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A Duality Principle for Groups II: Multi-frames Meet Super-Frames

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Abstract

The duality principle for group representations developed in Dutkay et al. (J Funct Anal 257:1133–1143, 2009), Han and Larson (Bull Lond Math Soc 40:685–695, 2008) exhibits a fact that the well-known duality principle in Gabor analysis is not an isolated incident but a more general phenomenon residing in the context of group representation theory. There are two other well-known fundamental properties in Gabor analysis: the biorthogonality and the fundamental identity of Gabor analysis. The main purpose of this this paper is to show that these two fundamental properties remain to be true for general projective unitary group representations. Moreover, we also present a general duality theorem which shows that that muti-frame generators meet super-frame generators through a dual commutant pair of group representations. Applying it to the Gabor representations, we obtain that $\{\pi_\Lambda(m,n)g_1 \oplus \cdots \oplus \pi_\Lambda(m,n)g_k\}_{m,n\in\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d) \oplus \cdots \oplus L^2(\mathbb{R}^d)$ if and only if $\cup_{i=1}^k \{\pi_{\Lambda^o}(m,n)g_i\}_{m,n\in\mathbb{Z}^d}$ is a Riesz sequence, and $\cup_{i=1}^k \{\pi_\Lambda(m,n)g_i\}_{m,n\in\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d)$ if and only if $\{\pi_{\Lambda^o}(m,n)g_1 \oplus \cdots \oplus \pi_{\Lambda^o}(m,n)g_k\}_{m,n\in\mathbb{Z}^d}$ is a Riesz sequence, where π_Λ and π_{Λ^o} is a pair of Gabor representations restricted to a time–frequency lattice Λ and its adjoint lattice Λ^o in $\mathbb{R}^d \times \mathbb{R}^d$.

Keywords Projective group representations · Frame vectors · Bessel vectors · Multi-frame vectors · Super-frame vectors duality principle · Time–frequency analysis · Gabor frames

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1 Introduction

In this paper we continue the investigation on the duality phenomenon for projective unitary group representations. The purpose of this paper is two-fold: first we prove that the Wexler–Raz biorthogonality and the Fundamental Identity in Gabor analysis also reflect a general phenomenon for more general projective unitary representations of any countable group. Secondly we establish a duality principle connecting the multi-frame generators and super-frame generators, which is new even in the context of Gabor analysis. We start by recalling some basic definitions, backgrounds and fundamental theorems in Gabor analysis.

Frames were introduced by Duffin and Schaeffer in the context of nonharmonic Fourier series, and nowadays they have applications in a wide range of areas including sampling theory, operator theory, nonlinear sparse approximation, pseudo-differential operators, wavelet theory, wireless communications, data transmission with erasures, signal processing and quantum computing. Frames provide robust, basis-like (but generally non-unique) representations of vectors in a Hilbert space. The potential redundancy of frames often allows them to be more easily constructible and to possess better properties than are achievable using bases. For example, redundant frames offer more resilience to the effects of noise or to erasures of frame elements compared to bases.

A *frame* for a Hilbert space H is a sequence $\{x_n\}_{n\in I}$ in H with the property that there exist positive constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{n\in I} |\langle x\,,\, x_n\rangle|^2 \leq B\|x\|^2 \tag{1.1}$$

holds for every $x \in H$. A *tight frame* refers to the case when $A = B$, and a *Parseval frame* refers to the case when $A = B = 1$. In the case that (1.1) holds only for all $x \in \overline{span}\{x_n\}$, then we say that $\{x_n\}$ is a *frame sequence*, i.e., it is a frame for its closed linear span. If we only require the right-hand side of the inequality (1.1) to hold, then $\{x_n\}$ is called a *Bessel sequence*. Similarly, a Riesz sequence is a sequence that is a Riesz basis for its closed linear span.

Given a sequence $\{x_n\}_{n\in I}$ in a Hilbert space H. The *analysis operator* $\Theta : H \to \ell^2(I)$ is defined by

$$\Theta(x) = \sum_{n\in I} \langle x\,,\, x_n\rangle e_n, \quad x \in H,$$

where $\{e_n\}_{n\in I}$ is the standard orthonormal basis for $\ell^2(I)$ and the domain of Θ is the set of all $x \in H$ such that $\{\langle x\,,\, x_n\rangle\}_{n\in I} \in \ell^2(I)$. Clearly the domain of Θ is H if $\{x_n\}_{n\in I}$ is a frame sequence and the range of Θ is $\ell^2(I)$ if $\{x_n\}_{n\in I}$ is a Riesz sequence.

Gabor frames are a particular type of frames whose elements are simply generated by time–frequency shifts of a single window function or atom, and the structure of Gabor frames makes them especially suitable for applications involving time-dependent frequency content. Let $\Lambda = A(\mathbb{Z}^d \times \mathbb{Z}^d)$ be a full-rank time–frequency lattices, where A is a $2d \times 2d$ invertible real matrix. The adjoint lattice of Λ is the full rank time–frequency lattice defined by

$$\Lambda^o = \{\lambda^o \in \mathbb{R}^d \times \mathbb{R}^d : \sigma(\lambda, \lambda^o) \in \mathbb{Z}, \forall \lambda \in \Lambda\},$$

where σ denotes the standard symplectic form on $\mathbb{R}^{2d}$. A *Gabor (or Weyl-Heisenberg) family* is a collection of functions in $L^2(\mathbb{R}^d)$

$$\mathbf{G}(g, \Lambda) = \{e^{2\pi i <\ell, x>} g(x - \kappa) : \lambda = (\ell, \kappa) \in \Lambda\},$$

where $g \in L^2(\mathbb{R}^d)$ is the generator of the Gabor family. A Gabor frame (with a single generator) is a frame of the form $\mathbf{G}(g, \Lambda)$. Let E_ℓ and T_κ be the modulation and translation unitary operators defined by $E_\ell f(x) = e^{2\pi i <\ell, x>} f(x)$ and $T_\kappa f(x) = f(x - \kappa)$ for all $f \in L^2(\mathbb{R}^d)$. Then we have $\mathbf{G}(g, \Lambda) = \{E_\ell T_\kappa g : \lambda = (\ell, \kappa) \in \Lambda\}$. Hence a Gabor frame is a frame induced by the Gabor representation π_Λ of the abelian group $\mathbb{Z}^d \times \mathbb{Z}^d$ defined by $\pi_\Lambda(m, n) \to E_\ell T_\kappa$, where $(\ell, \kappa) = A(m, n)$.

In Gabor analysis, there are several fundamental theorems: Probably the most well-known one is the **Density Theorem** which tells us that a Gabor frame exists if and only if the $vol(\Lambda) \leq 1$, i.e., the density of Λ is greater than or equal to one (c.f. [4,26,28,38]), where the density of Λ is $\frac{1}{vol(\Lambda)}$ and $vol(\Lambda)$ is the Lebesgue measure of a fundamental domain of Λ, which is equal to $|det(A)|$ if $\Lambda = A(\mathbb{Z}^d \times \mathbb{Z}^d)$.

The other well-known theorems include the duality principle, the Wexler–Raz biorthogonality and the Fundamental Identity of Gabor frames. The duality principle for Gabor frames was independently and essentially simultaneously discovered by Daubechies et al. [8], Janssen [29], and Ron and Shen [39], and the techniques used in these three articles to prove the duality principle are quite different from each other, see [28] for a survey treating the duality principles from the perspective of harmonic analysis.

