Solving Nonlinear p -Adic Pseudo-differential Equations: Combining the Wavelet Basis with the Schauder Fixed Point Theorem

Ehsan Pourhadi¹ · Andrei Yu. Khrennikov¹ · Klaudia Oleschko² ·
María de Jesús Correa Lopez³

Received: 25 December 2019 / Published online: 14 August 2020
© The Author(s) 2020

Abstract

Recently theory of p -adic wavelets started to be actively used to study of the Cauchy problem for nonlinear pseudo-differential equations for functions depending on the real time and p -adic spatial variable. These mathematical studies were motivated by applications to problems of geophysics (fluids flows through capillary networks in porous disordered media) and the turbulence theory. In this article, using this wavelet technique in combination with the Schauder fixed point theorem, we study the solvability of nonlinear equations with mixed derivatives, p -adic (fractional) spatial and real time derivatives. Furthermore, in the linear case we find the exact solution for the Cauchy problem. Some examples are provided to illustrate the main results.

Keywords Pseudo-differential equations · p -adic field · p -adic wavelet basis · Schauder fixed point theorem · Arzelà–Ascoli theorem

Communicated by Hans G. Feichtinger.

✉ Andrei Yu. Khrennikov
andrei.khrennikov@msi.vxu.se

Ehsan Pourhadi
epourhadi@alumni.iust.ac.ir

Klaudia Oleschko
olechko@unam.mx

María de Jesús Correa Lopez
maria.jesus.correa@pemex.com

- ¹ International Center for Mathematical Modelling in Physics and Cognitive Sciences MSI, Linnaeus University, 351 95 Växjö, Sweden
- ² Centro de Geociencias, Campus UNAM Juriquilla, Universidad Nacional Autonoma de Mexico (UNAM), Blvd. Juriquilla 3001, 76230 Querétaro, Mexico
- ³ Edificio Piramide, Boulevard Adolfo Ruiz Cortines 1202, Oropeza, 86030 Villahermosa, Tabasco, Mexico

Mathematics Subject Classification 35S10 · 42B35 · 47H10

1 Introduction

During recent 30 years, p -adic analysis has received a lot of attention through its applications to mathematical physics, string theory, quantum mechanics, dynamical systems, turbulence, cognitive sciences, and recently geophysics, see e.g. [5–7, 11, 13, 15, 16, 19–21, 26, 28, 31, 39, 40] and references therein. It is well-known that the theory of p -adic distributions (generalized functions) and the corresponding Fourier and wavelet analysis play an important role in solving mathematical problems and applications in aforementioned fields.

In p -adic analysis, which is associated with maps $\mathbb{Q}_p \rightarrow \mathbb{C}$, the operation of differentiation is well not defined. For such reason, p -adic modeling widely utilizes the calculus of pseudo-differential operators. In this calculus, the crucial role is played by the fractional differentiation operator D^α (the *Vladimirov operator*). The pseudo-differential equations over p -adic fields have been studied in numerous publications [3, 8, 12, 17–19, 22, 24, 25, 27, 32, 34–37]. But up to now, in almost all models, only linear and semilinear pseudo-differential equations have been considered (see also [2, 4, 9, 10, 28, 41, 42]). For instance, we can mention recent paper [30] devoted to the study of two classes of semi-linear pseudo-differential equations via the use of the p -adic wavelet functions and the Adomian decomposition method.

It seems that the first nonlinear p -adic pseudo-differential equation was studied by Kozirev [26] (at least an equation that is interesting for physical applications—modeling of turbulence). This was also the first application of the p -adic wavelet basis for study of nonlinear equations.

In paper [16] there was considered a p -adic analogue of one of the most important for applications to geophysics nonlinear equations, the *porous medium equation* (see [38]), that is the equation

$$\frac{\partial u}{\partial t} + D^\alpha(\varphi(u)), \quad u = u(t, x), \quad t > 0, \quad x \in \mathbb{Q}_p,$$

where φ is a strictly monotone increasing continuous real function satisfying $|\varphi(s)| \leq Cs^m$ for $s \in \mathbb{R}$ ($C > 0, m \geq 1$) and $D^\alpha, \alpha > 0$ is Vladimirov's fractional differentiation operator.

By the construction of Markov process in the balls, Antoniouk et al. [5] studied the Cauchy problem for p -adic nonlinear evolutionary pseudo-differential equations over the p -adic balls and gave a formula for the solution of these equations. With the help of Crandall–Liggett theorem together with the concept of m -accretive nonlinear operators, they also revealed a result in order to prove the existence of a unique mild solution for a nonlinear equation including a generator of the semigroup $T(t)$ in $L^1(\mathbb{Q}_p^n)$.

In 2019, Pourhadi et al. [31] studied a class of nonlinear p -adic pseudo-differential equation as the p -adic analogue of the Navier-Stokes equation (see Oleschko et al. [21] for derivation) using the Schauder fixed point theorem together with Adomian

decomposition method to find some initial terms of the solution. This equation models *the propagation of fluid's flow through Geo-conduits, including the mixture of fractures (as well as fracture's corridors) and capillary networks*.

To proceed our investigation, in the current paper we aim to employ the same technique but on the more generalized forms of pseudo-differential equations with nonlinear term $F(t, u, x)$ and initial conditions. This problem is also a generalized form of the model proposed by Chuong and Co [10]. It is worth pointing out that the term F has not been observed in the previous results with focus on the wavelet theory and existence results. We also present an explicit form for the solution in the terms of wavelet basis for the certain cases.

Throughout this paper, we investigate the solvability of following IVP problem for a class of nonlinear pseudo-differential equations¹ over the p -adic field in $I = [0, T]$ given as

$$\begin{cases} D^\alpha \frac{\partial^2 u(t, x)}{\partial t^2} + 2aD^\gamma \frac{\partial u(t, x)}{\partial t} + bD^\beta u(t, x) + cu(t, x) = F(t, u, x), \\ x \in \mathbb{Q}_p, t \in (0, T], \\ u(0, x) = f(x), \quad u_t'(0, x) = g(x), \quad x \in \mathbb{Q}_p, t = 0, \end{cases} \quad (1.1)$$

such that $D^\alpha, D^\beta, D^\gamma$ are the fractional operators with orders α, β, γ , respectively, and $a, b, c \geq 0$ where either $0 \leq b \leq a^2 < c$ or $0 \leq c \leq a^2 < b$ holds and not both. Besides, let us suppose that

$$\gamma := \gamma(\alpha, \beta) = \begin{cases} \frac{\alpha}{2} & a^2 < c, \\ \frac{\alpha + \beta}{2} & a^2 < b. \end{cases}$$

In this work, as special case, when F is independent from the term u , that is, $F(t, x)$, we present the exact solution for the Cauchy problem (1.1).

In Sect. 2, we give some fundamental and auxiliary facts in order to proceed with the development of our work and conclude our results. Section 3 deals with the present of the solution to the homogeneous form of the nonlinear pseudo-differential equation (1.1) over the p -adic field $\mathbb{Q}_p$. Section 4 dedicates to investigate the study of the solution for a linear pseudo-differential equation over the p -adic field $\mathbb{Q}_p$ which is also deduced by considering F independent from u . Finally, in Sect. 5 we establish the existence of the solution for IVP (1.1) as our main problem.

2 Preliminaries

In what follows, for a prime number p , we denote by $\mathbb{Q}_p$ the field of p -adic numbers and by $\mathbb{Z}_p$ the ring of p -adic integers. Considering $x \neq 0$ in $\mathbb{Q}_p$, $\text{ord}(x) \in \mathbb{Z} \cup \{+\infty\}$ stands for the valuation of x , i.e. p -adic order of x , and $|x|_p = p^{-\text{ord}(x)}$ its absolute value which possesses the following properties:

¹ The linear equations of this class were invented and studied by Chuong and Co [10]

- (i) $|x|_p \geq 0$ for every $x \in \mathbb{Q}_p$, and $|x|_p = 0$ if and only if $x = 0$;
- (ii) $|xy|_p = |x|_p |y|_p$ for every $x, y \in \mathbb{Q}_p$;
- (iii) $|x + y|_p \leq \max\{|x|_p, |y|_p\}$, for every $x, y \in \mathbb{Q}_p$, and when $|x|_p \neq |y|_p$, we have $|x + y|_p = \max\{|x|_p, |y|_p\}$,

which also shows that the norm $|\cdot|_p$ is non-Archimedean and the space $(\mathbb{Q}_p, |\cdot|_p)$ is an ultrametric space.

