Endpoint Estimates for the Maximal Function over Prime Numbers

Bartosz Trojan¹

Received: 22 August 2018 / Revised: 18 June 2019 / Published online: 3 September 2019
© The Author(s) 2019

Abstract

Given an ergodic dynamical system $(X, \mathcal{B}, \mu, T)$, we prove that for each function f belonging to the Orlicz space $L(\log L)^2(\log \log L)(X, \mu)$, the ergodic averages

$$\frac{1}{\pi(N)} \sum_{p \in \mathbb{P}_N} f(T^p x),$$

converge for μ -almost all $x \in X$, where $\mathbb{P}_N$ is the set of prime numbers not larger than N and $\pi(N) = \#\mathbb{P}_N$.

Keywords Weak maximal ergodic inequality · Orlicz space · Prime numbers · Pointwise convergence

Mathematics Subject Classification Primary 37A45 · Secondary 46E30, 42B25

1 Introduction

Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system, that is $(X, \mathcal{B}, \mu)$ is a probability space with a measurable and measure preserving transformation $T : X \rightarrow X$. The classical Birkhoff theorem [1] states that for any function f from $L^p(X, \mu)$ with $p \in [1, \infty)$, the ergodic averages

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$$

Communicated by Hans G. Feichtinger.

✉ Bartosz Trojan
btrojan@impan.pl

¹ Institute of Mathematics, Polish Academy of Science, ul. Śniadeckich 8, 00-656 Warsaw, Poland

converge for μ -almost all $x \in X$. This classical result, among others, motivates studying ergodic averages over subsequences of integers. In this article we are interested in pointwise convergence of the following averages,

$$\mathcal{A}_N f(x) = \frac{1}{\pi(N)} \sum_{p \in \mathbb{P}_N} f(T^p x)$$

where $\mathbb{P}_N$ is the set of prime numbers not larger than N and $\pi(N) = \#\mathbb{P}_N$. The problem of ergodic averages along prime numbers was initially studied by Bourgain in [3] where the case of functions belonging to $L^2(X, \mu)$ has been covered. It was extended by Wierdl in [21] to all $L^p(X, \mu)$, for $p > 1$, see also [5, Sect. 9]. However, the endpoint $p = 1$, was left open for more than twenty years. Following the method developed in [6] by Buczolich and Mauldin, LaVictoire in [12] has shown that for each ergodic dynamical system there exists $f \in L^1(X, \mu)$ such that the sequence $(\mathcal{A}_N f : N \in \mathbb{N})$ diverges on a set of positive measure.

The purpose of this article is to find an Orlicz space close to $L^1(X, \mu)$ where the almost everywhere convergence holds. We show the following theorem (see Theorem 7.4).

Theorem A *For each $f \in L(\log L)^2(\log \log L)(X, \mu)$, the limit*

$$\lim_{N \rightarrow \infty} \mathcal{A}_N f(x)$$

exists for μ -almost all $x \in X$.

In light of the pointwise convergence obtained by Bourgain in [4], see also [15], to prove Theorem A it suffices to show the weak maximal ergodic inequality for functions in Orlicz space $L(\log L)^2(\log \log L)(X, \mu)$. This inequality is deduced from the following restricted weak Orlicz estimate.

Theorem B *There is $C > 0$ such that for any subset $A \subset X$,*

$$\mu \left\{ x \in X : \sup_{N \in \mathbb{N}} \mathcal{A}_N (\mathbb{1}_A)(x) > \lambda \right\} \leq C \lambda^{-1} \log^2(e/\lambda) \mu(A)$$

for all $1 > \lambda > 0$.

By appealing to the Calderón transference principle, see [7], Theorem B is deduced from the corresponding result for integers $\mathbb{Z}$ with the counting measure and the shift operator. To be more precise, for a function $f : \mathbb{Z} \rightarrow \mathbb{C}$, we define

$$\mathcal{A}_N f(x) = \frac{1}{\pi(N)} \sum_{p \in \mathbb{P}_N} f(x + p).$$

Our main result is following theorem (see Theorem 6.3).

Theorem C *There is $C > 0$ such that for any subset $F \subset \mathbb{Z}$ of a finite cardinality*

$$\left| \left\{ x \in \mathbb{Z} : \sup_{N \in \mathbb{N}} \mathcal{A}_N(\mathbb{1}_F)(x) > \lambda \right\} \right| \leq C \lambda^{-1} \log^2(e/\lambda) |F|$$

for all $0 < \lambda < 1$.

Theorem C together with $\ell^2(\mathbb{Z})$ estimates are sufficiently strong to imply the maximal inequality for all $\ell^p(\mathbb{Z})$ spaces, for $p > 1$, giving an alternative proof of the Wierld's theorem [21].

Let us now give some details about the proof of Theorem C. Without loss of generality, we may restrict the supremum to dyadic numbers. It is more convenient to work with weighted averages $\mathcal{M}_N f$ instead of $\mathcal{A}_N f$ where

$$\mathcal{M}_N f(x) = \frac{1}{\vartheta(N)} \sum_{p \in \mathbb{P}_N} f(x+p) \log p,$$

and

$$\vartheta(N) = \sum_{p \in \mathbb{P}_N} \log p.$$

Given $t > 0$, for each $n \in \mathbb{N}$, we decompose the operator $\mathcal{M}_{2^n}$ into two parts A_n^t and B_n^t , in such a way that the maximal function associated with A_n^t has $\ell^{1,\infty}(\mathbb{Z})$ norm $\lesssim t \|f\|_{\ell^1}$, whereas the one corresponding to B_n^t has $\ell^2(\mathbb{Z})$ norm $\lesssim \exp(-c\sqrt{t}) \|f\|_{\ell^2}$. When applied to the distribution function $|\{ \sup_{n \in \mathbb{N}} \mathcal{M}_{2^n}(\mathbb{1}_F) > \lambda \}|$, we can optimize both estimates by taking $t \simeq \log^2(e/\lambda)$. This idea originated to Fefferman [9], see also Bourgain [2]. Ionescu introduced this technique in a related discrete context, see [10]. The decomposition of $\mathcal{M}_{2^n}$ uses the circle method of Hardy and Littlewood. However, to achieve the exponential decay of the error term, due to the Page's theorem, the approximating multiplier has to contain the second term of the asymptotic as well. Thus, the possible existence of the Siegel zero entails that in the neighborhood of the rational point a/q the approximating multiplier $\widehat{L}_{2^n}^{a,q}(\cdot - a/q)$ depends on the rational number a/q . We refer to Sects. 3 and 5 for details. Thanks to the log-convexity of $\ell^{1,\infty}(\mathbb{Z})$, the weak type estimates are reduced to showing

$$\left| \left\{ x \in \mathbb{Z} : \sup_{t \leq n} \sum_{a \in A_q} \mathcal{F}^{-1}(\widehat{L}_{2^n}^{a,q}(\cdot - a/q) \eta_s(\cdot - a/q) \hat{f})(x) > \lambda \right\} \right| \leq C \frac{1}{\lambda \varphi(q)} \|f\|_{\ell^1}$$

for $2^s \leq q < 2^{s+1}$ with $1 \leq s \leq \sqrt{t}$. At this stage we exploit the behavior of the Gauss sums described in Theorem 2.1.