We summarize here the four fundamental properties of Gabor representation in the following theorem, see Sect. 2 for notation and definitions:

Theorem 1.1 *Let $\Lambda = A\mathbb{Z}^{2d}$ be a lattice and Λ^0 be its adjoint lattice. Then we have*

(i) [*Density theorem*] *There exists a function $g \in \mathcal{L}^2(\mathbb{R}^d)$ such that $\{\pi_\Lambda(m, n)g\}$ is a frame for $L^2(\mathbb{R}^d)$ if and only if $|det(A)| \leq 1$.*
(ii) [*Duality principle*] *A Gabor family $\{\pi_\Lambda(m, n)g\}$ is a frame (resp. Parserval frame) for $L^2(\mathbb{R}^d)$ if and only if $\{\pi_{\Lambda^{(0)}}(m, n)g\}$ is a Riesz sequence (resp. orthogonal sequence).*
(iii) [*Wexler–Raz biorthogonality*] *If $\{\pi_\Lambda(m, n)g\}$ is a frame for $L^2(\mathbb{R}^d)$, then*

$$\langle \pi_{\Lambda^0}(m, n)g, S^{-1}g \rangle = |det A| \delta_{(m,n),(0,0)},$$

where S is the frame operator for $\{\pi_\Lambda(m, n)g\}$
(iv) [*Fundamental Identity of Gabor Analysis—Janssen representation*] *If f, g, h, k are Bessel vectors for π_Λ, then*

$$\sum_{m,n}\langle f, \pi_\Lambda(m,n)g\rangle\langle \pi_\Lambda(m,n)h, k\rangle$$
$$= vol(\Lambda)^{-1}\sum_{m,n}\langle f, \pi_{\Lambda^\circ}(m,n)k\rangle\langle \pi_{\Lambda^\circ}(m,n)h, g\rangle$$

i.e.

$$\langle \Theta_{\pi_\Lambda,g}(f), \Theta_{\pi_\Lambda,h}(k)\rangle = vol(\Lambda)^{-1}\langle \Theta_{\pi_{\Lambda^\circ},k}(f), \Theta_{\pi_{\Lambda^\circ},h}(g)\rangle,$$

where $\Theta_{\pi_\Lambda,g}$ (*similarly for* $\Theta_{\pi_{\Lambda^\circ},h}$ *etc.*) *is the analysis operator for* $\{\pi_\Lambda(m,n)g : m, n \in \mathbb{Z}^d\}$.

The Fundamental Identity of Gabor Analysis holds under weaker assumptions, see [16]. These basic properties of Gabor frames establish an intrinsic connection between the Gabor representations π_Λ and π_{Λ^0}, both are projective unitary representations of the abelian group $\mathbb{Z}^d\times\mathbb{Z}^d$. One might wonder if this holds as well for general projective representations of countable groups. Indeed, the density theorem for projective unitary representations has been obtained in [19,20], and the duality principle for general groups was also established in [27] and [12]. Let us summarize these results:

Theorem 1.2 *Let* π *be a frame representation and* (π,σ) *be a dual commutant pair* (*see Definition* 2.1) *of projective unitary representations of* G *on a Hilbert space* H. *Then* $\{\pi(g)\xi\}_{g\in G}$ *is a frame* (*respectively, a tight frame*) *for* H *if and only if* $\{\sigma(g)\xi\}_{g\in G}$ *is a Riesz sequence* (*respectively, an orthogonal sequence*).

One of the central problems concerning the duality principle is the existence problem of dual commutant pairs (π,σ) for a group G and/or for a given representations π. This turns out to be a very challenging problem due to the following result [12]:

Theorem 1.3 *Let* $\pi = \lambda|_P$ *be a subrepresentation of the left regular representation* λ *of an ICC* (*infinite conjugacy class*) *group* G, *where* P *is an orthogonal projection in the commutant* $\lambda(G)'$ *of* $\lambda(G)$. *Then the following are equivalent:*

(i) $\lambda(G)'$ *and* $P\lambda(G)'P$ *are isomorphic von Neumann algebras.*
(ii) *There exists a group representation* σ *such that* (π,σ) *form a dual commutant pair.*

For the free groups $\mathcal{F}_n$ with n-generators ($n\geq 2$), it is a longstanding problem whether all their group von Neumann algebras are $*$-isomorphic. It is well-known [15,37] that either all the von Neumann algebras $P\lambda(\mathcal{F}_n)'P$ ($0\neq P\in\lambda(\mathcal{F}_n)'$) are $*$-isomorphic, or no two of them are $*$-isomorphic. This implies that the classification problem is also equivalent to the question whether there exists a proper projection $P\in\lambda(\mathcal{F}_n)'$ such that $\lambda(\mathcal{F}_n)'$ and $P\lambda(\mathcal{F}_n)'P$ are isomorphic von Neumann algebras.The above Theorem 1.3 shows that the existence problem of dual commutant pairs for free groups is also equivalent to the longstanding classification problem for free group von Neumann algebras.

There are many groups admitting dual commutant pairs. For example, if G is either an abelian group or an amenable ICC group, then for every projection $0\neq P\in\lambda(G)'$, there exists a unitary representation σ of G such that $(\lambda|_P, \sigma)$ is a dual commutant pair, where $\lambda|_P$ is the subrepresentation of the left regular representation λ restricted to $range(P)$. On the other hand, there exists an ICC group (e.g., $G = \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$), such that none of the nontrivial subrepresentations $\lambda|_P$ admits a dual commutant pair (c.f. [7,12,15,34–37]). These examples demonstrate the complexity of the existence problem, which remains widely open in general.

In this paper we first prove that the Wexler–Raz biorthogonality and the Fundamental Identity in Gabor analysis remain to be true for more general projective unitary representations of any countable group G. Secondly we shall establish the duality principle connecting the multi-frame generators and super-frame generators, which is new even in the context of Gabor analysis. In order to state our main results we recall some necessary definitions, notations and terminologies related to frames and frame representations.

1.1 Related results

We would like to mention that there is a more general duality principle in frame theory [5,6,40]. However, it is usually very difficult (if at all possible) to use it to derive duality principles for well-structured frames. In particular we are not able to see how the general frame duality can be applied to obtain the duality principle for groups in the setting of this paper and [12,27].

Extensions of the duality principle for Gabor frames have been obtained to finite abelian groups [17] and to locally compact abelian groups [30] and to the setting of superframes in [23,31]. In [31] Jakobsen and Luef have also established one of our results, Corollary 3.3, using a completely different approach. Finally, [1] generalizes the Gabor duality principle to the setting of equivalence bimodules for Morita equivalent C^*-algebras which contains the one for Gabor frames for Heisenberg modules over twisted group C^*-algebras [31].

Let us stress that the focus of this paper is on establishing a general duality principle for arbitrary (discrete) groups building its connections with the theory of operator algebras and group representations. In this context the duality principle for Gabor frames is just a special case of a more general duality theory for arbitrary projective unitary representations of discrete groups. Our approach via dual commutant pairs does not allow us to deduce the relation between the frame and Riesz bounds like in the Gabor case.

2 Background

Recall (cf. [41]) that a *projective unitary representation* π for a countable group G is a mapping $g \to \pi(g)$ from G into the group $U(H)$ of all the unitary operators on a separable Hilbert space H such that $\pi(g)\pi(h) = \mu(g, h)\pi(gh)$ for all $g, h \in G$, where $\mu(g, h)$ is a scalar-valued function on $G \times G$ taking values in the circle group $\mathbb{T}$. This function $\mu(g, h)$ is then called a *multiplier or* 2-*cocycle* of π. In this case we also say that π is a μ-projective unitary representation. It is clear from the definition that we have

(i) $\mu(g_1, g_2g_3)\mu(g_2, g_3) = \mu(g_1g_2, g_3)\mu(g_1, g_2)$ for all $g_1, g_2, g_3 \in G$,
(ii) $\mu(g, e) = \mu(e, g) = 1$ for all $g \in G$, where e denotes the group unit of G.