A canonical form of any p -adic number $x \in \mathbb{Q}_p$, $x \neq 0$, is represented as follows

$$x = \sum_{j=\gamma}^{\infty} x_j p^j \quad (2.1)$$

where $\gamma = \gamma(x) \in \mathbb{Z}$, and $x_k = 0, 1, \dots, p-1$, $x_0 \neq 0$, $k = 0, 1, \dots$. This series converges in the p -adic norm $|\cdot|_p$ to $p^{-\gamma}$. The fractional part of a p -adic number $x \in \mathbb{Q}_p$ defined by (2.1) is given as

$$\{x\}_p = \begin{cases} 0, & \text{if } \gamma(x) \geq 0 \text{ or } x = 0, \\ p^\gamma(x_0 + x_1 p + x_2 p^2 + \dots + x_{|\gamma|-1} p^{|\gamma|-1}), & \text{if } \gamma(x) < 0. \end{cases} \quad (2.2)$$

The standard additive character χ_p of the field $\mathbb{Q}_p$ is given by

$$\chi_p(x) = e^{2\pi i \{x\}_p}, \quad x \in \mathbb{Q}_p.$$

For the topology induced by $|\cdot|_p$ in $\mathbb{Q}_p$ we assume that

$$\begin{aligned} B_\gamma(a) &= \{x \in \mathbb{Q}_p : |x - a|_p \leq p^\gamma\}, \\ S_\gamma(a) &= \{x \in \mathbb{Q}_p : |x - a|_p = p^\gamma\} \end{aligned}$$

are ball and sphere of radius p^γ with center at a , respectively. For the convenience, we suppose $B_\gamma(0) = B_\gamma$ and $S_\gamma(0) = S_\gamma$. Recall that any point of the ball is its center, besides, any two balls in $\mathbb{Q}_p$ are either disjoint or one is contained in the other. Moreover, sets of all balls and spheres are open and closed sets (i.e. clopen) in $\mathbb{Q}_p$.

The topological group $(\mathbb{Q}_p, +)$ is locally compact commutative and thus there is an additive Haar measure dx , which is positive and invariant under the translation, i.e., $d(x+a) = dx$, $a \in \mathbb{Q}_p$. This measure is unique by normalizing dx so that

$$\int_{B_0} dx = 1, \quad d(ax+b) = |a|_p dx, \quad a \in \mathbb{Q}_p^* = \mathbb{Q}_p - \{0\}.$$

Further, regarding with the additive normalized character $\chi_p(x)$ on $\mathbb{Q}_p$ we have

$$\int_{B_\gamma} \chi_p(\xi x) dx = p^\gamma \Omega(p^\gamma |\xi|_p),$$

where $\Omega(t)$ is the characteristic function of the interval $[0, 1] \subset \mathbb{R}$.

We say the complex-valued function f defined on $\mathbb{Q}_p$ is *locally constant* if for any $x \in \mathbb{Q}_p$, there exists an integer $l(x) \in \mathbb{Z}$ such that $f(x + y) = f(x)$, for every $y \in B_{l(x)}$. We signify by $\mathcal{E}(\mathbb{Q}_p)$ the linear space of such functions in $\mathbb{Q}_p$. By $D(\mathbb{Q}_p)$ we mean the subspace of $\mathcal{E}(\mathbb{Q}_p)$ consisting of locally constant functions with compact support (so-called test function). Besides, denote by $D'(\mathbb{Q}_p)$ the set of all linear functionals on $D(\mathbb{Q}_p)$ (see also [39, VI.3]).

The Fourier transform of test function $\varphi \in D(\mathbb{Q}_p)$ is given by

$$\hat{\varphi}(\xi) = F[\varphi](\xi) = \int_{\mathbb{Q}_p} \varphi(x) \chi_p(\xi x) dx.$$

Besides, $\hat{\varphi}(\xi) \in D(\mathbb{Q}_p)$ and $\varphi(x) = F^{-1}[\varphi](\xi) = \int_{\mathbb{Q}_p} \hat{\varphi}(\xi) \chi_p(-\xi x) d\xi$ as the inverse Fourier transform.

Suppose $L^2(\mathbb{Q}_p)$ is the set of measurable $\mathbb{C}$ -valued functions f on $\mathbb{Q}_p$ such that

$$\|f\|_{L^2(\mathbb{Q}_p)} = \left(\int_{\mathbb{Q}_p} |f(x)|^2 dx \right)^{\frac{1}{2}} < \infty,$$

which is clearly a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{\mathbb{Q}_p} f(x) \overline{g(x)} dx, \quad f, g \in L^2(\mathbb{Q}_p),$$

and $\|f\|_{L^2(\mathbb{Q}_p)}^2 = \langle f, f \rangle$.

Hence, there is a linear isomorphism taking $D(\mathbb{Q}_p)$ onto $D(\mathbb{Q}_p)$ which also can be uniquely extended to a linear isomorphism of $L^2(\mathbb{Q}_p)$. Furthermore, the Plancherel equality holds

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle, \quad f, g \in L^2(\mathbb{Q}_p).$$

In 1910, Haar [14] initially introduced the wavelet basis by presenting an orthonormal basis in $L^2(\mathbb{R})$ including dyadic translations and dilations of a single function:

$$\psi_{jn}^H(x) = 2^{-\frac{j}{2}} \psi^H(2^{-j}x - n), \quad x \in \mathbb{R}, \quad j, n \in \mathbb{Z} \quad (2.3)$$

where

$$\psi^H(x) = \chi_{[0, \frac{1}{2})}(x) - \chi_{[\frac{1}{2}, 1]}(x)$$

is called a *Haar wavelet* and χ_A denotes the characteristic function of a set $A \subset \mathbb{R}$. The generalization of Haar basis (2.3) has been studied in various results. In 2002, a basis of complex-valued wavelets with compact support in $L^2(\mathbb{Q}_p^m)$ has been initially

introduced by Kozyrev [23] (see also [24, 25, 27]). This basis is comparable to the Haar basis and takes the following form

$$\psi_{k;jn}(x) = p^{-\frac{mj}{2}} \chi_p(p^{-1}k \cdot (p^j x - n)) \Omega(|p^j x - n|_p), \quad x \in \mathbb{Q}_p^m \quad (2.4)$$

where $k \in J_p^m := J_p \times J_p \times \cdots \times J_p$, $J_p = \{1, 2, \dots, p-1\}$, $j \in \mathbb{Z}$, and n can be taken as an element of the m -direct product of factor group

$$\mathbb{Q}_p/\mathbb{Z}_p = \left\{ \sum_{i=a}^{-1} n_i p^i \mid n_i = 0, 1, \dots, p-1, a \in \mathbb{Z}^- \right\}$$

and here, χ_p and Ω are the standard additive character of $\mathbb{Q}_p$ and characteristic function of $[0, 1]$, respectively, as defined before.

Assume the following subspaces of the test functions from $D(\mathbb{Q}_p)$

$$\begin{aligned} \Psi &= \Psi(\mathbb{Q}_p) = \{\psi \in D(\mathbb{Q}_p), \psi(0) = 0\}, \\ \Phi &= \Phi(\mathbb{Q}_p) = \{\phi : \phi = F[\psi], \psi \in \Psi\}. \end{aligned}$$

It is obvious to see that $\Psi, \Phi \neq \emptyset$. Regarding with the fact that Fourier transform is a linear isomorphism $D(\mathbb{Q}_p)$ into $D(\mathbb{Q}_p)$, we get $\Psi, \Phi \in D(\mathbb{Q}_p)$. To describe the space Φ we remark that $\phi \in \Phi$ if and only if $\phi \in D(\mathbb{Q}_p)$ and $\int_{\mathbb{Q}_p} \phi(x) dx = 0$. The space Φ is called the *p-adic Lizorkin space of test functions of the first kind* which is a complete space under the topology of the space $D(\mathbb{Q}_p)$. Furthermore, the space $\Phi' = \Phi'(\mathbb{Q}_p)$ is said to be the *p-adic Lizorkin space of distributions of the first kind* which is the topological dual space of $\Phi(\mathbb{Q}_p)$ (see also [1]).

The fractional operator $D^\alpha : \varphi \rightarrow D^\alpha \varphi$ is defined as a convolution of the following functions:

$$D^\alpha \varphi(x) = f_{-\alpha}(x) * \varphi(x) = \langle f_{-\alpha}(x), \varphi(x - \xi) \rangle, \quad \varphi \in \Phi'(\mathbb{Q}_p), \alpha \in \mathbb{C},$$

where the distribution $f_\alpha \in \Phi'(\mathbb{Q}_p)$ is called the *Riesz kernel* given by

$$f_\alpha(x) = \begin{cases} \frac{|x|_p^{\alpha-1}}{\Gamma_p(\alpha)}, & \text{if } \alpha \neq 0, 1, \\ \delta(x), & \text{if } \alpha = 0, \\ \frac{p^{-1} - 1}{\log p} \log |x|_p, & \text{if } \alpha = 1, \end{cases} \quad x \in \mathbb{Q}_p$$

and $\Gamma_p(\alpha) = \frac{1-p^{\alpha-1}}{1-p^{-\alpha}}$ is the Γ -function (for more details see [39]).