Let us emphasize that under the Generalized Riemann Hypothesis we can obtain in Proposition 3.1, and consequently in Theorem 3.2, a better error estimate. However, it is not clear whether one can prove Theorem 6.1 with the bounds proportional to $\sqrt{t} \|f\|_{\ell^1}$.

The paper is organized as follows. In Sect. 2, we collect necessary facts about Dirichlet characters and the zero-free region. Then we evaluate the Gauss sum that appears in the approximating multiplier (Theorem 2.1). Section 3 is devoted to construction of the approximating multipliers. In Sects. 5 and 6, we show ℓ^2 and the weak type estimates, respectively. In Sect. 7, we give two applications of Theorem C. Namely, we show how to deduce the maximal ergodic inequality for functions from $\ell^p(\mathbb{Z})$, (Theorem 7.1). Next we apply the transference principle (Proposition 7.3) and show almost everywhere convergence of the ergodic averages $(\mathcal{A}_N f : N \in \mathbb{N})$ for $f \in L(\log L)^2(\log \log L)(X, \mu)$, (Theorem 7.4).

Notation

Throughout the whole article, we write $A \lesssim B$ ($A \gtrsim B$) if there is an absolute constant $C > 0$ such that $A \leq CB$, ($A \geq CB$). Moreover, C stands for a large positive constant which value may vary from occurrence to occurrence. If $A \lesssim B$ and $A \gtrsim B$ hold simultaneously then we write $A \simeq B$. The set of positive integers and the set of prime numbers are denoted by $\mathbb{N}$ and $\mathbb{P}$, respectively. For $x > 0$, we set $\mathbb{Z}_x = [1, x] \cap \mathbb{N}$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2 Gauss sums

We start by recalling some basic facts from number theory. A general reference here is the book [16].

A homomorphism

$$\chi : (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \mathbb{C}^\times,$$

is called a Dirichlet character modulo q . The simplest example, called the *principal* character modulo q , is defined as

$$\mathbb{1}_q(x) = \begin{cases} 1 & \text{if } \gcd(x, q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

A character χ modulo q is *primitive*, if q is the least integer d , such that $\chi(m) = \chi(n)$ for all $m \equiv n \pmod{d}$ and $(mn, q) = 1$. For each character χ there is the unique primitive character χ^* modulo q_0 for some $q_0 \mid q$, such that

$$\chi(n) = \begin{cases} \chi^*(n) & \text{if } \gcd(n, q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The character is *quadratic* if it takes only values $\{-1, 0, 1\}$ with at least one -1 . Recall that, if χ^* is a primitive quadratic character with modulus q_0 , then

- $q_0 \equiv 1 \pmod{4}$, and q_0 is square-free, or

- $4 \mid q_0$, $q_0/4 \equiv 2$ or $3 \pmod{4}$, and $q_0/4$ is square-free.

Given a Dirichlet character χ and $s \in \mathbb{C}$ with $\Re s > 1$, we define the Dirichlet L -function by the formula

$$L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}.$$

In fact, $L(\cdot, \chi)$ extends to the analytic function in $\{z \in \mathbb{C} : \Re z > 0\}$. There is an absolute constant $c > 0$, such that if χ is a Dirichlet character modulo q , then the region

$$\left\{ z \in \mathbb{C} : 1 - \frac{c}{\log q} < \Re z < 1 \right\} \quad (1)$$

contains at most one zero of $L(\cdot, \chi)$, which we denote by β_q . The zero β_q is real and the corresponding character is quadratic. The character having zero in (1) is called *exceptional*. Since $L(\beta, \chi) = 0$ implies that $L(1 - \beta, \chi) = 0$, we may assume that $\frac{1}{2} \leq \beta_q < 1$.

The Gauss sum of a Dirichlet character χ modulo q is defined as

$$G(\chi, n) = \frac{1}{\varphi(q)} \sum_{r \in A_q} \chi(r) e^{2\pi i r n / q}$$

where $A_q = \{1 \leq a \leq q : \gcd(a, q) = 1\}$, and $\varphi(q) = \#A_q$. Let us recall that for each $\epsilon > 0$ there is $C_\epsilon > 0$ such that

$$\varphi(q) \geq C_\epsilon q^{1-\epsilon}. \quad (2)$$

We set

$$\tau(\chi) = \varphi(q) G(\chi, 1).$$

Let us denote by μ the Möbius function, which is defined for $q = p_1^{\alpha_1} \dots p_n^{\alpha_n}$, where $p_1, \dots, p_n$ are distinct primes, as

$$\mu(q) = \begin{cases} (-1)^n & \text{if } \alpha_1 = \dots = \alpha_n = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and $\mu(1) = 1$. The following theorem plays the crucial role in Sect. 6.

Theorem 2.1 *Let χ be a quadratic Dirichlet character modulo q induced by χ^* having the conductor q_0 . For $x \in \mathbb{Z}$, we set $r = \gcd(q, x)$. Then*

$$\sum_{a \in A_q} G(\chi, a) e^{2\pi i x a / q} = \mu(r) q_0 \frac{\varphi(r)}{\varphi(q)} \chi^*(-x)$$

provided that q/q_0 is square-free, $\gcd(q/q_0, q_0) = 1$ and $r \mid q/q_0$. Otherwise the sum equals zero.