Any function $\mu : G \times G \to \mathbb{T}$ satisfying (i)–(ii) will be called a *multiplier* for G. It follows from (i) and (ii) that we also have

(iii) $\mu(g, g^{-1}) = \mu(g^{-1}, g)$ holds for all $g \in G$.

Similar to the group unitary representation case, the left and right regular projective representations with a prescribed multiplier μ for G can be defined by

$$\lambda_g \chi_h = \mu(g, h)\chi_{gh}, \quad h \in G,$$

and

$$\rho_g \chi_h = \mu(h, g^{-1})\chi_{hg^{-1}}, \quad h \in G,$$

where $\{\chi_g : g \in G\}$ is the standard orthonormal basis for $\ell^2(G)$. Clearly, λ_g and ρ_g are unitary operators on $\ell^2(G)$. Moreover, λ is a μ-projective unitary representation of G with multiplier μ and ρ is a projective unitary representation of G with multiplier $\overline{\mu}$. The representations λ and ρ are called the *left regular μ-projective representation* and the *right regular μ-projective representation* of G, respectively.

Given a projective unitary representation π of a countable group G on a Hilbert space H, a vector $\xi \in H$ is called a *complete frame vector (resp. complete tight frame vector, complete Parseval frame vector)* for π if $\{\pi(g)\xi\}_{g\in G}$ (here we view this as a sequence indexed by G) is a frame (resp. tight frame, Parseval frame) for the whole Hilbert space H, and is just called a *frame sequence vector (resp. tight frame sequence vector, Parseval sequence frame vector)* for π if $\{\pi(g)\xi\}_{g\in G}$ is a *frame sequence (resp. tight frame sequence, Parseval frame sequence)*. Riesz sequence vector and Bessel vector can be defined similarly. We will use $\mathcal{B}_\pi$ to denote the set of all Bessel vectors of π. A projective unitary representation that admits a complete frame vector is called a *frame representation*.

For Gabor representations, π_Λ and π_{Λ° are projective unitary representations of the group $\mathbb{Z}^d \times \mathbb{Z}^d$. Moreover, it is well-known that one of the two projective unitary representations π_Λ and π_{Λ^o} for the group $G = \mathbb{Z}^d \times \mathbb{Z}^d$ must be a frame representation and the other admits a Riesz vector. So we can always assume that π_Λ is a frame representation of $\mathbb{Z}^d \times \mathbb{Z}^d$ and hence π_{Λ^o} admits a Riesz vector. Moreover, we also have $\pi_\Lambda(G)' = \pi_{\Lambda^\circ}(G)''$, and both representations share the same Bessel vectors, where $\pi_\Lambda(G)'$ is the commutant of $\pi(G)$. This leads to the following definition:

Definition 2.1 [12] Let π and σ be two projective unitary representations of a countable group G on the same Hilbert space H. We say that (π, σ) is a *commutant pair* if $\pi(G)' = \sigma(G)''$, and a *dual commutant pair* if they satisfy the following two additional conditions:

(i) $\mathcal{B}_\pi = \mathcal{B}_\sigma$.

(ii) One of them admits a complete frame generator and the other one admits a Riesz sequence generator.

Remark 2.1 We point out that it seems that the condition $\mathcal{B}_\pi = \mathcal{B}_\sigma$ in the above definition may not be easy to verify. However, with the help of the parameterization results established in [13,25], it may not as difficult as it looks like to verify this condition. For example, assume that π and σ have finite cyclic multiplicity (the cyclic multiplicity of π is the smallest cardinality k such that there exist vectors $\xi_1, \ldots, \xi_k$ such that $\overline{span}\{\pi(g)\xi_i : g \in G, i = 1, \ldots, k\} = \mathrm{H}$). Then, by Theorem 2.10 in [13], the condition $\mathcal{B}_\pi = \mathcal{B}_\sigma$ can be verified by checking only finitely many Bessel vectors. In the case that π is a frame representation and ξ is a fixed frame vector, then from the parameterization theorem in [25] we have that $\mathcal{B}_\pi = \{T\xi : T \in \pi(G)''\}$. So if we can verify that $\xi \in \mathcal{B}_\sigma$, then we already have the inclusion:

$$\mathcal{B}_\pi = \{T\xi : T \in \pi(G)''\} = \{T\xi : T \in \sigma(G)'\} \subseteq \mathcal{B}_\sigma.$$

We conjecture that the conditions $\pi(G)' = \sigma(G)''$ and $\mathcal{B}_\pi \subseteq \mathcal{B}_\sigma$ automatically imply that $\mathcal{B}_\pi = \mathcal{B}_\sigma$.

For any projective representation π of a countable group G on a Hilbert space H and $x \in H$, the *analysis operator* $\Theta_{x,\pi}$ (or Θ_x if π is well-understood from the context) for x from $\mathcal{D}(\Theta_x)(\subseteq H)$ to $\ell^2(G)$ is defined by

$$\Theta_x(y) = \sum_{g\in G} \langle y, \pi(g)x\rangle \chi_g,$$

where $\mathcal{D}(\Theta_x) = \{y \in H : \sum_{g\in G} |\langle y, \pi(g)x\rangle|^2 < \infty\}$ is the domain space of Θ_x. Clearly, $\mathcal{B}_\pi \subseteq \mathcal{D}(\Theta_x)$ holds for every $x \in H$. In the case that $\mathcal{B}_\pi$ is dense in H, we have that Θ_x is a densely defined and closable linear operator from $\mathcal{B}_\pi$ to $\ell^2(G)$ (cf. [18]). Moreover, $x \in \mathcal{B}_\pi$ if and only if Θ_x is a bounded linear operator on H, which in turn is equivalent to the condition that $\mathcal{D}(\Theta_x) = H$. It is useful to note that $\Theta_\eta^*\Theta_\xi$ commutes with $\pi(G)$ if $\xi, \eta \in \mathcal{B}_\pi$. Thus, if ξ is a complete frame vector for π, then $\eta := S^{-1/2}\xi$ is a complete Parseval frame vector for π, where $S = \Theta_\xi^*\Theta_\xi$ and is called the *frame operator* for ξ (or *Bessel operator* if ξ is a Bessel vector).

It was proved in [19] that a complete Parseval frame vector η for π, $Tr_{\pi(G)'}(A) = \langle A\eta, \eta\rangle$ defines a faithful normal trace on $\pi(G)'$. In the case of the Gabor representation π_Λ we have that $Tr_{\pi(G)'}(I) = vol(\Lambda)$. Thus Theorem 3.1 may be viewed as generalizations of the Wexler–Raz biorthogonality and the Fundamental Identity of Gabor analysis for general frame representations.

3 Main results

We are now in the position to formulate the main theorems:

Theorem 3.1 *Let π be a frame representation and (π, σ) be a dual commutant pair of projective unitary representations of G on H.*

(i) *If* $\{\pi(g)\xi\}$ *is a frame for* H*, then*

$$\langle \sigma(g)\xi, S^{-1}\xi\rangle = Tr_{\pi(G)'}(I)\delta_{g,e},$$

where S *is the frame operator for* $\{\pi(g)\xi\}$*,* e *is the group unit of* G *and* $Tr_{\pi(G)'}(I) = ||S^{-1/2}\xi||^2$.

(ii) *If* ξ, η, x, y *are Bessel vectors for* π*, then*

$$\sum_{g\in G}\langle x, \pi(g)\xi\rangle\langle \pi(g)\eta, y\rangle = \frac{1}{Tr_{\pi(G)'}(I)}\sum_{g\in G}\sum_{g\in G}\langle x, \sigma(g)(y)\rangle\langle \sigma(g)\eta, \xi\rangle.$$

i.e. $\langle \Theta_{\xi,\pi}(x), \Theta_{\eta,\pi}(y)\rangle = \frac{1}{Tr_{\pi(G)'}(I)}\langle \Theta_{y,\sigma}(x), \Theta_{\eta,\sigma}(\xi)\rangle$.