The domain of D^α is given by

$$\mathcal{M}(D^\alpha) = \{\varphi \in L^2(\mathbb{Q}_p) \mid D^\alpha \varphi \in L^2(\mathbb{Q}_p)\}.$$

3 The Homogeneous Cauchy Problem

Throughout this section we are interested in the study of the solution to the homogeneous form of the nonlinear pseudo-differential equation (1.1).

Theorem 3.1 Suppose that $f \in \mathcal{M}(D^\beta)$, $g \in \mathcal{M}(D^\gamma)$ and $f_{k;jn}$ and $g_{k;jn}$ are the corresponding coefficients of the series of f, g in terms of the orthonormal functions $\{\psi_{k;jn}(x)\}$, respectively. Then the homogeneous form of pseudo-differential equation (1.1), that is,

$$\begin{cases} D^\alpha \frac{\partial^2 u(t, x)}{\partial t^2} + 2aD^\gamma \frac{\partial u(t, x)}{\partial t} + bD^\beta u(t, x) + cu(t, x) = 0, & x \in \mathbb{Q}_p, t \in (0, T], \\ u(0, x) = f(x), \quad u'_t(0, x) = g(x), & x \in \mathbb{Q}_p, t = 0, \end{cases} \quad (3.1)$$

where the constants a, b, c are the same as given for Eq. (1.1), possesses a unique solution of the form

$$u(t, x) = \sum_J u_{k;jn}(t) \psi_{k;jn}(x), \quad \text{s.t. } J := J_p \times \mathbb{Z} \times \mathbb{Q}_p/\mathbb{Z}_p$$

belonging to $\mathcal{U} = C(I, \mathcal{M}(D^\beta)) \cap C^1(I, \mathcal{M}(D^\gamma)) \cap C^2(I, \mathcal{M}(D^\alpha))$, in which

$$\begin{aligned} u_{k;jn}(t) &= \exp(-atp^{(\gamma-\alpha)(1-j)}) \\ &\left[f_{k;jn} \cos(A_j t) + \left(\frac{ap^{(\gamma-\alpha)(1-j)}}{A_j} f_{k;jn} + \frac{1}{A_j} g_{k;jn} \right) \sin(A_j t) \right] \\ &, \quad \forall (k, j, n) \in J, \end{aligned}$$

and

$$A_j := p^{-\alpha(1-j)} \sqrt{bp^{(\alpha+\beta)(1-j)} - a^2 p^{2\gamma(1-j)} + cp^{\alpha(1-j)}}.$$

Proof Assume that $u(t, x)$ is the solution what we are looking for. Considering u in terms of wavelet functions $\psi_{k;jn}(x)$ with coefficients $u_{k;jn}(t)$ one can easily arrive at the following linear differential equation.

$$\begin{aligned} p^{\alpha(1-j)} u''_{k;jn}(t) + 2ap^{\gamma(1-j)} u'_{k;jn}(t) \\ + bp^{\beta(1-j)} u_{k;jn}(t) + cu_{k;jn}(t) = 0, \quad \forall (k, j, n) \in J. \end{aligned} \quad (3.2)$$

The corresponding characteristic equation is as follows:

$$p^{\alpha(1-j)} \lambda^2 + 2ap^{\gamma(1-j)} \lambda + bp^{\beta(1-j)} + c = 0.$$

Moreover, the discriminant of above quadratic equation is given as

$$\Delta'_j = a^2 p^{2\gamma(1-j)} - bp^{(\alpha+\beta)(1-j)} - cp^{\alpha(1-j)}.$$

The definition of γ immediately implies that $\Delta'_j < 0$ and then it follows that

$$u_{k;jn}(t) = \exp(-atp^{(\gamma-\alpha)(1-j)})[M_{k;jn} \cos(A_j t) + N_{k;jn} \sin(A_j t)], \quad \forall (k, j, n) \in J, \quad (3.3)$$

where

$$A_j := p^{-\alpha(1-j)} \sqrt{bp^{(\alpha+\beta)(1-j)} - a^2 p^{2\gamma(1-j)} + cp^{\alpha(1-j)}} > 0. \quad (3.4)$$

Now, imposing the initial conditions in our obtained solution we derive

$$M_{k;jn} = f_{k;jn}, \quad N_{k;jn} = \frac{ap^{(\gamma-\alpha)(1-j)}}{A_j} f_{k;jn} + \frac{1}{A_j} g_{k;jn} \quad (3.5)$$

where $f_{k;jn}$ and $g_{k;jn}$ are respectively the components of f and g in the corresponding representations based on wavelet functions $\psi_{k;jn}(x)$.

Therefore, the solution $u(t, x)$ of the homogeneous equation (3.1) is defined by

$$u(t, x) = \sum_J \exp(-atp^{(\gamma-\alpha)(1-j)}) \left[f_{k;jn} \cos(A_j t) + \left(\frac{ap^{(\gamma-\alpha)(1-j)}}{A_j} f_{k;jn} + \frac{1}{A_j} g_{k;jn} \right) \sin(A_j t) \right] \psi_{k;jn}(x) \quad (3.6)$$

such that A_j is given as (3.4). Since $t \in I$ and $a > 0$ one can observe that

$$0 < \exp(-atp^{(\gamma-\alpha)(1-j)}) \leq 1. \quad (3.7)$$

This shows that the series (3.6) converges in $L^2(\mathbb{Q}_p)$ uniformly in $t \in I$.

Considering the hypotheses $f \in \mathcal{M}(D^\beta)$, $g \in \mathcal{M}(D^\gamma)$ we immediately derive that

$$\begin{aligned} D_x^\beta u(t, x) &= \sum_J \exp(-atp^{(\gamma-\alpha)(1-j)}) \\ &\times \left[p^{\beta(1-j)} f_{k;jn} \cos(A_j t) + p^{\beta(1-j)} \left(\frac{ap^{(\gamma-\alpha)(1-j)}}{A_j} f_{k;jn} + \frac{1}{A_j} g_{k;jn} \right) \sin(A_j t) \right] \psi_{k;jn}(x), \\ \frac{\partial u(t, x)}{\partial t} &= \sum_J \exp(-atp^{(\gamma-\alpha)(1-j)}) S_{k;jn}(t) \psi_{k;jn}(x), \end{aligned}$$

where

$$S_{k;jn}(t) = g_{k;jn} \cos(A_j t) - \frac{1}{A_j} \left(a^2 p^{2(\gamma-\alpha)(1-j)} f_{k;jn} + ap^{(\gamma-\alpha)(1-j)} g_{k;jn} \right) \sin(A_j t).$$

Furthermore, using (3.7) we get

$$D_x^\gamma \frac{\partial u(t, x)}{\partial t} = \sum_j p^{\gamma(1-j)} \exp(-at p^{(\gamma-\alpha)(1-j)}) S_{k;jn}(t) \psi_{k;jn}(x),$$

which means that all the series as above are convergent in $L^2(\mathbb{Q}_p)$ uniformly in $t \in I$. That is, $u \in C(I, \mathcal{M}(D^\beta)) \cap C^1(I, \mathcal{M}(D^\gamma))$. With similar reasoning, one can see that both series with respect to $\frac{\partial^2 u}{\partial t^2}$ and $D^\alpha \frac{\partial^2 u}{\partial t^2}$ converge in $L^2(\mathbb{Q}_p)$ uniformly in $t \in I$ and hence u given as (3.6) is unique and also belongs to $\mathcal{U}$. $\square$

Example 1 Suppose that the problem (3.1) takes the following form over $I \times \mathbb{Q}_p$:

$$\begin{cases} D^{\frac{3}{2}} \frac{\partial^2 u(t, x)}{\partial t^2} + 2D^1 \frac{\partial u(t, x)}{\partial t} + 2D^{\frac{1}{2}} u(t, x) + u(t, x) = 0, & x \in \mathbb{Q}_p, t \in (0, T], \\ u(0, x) = u'_t(0, x) = \Omega(|x|_p), & x \in \mathbb{Q}_p, t = 0. \end{cases}$$