Proof By [16, Theorem 9.12], if $r \mid q/q_0$ then

$$\sum_{a \in A_q} \chi(a) e^{2\pi i a x / q} = \frac{\varphi(q)}{\varphi(q/r)} \chi^*(x/r) \chi^*(q/(rq_0)) \mu(q/(rq_0)) \tau(\chi^*), \quad (3)$$

otherwise the sum equals zero. In particular, for $a \in A_q$, we have

$$G(\chi, a) = \frac{\mu(q/q_0)}{\varphi(q)} \chi^*(a) \chi^*(q/q_0) \tau(\chi^*). \quad (4)$$

Hence, $G(\chi, a) \neq 0$ entails that q/q_0 is square-free and $\gcd(q/q_0, q_0) = 1$. Next, using (4) and (3) we get

$$\begin{aligned} \sum_{a \in A_q} G(\chi, a) e^{2\pi i x a / q} &= \frac{\mu(q/q_0)}{\varphi(q)} \chi^*(q/q_0) \tau(\chi^*) \sum_{a \in A_q} \chi(a) e^{2\pi i x a / q} \\ &= \frac{\mu(r)}{\varphi(q/r)} \chi^*(q/q_0) \chi^*(x/r) \chi^*(q/(rq_0)) \tau(\chi^*)^2 \\ &= \frac{\mu(r)}{\varphi(q/r)} \chi^*(x) \tau(\chi^*)^2. \end{aligned}$$

Because $|\tau(\chi^*)| = \sqrt{q_0}$, we have $\tau(\chi^*)^2 = q_0 \chi^*(-1)$. Hence,

$$\sum_{a \in A_q} G(\chi, a) e^{2\pi i x a / q} = \frac{\mu(r)}{\varphi(q/r)} \chi^*(-x) q_0. \quad (5)$$

Finally, since q/q_0 is square-free, $\gcd(q/q_0, q_0) = 1$ and $r \mid q/q_0$, we deduce that $\gcd(q/r, r) = 1$. Therefore,

$$\varphi(q/r) \varphi(r) = \varphi(q),$$

which together with (5) completes the proof. $\square$

Let us observe that the identity (4) together with (2) imply that

$$|G(\chi, a)| \leq \frac{\sqrt{q_0}}{\varphi(q)} \leq C_\epsilon q^{-\frac{1}{2} + \epsilon}. \quad (6)$$

for any $\epsilon > 0$. Moreover, $G(\chi, a) \neq 0$ entails that q is square-free or $4 \mid q$ and $q/4$ is square-free.

3 Approximating Multipliers

Let us denote by $\mathcal{A}_N$ the averaging operator over prime numbers, that is for a function $f: \mathbb{Z} \rightarrow \mathbb{C}$ we have

$$\mathcal{A}_N f(x) = \frac{1}{\pi(N)} \sum_{p \in \mathbb{P}_N} f(x+p)$$

where $\mathbb{P}_N = [1, N] \cap \mathbb{P}$ and $\pi(N) = \#\mathbb{P}_N$. Since sums over primes are very irregular, it is more convenient to work with

$$\mathcal{M}_N f(x) = \frac{1}{\vartheta(N)} \sum_{p \in \mathbb{P}_N} f(x+p) \log p$$

where

$$\vartheta(N) = \sum_{p \in \mathbb{P}_N} \log p.$$

By the partial summation, we easily see that

$$\begin{aligned} \sum_{p \in \mathbb{P}_N} f(x+p) &= \sum_{n=2}^N \left(\vartheta(n) \mathcal{M}_n f(x) - \vartheta(n-1) \mathcal{M}_{n-1} f(x) \right) \frac{1}{\log n} \\ &= \vartheta(N) \mathcal{M}_N f(x) \frac{1}{\log N} + \sum_{n=2}^{N-1} \vartheta(n) \mathcal{M}_n f(x) \left(\frac{1}{\log n} - \frac{1}{\log(n+1)} \right), \end{aligned}$$

thus

$$\begin{aligned} |\mathcal{A}_N f(x)| &\leq \sup_{N' \in \mathbb{N}} |\mathcal{M}_{N'} f(x)| \frac{1}{\pi(N)} \left(\vartheta(N) \frac{1}{\log N} + \sum_{n=2}^{N-1} \vartheta(n) \left(\frac{1}{\log n} - \frac{1}{\log(n+1)} \right) \right) \\ &\leq \sup_{N' \in \mathbb{N}} |\mathcal{M}_{N'} f(x)|. \end{aligned} \quad (7)$$

To better understand the operators $\mathcal{M}_N$, we use the Hardy–Littlewood circle method. Let $\mathcal{F}$ denote the Fourier transform on $\mathbb{R}$ defined for any function $f \in L^1(\mathbb{R})$ as

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx.$$

If $f \in \ell^1(\mathbb{Z})$, we set

$$\hat{f}(\xi) = \sum_{n \in \mathbb{Z}} f(n) e^{-2\pi i \xi n}.$$

To simplify the notation we denote by $\mathcal{F}^{-1}$ the inverse Fourier transform on $\mathbb{R}$ or the inverse Fourier transform on the torus $\mathbb{T} \equiv [0, 1)$, depending on the context. Let m_N be the Fourier multiplier corresponding to $\mathcal{M}_N$, i.e.,

$$m_N(\xi) = \frac{1}{\vartheta(N)} \sum_{p \in \mathbb{P}_N} e^{2\pi i \xi p} \log p. \quad (8)$$

Then for a finitely supported function $f: \mathbb{Z} \rightarrow \mathbb{C}$, we have

$$\mathcal{M}_N f(x) = \mathcal{F}^{-1}(m_N \hat{f})(x).$$

For $\frac{1}{2} \leq \beta \leq 1$, we set

$$M_N^\beta = \frac{1}{N} \sum_{n=1}^N \frac{n^\beta - (n-1)^\beta}{\beta} \delta_n. \quad (9)$$

To simplify the notation we write M_N for M_N^1 . Let $M_0 \equiv 0$. Recall that

$$|\widehat{M_N}(\xi)| \lesssim \min \left\{ (N|\xi|)^{-1}, N|\xi| \right\}. \quad (10)$$

For $\beta < 1$, we notice that the operators M_N^β are not averaging operators. Moreover, by the partial summation and (10), we get

$$\begin{aligned} |\widehat{M_N^\beta}(\xi)| &= \frac{1}{\beta N} \left| \sum_{n=1}^N (n \widehat{M_n}(\xi) - (n-1) \widehat{M_{n-1}}(\xi)) (n^\beta - (n-1)^\beta) \right| \\ &\lesssim (N|\xi|)^{-1} N^{\beta-1} + (N|\xi|)^{-1} \sum_{n=1}^{N-1} (2n^\beta - (n-1)^\beta - (n+1)^\beta) \\ &\lesssim (N|\xi|)^{-1} N^{\beta-1} + (N|\xi|)^{-1} \sum_{n=1}^{N-1} n^{\beta-2}. \end{aligned}$$

