On a Characterization of Riesz Bases via Biorthogonal Sequences

Diana T. Stoeva¹

Received: 21 March 2020 / Published online: 30 July 2020
© The Author(s) 2020

Abstract

It is well known that a sequence in a Hilbert space is a Riesz basis if and only if it is a complete Bessel sequence with biorthogonal sequence which is also a complete Bessel sequence. Here we prove that the completeness of one (any one) of the biorthogonal sequences can be removed from the characterization.

Keywords Riesz basis · Biorthogonal sequence · Complete sequence · Bessel sequence · Gabor system

Mathematics Subject Classification 42C15 · 46B15

1 Introduction and motivation

A *Riesz basis* for a separable Hilbert space $\mathcal{H}$ is a sequence of the form $(Ve_k)_{k=1}^\infty$ with $(e_k)_{k=1}^\infty$ being an orthonormal basis for $\mathcal{H}$ and V being a bounded bijective operator from $\mathcal{H}$ onto $\mathcal{H}$. Riesz bases were introduced by Bari [3,4] and already in [4] many properties and equivalent characterizations were determined. Below we collect the standard equivalences of Riesz bases from [6,9,12], some of which appeared already in [4,7]:

Theorem 1.1 *For a sequence $(f_k)_{k=1}^\infty$ in a Hilbert space $\mathcal{H}$, the following conditions are equivalent:*

- ($\mathcal{R}_1$) $(f_k)_{k=1}^\infty$ forms a Riesz basis for $\mathcal{H}$.
- ($\mathcal{R}_2$) $(f_k)_{k=1}^\infty$ is a complete Bessel sequence in $\mathcal{H}$ and it has a biorthogonal sequence $(g_k)_{k=1}^\infty$ which is also a complete Bessel sequence in $\mathcal{H}$.

Communicated by Hans G. Feichtinger.

✉ Diana T. Stoeva
dstoeva@kfs.oew.ac.at

¹ Acoustics Research Institute, Austrian Academy of Sciences, Wohllebengasse 12-14, 1040 Vienna, Austria

- ($\mathcal{R}_3$) $(f_k)_{k=1}^\infty$ is complete in $\mathcal{H}$ and it has a complete biorthogonal sequence $(g_k)_{k=1}^\infty$ so that $\sum_{k=1}^\infty |\langle f, f_k \rangle|^2 < \infty$ and $\sum_{k=1}^\infty |\langle f, g_k \rangle|^2 < \infty$ for every $f \in \mathcal{H}$.
- ($\mathcal{R}_4$) $(f_k)_{k=1}^\infty$ is complete in $\mathcal{H}$ and there exist positive constants A and B so that

$$A \sum |c_k|^2 \leq \left\| \sum c_k f_k \right\|^2 \leq B \sum |c_k|^2 \quad (1.1)$$

for every finite scalar sequence (c_k) (and hence for every $(c_k)_{k=1}^\infty \in \ell^2$).

- ($\mathcal{R}_5$) $(f_k)_{k=1}^\infty$ is complete in $\mathcal{H}$ and its Gram matrix $(\langle f_k, f_j \rangle)_{j,k=1}^\infty$ determines a bounded bijective operator on ℓ^2 .
- ($\mathcal{R}_6$) $(f_k)_{k=1}^\infty$ is a bounded unconditional basis for $\mathcal{H}$.
- ($\mathcal{R}_7$) $(f_k)_{k=1}^\infty$ is a basis for $\mathcal{H}$ such that $\sum_{k=1}^\infty c_k f_k$ converges in $\mathcal{H}$ if and only if $\sum_{k=1}^\infty |c_k|^2 < \infty$.

The main purpose of this paper is to show that one may remove the condition for completeness of one (any one) of the sequences $(f_k)_{k=1}^\infty$ and $(g_k)_{k=1}^\infty$ in ($\mathcal{R}_2$), and thus in ($\mathcal{R}_3$) as well. In general, it is a more difficult task to check the lower Riesz basis condition in (1.1) compare to the upper one. From this point of view, ($\mathcal{R}_2$) is a useful characterization of Riesz bases, avoiding the verification of the lower condition. On the other hand, completeness might not be simple to check either. From this point of view, removing the verification of completeness of one of the sequences from ($\mathcal{R}_2$) is of significant importance when checking the Riesz basis property. It can also be used in further research, for example it applies to obtain conclusions about certain Gabor systems at the critical density (see the end of Sect. 2).