Then

$$\begin{aligned} f_{k;jn} &= g_{k;jn} = \langle g, \psi_{k;jn} \rangle = \int_{\mathbb{Q}_p} \Omega(|x|_p) \cdot \psi_{k;jn}(x) dx \\ &= p^{\frac{-3j}{2}} \int_{\mathbb{Q}_p} \Omega(|p^{-j}(\xi + n)|_p) \cdot \Omega(|\xi|_p) \cdot \chi_p(p^{-1}k\xi) d\xi \\ &= p^{\frac{-3j}{2}} \int_{\mathbb{Q}_p} \Omega(p^j \max\{|\xi|_p, |n|_p\}) \cdot \Omega(|\xi|_p) \cdot \chi_p(p^{-1}k\xi) d\xi \end{aligned}$$

where $k = 0, 1, 2, \dots, p-1$ and $n \in \mathbb{Q}_p/\mathbb{Z}_p$. Assuming $|n|_p = p^{-\gamma}$ for some integer $\gamma \leq -1$ together with the fact that $\Omega(|\xi|_p) \neq 0$ if and only if $\xi \in S_r$ for some $r \leq 0$, we derive

$$\begin{aligned} f_{k;jn} &= g_{k;jn} = p^{\frac{-3j}{2}} \Omega(p^{j-\gamma}) \sum_{r \leq 0} \int_{S_r} \chi_p(p^{-1}k\xi) d\xi \\ &= p^{\frac{-3j}{2}} \Omega(p^{j-\gamma}) \sum_{r \leq 0} \left(\int_{B_r} \chi_p(p^{-1}k\xi) d\xi - \int_{B_{r-1}} \chi_p(p^{-1}k\xi) d\xi \right). \end{aligned}$$

If $k = 0$ then

$$f_{0;jn} = g_{0;jn} = p^{\frac{-3j}{2}} \Omega(p^{j-\gamma}) = p^{\frac{-3j}{2}} \Omega(p^{j-\text{ord}_p(n)}).$$

Otherwise, for the case $1 \leq k \leq p-1$, using the formula

$$\int_{B_r} \chi_p(\xi x) dx = \begin{cases} p^r, & |\xi|_p \leq p^{-r}, \\ 0, & |\xi|_p \geq p^{-r+1}, \end{cases} \quad r \in \mathbb{Z},$$

we see that

$$f_{k;jn} = g_{k;jn} = p^{\frac{-3j}{2}} \Omega(p^{j-\text{ord}_p(n)}) \left(-p^{-1} + \sum_{r \leq -1} (p^r - p^{r-1}) \right) = 0.$$

Hence the solution of the problem is as follows

$$u(t, x) = \sum_{j \leq -1} \sum_{\substack{n \in \mathbb{Q}_p / \mathbb{Z}_p \\ j \leq \text{ord}_p(n)}} u_{0;jn}(t) \psi_{0;jn}(x),$$

where

$$u_{0;jn}(t) = \exp(-tp^{-0.5(1-j)}) p^{-1.5j} \left[\cos(A_j t) + p^{0.5(1-j)} \sqrt{1 + p^{-0.5(1-j)}} \sin(A_j t) \right],$$

and A_j is defined in

$$A_j := p^{-0.5(1-j)} \sqrt{1 + p^{-0.5(1-j)}}.$$

4 Cauchy Problem for a Linear Pseudo-differential Equation

This section dedicates to investigate the existence of solution for the following linear pseudo-differential equation over the p -adic field $\mathbb{Q}_p$ in $I = [0, T]$ given as

$$\begin{cases} D^\alpha \frac{\partial^2 u(t, x)}{\partial t^2} + 2aD^\gamma \frac{\partial u(t, x)}{\partial t} + bD^\beta u(t, x) + cu(t, x) = F(t, x), & t \in (0, T], \\ u(0, x) = f(x), \quad u'_t(0, x) = g(x), & x \in \mathbb{Q}_p, \quad t = 0, \end{cases} \quad (4.1)$$

where the coefficients are defined same as ones given for Eq. (1.1).

In order to present the result of this section we need the following lemma.

Lemma 1 For $\mathcal{M}(D^\alpha)$ the following inclusion holds true:

$$\mathcal{M}(D^\alpha) \subseteq \mathcal{M}(D^\beta), \quad 0 < \beta < \alpha.$$

Proof Suppose that $\varphi \in \mathcal{M}(D^\alpha)$, then

$$\int_{\mathbb{Q}_p} |\hat{\varphi}(\xi)|^2 d\xi < \infty \quad \text{and} \quad \int_{\mathbb{Q}_p} |\xi|_p^{2\alpha} |\hat{\varphi}(\xi)|^2 d\xi < \infty.$$

On the other hand, since $0 < \beta < \alpha$ we get

$$\int_{\mathbb{Q}_p} |\xi|_p^{2\beta} |\hat{\varphi}(\xi)|^2 d\xi \leq \max \left\{ \int_{\mathbb{Q}_p} |\hat{\varphi}(\xi)|^2 d\xi, \int_{\mathbb{Q}_p} |\xi|_p^{2\alpha} |\hat{\varphi}(\xi)|^2 d\xi \right\} < \infty,$$

that is,

$$D^\beta \varphi \in L^2(\mathbb{Q}_p), \quad \|D^\beta \varphi\|_{L^2(\mathbb{Q}_p)}^2 = \int_{\mathbb{Q}_p} |\xi|_p^{2\beta} |\hat{\varphi}(\xi)|^2 d\xi < \infty.$$

Hence, $\varphi \in \mathcal{M}(D^\beta)$. $\square$

Theorem 4.1 Suppose that $f \in \mathcal{M}(D^\beta)$, $g \in \mathcal{M}(D^\gamma)$, $f_{k;jn}$ and $g_{k;jn}$ are the corresponding coefficients of the series of f, g in terms of the orthonormal functions $\{\psi_{k;jn}(x)\}$, respectively. Further, assume that $|\gamma - \alpha| < \beta$. Then the non-homogeneous equation (4.1) has a unique solution of the form

$$u(t, x) = \sum_J u_{k;jn}(t) \psi_{k;jn}(x), \quad s.t. \quad J := J_p \times \mathbb{Z} \times \mathbb{Q}_p / \mathbb{Z}_p,$$

belonging to $\mathcal{U}$ where

$$\begin{aligned} u_{k;jn}(t) = & \exp(-atp^{(\gamma-\alpha)(1-j)}) \\ & \left(f_{k;jn} \cos(A_j t) + \frac{1}{A_j} \left(ap^{(\gamma-\alpha)(1-j)} f_{k;jn} + g_{k;jn} \right) \sin(A_j t) \right) \\ & + \frac{1}{A_j} \int_0^t \sin(A_j(t-r)) \exp(a(r-t)p^{(\gamma-\alpha)(1-j)}) \cdot F_{k;jn}(r) dr. \end{aligned}$$

Proof As shown in the proof of Theorem 3.1, we have

$$\begin{aligned} & p^{\alpha(1-j)} u''_{k;jn}(t) + 2ap^{\gamma(1-j)} u'_{k;jn}(t) \\ & + bp^{\beta(1-j)} u_{k;jn}(t) + cu_{k;jn}(t) = F_{k;jn}(t), \quad \forall(k, j, n) \in J \end{aligned}$$

where

$$F_{k;jn}(t) = \langle F(t, \cdot), \psi_{k;jn} \rangle_{L^2(\mathbb{Q}_p)} = \int_{\mathbb{Q}_p} F(t, x) \overline{\psi_{k;jn}(x)} dx, \quad t \in I.$$

Now, applying the variation of parameters method for the solutions (3.3) with constants (3.4) and (3.5) we present the solution of problem by the following

$$\begin{aligned} u_{k;jn}(t) = & \exp(-atp^{(\gamma-\alpha)(1-j)}) [M_{k;jn}(t) \cos(A_j t) + N_{k;jn}(t) \sin(A_j t)], \\ & \forall(k, j, n) \in J, \quad t \in I, \end{aligned}$$

where $M_{k;jn}(t)$, $N_{k;jn}(t)$ are the unknown functions which are found from the following system:

$$\begin{cases} M'_{k;jn}(t) \cos(A_j t) + N'_{k;jn}(t) \sin(A_j t) = 0, \\ M'_{k;jn}(t) \left(-A_j \sin(A_j t) - ap^{(\gamma-\alpha)(1-j)} \cos(A_j t) \right) + \\ \quad N'_{k;jn}(t) \left(A_j \cos(A_j t) - ap^{(\gamma-\alpha)(1-j)} \sin(A_j t) \right) \\ = \exp(atp^{(\gamma-\alpha)(1-j)}) \cdot F_{k;jn}(t) \end{cases}$$

which yields that

$$\begin{aligned} M_{k;jn}(t) &= \frac{-1}{A_j} \int_0^t \sin(A_j r) \exp(ar p^{(\gamma-\alpha)(1-j)}) \cdot F_{k;jn}(r) dr + \bar{m}_{k;jn}, \\ N_{k;jn}(t) &= \frac{1}{A_j} \int_0^t \cos(A_j r) \exp(ar p^{(\gamma-\alpha)(1-j)}) \cdot F_{k;jn}(r) dr + \bar{n}_{k;jn}. \end{aligned}$$

where

$$\bar{m}_{k;jn} = f_{k;jn}, \quad \bar{n}_{k;jn} = \frac{1}{A_j} \left(ap^{(\gamma-\alpha)(1-j)} f_{k;jn} + g_{k;jn} \right).$$

Hence, the unique solution of the problem is

$$u(t, x) = \sum_j u_{k;jn}(t) \psi_{k;jn}(x), \quad (4.2)$$

such that

$$\begin{aligned} u_{k;jn}(t) &= \exp(-atp^{(\gamma-\alpha)(1-j)}) \\ &\quad \left(f_{k;jn} \cos(A_j t) + \frac{1}{A_j} \left(ap^{(\gamma-\alpha)(1-j)} f_{k;jn} + g_{k;jn} \right) \sin(A_j t) \right) \\ &\quad + \frac{1}{A_j} \int_0^t \sin(A_j(t-r)) \exp(a(r-t)p^{(\gamma-\alpha)(1-j)}) \cdot F_{k;jn}(r) dr. \end{aligned}$$

Let us now show that the obtained solution belongs to $\mathcal{U}$.