Hence,

$$|\widehat{M_N^\beta}(\xi)| \lesssim (N|\xi|)^{-1}. \quad (11)$$

Moreover,

$$|\widehat{M_N^\beta}(\xi) - \beta^{-1} N^{\beta-1}| \lesssim N^\beta |\xi|,$$

thus

$$\begin{aligned} |\widehat{M_N^\beta}(\xi) - \widehat{M_{2N}^\beta}(\xi)| &\lesssim |\widehat{M_N^\beta}(\xi) - \beta^{-1}N^{\beta-1}| + |\widehat{M_{2N}^\beta}(\xi) - \beta^{-1}(2N)^{\beta-1}| \\ &\quad + |\beta^{-1}N^{\beta-1} - \beta^{-1}(2N)^{\beta-1}| \\ &\lesssim N^\beta|\xi| + (1-\beta)N^{\beta-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} |\widehat{M_N^\beta}(\xi) - \widehat{M_{2N}^\beta}(\xi)| &\lesssim \min \left\{ (N|\xi|)^{-1}, N^\beta|\xi| + (1-\beta)N^{\beta-1} \right\} \\ &\lesssim \min \left\{ (N|\xi|)^{-1}, N|\xi| \right\} + (1-\beta)N^{\beta-1}. \end{aligned} \quad (12)$$

Given $q \in \mathbb{N}$, and $a \in A_q$, we set

$$L_N^{a,q} = G(\mathbb{1}_q, a)M_N, \quad (13)$$

if there is no exceptional character modulo q , and

$$L_N^{a,q} = G(\mathbb{1}_q, a)M_N - G(\chi_q, a)M_N^{\beta_q}, \quad (14)$$

when there is an exceptional character χ_q modulo q and β_q is the corresponding zero.

Proposition 3.1 *There is $c > 0$ such that if $\xi \in \mathbb{T}$,*

$$\left| \xi - \frac{a}{q} \right| \leq N^{-1}Q$$

for some $1 \leq q \leq Q$, $a \in A_q$, and $1 \leq Q \leq \exp(c\sqrt{\log N})$, then

$$m_N(\xi) = \widehat{L_N^{a,q}}(\xi - a/q) + \mathcal{O}\left(Q \exp(-c\sqrt{\log N})\right).$$

Proof Observe that for a prime p , $p \mid q$ if and only if $(p \bmod q, q) > 1$. Hence,

$$\left| \sum_{\substack{r=1 \\ (r,q)>1}}^q \sum_{\substack{p \in \mathbb{P}_N \\ p \equiv r \bmod q}} e^{2\pi i \xi p} \log p \right| \leq \sum_{\substack{p \in \mathbb{P} \\ p|q}} \log p \leq q.$$

Let $\theta = \xi - a/q$. For $p \equiv r \pmod{q}$, we have

$$\xi p \equiv \theta p + ra/q \pmod{1},$$

thus

$$\sum_{r \in A_q} \sum_{\substack{p \in \mathbb{P}_N \\ p \equiv r \pmod{q}}} e^{2\pi i \xi p} \log p = \sum_{r \in A_q} e^{2\pi i r a/q} \sum_{\substack{p \in \mathbb{P}_N \\ p \equiv r \pmod{q}}} e^{2\pi i \theta p} \log p.$$

For $x \geq 2$, we set

$$\vartheta(x; q, r) = \sum_{\substack{p \in \mathbb{P}_x \\ p \equiv r \pmod{q}}} \log p.$$

Then, by the partial summation, we obtain

$$\begin{aligned} \sum_{\substack{p \in \mathbb{P}_N \\ p \equiv r \pmod{q}}} e^{2\pi i \theta p} \log p &= \sum_{\substack{p \in \mathbb{P}_N \setminus \mathbb{P}_{\sqrt{N}} \\ p \equiv r \pmod{q}}} e^{2\pi i \theta p} \log p + \mathcal{O}(\sqrt{N}) \\ &= \vartheta(N; q, r) e^{2\pi i \theta N} - \vartheta(\sqrt{N}; q, r) e^{2\pi i \theta \sqrt{N}} \\ &\quad - 2\pi i \theta \int_{\sqrt{N}}^N \vartheta(t; q, r) e^{2\pi i \theta t} dt + \mathcal{O}(\sqrt{N}). \end{aligned} \quad (15)$$

Analogously, for any $\frac{1}{2} \leq \beta \leq 1$, we can write

$$\begin{aligned} \sum_{n=1}^N \frac{n^\beta - (n-1)^\beta}{\beta} e^{2\pi i \theta n} &= \beta^{-1} N^\beta e^{2\pi i \theta N} - \beta^{-1} \sqrt{N}^\beta e^{2\pi i \theta \sqrt{N}} \\ &\quad - 2\pi i \theta \beta^{-1} \int_{\sqrt{N}}^N t^\beta e^{2\pi i \theta t} dt + \mathcal{O}(\sqrt{N}). \end{aligned} \quad (16)$$

By the Page's theorem, there is an absolute constant $c > 0$ such that for each $x \geq 2$, $1 \leq q \leq \exp(c\sqrt{\log x})$, and $r \in A_q$,

$$\left| \vartheta(x; q, r) - \frac{x}{\varphi(q)} \right| \lesssim x \exp(-c\sqrt{\log x}),$$

if there is no exceptional character modulo q , and

$$\left| \vartheta(x; q, r) - \frac{x}{\varphi(q)} + \frac{\chi(r)}{\varphi(q)} \beta^{-1} x^\beta \right| \lesssim x \exp(-c\sqrt{\log x}),$$

when there is an exceptional character χ modulo q , and β is the concomitant zero. Therefore, by (15) and (16), we obtain

$$\begin{aligned}
& \left| \sum_{\substack{p \in \mathbb{P}_N \\ p \equiv r \pmod{q}}} e^{2\pi i \theta p} \log p - \frac{1}{\varphi(q)} \sum_{n=1}^N e^{2\pi i \theta n} \left(1 - \chi(r) \frac{n^\beta - (n-1)^\beta}{\beta} \right) \right| \\
& \lesssim \sqrt{N} + \left| \vartheta(N; q, r) - \frac{N}{\varphi(q)} + \frac{\chi(r)}{\varphi(q)} \beta^{-1} N^\beta \right| \\
& \quad + \left| \vartheta(\sqrt{N}; q, r) - \frac{\sqrt{N}}{\varphi(q)} + \frac{\chi(r)}{\varphi(q)} \beta^{-1} \sqrt{N}^\beta \right| \\
& \quad + |\theta| \int_{\sqrt{N}}^N \left| \vartheta(t; q, r) - \frac{t}{\varphi(q)} + \frac{\chi(r)}{\varphi(q)} \beta^{-1} t^\beta \right| dt \\
& \lesssim N \exp(-c\sqrt{\log N}) + QN^{-1} \int_{\sqrt{N}}^N t \exp(-c\sqrt{\log t}) dt,
\end{aligned}$$

which is bounded by $NQ \exp(-c\sqrt{\log N})$. Finally, by the prime number theorem

$$\left| \frac{\vartheta(N) - N}{N} \right| \leq C \exp(-c\sqrt{\log N}),$$

and the proposition follows. $\square$

Next, we select $\eta : \mathbb{R} \rightarrow \mathbb{R}$, a smooth function such that $0 \leq \eta \leq 1$, and

$$\eta(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq \frac{1}{4}, \\ 0 & \text{if } |\xi| \geq \frac{1}{2}. \end{cases}$$

