}^1$, which is non-vanishing on the sphere, we show that $H(x, y) \gtrsim |y|^{\frac{1}{d_{L_1}}}$. There exists $t^* = t^*(x, y) \in I$ with $H(x, y) = H_{t^*}(x, y)$. By an appropriate choice of $1 \leq j_0 \leq d_{L_1}$, we see that

$$\begin{aligned} 1 &\lesssim |y|^{\frac{1}{d_{L_1}}} \leq |(x, y)|^{\frac{1}{d_{L_1}}} \lesssim \left(\widetilde{H}^1(x, y)\right)^{\frac{1}{d_{L_1}}} \lesssim \left(\widetilde{H}^1(x, y)\right)^{\frac{1}{j_0}} = \left(\inf_{t \in I} |\Phi^{(j_0)}(t)|\right)^{\frac{1}{j_0}} \\ &\leq \left|\Phi^{(j_0)}(t^*)\right|^{\frac{1}{j_0}} \leq \max_{j=1, \dots, d_{L_1}} \left|\Phi^{(j)}(t^*)\right|^{\frac{1}{j}} \leq H_{t^*}(x, y) = H(x, y). \end{aligned}$$

□

We require the use of an asymptotic expansion before applying the van der Corput estimate. As such, we introduce a cutoff function η , chosen such that $\eta(r) = 0$ for $r \leq \frac{1}{2}$ and $\eta(r) = 1$ for $r \geq 1$. We will later perform polar integration; let us denote by ϕ_0 the function on $[0, \infty)$ for which $\phi(\xi) = \phi_0(|\xi|)$.

Lemma 5.3 *With $\Phi(r) = \Phi_{x,y,\pm}(r) = \pm|x|r + \sum_{j=1}^{L_1} y_j \psi_j(r)$. We define $H(x, y) = \inf_{r \in [-\delta, \delta]} \sum_{j=1}^{d_{L_1}} |\Phi^{(j)}(r)|^{\frac{1}{j}}$. If $H(x, y) \geq \kappa$, then we have that*

$$\left| \int e^{2\pi i \Phi(r)} \frac{1}{|rx|^{\frac{n-1}{2}}} \eta(|x|r) \phi_0(r) r^{n-1} dr \right| \lesssim \kappa^{-1} \frac{1}{|x|^{\frac{n-1}{2}}}.$$

Proof When we apply Lemma 5.1, we see that, if $H(x, y) \geq \kappa$, then

$$\left| \int_0^\infty e^{2\pi i (\pm r|x| + \sum_{j=1}^{L_1} y_j \psi_j(r))} \frac{1}{|rx|^{\frac{n-1}{2}}} \eta(|x|r) \phi_0(r) r^{n-1} dr \right| \lesssim_d \|\tilde{\phi}'\|_{L^1} \min \{1, \kappa^{-1}\},$$

where $\tilde{\phi}(r) = \frac{1}{|rx|^{\frac{n-1}{2}}} \eta(|x|r) \phi_0(r) r^{n-1} = \frac{1}{|x|^{\frac{n-1}{2}}} \eta(|x|r) \phi_0(r) r^{\frac{n-1}{2}}$. We see that

$$\begin{aligned} \tilde{\phi}'(r) &= \frac{1}{|x|^{\frac{n-1}{2}}} \left(|x| \eta'(|x|r) \phi_0(r) r^{\frac{n-1}{2}} + \eta(|x|r) \phi_0'(r) r^{\frac{n-1}{2}} \right. \\ &\quad \left. + \frac{n-1}{2} \eta(|x|r) \phi_0(r) r^{\frac{n-3}{2}} \right) \end{aligned}$$

so that

$$\begin{aligned} \|\tilde{\phi}'\|_{L^1} &\leq \frac{1}{|x|^{\frac{n-1}{2}}} \left(\int \left| |x| \eta'(|x|r) \phi_0(r) r^{\frac{n-1}{2}} \right| dr + \int \left| \eta(|x|r) \phi_0'(r) r^{\frac{n-1}{2}} \right| dr \right) \\ &\quad + \frac{1}{|x|^{\frac{n-1}{2}}} \int \left| \frac{n-1}{2} \eta(|x|r) \phi_0(r) r^{\frac{n-3}{2}} \right| dr \\ &\lesssim \frac{1}{|x|^{\frac{n-1}{2}}} \left(|x| |x|^{-\frac{n-1}{2}} |x|^{-1} + 1 + \left(1 + |x|^{-\frac{n-1}{2}} \right) \right) \sim \frac{1}{|x|^{\frac{n-1}{2}}}. \end{aligned}$$

□

Proof of (6) Recall the product expression $K(x, y, z) = k(x, y) A_\alpha(y, z)$ from (10) and that the graphing function Ψ is given at (4). By combining bounds on A_α , (13) and $|A_\alpha| \leq C$, we see

$$|k(x, y) A_\alpha(y, z)| \lesssim |k(x, y)| (1 + |(y, z)|)^{-L-\alpha}.$$

It thus suffices to show that $k(x, y)(1 + |(y, z)|)^{-L-\alpha} \in L^p(\mathbb{R}^N, dx dy dz)$ for the desired range of p . We form a finite decomposition of $k(x, y)(1 + |(y, z)|)^{-L-\alpha}$ and provide an L^p bound on each of the resulting terms. Given such bounds, an application of the triangle inequality suffices to complete the proof. We show the result for $p \leq 2$, the full range follows because $L^p \cap L^\infty \subset L^q$ for all $q \in [2, \infty]$ when $p \leq 2$.