Our second main theorem deals with the duality principle for multi-frame and super-frame generators.

Definition 3.1 Let π be projective unitary representation of a countable group G on a Hilbert space H and let $\xi_1, \ldots, \xi_n \in H$. We say that $\vec{\xi} = (\xi_1, \ldots, \xi_n)$ is

(i) a multi-frame vector for π if $\{\pi(g)\xi_i : g \in G, i = 1, \ldots, n\}$ is a frame for H, and

(ii) a super-frame vector if each $\{\pi(g)\xi_i : g \in G\}$ is a frame for H and $\Theta_{\xi_i}(H) \perp \Theta_{\xi_j}(H)$ for $i \neq j$.

Parseval multi-frame vector and Parseval super-frame vector can be defined similarly. We remark that the concept of super-frames was first introduced and systematically studied by Balan [2,3], Han and Larson [25] in the 1990's, and since then it has received some attention (c.f. [9–11,13,18,20–22,24] and the references therein).

Theorem 3.2 *Let* π *be a frame representation and* (π, σ) *be a dual commutant pair of projective unitary representations of G on H, and* $\vec{\xi} = (\xi_1, \ldots, \xi_n) \in H$ *. Then we have*

(i) $\vec{\xi}$ *is a super-frame vector for* π *if and only if* $\{\sigma(g)\xi_j : g \in G, j = 1, \ldots, n\}$ *is Riesz sequence in* H.

(ii) $\vec{\xi}$ *is a multi-frame vector for* π *if and only if* $\{\sigma(g)\xi_1 \oplus \cdots \oplus \sigma(g)\xi_n : g \in G\}$ *is a Riesz sequence in* $H \oplus \cdots \oplus H$.

Since the Gabor representations π_Λ and π_{Λ^o} form a dual commutant pair, we immediately have the following consequences:

Corollary 3.3 *Let* Λ *be a time–frequency lattice and* Λ^o *be its dual lattice. Let* $g_1, \ldots, g_k \in L^2(\mathbb{R}^d)$*. Then*

(i) $\{\pi_\Lambda(m,n)g_1 \oplus \cdots \oplus \pi_\Lambda(m,n)g_k\}_{m,n\in\mathbb{Z}^d}$ *is a frame for* $L^2(\mathbb{R}^d) \oplus \cdots \oplus L^2(\mathbb{R}^d)$ *if and only if* $\cup_{i=1}^k\{\pi_{\Lambda^o}(m,n)g_i\}_{m,n\in\mathbb{Z}^d}$ *is a Riesz sequence in* $L^2(\mathbb{R}^d)$.

(ii) $\cup_{i=1}^k\{\pi_\Lambda(m,n)g_i\}_{m,n\in\mathbb{Z}^d}$ *is a frame for* $L^2(\mathbb{R}^d)$ *if and only if* $\{\pi_{\Lambda^o}(m,n)g_1 \oplus \cdots \oplus \pi_{\Lambda^o}(m,n)g_k\}_{m,n\in\mathbb{Z}^d}$ *is a Riesz sequence* $L^2(\mathbb{R}^d) \oplus \cdots \oplus L^2(\mathbb{R}^d)$.

4 Proof of Theorem 3.1

We refer to [14,32,33] for standard notions and basic properties about von Neumann algebras. Note that $[K]$ denotes the closed subspace generated by a subset K of a Hilbert space H. Theorem 1.2 and the following lemmas are needed in the proofs for both Theorems 3.1 and 3.2.

Lemma 4.1 [19] *Let π be a projective representation of a countable group G on a Hilbert space H such that $\mathcal{B}_\pi$ is dense in H. Then*

$$\pi(G)' = \overline{span}^{WOT}\{\Theta_\eta^*\Theta_\xi : \xi, \eta \in \mathcal{B}_\pi\},$$

where "WOT" denotes the closure in the weak operator topology.

Lemma 4.2 [19] *Let π be a projective representation of a countable group G on a Hilbert space H such that $\mathcal{B}_\pi$ is dense in H. If $\{\pi(g)\xi_i, g \in G, i = 1, \ldots, n\}$ is a Parseval frame for H, then*

$$Tr_{\pi(G)'}(A) = \sum_{i=1}^{n}\langle A\xi_i, \xi_i\rangle$$

defines a faithful trace on $\pi(G)'$, i.e. $Tr_{\pi(G)'}(A^*A) = 0$ implies $A = 0$. Moreover, this is independent of the choice of the Parseval multi-frame vector $\vec{\xi} = (\xi_1, \ldots, \xi_n)$.

Lemma 4.3 *Let π be a projective unitary representation π of a countable group G on a Hilbert space H. Then π is a frame representation if and only if π is unitarily equivalent to a subrepresentation of the left regular projective unitary representation of G. Consequently, if π is a frame representation, then both $\pi(G)'$ and $\pi(G)''$ are finite von Neumann algebras.*

Lemma 4.4 [19,25] *Let π be a projective representation of a countable group G on a Hilbert space H and $\{\pi(g)\xi\}_{g\in G}$ is a Parseval frame for H. Then*

- (i) *$\{\pi(g)\eta\}_{g\in G}$ is a Parseval frame for H if and only if there is a unitary operator $U \in \pi(G)''$ such that $\eta = U\xi$;*
- (ii) *$\{\pi(g)\eta\}_{g\in G}$ is a frame for H if and only if there is an invertible operator $U \in \pi(G)''$ such that $\eta = U\xi$;*
- (iii) *$\{\pi(g)\eta\}_{g\in G}$ is a Bessel sequence if and only if there is an operator $U \in \pi(G)''$ such that $\eta = U\xi$, i.e., $\mathcal{B}_\pi = \pi(G)''\xi$.*

Proof of Theorem 3.1 Let (π, σ) be a dual commutant pair of representations for G on a Hilbert space H.

- (i) Let $\{\pi(g)\xi\}$ be a frame for H and let S be its frame operator. We set $\eta = S^{-1/2}\xi$. Then $\{\pi(g)\eta\}_{g\in G}$ is a Parseval frame for H. By Lemma 4.2 we have that $Tr_{\pi(G)'}(A) := \langle A\eta, \eta\rangle$ defines a faithful trace on $w^*(\sigma(G))$, where $w^*(\sigma(G))$

is the von Neumann algebra generated by $\sigma(G)$ and it is equal to $\pi(G)'$. Note that $S, \sigma(g) \in \pi(G)'$. Thus we have

$$\begin{aligned}\langle \sigma(g)\eta, \eta\rangle &= Tr_{\pi(G)'}(\sigma(g)) = Tr_{\pi(G)'}(S^{-1/2}\sigma(g)S^{1/2})\\ &= \langle S^{-1/2}\sigma(g)S^{1/2}\eta, \eta\rangle = \langle \sigma(g)\xi, S^{-1}\xi\rangle.\end{aligned}$$

However, by Theorem 1.2, $\{\sigma(g)\eta\}_{g\in G}$ is an orthogonal sequence. Thus we have $\langle \sigma(g)\xi, S^{-1}\xi\rangle = 0$ for any $g \neq e$. Observe that $\langle \sigma(e)\xi, S^{-1}\xi\rangle = ||S^{-1/2}\xi||^2 = Tr_{\pi(G)'}(I)$. So we get the biorthogonality relation:

$$\langle \sigma(\xi), S^{-1}\xi\rangle = Tr_{\pi(G)'}(I)\delta_{g,e}.$$

(ii) Let φ be a Parserval frame vector for π. Then by Theorem 1.2 we get that $\{\frac{1}{\sqrt{Tr_{\pi(G)'}(I)}}\sigma(g)\varphi\}_{g\in G}$ is an orthonormal basis for $[\pi(G)'\varphi]$. Since $\Theta^*_{\xi,\pi}\Theta_{\eta,\pi} \in \pi(G)' = w^*(\sigma(G))$, we get that $\Theta^*_{\xi,\pi}\Theta_{\eta,\pi}\varphi \in [\sigma(G)\varphi]$. This implies that

$$\Theta^*_{\xi,\pi}\Theta_{\eta,\pi}\varphi = (Tr_{\pi(G)'}(I))^{-1}\sum_{g\in G} c_g\sigma(g)\varphi,$$

where $c_g = \langle \Theta^*_{\xi,\pi}\Theta_{\eta,\pi}\varphi, \sigma(g)\varphi\rangle$.