We may assume that η is a convolution of two smooth functions with supports contained in $(-\frac{1}{2}, \frac{1}{2})$. For $s \in \mathbb{N}_0$, we set

$$\eta_s(\xi) = \eta(2^{4s}\xi).$$

We define a family of approximating multipliers, by the formula

$$v_n^s(\xi) = \sum_{a/q \in \mathcal{R}_s} \widehat{L_{2^n}^{a,q}}(\xi - a/q) \eta_s(\xi - a/q) \quad (17)$$

where

$$\mathcal{R}_s = \left\{ a/q \in \mathbb{Q} \cap (0, 1] : a \in A_q, \text{ and } 2^s \leq q < 2^{s+1}, q \text{ is square-free or } 4 \mid q \text{ and } q/4 \text{ is square-free} \right\},$$

and $\mathcal{R}_0 = \{1\}$. We set $v_n = \sum_{s \geq 0} v_n^s$.

Theorem 3.2 *There are $C, c > 0$ such that for all $n \in \mathbb{N}_0$ and $\xi \in \mathbb{T}$,*

$$|\mathfrak{m}_{2^n}(\xi) - v_n(\xi)| \leq C \exp(-c\sqrt{n})$$

where $\mathfrak{m}_N$ is defined by (8).

Proof Let

$$Q_n = \exp\left(\frac{c}{2}\sqrt{n}\right)$$

where the constant c is determined in Proposition 3.1. By the Dirichlet's principle, there are coprime integers a and q , satisfying $1 \leq a \leq q \leq 2^n Q_n^{-1}$, and such that

$$\left| \xi - \frac{a}{q} \right| \leq \frac{1}{q} 2^{-n} Q_n.$$

Let us first consider the case when $1 \leq q \leq Q_n$. We select $s_1 \in \mathbb{N}_0$ satisfying

$$2^{s_1+1} < \frac{1}{2} 2^n Q_n^{-2} \leq 2^{s_1+2}.$$

For $s \leq s_1$ and $a'/q' \in \mathcal{R}_s$, with $a'/q' \neq a/q$, we have

$$\left| \xi - \frac{a'}{q'} \right| \geq \frac{1}{qq'} - \left| \xi - \frac{a}{q} \right| \geq Q_n^{-1} 2^{-s_1-1} - 2^{-n} Q_n \geq 2^{-n} Q_n.$$

Therefore, by (6) and (11),

$$\left| \widehat{L_{2^n}^{a',q'}}(\xi - a'/q') \eta_s(\xi - a'/q') \right| \lesssim 2^{-\frac{s}{4}} |2^n(\xi - a'/q')|^{-1} \leq 2^{-\frac{s}{4}} Q_n^{-1},$$

which implies that

$$\left| \sum_{s=0}^{s_1} \sum_{\substack{a'/q' \in \mathcal{R}_s \\ a'/q' \neq a/q}} \widehat{L_{2^n}^{a',q'}}(\xi - a'/q') \eta_s(\xi - a'/q') \right| \lesssim Q_n^{-1} \sum_{s \geq 0} 2^{-\frac{s}{4}}.$$

For $s > s_1$, by (6) we obtain

$$\left| \sum_{s > s_1} \sum_{\substack{a'/q' \in \mathcal{R}_s \\ a'/q' \neq a/q}} \widehat{L_{2^n}^{a',q'}}(\xi - a'/q') \eta_s(\xi - a'/q') \right| \lesssim \sum_{s > s_1} 2^{-\frac{s}{4}} \lesssim (2^n Q_n^{-2})^{-\frac{1}{4}} \lesssim Q_n^{-1}.$$

If q is square-free or $4 \mid q$ and $q/4$ is square-free then there is $s_0 \in \mathbb{N}_0$ such that $a/q \in \mathcal{R}_{s_0}$, thus

$$Q_n \geq 2^{s_0}.$$

By Proposition 3.1,

$$\left| m_{2^n}(\xi) - \widehat{L_{2^n}^{a,q}}(\xi - a/q) \eta_{s_0}(\xi - a/q) \right| \lesssim |2^n(\xi - a/q)|^{-1} (1 - \eta_{s_0}(\xi - a/q)) + Q_n^{-1}.$$

Since $1 - \eta_{s_0}(\xi - a/q) > 0$, implies that

$$\left| \xi - \frac{a}{q} \right| \geq \frac{1}{4} 2^{-4s_0} \gtrsim Q_n^{-4},$$

we obtain

$$\left| m_{2^n}(\xi) - \widehat{L_{2^n}^{a,q}}(\xi - a/q) \eta_{s_0}(\xi - a/q) \right| \lesssim 2^{-n} Q_n^4 + Q_n^{-1} \lesssim Q_n^{-1}.$$

Finally, if q and $q/4$ are not square-free then by Proposition 3.1,

$$\left| m_{2^n}(\xi) - \widehat{L_{2^n}^{a,q}}(\xi - a/q) \eta_{s_0}(\xi - a/q) \right| = |m_{2^n}(\xi)| \lesssim Q_n^{-1}.$$

It remains to deal with $Q_n \leq q \leq 2^n Q_n^{-1}$. By the Vinogradov's inequality (see [20, Theorem 1, Chap. IX] or [17, Theorem 8.5]), we get

$$|m_{2^n}(\xi)| \lesssim n^4 \left(q^{-\frac{1}{2}} + 2^{-\frac{1}{2}n} q^{\frac{1}{2}} + 2^{-\frac{1}{5}n} \right) \lesssim n^4 Q_n^{-\frac{1}{2}}.$$

Next, we show that

$$\left| \sum_{s \geq 0} \sum_{a'/q' \in \mathcal{R}_s} \widehat{L_{2^n}^{a',q'}}(\xi - a'/q') \eta_s(\xi - a'/q') \right| \lesssim Q_n^{-\frac{1}{8}}.$$

Select $s_2 \in \mathbb{N}_0$ such that

$$2^{s_2+1} \leq Q_n^{\frac{1}{2}} \leq 2^{s_2+2}. \quad (18)$$

For $s \leq s_2$, if $a'/q' \in \mathcal{R}_s$, then $1 \leq q' \leq Q_n^{\frac{1}{2}}$, and hence

$$\left| \xi - \frac{a'}{q'} \right| \geq \frac{1}{q'} 2^{-n} Q_n \geq 2^{-n} Q_n^{\frac{1}{2}}.$$