Taking into account that

$$0 < \exp(-atp^{(\gamma-\alpha)(1-j)}) \leq 1, \quad 0 \leq \left\{ |\sin(A_j t)|, |\cos(A_j t)| \right\} \leq 1,$$

it is clear that the series (4.2) is convergent in $L^2(\mathbb{Q}_p)$ uniformly in I . We also note that since $|\gamma - \alpha| < \beta$, then $f \in \mathcal{M}(D^{|\gamma-\alpha|})$ (see Lemma 1).

On the other hand, since $f \in \mathcal{M}(D^\beta)$, $g \in \mathcal{M}(D^\gamma)$, then the series corresponding to $D^\beta u(t, x)$ converges in $L^2(\mathbb{Q}_p)$ uniformly in I . Hence $u \in C(I, \mathcal{M}(D^\beta))$.

Moving forward, we have

$$\frac{\partial u(t, x)}{\partial t} = \sum_J S_{k; jn}(t) \psi_{k; jn}(x),$$

where

$$\begin{aligned} S_{k; jn}(t) = & \exp(-atp^{(\gamma-\alpha)(1-j)}) \left(-ap^{(\gamma-\alpha)(1-j)} \left[f_{k; jn} \cos(A_j t) \right. \right. \\ & + \frac{1}{A_j} \left(ap^{(\gamma-\alpha)(1-j)} f_{k; jn} + g_{k; jn} \right) \sin(A_j t) \Big] \\ & - A_j f_{k; jn} \sin(A_j t) + \left(ap^{(\gamma-\alpha)(1-j)} f_{k; jn} + g_{k; jn} \right) \cos(A_j t) \Big) \\ & + \frac{1}{A_j} \int_0^t \exp(a(r-t)p^{(\gamma-\alpha)(1-j)}) \left[A_j \cos(A_j(t-r)) \right. \\ & \left. \left. - ap^{(\gamma-\alpha)(1-j)} \sin(A_j(t-r)) \right] \cdot F_{k; jn}(r) dr \right) \end{aligned}$$

converging in $L^2(\mathbb{Q}_p)$ uniformly over the interval I since $f, g \in \mathcal{M}(D^{|\gamma-\alpha|})$ and

$$0 < \left\{ \exp(-atp^{(\gamma-\alpha)(1-j)}), \quad \exp(a(r-t)p^{(\gamma-\alpha)(1-j)}) \right\} < 1.$$

Besides,

$$D^\gamma \frac{\partial u(t, x)}{\partial t} = \sum_J p^{\gamma(1-j)} S_{k; jn}(t) \psi_{k; jn}(x)$$

is convergent in $L^2(\mathbb{Q}_p)$ uniformly in I , which means $u \in C^1(I, \mathcal{M}(D^\gamma))$. Similarly, the series

$$D^\alpha \frac{\partial^2 u(t, x)}{\partial t^2} = \sum_J p^{\alpha(1-j)} \bar{S}_{k; jn}(t) \psi_{k; jn}(x)$$

converges in $L^2(\mathbb{Q}_p)$ uniformly in I where

$$\begin{aligned} \bar{S}_{k; jn}(t) = & \exp(-atp^{(\gamma-\alpha)(1-j)}) \left(a^2 p^{2(\gamma-\alpha)(1-j)} \left[f_{k; jn} \cos(A_j t) \right. \right. \\ & + \frac{1}{A_j} \left(ap^{(\gamma-\alpha)(1-j)} f_{k; jn} + g_{k; jn} \right) \sin(A_j t) \Big] \\ & \left. - 2ap^{(\gamma-\alpha)(1-j)} \left[-f_{k; jn} A_j \sin(A_j t) + \left(ap^{(\gamma-\alpha)(1-j)} f_{k; jn} + g_{k; jn} \right) \cos(A_j t) \right] \right) \end{aligned}$$

$$\begin{aligned}
& -A_j^2 f_{k;jn} \cos(A_j t) - A_j \left(a p^{(\gamma-\alpha)(1-j)} f_{k;jn} + g_{k;jn} \right) \sin(A_j t) + F_{k;jn}(t) \\
& + \frac{1}{A_j} \int_0^t \exp(a(r-t) p^{(\gamma-\alpha)(1-j)}) \left[-2a p^{(\gamma-\alpha)(1-j)} A_j \cos(A_j(t-r)) \right. \\
& \left. + (a^2 p^{2(\gamma-\alpha)(1-j)} - A_j^2) \sin(A_j(t-r)) \right] \cdot F_{k;jn}(r) dr, \quad \forall t \in I,
\end{aligned}$$

and $u \in C^2(I, \mathcal{M}(D^\alpha))$, that is, $u \in \mathcal{U}$. Therefore, the function (4.2) is the unique solution to the problem (4.1) which satisfies the mentioned initial conditions. $\square$

Example 2 Consider the problem (4.1) with a similar constants given in Example 1. Suppose that $f(x) = g(x) = \ln |x|_p$ and non-homogeneity term $F(t, x) = t \ln |x|_p$. To write the function F in terms of basis $\psi_{k;jn}$, using notation $\xi = p^j x - n$ we have

$$\begin{aligned}
F_{k;jn}(t) &= \langle F, \psi_{k;jn} \rangle = \int_{\mathbb{Q}_p} t \ln |x|_p \cdot \psi_{k;jn}(x) dx \\
&= t p^{\frac{-3j}{2}} \int_{\mathbb{Q}_p} \ln(|p^{-j}(\xi + n)|_p) \cdot \Omega(|\xi|_p) \cdot \chi_p(p^{-1}k\xi) d\xi \\
&= t p^{\frac{-3j}{2}} \int_{\mathbb{Q}_p} \ln(p^j \max\{|\xi|_p, |n|_p\}) \cdot \Omega(|\xi|_p) \cdot \chi_p(p^{-1}k\xi) d\xi \\
&= t p^{\frac{-3j}{2}} (j \ln p + \ln |n|_p) \int_{\mathbb{Q}_p} \Omega(|\xi|_p) \cdot \chi_p(p^{-1}k\xi) d\xi.
\end{aligned}$$

If $k = 0$ then

$$F_{0;jn}(t) = t p^{\frac{-3j}{2}} \ln p \cdot (j - \text{ord}_p(n)),$$

otherwise $F_{k;jn}(t) = 0$. Similarly, we can discuss on the values of $f_{k;jn}$, $g_{k;jn}$ and derive

$$f_{k;jn} = g_{k;jn} = p^{\frac{-3j}{2}} \ln p \cdot (j - \text{ord}_p(n)),$$

for $k = 0$, otherwise $f_{k;jn} = g_{k;jn} = 0$. Consequently, the solution is given by the following form:

$$u(t, x) = \sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{Q}_p / \mathbb{Z}_p} u_{0;jn}(t) \psi_{0;jn}(x),$$

where

$$u_{0;jn}(t) = \exp(-tp^{-0.5(1-j)})p^{\frac{-3j}{2}} \ln p \cdot (j - \text{ord}_p(n)) \\ \left(\cos(A_j t) + \frac{1}{A_j} \left(p^{-0.5(1-j)} + 1 \right) \sin(A_j t) \right) \\ + \frac{1}{A_j} p^{\frac{-3j}{2}} \ln p \cdot (j - \text{ord}_p(n)) \int_0^t r \sin(A_j(t-r)) \exp((r-t)p^{-0.5(1-j)}) dr,$$

and A_j is defined as Example 1.