By Lemma 4.4 there is an operator $A \in w^*(\pi(G))$ such that $y = A\varphi$. Thus we have

$$\begin{aligned}\Theta^*_{\xi,\pi}\Theta_{\eta,\pi}(y) &= \Theta^*_{\xi,\pi}\Theta_{\eta,\pi}(A\varphi) = A\Theta^*_{\xi,\pi}\Theta_{\eta,\pi}(\varphi)\\ &= (Tr_{\pi(G)'}(I))^{-1}\sum_{g\in G} c_g A\sigma(g)\varphi\\ &= (Tr_{\pi(G)'}(I))^{-1}\sum_{g\in G} c_g\sigma(g)A\varphi\\ &= (Tr_{\pi(G)'}(I))^{-1}\sum_{g\in G} c_g\sigma(g)y\end{aligned}$$

Therefore we get

$$\langle \Theta_{\xi,\pi}(x), \Theta_{\eta,\pi}(y)\rangle = \langle x,\ \Theta^*_{\xi,\pi}\Theta_{\eta,\pi}(y)\rangle = (Tr_{\pi(G)'}(I))^{-1}\sum_{g\in G}\overline{c}_g\langle x, \sigma(g)y\rangle.$$

Now we compute c_g:

$$\begin{aligned}c_g &= \langle \Theta^*_{\xi,\pi}\Theta_{\eta,\pi}\varphi, \sigma(g)\varphi\rangle = \langle \Theta_{\eta,\pi}(\varphi), \Theta_{\xi,\pi}(\sigma(g)\varphi)\rangle\\ &= \sum_{h\in G}\langle \varphi, \pi(h)\eta\rangle \cdot \overline{\langle \sigma(g)\varphi, \pi(h)\xi\rangle}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{h\in G} \langle \varphi, \pi(h)\eta\rangle \cdot \langle \pi(h)\xi, \sigma(g)\varphi\rangle \\
&= \sum_{h\in G} \langle \pi(h^{-1})\varphi, \eta\rangle \cdot \langle \sigma(g^{-1})\xi, \pi(h^{-1})\varphi\rangle \\
&= \sum_{h\in G} \langle \sigma(g^{-1})\xi, \pi(h^{-1})\varphi\rangle \cdot \langle \pi(h^{-1})\varphi, \eta\rangle \\
&= \sum_{h\in G} \langle \sigma(g^{-1})\xi, \pi(h)\varphi\rangle \cdot \langle \pi(h)\varphi, \eta\rangle \\
&= \langle \sigma(g^{-1})\xi, \eta\rangle = \langle \xi, \sigma(g)\eta\rangle,
\end{aligned}
$$

where we used the fact that $\sigma(g)$ and $\pi(h)$ commute for all $g, h \in G$, and that $\{\pi(h)\varphi\}_{h\in G}$ is a Parserval frame H.

Finally, we have

$$
\begin{aligned}
\langle \Theta_{\xi,\pi}(x), \Theta_{\eta,\pi}(y)\rangle &= \langle x,\ \Theta^*_{\xi,\pi}\Theta_{\eta,\pi}(y)\rangle \\
&= (Tr_{\pi(G)'}(I))^{-1} \sum_{g\in G} \overline{c}_g \langle x, \sigma(g)y\rangle \\
&= (Tr_{\pi(G)'}(I))^{-1} \sum_{g\in G} \langle \sigma(g)\eta, \xi\rangle, \langle x, \sigma(g)y\rangle \\
&= (Tr_{\pi(G)'}(I))^{-1} \langle \Theta_{y,\sigma}(x), \Theta_{\eta,\sigma}(\xi)\rangle.
\end{aligned}
$$

This completes the proof. □

5 Proof of Theorem 3.2

The proof of Theorem 3.2 is much more subtle and involved. While Theorem 1.2 will be needed, it is not a direct consequence of the theorem. For the sake of clarity we divide the proof into two theorems with one of them concerning the duality for multi-frame generators and the other one dealing with the duality for super-frame generators. We need a series of lemmas for both cases. In what follows we use $H^{(k)}$ to denote the orthogonal direct sum of a Hilbert space H and $\pi^{(k)}$ to denote the k-fold direct sum of the representation π of G on $H^{(k)}$. So for any vector $\vec{\xi} = (\xi_1, \ldots, \xi_k) \in H^{(k)}$, we have $\pi^{(k)}(g)\vec{\xi} = (\pi(g)\xi_1, \ldots, \pi(g)\xi_k) = \pi(g)\xi_1 \oplus \cdots \oplus \pi(g)\xi_k$. We will use the following notations: Let π be a projective unitary representation of G on a Hilbert space H.

(i) For any $\xi \in H$, $\Theta_{\xi,\pi} : H \to \ell^2(G)$ is the analysis operator for the sequence $\{\pi(g)\xi\}_G$.

(ii) For $\xi = (\xi_1, \ldots, \xi_k) \in H^{(k)}$, $\Theta_{\vec{\xi},\pi} : H \to (\ell^2(G))^{(k)}$ is the analysis operator for the sequence $\cup_{i=1}^k \{\pi(g)\xi_i\}_{g\in G}$ defined by

$$
\Theta_{\vec{\xi},\pi}(x) = \Theta_{\xi_1,\pi}(x) \oplus \cdots \oplus \Theta_{\xi_k,\pi}(x).
$$

(iii) For $\xi = (\xi_1, \ldots, \xi_k) \in H^{(k)}$, $\Theta_{\vec{\xi},\pi^{(k)}} : H^{(k)} \to (\ell^2(G))^{(k)}$ is the analysis operator for the sequence $\{\pi^{(k)}(g)\vec{\xi}\}_{g\in G}$.

Clearly $\Theta_{\vec{\xi},\pi}$ can be viewed as the restriction of $\Theta_{\vec{\xi},\pi^{(k)}}$ to the subspace $\{x\oplus\cdots\oplus x : x \in H\}$ of $H^{(k)}$.

Lemma 5.1 *Let π be a projective unitary representation of a countable group G on a Hilbert space H such that $\pi(G)'$ is finite. Assume that $\cup_{i=1}^{k}\{\pi(g)\xi_i\}_{g\in G}$ is a frame for H. If $A \in \pi(G)'$ such that $\xi_i = A\eta_i$ and each η_i is a Bessel vector for π, then A is invertible and $\cup_{i=1}^{k}\{\pi(g)\eta_i\}_{g\in G}$ is also a frame for H.*

Proof Let D and C be the frame bounds for $\cup_{i=1}^{k}\{\pi(g)\xi_i\}_{g\in G}$ and $\cup_{i=1}^{k}\{\pi(g)\eta_i\}_{g\in G}$, respectively. Then for every $x \in H$ we

$$\begin{aligned} D||x||^2 &\le \sum_{i=1}^{k}\sum_{g\in G}|\langle x, \pi(g)\xi_i\rangle|^2 \\ &= \sum_{i=1}^{k}\sum_{g\in G}|\langle x, \pi(g)A\eta_i\rangle|^2 \\ &= \sum_{i=1}^{k}\sum_{g\in G}|\langle A^*x, \pi(g)\eta_i\rangle|^2 \\ &\le C||A^*x||^2, \end{aligned}$$

This implies that A^* is bounded from below. Since $A^* \in \pi(G)'$ and $\pi(G)'$ is a finite von Neumann algebra, it follows that A^* must be invertible. Hence A is invertible. □

Lemma 5.2 [18,19] *Let π be a projective unitary representations of a countable group G on a Hilbert space H. If $x \in \mathcal{B}_\pi$, then there exists a vector $\xi \in M := \overline{span}\{\pi(g)x : g \in G\}$ such that $\{\pi(g)\xi\}_{g\in G}$ is a Parseval frame for M. Moreover, $\Theta_\xi(H) = \overline{\Theta_x(H)}$.*

Lemma 5.3 *Assume that (π, σ) is a commutant pair of projective representations of a countable group G on a Hilbert space H and $\pi(G)'$ is finite. If $\cup_{i=1}^{k}\{\pi(g)\xi_i\}_{g\in G}$ is a frame for H, then*

$$\{\sigma(g)\xi_1 \oplus \cdots \oplus \sigma(g)\xi_k\}_{g\in G}$$

is frame sequence in $H^{(k)}$.