Therefore, by (6) and (11),

$$\left| \widehat{L_{2^n}^{a',q'}}(\xi - a'/q') \eta_s(\xi - a'/q') \right| \lesssim 2^{-\frac{s}{4}} Q_n^{-\frac{1}{2}},$$

which entails that

$$\left| \sum_{s=0}^{s_2} \sum_{a'/q' \in \mathcal{R}_s} \widehat{L_{2^n}^{a',q'}}(\xi - a'/q') \eta_s(\xi - a'/q') \right| \lesssim Q_n^{-\frac{1}{2}} \sum_{s \geq 0} 2^{-\frac{s}{4}}.$$

If $s > s_2$, then by (6), we get

$$\left| \widehat{L_{2^n}^{a',q'}}(\xi - a'/q') \eta_s(\xi - a'/q') \right| \lesssim 2^{-\frac{s}{4}},$$

hence by (18),

$$\left| \sum_{s > s_2} \sum_{a'/q' \in \mathcal{R}_s} \widehat{L_{2^n}^{a',q'}}(\xi - a'/q') \eta_s(\xi - a'/q') \right| \lesssim \sum_{s > s_2} 2^{-\frac{s}{4}} \lesssim Q_n^{-\frac{1}{8}},$$

and the theorem follows. $\square$

4 Equidistribution of Weak ℓ^1 Norms

In this section we prove that the maximal function associated with kernels $(M_{2^n}^\beta : n \in \mathbb{N}_0)$ has weak $\ell^1(\mathbb{Z})$ -norm equidistributed in residue classes. Before embarking on the proof, let us recall two lemmas essential for the argument.

Lemma 4.1 [13, Lemma 1] *There is $C > 0$ such that for all $s \in \mathbb{N}$ and $u \in \mathbb{R}$,*

$$\left\| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i \xi x} \eta_s(\xi) d\xi \right\|_{\ell^1(x)} \leq C,$$

$$\left\| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i \xi x} (1 - e^{2\pi i \xi u}) \eta_s(\xi) d\xi \right\|_{\ell^1(x)} \leq C|u|2^{-4s}.$$

Lemma 4.2 [13, Lemma 2] *For all $p \geq 1$, any $1 \leq Q \leq 2^{2s}$ with $s \in \mathbb{N}$, $r \in \{1, \dots, Q\}$, and any finitely supported function $f : \mathbb{Z} \rightarrow \mathbb{C}$,*

$$\|\mathcal{F}^{-1}(\eta_s \hat{f})(Qx + r)\|_{\ell^p(x)} \simeq Q^{-\frac{1}{p}} \|\mathcal{F}^{-1}(\eta_s \hat{f})\|_{\ell^p}.$$

The following theorem is the main result of this section.

Theorem 4.3 *There is $C > 0$ such that for any $1 \leq Q \leq 2^{2s}$ with $s \in \mathbb{N}$, $r \in \{1, \dots, Q\}$, $\frac{1}{2} \leq \beta \leq 1$, and any finitely supported function $f : \mathbb{Z} \rightarrow \mathbb{C}$,*

$$\sup_{\lambda > 0} \lambda \cdot \left| \left\{ x \in \mathbb{Z} : \sup_{n \in \mathbb{N}_0} |M_{2^n}^\beta * \mathcal{F}^{-1}(\eta_s \hat{f})(Qx + r)| > \lambda \right\} \right| \leq C \|\mathcal{F}^{-1}(\eta_s \hat{f})(Qx + r)\|_{\ell^1(x)}.$$

Proof Observe that, by the mean value theorem, for $x \in \mathbb{N}$,

$$\frac{x^\beta - (x-1)^\beta}{\beta} \leq x^{\beta-1} \leq 1,$$

thus

$$M_N^\beta(x) \leq M_N(x).$$

In particular, by the Hardy–Littlewood maximal theorem, there is $C > 0$ such that for all $\frac{1}{2} \leq \beta \leq 1$, and any $f \in \ell^1(\mathbb{Z})$,

$$\sup_{\lambda > 0} \lambda \cdot \left| \left\{ x \in \mathbb{Z} : \sup_{n \in \mathbb{N}_0} |M_{2^n}^\beta * f(x)| > \lambda \right\} \right| \leq C \|f\|_{\ell^1}. \quad (19)$$

For $r \in \{1, \dots, Q\}$ and $\lambda > 0$, we set

$$J_r(\lambda) = \left| \left\{ x \in \mathbb{Z} : \sup_{n \in \mathbb{N}_0} |M_{2^n}^\beta * \mathcal{F}^{-1}(\eta_s \hat{f})(Qx + r)| > \lambda \right\} \right|.$$

Then, by (19), we have

$$\begin{aligned} J_1(\lambda) + \dots + J_Q(\lambda) &= \left| \left\{ x \in \mathbb{Z} : \sup_{n \in \mathbb{N}_0} |M_{2^n}^\beta * \mathcal{F}^{-1}(\eta_s \hat{f})(x)| > \lambda \right\} \right| \\ &\leq C \lambda^{-1} \|\mathcal{F}^{-1}(\eta_s \hat{f})\|_{\ell^1}. \end{aligned} \quad (20)$$

Moreover, for any $r, r' \in \{1, \dots, Q\}$, we have

$$\begin{aligned} &\left| \left\{ x \in \mathbb{Z} : \sup_{n \in \mathbb{N}_0} \left| \int_0^1 e^{2\pi i \xi(Qx+r)} \left(1 - e^{2\pi i \xi(r'-r)}\right) \widehat{M_{2^n}^\beta}(\xi) \eta_s(\xi) \hat{f}(\xi) \, d\xi \right| > \frac{1}{2} \lambda \right\} \right| \\ &\leq C \lambda^{-1} \left\| \int_0^1 e^{2\pi i \xi x} \left(1 - e^{2\pi i \xi(r'-r)}\right) \eta_s(\xi) \hat{f}(\xi) \, d\xi \right\|_{\ell^1(x)}. \end{aligned}$$

Since $\eta_s = \eta_s \eta_{s-1}$, by Young's convolution inequality and Lemma 4.1, we obtain

$$\begin{aligned} & \left\| \int_0^1 e^{2\pi i \xi x} \left(1 - e^{2\pi i \xi(r'-r)}\right) \eta_s(\xi) \hat{f}(\xi) d\xi \right\|_{\ell^1(x)} \\ & \leq \left\| \int_0^1 e^{2\pi i \xi x} \left(1 - e^{2\pi i \xi(r'-r)}\right) \eta_{s-1}(\xi) \right\|_{\ell^1(x)} \|\mathcal{F}^{-1}(\eta_s \hat{f})\|_{\ell^1} \\ & \leq C Q 2^{-4s} \|\mathcal{F}^{-1}(\eta_s \hat{f})\|_{\ell^1}. \end{aligned}$$