We first consider regions R_j for which it is easily shown

$$\|k(x, y)(1 + |(y, z)|)^{-(L+\alpha)}\|_{L^p(R_j)} < \infty.$$

Where we have a comparison of the form $W \ll Z$, the relevant large constants are to be later fixed. The remaining comparisons $W \gtrsim Z$ are chosen so that the whole of $\mathbb{R}^N$ is covered. The first of the regions is $R_0 \subset \mathbb{R}^N$, where $|x| \gg |y|$. The next region is $R_1 \subset \mathbb{R}^N$, where $|x| \lesssim |y| \lesssim 1$. The remaining region, where $1 \lesssim |x| \lesssim |y|$ and $|y| \gg 1$, is the main region $R \subset \mathbb{R}^N$. Trivially, we always have the estimate $|k(x, y)| \leq C$.

In the region R_0 , we consider the phase function

$$\widetilde{\Phi_{x,y}}(t) = \frac{1}{|x|} (x \cdot t + y \cdot \Psi(t)),$$

for which $\|\widetilde{\Phi_{x,y}}\|_{C^{M+1}(\overline{B(0,\delta)})} \lesssim_M 1$ and $|\nabla \widetilde{\Phi_{x,y}}(t)| \gtrsim 1$, for $|t| \leq \delta$. Therefore, by the non-stationary phase lemma,

$$|k(x, y)| = \left| \int e^{2\pi i |x| \widetilde{\Phi_{x,y}}(t)} \phi(t) dt \right| \lesssim C_M \frac{1}{|x|^M}.$$

Integrating the resulting estimate, we see

$$\begin{aligned} & \int \int \int_{R_0} |k(x, y)|^p (1 + |(y, z)|)^{-p(L+\alpha)} dx dy dz \\ & \lesssim \int \int \int_{R_0} \frac{1}{(1 + |x|)^{Mp}} \frac{1}{(1 + |(y, z)|)^{p(L+\alpha)}} dx dy dz, \end{aligned}$$

which is finite if we take $M > n$.

Next, for R_1 , we see that

$$\begin{aligned} & \int \int \int_{R_1} |k(x, y)|^p (1 + |(y, z)|)^{-p(L+\alpha)} dx dy dz \\ & \lesssim \int_{|x| \lesssim 1} 1 dx \int \int (1 + |(y, z)|)^{-p(L+\alpha)} dy dz < \infty. \end{aligned}$$

We have thus established that $\|k(x, y)(1 + |(y, z)|)^{-p(L+\alpha)}\|_{L^p(R_j, dx dy dz)} < \infty$ for $j \in \{0, 1\}$. It remains to show that $\|k(x, y)(1 + |(y, z)|)^{-p(L+\alpha)}\|_{L^p(R, dx dy dz)} < \infty$.

We can perform polar integration on the expression for k . Using the fact that Ψ is radial with $\Psi(t) = \Psi_0(|t|)$ and $\phi(t) = \phi_0(|t|)$, we expand

$$\begin{aligned} k(x, y) &= \int e^{2\pi i (x \cdot t + y \cdot \Psi(t))} \phi(t) dt \\ &= \int \int e^{2\pi i (x \cdot r\omega + y \cdot \Psi_0(r))} \phi_0(r) d\sigma(\omega) r^{n-1} dr \\ &= \int e^{2\pi i y \cdot \Psi_0(r)} \int e^{2\pi i r x \cdot \omega} d\sigma(\omega) \phi_0(r) r^{n-1} dr \\ &= \int e^{2\pi i y \cdot \Psi_0(r)} \hat{\sigma}(rx) \phi_0(r) r^{n-1} dr. \end{aligned}$$

For $n > 1$ we use the asymptotic expansion for $\hat{\sigma}$

$$\hat{\sigma}(rx) = \frac{1}{|rx|^{\frac{n-1}{2}}} \left(b e^{2\pi i r |x|} + a e^{-2\pi i r |x|} + \varepsilon_{\hat{\sigma}}(rx) \right),$$

for $|rx| \gtrsim 1$, where the a and b are constants and $|\varepsilon_{\hat{\sigma}}(rx)| \leq C|rx|^{-1}$; see, for example, Chapter 8 of [12]. For $n = 1$ the expansion holds trivially with $\varepsilon_{\hat{\sigma}}(rx) = 0$, since we have that $\hat{\sigma}(rx) = 2 \cos(2\pi r |x|) = e^{2\pi i r |x|} + e^{-2\pi i r |x|}$ is the Fourier transform of the counting measure on $\{-1, 1\}$. To make use of the expansion, we

introduce a cutoff function, η . We choose η such that $\eta(r) = 0$ for $r \leq \frac{1}{2}$ and $\eta(r) = 1$ for $r \geq 1$. We see that

$$k(x, y) = \varepsilon_1(x, y) + I(x, y),$$

where

$$\varepsilon_1(x, y) = \int_0^\infty e^{2\pi i \left(\sum_{j=1}^{L_1} y_j \psi_j(r) \right)} \widehat{\sigma}(rx) (1 - \eta(|x|r)) \phi_0(r) r^{n-1} dr$$

and

$$I(x, y) = \int_0^\infty e^{2\pi i \left(\sum_{j=1}^{L_1} y_j \psi_j(r) \right)} \widehat{\sigma}(rx) \eta(|x|r) \phi_0(r) r^{n-1} dr.$$