Proof Let

$$M = \overline{span}\{\sigma(g)\xi_1 \oplus \cdots \oplus \sigma(g)\xi_k\}_{g\in G}.$$

Then M is $\sigma^{(k)}$-invariant. Note that

$$w^*(\sigma^{(k)}(G)) = \{A \oplus \cdots \oplus A : A \in w^*(\sigma(G))\}.$$

So we have that

$$w^*(\sigma^{(k)}(G)|_M) = \{A^{(k)}|_M : A \in w^*(\sigma(G))\}.$$

Since $\vec{\xi} = (\xi_1, \ldots, \xi_k)$ is a Bessel vector for $\sigma^{(k)}$, by Lemma 5.2 we get that there exists a vector $\vec{\eta} = (\eta_1, \ldots, \eta_k) \in M$ such that

$$\{\sigma^{(k)}(g)\vec{\eta}\}_{g\in G}$$

is a Parseval frame for M. Now by Lemma 4.4 there exists an operator T in $w^*(\sigma^{(k)}(G)|_M)$ such that $T\vec{\eta} = \vec{\xi}$. Write $T = A^{(k)}|_M$ for some $A \in w^*(\sigma(G))$. Then we get that $A\eta_i = \xi_i$ for $i = 1, \ldots, k$ and $A \in \pi(G)'$. Thus, by Lemma 5.1, we have that A is invertible, which implies that T is invertible. Hence, by Lemma 4.4 again, $\{\sigma^{(k)}(g)\vec{\xi}\}_{g\in G}$ is a frame for M, which completes the proof. □

We also need the following generalization of Lemma 5.2. Although it is not a consequence of Lemma 5.2, the proof is very similar and we include a sketch of the proof for the reader's convenience.

Lemma 5.4 *Assume that π is a projective unitary representation of a countable group G on a Hilbert space H. Suppose that $\cup_{i=1}^k \{\pi(g)\xi_i\}_{g\in G}$ is a Bessel sequence and let*

$$M = \overline{span} \cup_{i=1}^k \{\pi(g)\xi_i\}_{g\in G}.$$

Then there exists a vector $\vec{\eta}$ such that

(i) $\cup_{i=1}^k \{\pi(g)\eta_i\}_{g\in G}$ *is a Parseval frame for M, and*
(ii) $\Theta_{\vec{\eta},\pi}(H) = [\Theta_{\vec{\xi},\pi}(H)]$.

Proof It is sufficient to consider the case when $M = H$. Write $T = \Theta_{\vec{\xi},\pi}$ and let $T = U|T|$ be its polar decomposition. Then U is an isometry from H into $\ell^2(G)^{(k)}$ since the range of T^* is dense in H. It can be easily verified that $T\pi(g) = \lambda^{(k)}(g)T$ for every $g \in G$, where λ is the left regular representation for G with the same multiplier as π. This implies that $U\pi(g) = \lambda^{(k)}(g)U$ for all $g \in G$. Let $\psi_i = 0\oplus\cdots\oplus 0\oplus\chi_e\oplus 0\cdots\oplus 0$, where χ_e appears in the i-th component, and let $\eta_i = U^*\psi_i$. Then we have

$$U\pi(g)\eta_i = U\pi(g)U^*\psi_i = UU^*\lambda^{(k)}(g)\psi = P\lambda^{(k)}(g)\psi_i,$$

where P is the orthogonal projection from $\ell^2(G)^{(k)}$ onto $[\Theta_{\vec{\xi},\pi}(H)]$. Since

$$\{\lambda^{(k)}(g)\psi_i : g \in G, i = 1, \ldots, k\}$$

is an orthonormal basis for $\ell^2(G)^{(k)}$, we get that $\{U\pi(g)\eta_i : g \in G, i = 1, \ldots, k\}$ is a Parserval frame for $[\Theta_{\vec{\xi},\pi}(H)]$ and the range space of its analysis operator is $[\Theta_{\vec{\xi},\pi}(H)]$. Since U is an isometry, we obtain that $\cup_{i=1}^k \{\pi(g)\eta_i\}_{g\in G}$ is a Parseval frame for H and $\Theta_{\vec{\eta},\pi}(H) = [\Theta_{\vec{\xi},\pi}(H)]$. □

Let π be a projective unitary representation of G on a Hilbert space H such that $\mathcal{B}_\pi$ is dense in H. Recall from [12] that two vectors ξ and η in H are called *π-orthogonal* if $range(\Theta_\xi) \perp range(\Theta_\eta)$, and *$\pi$-weakly equivalent* if $[range(\Theta_\xi)] = [range(\Theta_\eta]$.

The following result obtained in [27] characterizes the π-orthogonality and π-weakly equivalence in terms of the commutant of $\pi(G)$.

Lemma 5.5 *Let π be a projective representation of a countable group G on a Hilbert space H such that $\mathcal{B}_\pi$ is dense in H. Then two vectors $\xi, \eta \in H$ are*

(i) *π-orthogonal if and only if $[\pi(G)'\xi] \perp [\pi(G)'\eta]$, and*
(ii) *π-weakly equivalent if and only if $[\pi(G)'\xi] = [\pi(G)'\eta]$.*

We need the following (partial) generalization of Lemma 5.5 (ii).

Lemma 5.6 *Let (π, σ) be a commutant pair of projective unitary representations of a countable group G on a Hilbert space H such that $\mathcal{B}_\pi$ is dense in H. Let $\xi_i, \eta_i \in H$ $(i = 1, \dots, k)$ be Bessel vectors for π. If $[\Theta_{\vec{\xi},\pi}(H)] = [\Theta_{\vec{\eta},\pi}(H)]$, then $[\sigma^{(k)}(G)\vec{\xi}\,] = [\sigma^{(k)}(G)\vec{\eta}\,]$.*

Proof By Lemma 4.1, we know that $w^*(\sigma(G)) = \pi(G)'$ is the closure of the linear span of

$$\{\Theta^*_{u,\pi}\Theta_{v,\pi} : u, v \in \mathcal{B}_\pi\}$$

in the weak operator topology. Hence $w^*(\sigma^{(k)}(G))$ is the (wot)-closure of the linear span of

$$\{\Theta^*_{u,\pi}\Theta_{v,\pi} \oplus \cdots \oplus \Theta^*_{u,\pi}\Theta_{v,\pi} : u, v \in \mathcal{B}_\pi\}.$$