Thus

$$J_r(\lambda) \leq J_{r'}(\lambda/2) + C \lambda^{-1} Q 2^{-4s} \|\mathcal{F}^{-1}(\eta_s \hat{f})\|_{\ell^1},$$

which together with (20) imply that

$$\begin{aligned} Q J_r(\lambda) & \leq J_1(\lambda/2) + \dots + J_Q(\lambda/2) + C \lambda^{-1} Q^2 2^{-4s} \|\mathcal{F}^{-1}(\eta_s \hat{f})\|_{\ell^1} \\ & \lesssim \lambda^{-1} (1 + Q^2 2^{-4s}) \|\mathcal{F}^{-1}(\eta_s \hat{f})\|_{\ell^1} \\ & \lesssim \lambda^{-1} \|\mathcal{F}^{-1}(\eta_s \hat{f})\|_{\ell^1}, \end{aligned}$$

where the last inequality is a consequence of $1 \leq Q \leq 2^{2s}$. Therefore, in view of Lemma 4.2, we immediately get

$$J_r(\lambda) \lesssim \lambda^{-1} \|\mathcal{F}^{-1}(\eta_s \hat{f})(Qx + r)\|_{\ell^1(x)},$$

which is the desired conclusion. $\square$

Essentially the same reasoning as in the proof of Theorem 4.3 leads to the following theorem.

Theorem 4.4 *There is $C > 0$ such that for all $1 \leq Q \leq 2^{2s}$ with $s \in \mathbb{N}$, $r \in \{1, \dots, Q\}$, $\frac{1}{2} \leq \beta \leq 1$, and any finitely supported function $f : \mathbb{Z} \rightarrow \mathbb{C}$,*

$$\left\| \sup_{n \in \mathbb{N}_0} \left| \mathcal{F}^{-1}(\widehat{M_{2^n}^\beta \eta_s \hat{f}})(Qx + r) \right| \right\|_{\ell^2(x)} \leq C \|\mathcal{F}^{-1}(\eta_s \hat{f})(Qx + r) f\|_{\ell^2(x)}.$$

5 ℓ^2 Theory

We are now in the position to prove $\ell^2(\mathbb{Z})$ boundedness of the maximal function associated to the multipliers $(v_n^s : n \in \mathbb{N})$.

Theorem 5.1 *For each $\epsilon > 0$ there is $C > 0$ such that for all $s \in \mathbb{N}_0$, and any finitely supported function $f : \mathbb{Z} \rightarrow \mathbb{C}$,*

$$\left\| \sup_{n \in \mathbb{N}} |\mathcal{F}^{-1}(v_n^s \hat{f})| \right\|_{\ell^2} \leq C 2^{-s(\frac{1}{2}-\epsilon)} \|f\|_{\ell^2}.$$

Proof We divide the supremum into two parts: $0 \leq n < 2^{s+4}$ and $2^{s+4} \leq n$. Then the following holds true.

Claim 5.2 For each $\epsilon > 0$ there is $C > 0$ such that for all $s \in \mathbb{N}_0$, and any finitely supported function $f : \mathbb{Z} \rightarrow \mathbb{C}$,

$$\left\| \sup_{0 \leq n \leq 2^{s+4}} |\mathcal{F}^{-1}(v_n^s \hat{f})| \right\|_{\ell^2} \leq C(s+1)2^{-s(\frac{1}{2}-\epsilon)} \|f\|_{\ell^2}. \quad (21)$$

For the proof, we apply [14, Lemma 1] to write

$$\sup_{0 \leq n < 2^{s+4}} |\mathcal{F}^{-1}(v_n^s \hat{f})| \leq |\mathcal{F}^{-1}(v_0^s \hat{f})| + \sqrt{2} \sum_{i=0}^{s+4} \left(\sum_{j=0}^{2^{s+4-i}-1} |\mathcal{F}^{-1}((v_{(j+1)2^i}^s - v_{j2^i}^s) \hat{f})|^2 \right)^{\frac{1}{2}}. \quad (22)$$

Let us fix $i \in \{0, \dots, s+4\}$. Then by the Plancherel's theorem we get

$$\begin{aligned} & \sum_{j=0}^{2^{s+4-i}-1} \left\| \mathcal{F}^{-1}((v_{(j+1)2^i}^s - v_{j2^i}^s) \hat{f}) \right\|_{\ell^2}^2 \\ &= \sum_{j=0}^{2^{s+4-i}-1} \sum_{a/q \in \mathcal{R}_s} \int_0^1 \left| \sum_{m \in I_j^i} \widehat{L_{2^m}^{a,q}}(\xi - a/q) - \widehat{L_{2^{m-1}}^{a,q}}(\xi - a/q) \right|^2 \\ & \quad \times \eta_s(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi \end{aligned}$$

where $I_j^i = \{j2^i + 1, j2^i + 2, \dots, (j+1)2^i\}$. By (6), we obtain

$$\begin{aligned} & \sum_{j=0}^{2^{s+4-i}-1} \sum_{a/q \in \mathcal{R}_s} \int_0^1 \left| \sum_{m \in I_j^i} \widehat{L_{2^m}^{a,q}}(\xi - a/q) - \widehat{L_{2^{m-1}}^{a,q}}(\xi - a/q) \right|^2 \eta_s(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi \\ & \lesssim 2^{-s(1-\epsilon)} \sum_{a/q \in \mathcal{R}_s} \sum_{j=0}^{2^{s+4-i}-1} \sum_{m, m' \in I_j^i} \int_0^1 \Delta_m^q(\xi - a/q) \cdot \Delta_{m'}^q(\xi - a/q) \\ & \quad \cdot \eta_s(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi, \end{aligned}$$

where $\Delta_m^q = |\widehat{M_{2^m}^{\beta_q}} - \widehat{M_{2^{m-1}}^{\beta_q}}| + |\widehat{M_{2^m}^{\beta_q}} - \widehat{M_{2^{m-1}}^{\beta_q}}|$. In view of (12), we have

$$\sum_{n \in \mathbb{N}_0} \Delta_m^q(\xi) \lesssim \sum_{n \in \mathbb{N}_0} \min \{ (2^n |\xi|)^{-1}, 2^n |\xi| \} + (1 - \beta_q) 2^{-n(1-\beta_q)} \lesssim 1,$$

uniformly with respect to $\xi \in \mathbb{T}$, $q \in \mathbb{N}$, and $\frac{1}{2} \leq \beta_q \leq 1$. Since supports of $\eta_s(\cdot - a/q)$ are disjoint while a/q varies over $\mathcal{R}_s$, we obtain

$$\begin{aligned} \sum_{j=0}^{2^{s+4-i}-1} \left\| \mathcal{F}^{-1}((v_{(j+1)2^i}^s - v_{j2^i}^s)\hat{f}) \right\|_{\ell^2}^2 &\lesssim 2^{-s(1-\epsilon)} \sum_{a/q \in \mathcal{R}_s} \int_0^1 \eta_s(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi \\ &\lesssim 2^{-s(1-\epsilon)} \|f\|_{\ell^2}^2, \end{aligned}$$

which together with (22) imply (21).