The question of completeness of biorthogonal sequences is of interest in itself as well. As it is well known (see Example 2.2), completeness of a minimal sequence does not imply completeness of the biorthogonal one. Throughout the years it has been of interest to determine classes of complete minimal systems, whose biorthogonal systems are automatically complete, see e.g. [2, 5, 13]. In this paper we give a simple relation between the Bessel property and completeness (Proposition 2.3), which is of interest in itself and also contributes to the main purpose of the paper.

Let us end the section with some notation and results that will be needed in the sequel. In the entire paper, $\mathcal{H}$ denotes a separable Hilbert space. Recall that a sequence $(f_k)_{k=1}^\infty$ with elements from $\mathcal{H}$ is called: a *Bessel sequence in $\mathcal{H}$* if there exists a positive constant B so that $\sum_{k=1}^\infty |\langle h, f_k \rangle|^2 \leq B \|h\|^2$ for every $h \in \mathcal{H}$; *minimal* if for every $n \in \mathbb{N}$, the element f_n does not belong to the closed linear span of $(f_k)_{k \neq n}$. Given a Bessel sequence $F = (f_k)_{k=1}^\infty$ in $\mathcal{H}$, the synthesis operator T_F given by $T_F(c_k) = \sum_{k=1}^\infty c_k f_k$ is well defined and bounded from ℓ^2 into $\mathcal{H}$ (see, e.g., [6, Theorem 3.2.3]).

2 Removing a Condition from ($\mathcal{R}_2$)

In this section we obtain a characterization of Riesz bases with relaxed conditions compare to the ones in ($\mathcal{R}_2$). We show that one (any one) of the assumptions for

completeness in $(\mathcal{R}_2)$ can be omitted. First let us prove that one can remove the assumption for completeness of $(f_k)_{k=1}^\infty$:

Theorem 2.1 *For a sequence $(f_k)_{k=1}^\infty$ in $\mathcal{H}$, the following conditions are equivalent:*

- (i) $(f_k)_{k=1}^\infty$ is a Riesz basis for H .
- (ii) $(f_k)_{k=1}^\infty$ is a Bessel sequence in $\mathcal{H}$ and it has a biorthogonal sequence $(g_k)_{k=1}^\infty$ which is a complete Bessel sequence in $\mathcal{H}$.

Proof (ii) $\Rightarrow$ (i): let T_F and T_G denote the synthesis operators of the Bessel sequences $(f_k)_{k=1}^\infty$ and $(g_k)_{k=1}^\infty$, respectively.

First observe that T_F is injective. Indeed, if $T_F(c_k)_{k=1}^\infty = 0$ for some $(c_k)_{k=1}^\infty \in \ell^2$, then for every $j \in \mathbb{N}$ we have $0 = \langle \sum_{k=1}^\infty c_k f_k, g_j \rangle = c_j$.

For the surjectivity of T_F , take an arbitrary $f \in \mathcal{H}$ and observe that for every $j \in \mathbb{N}$ one has

$$\langle T_F T_G^* f, g_j \rangle = \sum_{k=1}^\infty \langle f, g_k \rangle \langle f_k, g_j \rangle = \langle f, g_j \rangle, \quad (2.1)$$

which by the completeness of $(g_j)_{j=1}^\infty$ implies that $f = T_F T_G^* f$. Thus, T_F is surjective.

For every $k \in \mathbb{N}$, $f_k = T_F \delta_k$. Then $(f_k)_{k=1}^\infty$ is the image of an orthonormal basis under a bounded bijective operator, which is precisely the definition of a Riesz basis.