Let us look first to $I(x, y)$. Using the asymptotic expansion, we see that

$$I(x, y) = I_+(x, y) + I_-(x, y) + \varepsilon_2(x, y),$$

where

$$I_+(x, y) = b \int_0^\infty e^{2\pi i \left(r|x| + \sum_{j=1}^{L_1} y_j \psi_j(r) \right)} \frac{1}{|rx|^{\frac{n-1}{2}}} \eta(|x|r) \phi_0(r) r^{n-1} dr,$$

$$I_-(x, y) = a \int_0^\infty e^{2\pi i \left(-r|x| + \sum_{j=1}^{L_1} y_j \psi_j(r) \right)} \frac{1}{|rx|^{\frac{n-1}{2}}} \eta(|x|r) \phi_0(r) r^{n-1} dr,$$

and

$$\varepsilon_2(x, y) = \int_0^\infty e^{2\pi i \left(\sum_{j=1}^{L_1} y_j \psi_j(r) \right)} \frac{1}{|rx|^{\frac{n-1}{2}}} \varepsilon_{\widehat{\sigma}}(rx) \eta(|x|r) \phi_0(r) r^{n-1} dr.$$

The main terms are I_+ and I_- . The phases we consider are thus

$$\Phi_{x,y,+}(r) = r|x| + \sum_{j=1}^{L_1} y_j \psi_j(r) \text{ and } \Phi_{x,y,-}(r) = -r|x| + \sum_{j=1}^{L_1} y_j \psi_j(r).$$

We estimate the contributions of the terms $I_+(x, y)$ and $I_-(x, y)$ to our estimate on $\|K\|_{L^p(R, dx dy dz)}$ separately. Since the analysis is the same in either case, we only present the argument for $I_+(x, y)$, with the relevant phase denoted by $\Phi = \Phi_{x,y} = \Phi_{x,y,+}$. We apply Lemma 5.3 to analyse the contribution of $I_+(x, y)$ to the L^p norm and work with reference to the quantity

$$H(x, y) = \inf_{|r| \leq \delta} \sum_{j=1}^{d_{L_1}} \left| \frac{\Phi_{x,y}^{(j)}(r)}{j!} \right|^{\frac{1}{j}}.$$

To carry out the L^p integration of $I_+(x, y)(1 + |(y, z)|)^{-(L+\alpha)}$ over R , we perform a dyadic partition on scales $H \sim 2^l$ and $|y| \sim 2^m$. By Lemma 5.2, we have $H(x, y) \gtrsim |y|^{\frac{1}{d_{L_1}}}$ on R . It follows that $l, m \geq 0$, provided we choose the defining inequality $|y| \gg 1$ with a large enough constant. When $H = H(x, y) \sim 2^l$, we have as a consequence of Lemma 5.3 that $|I_+(x, y)|^p \lesssim \frac{1}{|x|^{\frac{p(n-1)}{2}}} 2^{-lp}$. Thus we see

$$\begin{aligned} & \int \int \int_R |I_+(x, y)|^p \frac{1}{(1 + |(y, z)|)^{p(L+\alpha)}} dx dy dz \\ & \lesssim \sum_{m \geq 0} \sum_{l \geq 0} 2^{-lp} \int \int \int_{\{(x, y, z); H \sim 2^l, |x| \lesssim |y|, |z| \lesssim |y|, |y| \sim 2^m\}} \frac{1}{|x|^{\frac{p(n-1)}{2}}} \frac{1}{(1 + |y|)^{p(L+\alpha)}} dx dy dz \\ & \quad + \sum_{m \geq 0} \sum_{l \geq 0} 2^{-lp} \int \int \int_{\{(x, y, z); H \sim 2^l, |x| \lesssim |y|, |z| \lesssim |y|, |y| \sim 2^m\}} \frac{1}{|x|^{\frac{p(n-1)}{2}}} \frac{1}{(1 + |z|)^{p(L+\alpha)}} dx dy dz \\ & \lesssim \sum_{m \geq 0} \sum_{l \geq 0} 2^{-lp} 2^{-mp(L+\alpha)} 2^{L_2 m} \int \int_{\{(x, y); H(x, y) \sim 2^l, |x| \lesssim |y|, |y| \sim 2^m\}} \frac{1}{|x|^{\frac{p(n-1)}{2}}} dx dy \\ & = \sum_{m \geq 0} \sum_{l \geq 0} 2^{-lp} 2^{-mp(L+\alpha)} 2^{L_2 m} J_{l, m}, \end{aligned}$$

where

$$J_{l, m} = \int \int_{\{(x, y); H(x, y) \sim 2^l, |x| \lesssim |y|, |y| \sim 2^m\}} \frac{1}{|x|^{\frac{p(n-1)}{2}}} dx dy.$$

We split the sum into two and write

$$\sum_{m \geq 0} \sum_{l \geq 0} 2^{-lp} 2^{-mp(L+\alpha)} 2^{L_2 m} J_{l, m} = S_1 + S_2,$$

where

$$S_1 = \sum_{l \geq 0} 2^{-pl} \sum_{m \geq 2l} 2^{-pm(L+\alpha)} 2^{L_2 m} J_{l, m}$$

and

$$S_2 = \sum_{l \geq 0} 2^{-pl} \sum_{2l > m \geq 0} 2^{-pm(L+\alpha)} 2^{L_2 m} J_{l, m}.$$