5 Cauchy Problem for a Nonlinear Pseudo-differential Equation

Throughout this section we study the existence of solution for the following class of nonlinear pseudo-differential equation over the p -adic field $\mathbb{Q}_p$ in $I = [0, T]$ given as

$$\begin{cases} D^\alpha \frac{\partial^2 u(t, x)}{\partial t^2} + 2aD^\gamma \frac{\partial u(t, x)}{\partial t} + bD^\beta u(t, x) + cu(t, x) = F(t, u, x), & x \in \mathbb{Q}_p, t \in (0, T], \\ u(0, x) = f(x), \quad u'_t(0, x) = g(x), & x \in \mathbb{Q}_p, t = 0, \end{cases} \quad (5.1)$$

where the coefficients are defined same as before. Moreover, suppose that the constants of Eq. (5.1) are chosen in the way that $A_j T < x^*$ where x^* is the only root of

$$h(x) := x + \arctan(e^{-10x}) - \frac{\pi}{2}, \quad (5.2)$$

(see Fig. 1) Considering u in terms of wavelet functions $\psi_{k;jn}(x)$ with coefficients $u_{k;jn}(t)$ one can simply see the following quasilinear differential equation.

$$p^{\alpha(1-j)} u''_{k;jn}(t) + 2ap^{\gamma(1-j)} u'_{k;jn}(t) + bp^{\beta(1-j)} u_{k;jn}(t) + cu_{k;jn}(t) \\ = F_{k;jn}(t, u), \quad \forall (k, j, n) \in J, \quad (5.3)$$

where the function u as the solution subjected to the problem is given as

$$u(t, x) = \sum_{\iota \in J} u_\iota(t) \psi_\iota(x), \quad \iota = (k, j, n) \in J := J_p \times \mathbb{Z} \times \mathbb{Q}_p/\mathbb{Z}_p,$$

and suppose that the nonlinear term F takes the following form

$$F(t, u, x) = \sum_{\iota \in J} F_\iota(t, \hat{u}) \psi_\iota(x), \quad \hat{u} = (u_\iota)_{\iota \in J}, \quad \iota = (k, j, n) \in J. \quad (5.4)$$

Fig. 1 Function h and x^* as its only real root

For the convenience of reader, let us remove the index of symbols in infinite system (5.3) and rewrite it as the following matrix differential equation. Taking $j \in \mathbb{Z}$ as arbitrarily fixed we derive

$$p^{\alpha(1-j)}\hat{u}''(t) + 2ap^{\gamma(1-j)}\hat{u}'(t) + (bp^{\beta(1-j)} + c)\hat{u}(t) = \mathcal{F}(t, \hat{u}), \quad \mathcal{F} = (F_i)_{i \in J}. \quad (5.5)$$

It is worth mentioning that all solutions of (5.5) depend on $j \in \mathbb{Z}$.

On the other hand, by initial conditions we easily see that

$$u_i(0) = \langle f, \psi_i \rangle, \quad u'_i(0) = \langle g, \psi_i \rangle, \quad i \in J. \quad (5.6)$$

Let $\hat{u}_1(t)$ and $\hat{u}_2(t)$ form a fundamental system of solutions of the truncated linear equation corresponding to $\mathcal{F} = 0$, that is,

$$\begin{aligned} \hat{u}_1(t) &:= (u_{1i}(t))_{i \in J} = \left(\exp(-atp^{(\gamma-\alpha)(1-j)}) \cos(A_j t) \right)_{i \in J}, \\ \hat{u}_2(t) &:= (u_{2i}(t))_{i \in J} = \left(\exp(-atp^{(\gamma-\alpha)(1-j)}) \sin(A_j t) \right)_{i \in J}. \end{aligned} \quad (5.7)$$

Then, considering Eq. (5.5) by its component for any $\iota \in J$, we have the following transformation

$$\begin{aligned}\xi_{\iota} &:= r(t) = \frac{u_{2\iota}(t)}{u_{1\iota}(t)} = \frac{\sin(A_j t)}{\sin(A_j t) - e^{10A_j T} \cos(A_j t)} \\ &= \frac{\sin(A_j t)}{B_j \cos(A_j t + r_j)}, \quad t \in I, \quad \omega_{\iota} = \frac{u_{\iota}}{u_{1\iota}(t)},\end{aligned}\quad (5.8)$$

where

$$B_j = \sqrt{1 + e^{20A_j T}}, \quad r_j = \arctan(e^{-10A_j T}). \quad (5.9)$$

Here, for some purpose, $u_{1\iota}$ is defined as arbitrary special combination of $u_{1\iota}$ and $u_{2\iota}$ given by (5.7). Remark that in definition of ξ_{ι} , $\cos(A_j t + r_j) \neq 0$ since

$$0 < r_j \leq A_j t + r_j \leq A_j T + r_j < \frac{\pi}{2}, \quad t \in I.$$

More precisely, this is concluded by the fact that $h(x) < 0$ for $x < x^* \cong 3.1095$ where x^* is the only root of h . And that is satisfied since $A_j T < x^*$.

Based on the imposed condition $A_j T < x^*$, we notice that $\xi_{\iota} = r(t)$ is increasing with respect to $t \in I$,

$$\xi_{\iota} \in \bar{I} := [0, r(T)].$$

The substitutions (5.8) convert Eq. (5.5) into a simpler form

$$\omega''_{\xi\xi} = \Phi_{\iota}(\xi_{\iota}, \omega_{\iota}), \quad \text{where}$$

$$\Phi_{\iota}(\xi_{\iota}, \omega_{\iota}) = \frac{u_{1\iota}^3(t)}{[W(u_{1\iota}, u_{2\iota})(t)]^2} F_{\iota}(t, u_{1\iota}(t)\omega_{\iota}), \quad (\text{see also [28, 0.3.2 - 9]}) \quad (5.10)$$

where $W(u_{1\iota}, u_{2\iota})(t)$ is the Wronskian of linearly independent functions $u_{1\iota}, u_{2\iota}$. For the convenience let us ignore the index $\iota \in J$, and substitute $\rho := \omega'_{\xi}$ then we have the following integral equation:

$$\begin{cases} \rho(\xi) = \rho(0) + \int_0^{\xi} \Phi(s, \omega) ds, & \xi \in \bar{I}, \\ \omega(\xi) = \omega(0) + \int_0^{\xi} \rho(s) ds, & \xi \in \bar{I}. \end{cases} \quad (5.11)$$

Since $t = 0$ if and only if $\xi = 0$ then

$$\begin{aligned}\rho(0) &= \omega'_{\xi}(0) = [\omega'_{\iota} \cdot t'_{\xi}](0) = \frac{u'(0)u_1(0) - u'_1(0)u(0)}{u_1^2(0)} \cdot \frac{-e^{10A_j T}}{A_j} \\ &= e^{-10A_j T} \langle f, \psi \rangle + \frac{1}{A_j} \left(\langle g, \psi \rangle + ap^{(\gamma-\alpha)(1-j)} \langle f, \psi \rangle \right),\end{aligned}$$

and

$$\omega(0) = \frac{u(0)}{u_1(0)} = \langle f, \psi \rangle.$$

which both values should be replaced in Eq. (5.11).

Remark 1 To study the solvability of nonlinear system of differential equations (5.3) it only needs to investigate the existence of $\omega(t)$ from Eq. (5.11). To do this let us first present the following well-known fixed point result.

Theorem 5.1 (Schauder Fixed Point Theorem [33, Theorem 4.1.1]) *Let U be a nonempty and convex subset of a normed space E . Let T be a continuous mapping of U into a compact set $K \subset U$. Then T has a fixed point.*

Theorem 5.2 *Suppose that the following conditions hold:*

(i)

$$|F_t(t, u_t) - F_t(t, v_t)| \leq \varphi_t(t)|u_t - v_t|, \quad (t, t) \in I \times J \quad (5.12)$$

where $\varphi_t \in L^1(I)$ with maximum value $\bar{\varphi}_t$ and $\bar{F}_t = \max_{t \in I} |F_t(t, 0)|$.

(ii) *There exists a function $\mathcal{H}_j : \bar{I} \rightarrow \mathbb{R}^+$ for $j \in \mathbb{Z}$ belonging to $L^1(\bar{I})$ such that*

$$\left| e^{-10A_j T} \langle f, \psi \rangle + \frac{1}{A_j} \left(\langle g, \psi \rangle + ap^{(\gamma-\alpha)(1-j)} \langle f, \psi \rangle \right) + \int_0^\xi \Phi(s, \langle f, \psi \rangle) + \int_0^s \rho(r) dr ds \right| \leq \mathcal{H}_j(\xi), \quad \xi \in \bar{I},$$

whenever $|\rho(\xi)| \leq \mathcal{H}_j(\xi)$ for any $\xi \in \bar{I}$. Furthermore, suppose that $A_j T < x^*$ where x^* is the only real root of (5.2). Then Eq. (5.11) has a solution $\rho(\xi)$ in $C(\bar{I}, \mathbb{R})$ bounded above by $\mathcal{H}_j$.