(i) $\Rightarrow$ (ii): follows from Theorem 1.1. $\square$

Let us now consider relations between biorthogonality and completeness. It is well known that completeness of one of two biorthogonal sequences does not imply completeness of the other one:

Example 2.2 [13] Let $\mathcal{H}$ be infinite-dimensional. Take an infinite-dimensional closed proper subspace K of $\mathcal{H}$, an orthonormal basis $(g_k)_{k=1}^\infty$ for K , and an orthonormal basis $\mathcal{B}$ for $K^\perp$. Let $(y_k)_{k=1}^\infty$ be a sequence which contains every element of $\mathcal{B}$ infinitely many times. Consider $f_k := g_k + y_k$, $k \in \mathbb{N}$. Then $(f_k)_{k=1}^\infty$ is complete in $\mathcal{H}$, $(f_k)_{k=1}^\infty$ and $(g_k)_{k=1}^\infty$ are biorthogonal, and clearly $(g_k)_{k=1}^\infty$ is not complete in $\mathcal{H}$. As a simple concrete case, consider (g_k) being $(e_n)_{n=2}^\infty$ and (f_k) being $(e_n + e_1)_{n=2}^\infty$, where $(e_k)_{k=1}^\infty$ denotes an orthonormal basis for $\mathcal{H}$.

Here we observe that adding the Bessel assumption leads to a different conclusion regarding completeness. We show that two biorthogonal sequences which are both Bessel sequences are either both complete or both incomplete:

Proposition 2.3 *Let $(f_k)_{k=1}^\infty$ and $(g_k)_{k=1}^\infty$ be Bessel sequences in $\mathcal{H}$ which are biorthogonal. If one of them is complete in $\mathcal{H}$, then the other one is also complete in $\mathcal{H}$.*

Proof Without loss of generality, assume that $(g_k)_{k=1}^\infty$ is complete in $\mathcal{H}$. Fix an arbitrary $f \in \mathcal{H}$. As in the proof of Theorem 2.1, for every $f \in \mathcal{H}$ and every $j \in \mathbb{N}$, the biorthogonality assumption leads to validity of (2.1) and then the completeness of $(g_k)_{k=1}^\infty$ in $\mathcal{H}$ implies that $f = T_F T_G^* f = \sum_{k=1}^\infty \langle f, g_k \rangle f_k$. Therefore, $(f_k)_{k=1}^\infty$ is complete in $\mathcal{H}$. $\square$

Note that the proof of the above statement also leads to the conclusion that $(f_k)_{k=1}^\infty$ and $(g_k)_{k=1}^\infty$ are dual frames.

Let us stress that in Proposition 2.3 it is essential to assume the Bessel condition for both sequences. If only one of the biorthogonal sequences is assumed to be Bessel, the conclusion of Proposition 2.3 does not hold in general. Consider for example the Bessel sequence $(g_k)_{k=1}^\infty$ and the non-Bessel sequence $(f_k)_{k=1}^\infty$ in Example 2.2. Note, however, that the Bessel property is not necessary for completeness of two biorthogonal sequences. Consider the following simple examples:

Example 2.4 Let $(e_k)_{k=1}^\infty$ denote an orthonormal basis for $\mathcal{H}$. The biorthogonal sequences: $(\frac{1}{k}e_k)_{k=1}^\infty$ and $(ke_k)_{k=1}^\infty$ are both complete in $\mathcal{H}$, while only one of them is a Bessel sequence in $\mathcal{H}$; $(e_1, 2e_2, \frac{1}{3}e_3, 4e_4, \frac{1}{5}e_5, \dots)$ and $(e_1, \frac{1}{2}e_2, 3e_3, \frac{1}{4}e_4, 5e_5, \dots)$ are both complete in $\mathcal{H}$, while none of them is a Bessel sequence in $\mathcal{H}$.

As a consequence of Proposition 2.3 and Theorem 2.1, one can now write the following characterization of Riesz bases, removing one (any one) of the completeness conditions from $(\mathcal{R}_2)$.

Theorem 2.5 *For a sequence $(f_k)_{k=1}^\infty$ in $\mathcal{H}$, the following conditions are equivalent:*

- (i) $(f_k)_{k=1}^\infty$ is a Riesz basis for H .
- (ii) $(f_k)_{k=1}^\infty$ is a Bessel sequence in $\mathcal{H}$, it has a biorthogonal sequence $(g_k)_{k=1}^\infty$ which is also a Bessel sequence in $\mathcal{H}$, and one of $(f_k)_{k=1}^\infty$ and $(g_k)_{k=1}^\infty$ is complete in $\mathcal{H}$.

Applications. For many structured systems, completeness is not a simple property to check. From this point of view, the removal of a verification for completeness from $(\mathcal{R}_2)$ can be very useful for simpler verification of the Riesz basis property.