We provide two different estimates on the size of $J_{l, m}$. The first estimate is the trivial estimate

$$J_{l, m} \leq \int \int_{\{(x, y); |y| \lesssim 2^m, |x| \lesssim 2^m\}} \frac{1}{|x|^{\frac{p(n-1)}{2}}} dx dy \leq 2^{mL_1} 2^{mn-p\left(\frac{n-1}{2}\right)m}. \quad (14)$$

The second estimate requires a little more care and it is here we work by the method from [1]. Since $H(x, y) \sim 2^l$ we will show that there exists some

$$r_u \in \left\{ \frac{1}{2^l}[-2^l\delta], \frac{1}{2^l}[-2^l\delta] + \frac{1}{2^l}, \dots, \frac{1}{2^l}[2^l\delta] - \frac{1}{2^l}, \frac{1}{2^l}[2^l\delta] \right\} \text{ such that } |\Phi'(r_u)| \lesssim 2^l,$$

which is pivotal to our estimate. We can choose $r_u = \frac{1}{2^l} [2^l r^*]$, where

$$\sum_{j=1}^{d_{L_1}} |\Phi^{(j)}(r^*)|^{\frac{1}{j}} = \inf_{r \in [-\delta, \delta]} \sum_{j=1}^{d_{L_1}} |\Phi^{(j)}(r)|^{\frac{1}{j}} = H(x, y).$$

We have that

$$\Phi'(r_u) = \sum_{j=0}^{d_{L_1}-1} \frac{1}{j!} \Phi^{(1+j)}(r^*)(r_u - r^*)^j.$$

Using the inequalities $|r_u - r^*| \leq 2^{-l}$ and $H(x, y) \sim 2^l$, we then see that

$$|\Phi'(r_u)| \lesssim \sum_{j=0}^{d_{L_1}-1} |\Phi^{(1+j)}(r^*)| 2^{-lj} \lesssim \sum_{j=0}^{d_{L_1}-1} 2^{l(j+1)} 2^{-lj} \sim 2^l.$$

As a consequence, we have the inclusion

$$\{(x, y); H(x, y) \sim 2^l\} \subset \bigcup_{u=[-2^l\delta]}^{[-2^l\delta]} \{(x, y); |\Phi'(r_u)| \lesssim 2^l\}.$$

For $r_u \in \left\{ \frac{1}{2^l}[-2^l\delta], \frac{1}{2^l}[-2^l\delta] + \frac{1}{2^l}, \dots, \frac{1}{2^l}[2^l\delta] - \frac{1}{2^l}, \frac{1}{2^l}[2^l\delta] \right\}$, we will make the change of variables $s \mapsto \tilde{s} = s + \sum_{j=1}^{L_1} y_j \psi'_j(r_u) = \Phi'(r_u)$, where $s = |x|$. The change of variables has a unit Jacobian. The variables $\omega \in S^{n-1}$, $s \in \mathbb{R}$, and $y \in \mathbb{R}^{L_1}$ are to be taken with $s\omega = x$. We denote by e_1 the vector $(1, 0, \dots, 0) \in \mathbb{R}^n$ and note $H(s\omega, y) = H(se_1, y)$ to see that

$$\begin{aligned} J_{l,m} &= \iint \int_{\{(\omega, s, y); H(s\omega, y) \sim 2^l, |y| \sim 2^m, s \lesssim |y|\}} \frac{1}{s p^{\frac{n-1}{2}}} d\sigma(\omega) \chi_{[0, \infty)}(s) s^{n-1} ds dy \\ &\lesssim \sum_{u=[-2^l\delta]}^{[2^l\delta]} \iint_{\{(s, y); |\Phi'_{se_1, y}(r_u)| \lesssim 2^l, |y| \lesssim 2^m\}} s^{n-1-p\frac{n-1}{2}} \chi_{[0, \infty)}(s) ds dy \\ &\lesssim \sum_{u=[-2^l\delta]}^{[2^l\delta]} \iint_{\{(\tilde{s}, y); |\tilde{s}| \lesssim 2^l, |y| \lesssim 2^m\}} \left| \tilde{s} - \sum_{j=1}^{L_1} y_j \psi'_j(r_u) \right|^{n-1-p\frac{n-1}{2}} d\tilde{s} dy \end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{u=[-2^l\delta]}^{[2^l\delta]} \int \int_{\{(\tilde{s}, y); |\tilde{s}| \lesssim 2^l, |y| \lesssim 2^m\}} \left(|\tilde{s}|^{n-1-p\frac{n-1}{2}} + \left| \sum_{j=1}^{L_1} y_j \psi'_j(r_u) \right|^{n-1-p\frac{n-1}{2}} \right) d\tilde{s} dy \\
&\lesssim 2^{2l} \int_{\{y; |y| \lesssim 2^m\}} \left(2^{l(n-1-p\frac{n-1}{2})} + 2^{m(n-1-p\frac{n-1}{2})} \right) dy \\
&\lesssim 2^{2l} 2^{L_1 m} \left(2^{l(n-1-p\frac{n-1}{2})} + 2^{m(n-1-p\frac{n-1}{2})} \right).
\end{aligned}$$

We use this last estimate in the case that $2l \leq m$, in which case we see that

$$J_{l,m} \lesssim 2^{2l} 2^{L_1 m} 2^{m(n-1-p\frac{n-1}{2})}. \quad (15)$$

We use this bound to see that

$$\begin{aligned}
S_1 &= \sum_{l \geq 0} 2^{-pl} \sum_{m \geq 2l} 2^{-pm(L+\alpha)} 2^{L_2 m} J_{l,m} \\
&\lesssim \sum_{m \geq 0} 2^{-pm(L+\alpha)} 2^{Lm} 2^{m(n-1-p\frac{n-1}{2})} \sum_{0 \leq l \leq \frac{m}{2}} 2^{2l} 2^{-pl} \\
&\lesssim \sum_{m \geq 0} 2^{-pm(L+\alpha)} 2^{Lm} 2^{m(n-1-p\frac{n-1}{2})} 2^m 2^{-p\frac{m}{2}},
\end{aligned}$$

which is finite if $p > \frac{L+n}{L+\alpha+\frac{n}{2}}$.