Proof Following condition (ii) let us first consider the set $\mathcal{S} \subset C(\bar{I}, \mathbb{R})$ defined by

$$\mathcal{S} = \{\rho : \rho \in C(\bar{I}, \mathbb{R}) \text{ and } |\rho(\xi)| \leq \mathcal{H}_j(\xi) \text{ for all } \xi \in \bar{I}\}.$$

Obviously, the set $\mathcal{S}$ is a nonempty, closed, bounded and convex subset of $\mathcal{U}$. Furthermore, suppose that

$$(\Gamma\rho)(\xi) := e^{-10A_j T} \langle f, \psi \rangle + \frac{1}{A_j} \left(\langle g, \psi \rangle + ap^{(\gamma-\alpha)(1-j)} \langle f, \psi \rangle \right) + \int_0^\xi \Phi(s, \omega) ds$$

where ω is given by the second relation in (5.11). To prove that Eq. (5.11) has a solution it only needs to show that the operator Γ has a fixed point in $\mathcal{S}$. First, we show that $\mathcal{S}$ is Γ -invariant, that is, $\Gamma\mathcal{S} \subset \mathcal{S}$. This is easily implied by condition (ii).

For the fixed $\xi \in \bar{I}$, suppose $\rho_n \rightarrow \rho$ as $n \rightarrow \infty$, we get

$$\left| (\Gamma \rho_n)(\xi) - (\Gamma \rho)(\xi) \right| \leq \int_0^\xi \left| \Phi(s, \omega_n) - \Phi(s, \omega) \right| ds \quad (5.13)$$

where

$$\omega_n(s) = \langle f, \psi \rangle + \int_0^s \rho_n(r) dr, \quad \omega(s) = \langle f, \psi \rangle + \int_0^s \rho(r) dr, \quad s \in \bar{I}.$$

On the other hand, using (5.10) and (5.12) we see that

$$\begin{aligned} \left| \Phi(s, \omega_n) - \Phi(s, \omega) \right| &= \left| \frac{u_1^3(t)}{[W(u_1, u_2)(t)]^2} \right| \cdot \left| F(t, u_1(t)\omega_n) - F(t, u_1(t)\omega) \right| \\ &= \left| \frac{[\sin(A_j t) - e^{10A_j T} \cos(A_j t)]^3}{A_j^2 \exp(-atp^{(\gamma-\alpha)(1-j)}) \cdot e^{20A_j T}} \right| \cdot \left| F(t, u_1(t)\omega_n) - F(t, u_1(t)\omega) \right| \\ &\leq \frac{\varphi(t)}{A_j^2 \cdot e^{20A_j T}} \left(\sin(A_j t) - e^{10A_j T} \cos(A_j t) \right)^4 \cdot |\omega_n(s) - \omega(s)| \\ &\leq \frac{\bar{\varphi}}{A_j^2 \cdot e^{20A_j T}} \left(1 + e^{20A_j T} \right)^2 \cdot |\omega_n(s) - \omega(s)| \\ &\leq \frac{\bar{\varphi} \cdot \ell(\bar{I})}{A_j^2 \cdot e^{20A_j T}} \left(1 + e^{20A_j T} \right)^2 \cdot \|\rho_n - \rho\| \end{aligned} \quad (5.14)$$

for any $t \in I$ and $s \in \bar{I}$, and $\|\cdot\|$ is the supremum norm on $C(\bar{I}, \mathbb{R})$. This together with (5.13) implies that

$$\|\Gamma \rho_n - \Gamma \rho\| \leq \frac{\bar{\varphi} \cdot \ell^2(\bar{I})}{A_j^2 \cdot e^{20A_j T}} \left(1 + e^{20A_j T} \right)^2 \cdot \|\rho_n - \rho\|.$$

Therefore, the continuity of Γ is proven. Next, we need to establish that $\Gamma(\mathcal{S})$ is equicontinuous. Assume that $\epsilon > 0$ is given, without loss of generality, $\xi_1 < \xi_2$ are arbitrarily taken from $\bar{I}$ and $\rho \in \mathcal{S}$. Applying condition (i) we have

$$\begin{aligned} \left| (\Gamma \rho)(\xi_2) - (\Gamma \rho)(\xi_1) \right| &\leq \int_{\xi_1}^{\xi_2} \left(|\Phi(s, \omega) - \Phi(s, 0)| + |\Phi(s, 0)| \right) ds \\ &\leq \frac{\bar{\varphi}}{A_j^2 \cdot e^{20A_j T}} \left(1 + e^{20A_j T} \right)^2 \int_{\xi_1}^{\xi_2} \left[|\langle f, \psi \rangle| + \int_0^s |\rho(r)| dr + \bar{F} \right] ds \\ &\leq \frac{\bar{\varphi}}{A_j^2 \cdot e^{20A_j T}} \left(1 + e^{20A_j T} \right)^2 \left[|\langle f, \psi \rangle| + \int_{\bar{I}} |\mathcal{H}_j(r)| dr + \bar{F} \right] (\xi_2 - \xi_1) \end{aligned}$$

which vanishes as $\xi_1 \rightarrow \xi_2$. Consequently, we conclude that $\Gamma(\mathcal{S})$ is equicontinuous on the compact interval $\bar{I}$. Moreover, in view of Arzelà–Ascoli theorem one can see that $\Gamma(\mathcal{S})$ is relatively compact. Therefore, all the conditions of Schauder fixed point

theorem are fulfilled and the operator Γ , as a self-map on $\mathcal{S}$, has a fixed point in this set. This fact implies that Eq. (5.11) has at least one solution in $\mathcal{S}$. $\square$

An immediate consequence of Theorem 5.2 is given as follows.

Theorem 5.3 *Suppose that all the conditions of Theorem 5.2 are satisfied. Then the problem (5.1) has a solution in $\mathcal{U}$ given in Theorem 3.1.*

Proof From Theorem 5.2, it is possible to find the solutions $\rho := \omega'_\xi$ and since then ω for Eq. (5.11). This together with (5.8) yields $u_t = u_{1t} \cdot \omega_t$ exists which means that the function

$$u(t, x) = \sum_{\iota \in J} u_\iota(t) \psi_\iota(x), \quad \iota = (k, j, n) \in J := J_p \times \mathbb{Z} \times \mathbb{Q}_p / \mathbb{Z}_p,$$

as a solution of Eq. (5.3) exists under the imposing hypotheses. This completes the proof. $\square$

Example 3 Consider the problem (5.1) with $c = 0$, $b > a^2$, $T < \frac{x^*}{\sqrt{b-a^2}}$ where x^* is the root of (5.2), and $f = g = 0$. We remark that

$$A_j T = \sqrt{b - a^2} \cdot T < x^*.$$

Assume F_t , as given in (5.4), has the form $F_t(t, u) = \phi_t(t) \sigma_t(u)$ where ϕ_t is continuous on I and σ_t belongs to Σ as the class of all increasing convex functions satisfying

$$\|\sigma'_t\| < \infty, \quad \left(\sigma_t(\alpha t) \leq \alpha \sigma_t(t), \quad \forall t \in \mathbb{R} \iff \alpha \geq 0 \right).$$

Obviously, $\Sigma \neq \emptyset$ since any increasing linear function is contained in Σ . We note that

$$|F_t(t, u_t) - F_t(t, v_t)| \leq |\phi_t(t)| \cdot \|\sigma'_t\| \cdot |u_t - v_t|, \quad (t, \iota) \in I \times J,$$

that is, the condition (i) in Theorem 5.2 is fulfilled. To check the condition (ii), for the variable $s = r(t)$ as given in (5.8), we derive

$$\begin{aligned} \left| \int_0^\xi \Phi(s, \int_0^s \rho(r) dr) ds \right| &= \left| \int_0^\xi \frac{u_{1t}^3(t) \phi_t(t)}{[W(u_{1t}, u_{2t})(t)]^2} \sigma_t(u_{1t}(t)) \int_0^s \rho(r) dr ds \right| \\ &\leq \left| \frac{u_{1t}^4(t_0) \phi_t(t_0)}{[W(u_{1t}, u_{2t})(t_0)]^2} \right| \cdot \left| \int_0^\xi \sigma_t \left(\int_0^s \rho(r) dr \right) ds \right|, \\ &\quad \text{for } t_0 \in I \text{ with } r(t_0) \in [0, \xi], \end{aligned}$$

followed by a variant of the Mean Value Theorem and the fact that $\sigma_t \in \Sigma$. Now, for any $\xi \in \tilde{I}$ if $|\rho(\xi)| \leq \mathcal{H}_j(\xi)$ then using Jensen's inequality we conclude that

$$\left| \int_0^\xi \Phi(s, \int_0^s \rho(r) dr) ds \right| \leq \frac{B_j^2 \|\phi_t\|}{A_j^2 \cos^2(r_j)} \cdot \int_0^\xi \int_0^s |(\sigma_t \circ \mathcal{H}_j)(r)| dr ds$$

where the constants are defined by (5.9). Therefore, for given ϕ_i and σ_i if one can find a function $\mathcal{H}_j$ with nonnegative values on $\bar{I}$ satisfying the following inequality

$$\frac{B_j^2 \|\phi_i\|}{A_j^2 \cos^2(r_j)} \cdot \int_0^\xi \int_0^s |(\sigma_i \circ \mathcal{H}_j)(r)| dr ds \leq \mathcal{H}_j(\xi), \quad (5.15)$$

then condition (ii) is fulfilled. To illustrate this, for instance one can take $\sigma_i = id$ and $\mathcal{H}_j(x) = e^{\mu x}$ with $\mu = \frac{A_j \cos(r_j)}{B_j \|\phi_i\|}$ and derive that (5.15) holds. Now, applying Theorem 5.2 we conclude the problem (5.1) with the imposed conditions has a solution in $\mathcal{U}$.