Assume that $\vec{z} = (z_1, \dots, z_k) \in [\sigma^{(k)}(G)\vec{\xi}\,]^\perp$. Then for any $u, v \in \mathcal{B}_\pi$ we have

$$\begin{aligned} 0 &= \sum_{i=1}^k \langle z_i, \Theta^*_{u,\pi}\Theta_{v,\pi}(\xi_i)\rangle = \sum_{i=1}^k \langle \Theta_{u,\pi}(z_i), \Theta_{v,\pi}(\xi_i)\rangle \\ &= \sum_{i=1}^k \langle \Theta_{\xi_i,\pi}(v), \Theta_{z_i,\pi}(u)\rangle = \langle \Theta_{\vec{\xi},\pi}(v), \Theta_{\vec{z},\pi}(u)\rangle. \end{aligned}$$

This implies $\Theta_{\vec{z},\pi}(u) \perp \Theta_{\vec{\xi},\pi}(v)$. Since $v \in \mathcal{B}_\pi$ is arbitrary and $\mathcal{B}_\pi$ is dense in H, we get that $\Theta_{\vec{z},\pi}(u) \perp [\Theta_{\vec{\xi},\pi}(H)]$, which implies that $\Theta_{\vec{z},\pi}(u) \perp [\Theta_{\vec{\eta},\pi}(H)]$. Therefore we obtain that

$$\sum_{i=1}^k \langle z_i, \Theta^*_{u,\pi}\Theta_{v,\pi}(\eta_i)\rangle = \langle \Theta_{\vec{\eta},\pi}(v), \Theta_{\vec{z},\pi}(u)\rangle = 0.$$

This implies that $\vec{z} \in [\sigma^{(k)}(G)\vec{\eta}\,]^\perp$. Hence $[\sigma^{(k)}(G)\vec{\xi}\,] \subseteq [\sigma^{(k)}(G)\vec{\eta}\,]$. Similarly, we also have the reversed inclusion. Therefore we obtain $[\sigma^{(k)}(G)\vec{\xi}\,] = [\sigma^{(k)}(G)\vec{\eta}\,]$, as claimed. ◻

Lemma 5.7 *Let (π, σ) be a commuting pair of projective unitary representations of a countable group G on a Hilbert space H. If $\{\sigma^{(k)}(g)\vec{\xi}\}_{g\in G}$ is a Riesz sequence, then*

$$\overline{span} \cup_{i=1}^{k} \{\pi(g)\xi_i\}_{g\in G} = H.$$

Proof Assume that $x \perp \pi(g)\xi_i$ for all $g \in G$ and $i = 1, \ldots, k$. We need to show that $x = 0$. Since $w^*(\pi(G)) = \sigma(G)'$, we get that $[\sigma(G)'x] \perp [\sigma(G)'\xi_i]$. Applying Lemma 5.5 (i) to σ, we get that x and ξ_i are σ-orthogonal. This implies that $range(\Theta_{x,\sigma}) \perp range(\Theta_{\xi_i,\sigma})$ for every i. Let $\vec{x} = (x, 0, \ldots, 0) \in H^{(k)}$. Then we have $range(\Theta_{\vec{x},\sigma^{(k)}}) \perp range(\Theta_{\vec{\xi},\sigma^{(k)}})$. Since $\{\sigma^{(k)}(g)\vec{\xi}\}_{g\in G}$ is a Riesz sequence, we know that $range(\Theta_{\vec{\xi},\sigma^{(k)}}) = \ell^2(G)$. Thus $range(\Theta_{\vec{x},\sigma^{(k)}}) = \{0\}$, which implies $\vec{x} = 0$ and hence $x = 0$, as claimed. □

Lemma 5.8 *Let (π, σ) be a commutant pair of projective unitary representations of a countable group G on a Hilbert space H. If $\{\sigma^{(k)}(g)\vec{\xi}\}_{g\in G}$ is a Riesz sequence, then $\cup_{i=1}^{k}\{\pi(g)\xi_i\}_{g\in G}$ is a frame for H.*

Proof From Lemma 5.7 we know that $\overline{span} \cup_{i=1}^{k} \{\pi(g)\xi_i\}_{g\in G} = H$. By using Lemma 5.4, there exists a vector $\vec{\eta}$ such that $\cup_{i=1}^{k}\{\pi(g)\eta_i\}_{g\in G}$ is a Parseval frame for H, and $\Theta_{\vec{\eta},\pi}(H) = [\Theta_{\vec{\xi},\pi}(H)]$. By Lemma 5.6 we get that $[\sigma^{(k)}(G)\vec{\eta}\,] = [\sigma^{(k)}(G)\vec{\xi}\,]$. Let $M = [\sigma^{(k)}(G)\vec{\xi}\,]$. Since $\vec{\xi}$ is a frame vector and $\vec{\eta}$ is a Bessel vector for $\sigma^{(k)}|_M$, we have by Lemma 4.4 that there is an operator T in $w^*(\sigma^{(k)}(G)|_M)$ such that $\vec{\eta} = T\vec{\xi}$. Write $T = (A \oplus \cdots \oplus A)|_M$ with $A \in w^*(\sigma(G)) = \pi(G)'$. Then we have $A\xi_i = \eta_i$ for $i = 1, \ldots, k$. From Lemma 5.1 we get that A is invertible and $\cup_{i=1}^{k}\{\pi(g)\xi_i\}_{g\in G}$ is a frame for H □

Lemma 5.9 *Let π be a projective unitary representations of a countable group G on a Hilbert space H such that $\mathcal{B}_\pi$ is dense in H and it admits a Riesz sequence vector. Suppose that $\xi \in H$ such that $range(\Theta_{\xi,\pi})$ is not dense in $\ell^2(G)$. Then there exits a nonzero vector $x \in H$ such that $range(\Theta_{x,\pi})$ and $range(\Theta_{\xi,\pi})$ are orthogonal.*

Proof Let $\psi \in H$ be such that $\{\pi(g)\psi\}_{g\in G}$ is a Riesz sequence and $\Theta_{\psi,\pi} = V|\Theta_{\psi,\pi}|$ be the polar decomposition of its analysis operator. Since $range(\Theta_{\psi,\pi}) = \ell^2(G)$, we have that V is a co-isometry. It can be verified that $V\pi(g) = \lambda(g)V$ and hence we get $\pi(g)V^* = V^*\lambda(g)$ for every $g \in G$, where λ is the left regular projective unitary representation associated with the same multiplier as π. Now let P be the orthogonal projection onto $[range(\Theta_{\xi,\pi})]$ and $x = V^*P^\perp\chi_e$, where $P^\perp = I - P$. Then P commutes with λ and so $x \neq 0$. Now for any $y \in H$ we get

$$\begin{aligned}\Theta_{x,\pi}(y) &= \sum_{g\in G}\langle y, \pi(g)x\rangle\chi_g \\ &= \sum_{g\in G}\langle y, \pi(g)V^*P^\perp\chi_e\rangle\chi_g \\ &= \sum_{g\in G}\langle y, V^*\lambda(g)P^\perp\chi_e\rangle\chi_g\end{aligned}$$

$$= \sum_{g \in G} \langle y, V^* P^{\perp} \lambda(g) \chi_e \rangle \chi_g$$
$$= \sum_{g \in G} \langle P^{\perp} V y, \chi_g \rangle \chi_g$$
$$= P^{\perp} V y.$$

Thus $range(\Theta_{x,\pi})$ is contained in the range space of $P^{\perp}$, and therefore we get that $range(\Theta_{x,\pi})$ and $range(\Theta_{\xi,\pi})$ are orthogonal. □

Now we are in the position to prove Theorem 3.2, which follows from the next two theorems.