Using the trivial bound (14) we see that

$$\begin{aligned}
S_2 &= \sum_{l \geq 0} 2^{-pl} \sum_{2l > m \geq 0} 2^{-pm(L+\alpha)} 2^{L_2 m} J_{l,m} \\
&= \sum_{m \geq 0} 2^{-pm(L+\alpha)} 2^{mL} 2^{mn-p\left(\frac{n-1}{2}\right)} \sum_{2l > m \geq 0} 2^{-pl} \\
&\lesssim \sum_{m \geq 0} 2^{-pm(L+\alpha)} 2^{mL} 2^{mn-p\left(\frac{n-1}{2}\right)} 2^{-p\frac{m}{2}},
\end{aligned}$$

which is finite if $p > \frac{L+n}{L+\alpha+\frac{n}{2}}$.

The analysis may be repeated to show that $\|I_-(x, y)A_\alpha(y, z)\|_{L^p(R, dx dy dz)} < \infty$.

To complete the proof, we must show that $\|\varepsilon_i(x, y)(1 + |(y, z)|)^{-L-\alpha}\|_{L^p(R, dx dy dz)} < \infty$ for each $i \in \{1, 2\}$ when $p > \frac{n+L}{L+\alpha+\frac{n}{2}}$. For ε_2 , using $|\varepsilon_\delta(rx)| \lesssim |rx|^{-1}$, we have

$$\begin{aligned}
|\varepsilon_2(x, y)| &= \left| \int_0^\infty e^{2\pi i \left(\sum_{j=1}^{L_1} y_j \psi_j(r) \right)} \frac{1}{|rx|^{\frac{n-1}{2}}} \varepsilon_\delta(rx) \eta(|x|r) \phi_0(r) r^{n-1} dr \right| \\
&\leq C \frac{1}{|x|^{\frac{n-1}{2}}} \frac{1}{|x|} \int_{\frac{1}{2}|x|^{-1}}^1 r^{-1} r^{\frac{n-1}{2}} dr \sim \frac{1}{|x|^{\frac{n+1}{2}}}, \text{ for } (x, y, z) \in R.
\end{aligned}$$

For ε_1 , we find

$$\begin{aligned} |\varepsilon_1(x)| &= \left| \int_0^\infty e^{2\pi i \left(\sum_{j=1}^{L_1} y_j \psi_j(r) \right)} \widehat{\sigma}(rx) (1 - \eta(|x|r)) \phi_0(r) r^{n-1} dr \right| \\ &\lesssim \int_0^{|x|^{-1}} r^{n-1} dr \sim \frac{1}{|x|^n}, \text{ for } (x, y, z) \in R. \end{aligned}$$

In either case, we know that $|x| \gtrsim 1$ so that $|\varepsilon_i(x, y)| \lesssim \frac{1}{|x|^{\frac{n+1}{2}}}$, for $(x, y, z) \in R$.

Of course, we also have the bound $|\varepsilon_i(x, y)| \leq C$ for each i . By carrying out polar integration, we find that

$$\begin{aligned} &\|\varepsilon_i(x, y)(1 + |(y, z)|)^{-L-\alpha}\|_{L^p(R, dx dy dz)}^p \\ &\lesssim \int \int \int (1 + |(y, z)|)^{-p(L+\alpha)} (1 + |x|)^{-p\frac{n+1}{2}} dx dy dz \\ &\sim \int_1^\infty r^{(n+L-1)-p(L+\alpha+\frac{n+1}{2})} dr, \end{aligned}$$

which is finite if $p > \frac{n+L}{L+\alpha+\frac{n+1}{2}}$. In particular, this is true if $p > \frac{n+L}{L+\alpha+\frac{n}{2}}$. This completes the proof that $\|K\|_{L^p(\mathbb{R}^N)} < \infty$. $\square$

6 A Necessary Condition for $K \in L^p$; Proof of (5)

We turn to the proof of the remaining implication of Theorem 1.1. That is the necessary condition for the Bochner–Riesz kernel to be in L^p , (5). We will use the product expression, (10), for the kernel K . Since we have the lower bound $|A_\alpha(y, z)| \geq c_\alpha |(y, z)|^{-\alpha-L}$ for large $|(y, z)|$ from (13), it will suffice to establish a lower bound on the size of the oscillatory integral $k(x, y)$ in a sufficiently large region. We later prove the following.