Acknowledgements Open access funding provided by Linnaeus University.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Albeverio, S., Khrennikov, A.Yu., Shelkovich, V.M.: Harmonic analysis in the p -adic Lizorkin spaces: fractional operators, pseudo-differential equations, p -adic wavelets, Tauberian theorems. *J. Fourier Anal. Appl.* **12**(4), 393–425 (2006)
2. Albeverio, S., Khrennikov, A.Yu., Shelkovich, V.M.: p -Adic semi-linear evolutionary pseudo-differential equations in the Lizorkin space, *Dokl. Ross. Akad. Nauk* **415**(3), 295–299 (2007); English transl. in: *Russian Dokl. Math.* **76**(1), 539–543 (2007)
3. Albeverio, S., Khrennikov, A.Yu., Shelkovich, V.M.: *Theory of p -adic Distributions: Linear and Non-linear Models*. Cambridge University Press, Cambridge (2010)
4. Albeverio, S., Khrennikov, A.Yu., Shelkovich, V.M.: The Cauchy problems for evolutionary pseudo-differential equations over p -adic field and the wavelet theory. *J. Math. Anal. Appl.* **375**, 82–98 (2011)
5. Antoniouk, A.V., Khrennikov, A.Yu., Kochubei, A.N.: Multidimensional nonlinear pseudo-differential evolution equation with p -adic spatial variables. *J. Pseudo-Differ. Oper. Appl.* (2019). <https://doi.org/10.1007/s11868-019-00320-3>
6. Aref'eva, I.Y., Dragovich, B.G., Volovich, I.V.: On the p -adic summability of the anharmonic oscillator. *Phys. Lett. B* **200**, 512–514 (1988)
7. Avantaggiati, A. (ed.): *Pseudodifferential Operators with Applications*. Springer, Berlin (2010)
8. Casas-Sánchez, O.F., Zúñiga-Galindo, W.A.: p -Adic elliptic quadratic forms, parabolic-type pseudodifferential equations with variable coefficients and Markov processes. *p-Adic Numbers Ultramet. Anal. Appl.* **6**(1), 1–20 (2014)
9. Casas-Sánchez, O.F., Galeano-Peñaloza, J., Rodríguez-Vega, J.J.: Parabolic-type pseudodifferential equations with elliptic symbols in dimension 3 over p -adics. *p-Adic Numbers Ultramet. Anal. Appl.* **7**(1), 1–16 (2015)
10. Chuong, N.M., Co, N.V.: The Cauchy problem for a class of pseudodifferential equations over p -adic field. *J. Math. Anal. Appl.* **340**, 629–645 (2008)
11. Dragovich, B.G.: Adelic harmonic oscillator. *Int. J. Mod. Phys. A* **10**, 2349–2359 (1995)
12. Dragovich, B.G.: Path integrals for quadratic Lagrangians on p -adic and adelic spaces. *p-Adic Numbers Ultramet. Anal. Appl.* **2**(4), 322–340 (2010)

13. Dragovich, B.G., Misic, N.Z.: p -Adic hierarchical properties of the genetic code. *Biosystems* **185**, 104017 (2019)
14. Haar, A.: Zur Theorie der orthogonalen Funktionensysteme. *Math. Ann.* **69**, 331–371 (1910)
15. Khrennikov, A.Yu.: p -Adic Valued Distributions in Mathematical Physics. Kluwer Academic Publishers, Dordrecht (1994)
16. Khrennikov, A.Yu., Kochubei, A.N.: p -Adic analogue of the porous medium equation. *J. Fourier Anal. Appl.* **24**, 1401–1424 (2018)
17. Khrennikov, A.Yu., Shelkovich, V.M.: Non-Haar p -adic wavelets and their application to pseudo-differential operators and equations. *Appl. Comput. Harmonic Anal.* **28**, 1–23 (2010)
18. Khrennikov, A.Yu., Shelkovich, V.M., Van Der Walt, J.H.: Adelic multiresolution analysis, construction of wavelet bases and pseudo-differential operators. *J. Fourier Anal. Appl.* **19**, 1323–1358 (2013)
19. Khrennikov, A., Oleschko, K., López, M.J.C.: Application of p -adic wavelets to model reaction-diffusion dynamics in random porous media. *J. Fourier Anal. Appl.* **22**, 809–822 (2016)
20. Khrennikov, A., Oleschko, K., López, M.J.C.: Modeling fluid's dynamics with master equations in ultrametric spaces representing the treelike structure of capillary networks. *Entropy* **18**(7), art. 249 (2016)
21. Khrennikov, A., Oleschko, K., Correa Lopez, M.J.: p -Adic analog of Navier-Stokes equations: dynamics of fluid's flow in percolation networks (from discrete dynamics with hierarchic interactions to continuous universal scaling model). *Entropy* **19**, 161 (2017)
22. Khrennikov, A.Yu., Kozyrev, S.V., Zúñiga-Galindo, W.A.: *Ultrametric Pseudodifferential Equations and Applications*. Cambridge University Press, Cambridge (2018)
23. Kozyrev, S.V.: Wavelet theory as p -adic spectral analysis. *Izv. Math.* **66**(2), 367–376 (2002)
24. Kozyrev, S.V.: p -adic pseudodifferential operators and p -adic wavelets. *Theor. Math. Phys.* **138**, 322–332 (2004)
25. Kozyrev, S.V.: Wavelets and spectral analysis of ultrametric pseudodifferential operators. *Sb. Math.* **198**, 97–116 (2007)
26. Kozyrev, S.V.: Toward an ultrametric theory of turbulence. *Theor. Math. Phys.* **157**, 1713–1722 (2008)
27. Kozyrev, S.V.: Dynamics on rugged landscapes of energy and ultrametric diffusion. *p -Adic Numbers Ultramet. Anal. Appl.* **2**, 122–132 (2010)
28. Oleschko, K., Khrennikov, A.: Applications of p -adics to geophysics: linear and quasilinear diffusion of water-in-oil and oil-in-water emulsions. *Theor. Math. Phys.* **190**, 154–163 (2017)
29. Polyanin, A.D., Zaitsev, V.F.: *Handbook of Exact Solutions for Ordinary Differential Equations*. CRC Press, New York (1999)
30. Pourhadi, E., Khrennikov, A.Yu.: On the solutions of Cauchy problem for two classes of semilinear pseudo-differential equations over p -adic field, p -Adic Numbers. *Ultramet. Anal. Appl.* **10**(4), 322–343 (2018)
31. Pourhadi, E., Khrennikov, A., Saadati, R., Oleschko, K., Correa Lopez, M.J.: Solvability of the p -Adic analogue of Navier-Stokes equation via the wavelet theory. *Entropy* **21**(11), 1129 (2019)
32. Qiu, H., Su, W.Y.: Pseudo-differential operators over p -adic fields. *Sci. China Ser. A* **41**(4), 323–336 (2011)
33. Smart, D.R.: *Fixed Point Theorems*. Cambridge University Press, London (1974)
34. Su, W.Y.: Pseudo-differential operators and derivatives on locally compact Vilenkin groups. *Sci. China Ser. A* **35**(7), 826–836 (1992)
35. Taibleson, M.H.: *Fourier Analysis on Local Fields*. Princeton University Press, Princeton (1975)
36. Taylor, M.E.: *Pseudodifferential Operators and Nonlinear PDE*. Birkhäuser, Boston (1991)
37. Taylor, M.E.: *Tools for PDE: Pseudodifferential Operators, Paradifferential Operators, and Layer Potentials*, Mathematical Surveys and Monographs, 81. American Mathematical Society, Providence, RI (2000)
38. Vázquez, J.L.: *The Porous Medium Equation: Mathematical Theory*. Clarendon Press, Oxford (2007)
39. Vladimirov, V.S., Volovich, I.V., Zelenov, E.I.: *p -Adic Analysis and Mathematical Physics*. World Scientific, Singapore (1994)
40. Volovich, I.V.: P -adic space-time and string theory. *Theor. Math. Phys.* **71**, 574–576 (1987)
41. Zúñiga-Galindo, W.A.: Fundamental solutions of pseudo-differential operators over p -adic fields. *Rend. Semin. Mat. Univ. Padova* **109**, 241–245 (2003)
42. Zúñiga-Galindo, W.A.: Parabolic equations and Markov processes over p -adic fields. *Potential Anal.* **28**, 185–200 (2008)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.