On the other hand, Theorem 2.5 can be used to get conclusions which can not be derived from the equivalence of $(\mathcal{R}_1)$ and $(\mathcal{R}_2)$. Consider for example the Gaussian $g(x) = e^{-\pi x^2}$, a sequence Λ of uniformly separated points from $\mathbb{R}^2$, and the associated Gabor system $G_{g,\Lambda} = (e^{2\pi i \mu \cdot} g(\cdot - \tau))_{(\tau, \mu) \in \Lambda}$. It is well known that $G_{g,\Lambda}$ is a Bessel sequence (follows e.g. from [8, Sect. 5.1]) but not a Riesz basis for $L^2(\mathbb{R})$, see, e.g. [11]. However, for certain Λ , $G_{g,\Lambda}$ is a complete minimal system in $L^2(\mathbb{R})$ —this is for example the case when $\Lambda = \{(-1, 0), (1, 0)\} \cup \{(0, \pm\sqrt{2n}) : n = 1, 2, 3, \dots\} \cup \{(\pm\sqrt{2n}, 0) : n = 1, 2, 3, \dots\}$ [1], or when Λ is the lattice $\mathbb{Z} \times \mathbb{Z}$ without the point $(1, 0)$, see, e.g. [10]. For such Λ 's, based on Theorem 1.1, one can conclude that the biorthogonal system of $G_{g,\Lambda}$ is non-Bessel or non-complete, while Theorem 2.5 immediately implies that the biorthogonal system is non-Bessel. We also expect further applications of Theorem 2.5 for other structured systems.

Acknowledgements The author is grateful to the reviewer for the valuable suggestions that improved the paper. She also acknowledges support from the Austrian Science Fund (FWF) through the Project “FLAME” Y 551-N13 and from the Vienna Science and Technology Fund (WWTF) through the Project VRG12-009.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If

material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Ascensi, G., Lyubarskii, Y., Seip, K.: Phase space distribution of Gabor expansions. *Appl. Comput. Harmon. Anal.* **26**(2), 277–282 (2009)
2. Baranov, A., Belov, Y.: Systems of reproducing kernels and their biorthogonal: completeness or incompleteness? *Int. Math. Res. Not.* **2011**(22), 5076–5108 (2011)
3. Bari, N.: Sur les bases dans l'espace de Hilbert. *C. R. (Dokl.) Acad. Sci. URSS* **54**, 379–382 (1946)
4. Bari, N.K.: Biorthogonal systems and bases in Hilbert space. In: *Mathematics*, Vol. IV, Uch. Zap. Mosk. Gos. Univ., issue 148, pp. 69–107. Moscow University Press, Moscow (1951)
5. Belov, Y.: Uniqueness of Gabor series. *Appl. Comput. Harmon. Anal.* **39**(3), 545–551 (2015)
6. Christensen, O.: *An Introduction to Frames and Riesz Bases*. Applied and Numerical Harmonic Analysis Series, Second Expanded Edition. Birkhäuser, Boston (2016)
7. Gel'fand, I.M.: Remark on the work of N.K. Bari "Biorthogonal systems and bases in Hilbert space". In: *Mathematics*, Vol. IV, Uch. Zap. Mosk. Gos. Univ., issue 148, pp. 224–225. Moscow University Press, Moscow (1951)
8. Heil, C.: History and evolution of the density theorem for Gabor frames. *J. Fourier Anal. Appl.* **13**(2), 113–166 (2007)
9. Heil, C.: *A Basis Theory Primer*, Expanded edition. Birkhäuser, Basel (2011)
10. Lyubarskii, Yu., Seip, K.: Convergence and summability of Gabor expansions at the Nyquist density. *J. Fourier Anal. Appl.* **5**, 127–157 (1999)
11. Seip, K.: Density theorems for sampling and interpolation in the Bargmann–Fock space I. *J. Reine Angew. Math.* **429**, 91–106 (1992)
12. Young, R.M.: *An Introduction to Nonharmonic Fourier Series*. Academic, New York (1980)
13. Young, R.M.: On complete biorthogonal systems. *Proc. Am. Math. Soc.* **83**(3), 537–540 (1981)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.