Lemma 6.1 *Let*

$$\begin{aligned} R = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^{L_1}; \frac{\delta_1}{2} \leq \frac{|x|}{|y_1|} \leq \delta_1, \frac{|y_i|}{|y_1|} \leq \delta_1 \text{ for } i \geq 2, \right. \\ \left. y_1 \geq E, y_i \geq 1 \text{ for } i \geq 2, \text{ and } |x| \geq 1 \right\}, \end{aligned} \quad (16)$$

where the parameter δ_1 is some sufficiently small constant and the parameter E is some sufficiently large constant. For k , (11), we have the bound

$$|k(x, y)| \gtrsim |x|^{-\frac{n}{2}}, \text{ for } (x, y) \in R.$$

By an application of this lemma, we can now prove part of Theorem 1.1, the necessary condition for $K \in L^p$.

Proof of (5) In the case $p < 2$ we see, using the bounds on $|A_\alpha|$ from Lemma 3.1,

$$\begin{aligned} & \int \int \int_{(x,y,z) \in R \times \mathbb{R}^{L_2}} |K(x, y, z)|^p dx dy dz \\ & \gtrsim \int_E \int_E^{\min\{E, |z|\}} (1 + |z|)^{-p(L+\alpha)} y_1^{L_1-1} \int_{\frac{\delta_1 y_1}{2} \leq r \leq \delta_1 y_1} r^{-\frac{pn}{2}+n-1} dr dy_1 dz \\ & \quad + \int_E (1 + y_1)^{-p(L+\alpha)} y_1^{L_1-1} y_1^{L_2} \int_{\frac{\delta_1 y_1}{2} \leq r \leq \delta_1 y_1} r^{-\frac{pn}{2}+n-1} dr dy_1 \\ & \sim \int_E y_1^{-p(L+\alpha)-\frac{pn}{2}+n+L-1} dy_1. \end{aligned}$$

This last expression is finite if and only if

$$L - \frac{pn}{2} - p(L + \alpha) + n < 0 \iff \frac{L + n}{p} - \frac{L + n}{2} - \frac{L}{2} < \alpha.$$

□

Proof of Lemma 6.1 From the definition at (4), we know that d_1 is even. This will allow us to find a unique critical point of the relevant phase function.

Using polar coordinates, we have

$$k(x, y) = \int_0^\infty e^{2\pi i \Psi_y(r)} r^{n-1} \phi_0(r) \left[\int_{\mathbb{S}^{n-1}} e^{2\pi i r x \cdot \omega} d\sigma(\omega) \right] dr$$

where $\Psi_y(r) = y_1 r^{d_1} + y_2 r^{d_2} + \dots + y_{L_1} r^{d_{L_1}}$ and the inner integral is the Fourier transform of the surface measure σ on $\mathbb{S}^{n-1}$, $\hat{\sigma}(rx)$. Our aim is to decompose $k(x, y) = m(x, y) + \varepsilon(x, y)$ where the main term, m , satisfies

$$|m(x, y)| \gtrsim |x|^{-\frac{n}{2}}, \text{ for } (x, y) \in R \quad (17)$$

and the error term ε satisfies $|\varepsilon(x, y)| \lesssim \epsilon |x|^{-\frac{n}{2}}$, for $(x, y) \in R$, where ϵ is some suitable small parameter.

We first decompose $k = k_1 + \varepsilon_1$, where

$$k_1(x, y) = \int_{|x|^{-1}}^\infty e^{2\pi i \Psi_y(r)} \hat{\sigma}(rx) r^{n-1} \phi_0(r) dr$$

and

$$\varepsilon_1(x, y) = \int_0^{|x|^{-1}} e^{2\pi i \Psi_y(r)} \hat{\sigma}(rx) r^{n-1} \phi_0(r) dr.$$

We see that

$$|\varepsilon_1(x, y)| \lesssim |x|^{-n} \leq \frac{\epsilon}{3} |x|^{-\frac{n}{2}}, \text{ for } (x, y) \in R, \quad (18)$$

provided E is chosen large enough after fixing δ_1 , since $|x| \geq \delta_1 |y_1|/2 \geq \delta_1 E/2$.

In the case $n \geq 2$, for k_1 where $r \geq |x|^{-1}$, we use the following well-known asymptotic formula; see, for example, Chapter 8 of [12]. For $r|x| \geq 1$,

$$\hat{\sigma}(rx) = \frac{1}{(r|x|)^{\frac{n-1}{2}}} \left(ae^{-2\pi i r|x|} + be^{2\pi i r|x|} + \varepsilon_{\hat{\sigma}}(r|x|) \right),$$

where $a, b \neq 0$ and $|\varepsilon_{\hat{\sigma}}(r|x|)| \lesssim (r|x|)^{-1}$. In the case $n = 1$ the same formula holds trivially with $a = b = 1$ and $\varepsilon_{\hat{\sigma}} = 0$, since σ is the counting measure on $\{-1, 1\}$. This gives a corresponding decomposition for $k_1 = k_{1,1} + \varepsilon_2 + \varepsilon_3$.

We have that

$$\begin{aligned} |\varepsilon_3(x, y)| &= \left| \int_{|x|^{-1}}^{\infty} e^{2\pi i \Psi_y(r)} \varepsilon_{\hat{\sigma}}(r|x|) r^{n-1} \phi_0(r) dr \right| \\ &\lesssim |x|^{-\frac{n+1}{2}} \int_0^1 r^{\frac{n-3}{2}} dr \leq \frac{\epsilon}{3} |x|^{-\frac{n}{2}}, \text{ for } (x, y) \in R, \end{aligned} \quad (19)$$

provided E is chosen large enough after fixing δ_1 , as previously.