Theorem 5.10 *Let π be a frame representation and (π, σ) be a dual commutant pair of projective unitary representations of G on H and $\vec{\xi} = (\xi_1, \ldots, \xi_k) \in H^{(k)}$. Then $\cup_{i=1}^{k} \{\pi(g)\xi_i\}_{g \in G}$ is a frame for H if and only if $\{\sigma(g)\xi_1 \oplus \cdots \oplus \sigma(g)\xi_k\}_{g \in G}$ is a Riesz sequence.*

Proof The sufficient part has been established in Lemma 5.8. To prove the necessary part, let us assume that $\cup_{i=1}^{k} \{\pi(g)\xi_i\}_{g \in G}$ is a frame for H. Then, by Lemma 5.3, we have that $\{\sigma(g)\xi_1 \oplus \cdots \oplus \sigma(g)\xi_k\}_{g \in G}$ is a frame sequence in $H^{(k)}$. Thus in order to show that it is a Riesz sequence, it suffices to show that the range space (which is already closed) of its analysis operator $\Theta_{\vec{\xi}, \sigma^{(k)}}$ is the entire space $\ell^2(G)$.

Assume to the contrary that $\Theta_{\vec{\xi}, \sigma^{(k)}}(H^{(k)}) \neq \ell^2(G)$. By the assumption on dual commutant pairs we know that $\sigma^{(k)}$ admits a Riesz sequence vector, and that the set of its Bessel vectors is dense in $H^{(k)}$, we obtain by Lemma 5.9 that there exists a nonzero vector $\vec{x} \in H^{(k)}$ such that $range(\Theta_{\vec{x}, \sigma^{(k)}}) \perp range(\Theta_{\vec{\xi}, \sigma^{(k)}})$.

Let $H_i = 0 \oplus \cdots \oplus 0 \oplus H \oplus 0 \cdots \oplus 0$, where H appears in the i-th component. Then we get in particular that $\Theta_{\vec{x}, \sigma^{(k)}}(H_i) \perp \Theta_{\vec{\xi}, \sigma^{(k)}}(H_j)$. Note that $\Theta_{\vec{x}, \sigma^{(k)}}(H_i) = \Theta_{x_i, \sigma}(H)$ and $\Theta_{\vec{\xi}, \sigma^{(k)}}(H_j) = \Theta_{\xi_j, \sigma}(H)$. So we have that $\Theta_{x_i, \sigma}(H) \perp \Theta_{\xi_j, \sigma}(H)$ for all $i, j = 1, \ldots, k$. Thus x_i and ξ_j are σ-orthogonal for all $i, j = 1, \ldots, k$. By Lemma 5.5 we get that $[\sigma(G)' x_i] \perp [\sigma(G)' \xi_j]$ for all i, j. Since $\sigma(G)' = w^*(\pi(G))$ we get that for each i, $x_i \perp [\pi(G)\xi_j]$ for $j = 1, \ldots k$. Hence $x_i = 0$ for each i and so $\vec{x} = 0$, which is a contradiction. Therefore we have that $\Theta_{\vec{\xi}, \sigma^{(k)}}(H^{(k)}) = \ell^2(G)$, and hence $\{\sigma^{(k)}(g)\vec{\xi}\}_{g \in G}$ is a Riesz sequence, as claimed. □

Theorem 5.11 *Let π be a frame representation and (π, σ) be a dual commutant pair of projective unitary representations of G on H. Let $\vec{\xi} = (\xi_1, \cdots, \xi_k)$. Then we have*

- (i) *$\vec{\xi}$ is a Parserval super-frame vector for π if and only if $\{\sigma(g)\xi_j : g \in G, j = 1, \ldots, k\}$ is an orthogonal sequence in H and $||\xi_i||^2 = Tr_{\pi(G)'}(I)$.*
- (ii) *$\vec{\xi}$ is a super-frame vector for π if and only if $\{\sigma(g)\xi_j : g \in G, j = 1, \ldots, k\}$ is Riesz sequence in H.*

Proof (i) First assume that $\vec{\xi}$ is a complete Parserval super-frame vector for π. Then each ξ_i is a complete Parserval frame vector for π, and ξ_i and ξ_j are π-orthogonal for $i \neq j$. Thus, by Theorem 1.2 and Lemma 5.5, we get that $\{\sigma(g)\xi_i\}_{g \in G}$ is an orthogonal sequence, and $[\pi(G)'\xi_i] \perp [\pi(G)'\xi_j]$ for $i \neq j$. Therefore we have that $\{\sigma(g)\xi_j : g \in G, j = 1, \ldots, k\}$ is an orthogonal sequence in H. The identity follows from Lemma 4.2 and the fact that each ξ_i is a complete Parseval frame vector for π. Clearly the above argument is reversible, and so we also get the sufficiency part of the proof.

(ii) First assume that $\vec{\xi}$ is super-frame vector for π. By Lemma 4.4, there exists an invertible operator $B = A \oplus \cdots . \oplus A \in w^*(\pi^{(k)}(G))$ such that $B\vec{\xi}$ is a Parserval super-frame vector for π. This implies by (i) that $\cup_{i=1}^{k}\{\sigma(g)A\xi_i\}_{g\in G}$ is an orthogonal sequence. Since $A \in w^*(\pi(G)) = \sigma(G)'$ is invertible, we get that $\sigma(g)\xi_i = A^{-1}\sigma A\xi_i$, and therefore

$$\{\sigma(g)\xi_i : g \in G, i = 1, \cdots, k\} = A^{-1}\{\sigma(g)A\xi_i : g \in G, i = 1, \cdots, k\}$$

is a Riesz sequence.

Conversely assume that $\{\sigma(g)\xi_j : g \in G, j = 1, \cdots, k\}$ is Riesz sequence in H. Let K be the closed subspace generated by $\{\sigma(g)\xi_j : g \in G, j = 1, \ldots, k\}$ and let $S = \Theta^*_{\vec{\xi},\sigma}\Theta_{\vec{\xi},\sigma}$.

$$S = \sum_{i=1}^{k} \Theta^*_{\xi_i,\sigma}\Theta_{\xi_i,\sigma}.$$

Note that since $\Theta^*_{\xi_i,\sigma}\Theta_{\xi_i,\sigma} \in \sigma(G)' = w^*(\pi(G))$, we obtain that $S \in w^*(\pi(G))$. Moreover, $S|_K : K \to K$ is positive invertible. Write $T = (S|_K)^{-1/2}$. Then T commutes with $\sigma(g)$ when restricted to K for all $g \in G$. Thus we obtain that

$$\cup_{i=1}^{k}\{\sigma(g)T\xi_i : g \in G\} = T \cup_{i=1}^{k} \{\sigma(g)\xi_i : g \in G\}$$

is an Parseval frame for K. Since T is invertible and $\{\sigma(g)\xi_j : g \in G, j = 1, \ldots, k\}$ is a Riesz basis for K, we get that $\cup_{i=1}^{k}\{\sigma(g)T\xi_i : g \in G\}$ is an orthogonal basis for K. Select $c_i > 0$ such that $c_i^2||T\xi_i||^2 = Tr_{\pi(G)'}(I)$ and write $\eta_i = c_iT\xi_i$. Then $\cup_{i=1}^{k}\{\sigma(g)\eta_i : g \in G\}$ is an orthogonal sequence with $||\eta_i||^2 = Tr_{\pi(G)'}(I)$. Thus, by (i) we get that $\eta = (\eta_1, \cdots, \eta_k)$ is a complete Parseval super-frame vector for π. This implies that $\vec{\phi} := (T\xi_1, \ldots, T\xi_k)$ is also a complete super-frame vector for π.

Let P be the orthogonal projection from H onto K. Then $P \in w^*(\pi(G))$ since K is invariant under $\sigma(G)$. Define $A = T \oplus P^{\perp}$ and

$$B = A \oplus \cdots \oplus A.$$

Then $B \in w^*(\pi^{(k)}(G))$ is invertible and $B\vec{\xi} = \vec{\phi}$. This implies by Lemma 4.4 that $\vec{\xi} = B^{-1}\vec{\phi}$ is a complete frame vector for $\pi^{(k)}$, i.e., ξ is super-frame vector for π. □

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