Now we consider

$$\varepsilon_2(x, y) = b \frac{1}{|x|^{\frac{n-1}{2}}} \int_{|x|^{-1}}^{\infty} e^{2\pi i(|x|r + \Psi_y(r))} r^{\frac{n-1}{2}} \phi_0(r) dr.$$

Because $y_1 \geq 0$, for $r \in \text{supp } \phi_0$,

$$\begin{aligned} ||x| + \Psi'_y(r)| &= ||x| + k_1 y_1 r^{d_1-1} + O(\delta_1 y_1 r^{d_1-1})| \\ &\geq |x| + d_1 y_1 r^{d_1-1} (1 - C\delta_1) \geq |x|, \end{aligned}$$

when $\delta_1 > 0$ is sufficiently small. Hence integrating by parts shows that

$$|\varepsilon_2(x, y)| \lesssim |x|^{-\frac{n+1}{2}} \leq \frac{\epsilon}{3} |x|^{-\frac{n}{2}} \text{ for } (x, y) \in R, \quad (20)$$

as before, provided E is chosen large enough after δ_1 has been fixed.

Finally, we have

$$k_{1,1}(x, y) = a \frac{1}{|x|^{\frac{n-1}{2}}} \int_{|x|^{-1}}^{\infty} e^{2\pi i(-|x|r + \Psi_y(r))} r^{\frac{n-1}{2}} \phi_0(r) dr.$$

We make the change of variables $r = \sigma s$ where $\sigma^{d_1-1} = \frac{|x|}{y_1} \sim \delta_1$ so that

$$k_{1,1}(x, y) = g(x, y) \sqrt{\lambda} \int_{\lambda^{-1}}^{\infty} e^{2\pi i \lambda \Phi_{x,y}(s)} s^{\frac{n-1}{2}} \phi_0(\sigma s) ds,$$

where $\lambda = \sigma|x|$ and $g(x, y) = a|x|^{-\frac{n}{2}}\sigma^{\frac{n}{2}} \sim_{\delta_1} |x|^{-\frac{n}{2}}$ on R . Here the phase is given by

$$\Phi_{x,y}(s) = -s + s^{d_1} + \epsilon_2 s^{d_2} + \dots + \epsilon_{L_1} s^{d_{L_1}}, \text{ where } \epsilon_j = \epsilon_j(x, y) = \frac{y_j}{y_1} \sigma^{d_j - d_1}.$$

We see that $|\epsilon_j s^{d_j-1}| \ll |s^{d_1-1}|$ for $|s| \leq \sigma^{-1}$ and $(x, y) \in R$. Hence, for $s \in \text{supp}_0(\sigma \cdot)$, the phase $\Phi_{x,y}(s)$ has a unique nondegenerate critical point, s_* . Note that $\lambda \gtrsim \delta_1^{\frac{1}{d_1-1}} \delta_1 E \gg 1$, provided E is chosen large enough after δ_1 has been fixed. The critical point $s_* \sim 1$ and the classical stationary phase argument (see [16]) shows that

$$\left| \int_{\lambda^{-1}}^{\infty} e^{2\pi i \lambda \Phi_{x,y}(s)} s^{\frac{n-1}{2}} \phi_0(\sigma s) ds \right| \sim \lambda^{-1/2},$$

implying that

$$|k_{1,1}(x, y)| \gtrsim_{\delta_1} |x|^{-\frac{n}{2}}, \text{ for } (x, y) \in R. \quad (21)$$

Hence (18), (19), (20) and (21) show that, with $m(x, y) := k_{1,1}(x, y)$, we can write $k(x, y) = m(x, y) + \varepsilon(x, y)$ where $|m(x, y)| \gtrsim_{\delta_1} |x|^{-\frac{n}{2}}$ and $|\varepsilon(x, y)| \lesssim \epsilon |x|^{-\frac{n}{2}}$. A suitably small choice of ϵ will ensure that $|\varepsilon(x, y)| \leq \frac{1}{2} |m(x, y)|$ so that

$$|k(x, y)| \gtrsim_{\delta_1} |x|^{-\frac{n}{2}}, \text{ for } (x, y) \in R,$$

as required. Recall that the bounds featuring small ϵ were obtained by making a large choice of E after a small δ_1 had been chosen. $\square$

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Appendix A: Proof of Proposition 1.5

For completeness, we sketch a proof of Proposition 1.5. The proof is due to Jim Wright, [17]. We work following the classical fractional integration proof of restriction; see, for example, [10]. We make use of the weak-type Young's inequality; see, for instance, [13].

Proof To apply the weak-type Young's inequality it will suffice to verify the following decay on the surface measure

$$|\hat{\mu}(x, y)| \leq C \min_{j=1, \dots, L} |y_j|^{-\frac{n}{d_j}}. \quad (22)$$

It is simply verified that $\min_{j=1, \dots, L} |y_j|^{-\theta \frac{n}{d_j}} \in L^{r, \infty}$ for $r = \theta^{-1} \frac{D}{n}$ for $\theta \in [0, 1]$. As a consequence of this, we establish the estimate

$$\|\hat{\mu} * f\|_{L^{p'}} \leq C \|f\|_{L^p},$$

which is equivalent to $L^p \rightarrow L^2$ Fourier restriction. Indeed, we write the convolution

$$\hat{\mu} * f(x, y) = \int_{\mathbb{R}^n} K_{y-z} * f_z(x) dz,$$

where $K_y(x) = \hat{\mu}(x, y)$ and $f_z(x) = f(x, z)$. For $p \in [1, 2]$ we may interpolate between $L^1 \rightarrow L^\infty$ and $L^2 \rightarrow L^2$ bounds¹ to show that

$$\|K_y * g\|_{L^{p'}(\mathbb{R}^n)} \leq C \left(\min_{j=1, \dots, L} |y_j|^{-\theta \frac{n}{d_j}} \right)^{\frac{2-p}{p}} \|g\|_{L^p}.$$

Given this, we may apply Minkowski's inequality and the weak-type Young's inequality to see that

$$\begin{aligned} \|\hat{\mu} * f\|_{L^{p'}} &= \left\| \left\| \int_{\mathbb{R}^n} K_{y-z} * f_z(x) dz \right\|_{L_x^{p'}} \right\|_{L_y^{p'}} \leq \left\| \int_{\mathbb{R}^n} \|K_{y-z} * f_z(x)\|_{L_x^{p'}} dz \right\|_{L_y^{p'}} \\ &\leq C \left\| \int_{\mathbb{R}^n} \left| \left(\min_{j=1, \dots, L} |y_j - z_j|^{-\theta \frac{n}{d_j}} \right)^{\frac{2-p}{p}} \right|^{-\frac{n}{D} \left(\frac{2-p}{p} \right)} \|f_z\|_{L^p} dz \right\|_{L_y^{p'}} \\ &\leq C \|f\|_{L^p(\mathbb{R}^N)}, \end{aligned}$$

¹ The $L^1 \rightarrow L^\infty$ bound is obtained from Young's inequality and the decay estimate and the $L^2 \rightarrow L^2$ bound has a Fourier multiplier which is uniformly bounded in y .

provided $1 + \frac{1}{p'} = \frac{1}{p} + \frac{n}{D} \left(\frac{2-p}{p} \right)$, i.e. $\frac{2}{p'} \geq 1 + \frac{D}{n}$. By interpolation with the trivial $L^1 \rightarrow L^\infty$ estimate we have that $f \mapsto f * \hat{\mu}$ is bounded from L^p to $L^{p'}$ for all $\frac{2}{p'} \geq 1 + \frac{D}{n}$.

To prove the key decay estimate (22) we perform polar integration to see that

$$\begin{aligned}\hat{\mu}(x, y) &= \int e^{2\pi i(x \cdot \xi + y \cdot \Psi(\xi))} \phi(\xi) d\xi = \int \int e^{2\pi i(x \cdot r\omega + y \cdot \Psi(r\omega))} \phi(r\omega) dr d\sigma(\omega) \\ &= \int \int e^{2\pi i(r b_0(\omega) + \sum_{j=1}^L y_j r^{d_j} b_j(\omega))} \phi(r\omega) dr d\sigma(\omega),\end{aligned}$$

where $b_0(\omega) = x \cdot \omega$ and $b_j(\omega) = \psi_j(\omega)$ for $1 \leq j \leq L$. The desired estimate follows from the one dimensional estimate

$$\left| \int e^{2\pi i(r b_0 + \sum_{j=1}^L y_j r^{d_j} b_j)} \phi_0(r) r^{n-1} dr \right| \leq C \min_{1 \leq j \leq L} |b_j|^{-\frac{n}{d_j}}. \quad (23)$$

Given this one dimensional estimate we find that

$$\begin{aligned}|\hat{\mu}(x, y)| &\leq C \int_{S^{n-1}} \min_{j=1, \dots, L} \left\{ 1, \frac{1}{|y_j \psi_j(\omega)|^{\frac{n}{d_j}}} \right\} d\sigma(\omega) \\ &= \left| \left\{ \xi; |\xi| \leq 1, |\psi_j(\xi)| \leq |y_j|^{-1}, 1 \leq j \leq L \right\} \right|,\end{aligned}$$

where the last equality is obtained by homogeneity and polar integration. Thus it remains to verify the sublevel set estimate

$$\left| \left\{ \xi; |\xi| \leq 1, |\psi_j(\xi)| \leq \delta_j, 1 \leq j \leq L \right\} \right| \leq C \min_{j=1, \dots, L} \delta_j^{\frac{n}{d_j}}.$$

This is trivially verified in the case that $\psi_j(\xi) = |\xi|^{d_j}$.

To obtain the bound (23) we may apply classical oscillatory integral estimates. In particular, we use van der Corput's Lemma and the method of non-stationary phase. We rescale, choosing $\sigma = \min_j \frac{1}{|b_j|^{\frac{1}{d_j}}}$ and setting $\sigma s = r$. Then we find

$$\begin{aligned}&\left| \int e^{2\pi i(r b_0 + \sum_{j=1}^L y_j r^{d_j} b_j)} \phi_0(r) r^{n-1} dr \right| \\ &= \sigma^n \left| \int e^{2\pi i(s \sigma b_0 + \sum_{j=1}^L b_j \sigma^{d_j} s^{d_j})} \phi_0(s) s^{n-1} ds \right|.\end{aligned}$$

We now require a constant bound on this last integral expression. We are able to obtain a uniform lower bound on $\sum_{j=1}^{L+1} |\Phi^{(j)}(r)|$. This bound gives oscillatory integral estimates for integration over subintervals of $\text{supp } \phi_0$. We sum the estimates over each of the subintervals using the triangle inequality to complete the proof. To establish the uniform lower bound on $\sum_{j=1}^{L+1} |\Phi^{(j)}(r)|$ we require that $\min_j d_j \geq n(L+1)$. $\square$

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