It remains now to treat supremum over $n \geq 2^{s+4}$. For each $\frac{1}{2} \leq \beta < 1$ we set

$$\mathcal{R}_s^\beta = \{a/q \in \mathcal{R}_s : \beta_q = \beta\}.$$

and $\mathcal{R}_s^1 = \mathcal{R}_s$. In view of the Landau's theorem [16, Corollary 11.9], there are $\mathcal{O}(\log s)$ distinct β 's. Therefore, it suffices to show the following claim.

Claim 5.3 *For each $\epsilon > 0$ there is $C > 0$ such that for all $s \in \mathbb{N}_0$, $\frac{1}{2} \leq \beta \leq 1$, any finitely supported function $f : \mathbb{Z} \rightarrow \mathbb{C}$,*

$$\left\| \sup_{2^{s+4} \leq n} \left| \sum_{a/q \in \mathcal{R}_s^\beta} G(\chi_q, a) \mathcal{F}^{-1}(\widehat{M_{2^n}^\beta}(\cdot - a/q) \eta_s(\cdot - a/q) \hat{f}) \right| \right\|_{\ell^2} \leq C 2^{-s(\frac{1}{2}-\epsilon)} \|f\|_{\ell^2}. \quad (23)$$

Let us fix $\frac{1}{2} \leq \beta \leq 1$. We define

$$I(x, y) = \sup_{2^{s+4} \leq n} \left| \sum_{a/q \in \mathcal{R}_s^\beta} G(\chi_q, a) e^{2\pi i x a/q} \mathcal{F}^{-1}(\widehat{M_{2^n}^\beta} \eta_s \hat{f}(\cdot + a/q))(y) \right|,$$

and

$$J(x, y) = \sum_{a/q \in \mathcal{R}_s^\beta} G(\chi_q, a) e^{2\pi i x a/q} \mathcal{F}^{-1}(\eta_s \hat{f}(\cdot + a/q))(y).$$

Observe that the functions $x \mapsto I(x, y)$ and $x \mapsto J(x, y)$ are Q_s periodic where

$$Q_s = 4 \prod_{p \in \mathbb{P}_{2^{s+1}}} p \lesssim e^{2^{s+2}}.$$

By the Plancherel's theorem, for $u \in \mathbb{Z}_{Q_s}$, we have

$$\begin{aligned} & \left\| \mathcal{F}^{-1}(\widehat{M_{2^n}^\beta \eta_s \hat{f}}(\cdot + a/q))(x+u) - \mathcal{F}^{-1}(\widehat{M_{2^n}^\beta \eta_s \hat{f}}(\cdot + a/q))(x) \right\|_{\ell^2(x)} \\ &= \left\| (1 - e^{2\pi i \xi u}) \widehat{M_{2^n}^\beta}(\xi) \eta_s(\xi) \hat{f}(\xi + a/q) \right\|_{L^2(d\xi)} \\ &\lesssim 2^{-n} |u| \cdot \left\| \eta_s \hat{f}(\cdot + a/q) \right\|_{L^2}, \end{aligned}$$

because by (11),

$$\sup_{\xi \in \mathbb{T}} |\xi| \cdot |\widehat{M_{2^n}^\beta}(\xi)| \lesssim 2^{-n}.$$

Therefore, by the triangle inequality

$$\left| \left\| I(x, x+u) \right\|_{\ell^2(x)} - \left\| I(x, x) \right\|_{\ell^2(x)} \right| \lesssim Q_s \sum_{n \geq 2^{s+4}} 2^{-n} \sum_{a/q \in \mathcal{R}_s} \left\| \eta_s \hat{f}(\cdot + a/q) \right\|_{L^2}.$$

Since $\mathcal{R}_s$ contains at most $2^{2(s+1)}$ rational numbers, by the Cauchy–Schwarz inequality we get

$$\sum_{a/q \in \mathcal{R}_s} \left\| \eta_s \hat{f}(\cdot + a/q) \right\|_{L^2} \leq 2^{s+1} \|f\|_{\ell^2}.$$

Observe that

$$Q_s \cdot 2^{-2^{s+4}} \cdot 2^{s+1} \leq 2^{2^{s+3} - 2^{s+4} + s + 1} \leq 2^{-s},$$

thus

$$\left\| I(x, x) \right\|_{\ell^2(x)} \lesssim \left\| I(x, x+u) \right\|_{\ell^2(x)} + 2^{-s} \|f\|_{\ell^2}.$$

Hence,

$$\left\| I(x, x) \right\|_{\ell^2(x)}^2 \lesssim \frac{1}{Q_s} \sum_{u=1}^{Q_s} \left\| I(x, x+u) \right\|_{\ell^2(x)}^2 + 2^{-2s} \|f\|_{\ell^2}^2. \quad (24)$$

Now, by multiple change of variables and periodicity we get

$$\begin{aligned} \sum_{u=1}^{Q_s} \left\| I(x, x+u) \right\|_{\ell^2(x)}^2 &= \sum_{u=1}^{Q_s} \sum_{x \in \mathbb{Z}} I(x-u, x)^2 = \sum_{x \in \mathbb{Z}} \sum_{u=1}^{Q_s} I(u, x)^2 \\ &= \sum_{u=1}^{Q_s} \left\| I(u, x) \right\|_{\ell^2(x)}^2. \end{aligned}$$

Using Theorem 4.4, we can estimate

$$\|I(u, x)\|_{\ell^2(x)} = \left\| \sup_{2^{s+4} \leq n} \left| \mathcal{F}^{-1}(\widehat{M_{2^n}^\beta \eta_s J(u, \cdot)}) \right| \right\|_{\ell^2} \lesssim \|J(u, x)\|_{\ell^2(x)}.$$

Notice that

$$\begin{aligned} \sum_{u=1}^{Q_s} \|J(u, x)\|_{\ell^2(x)}^2 &= \sum_{x \in \mathbb{Z}} \sum_{u=1}^{Q_s} J(u, x)^2 = \sum_{u=1}^{Q_s} \sum_{x \in \mathbb{Z}} J(x - u, x)^2 \\ &= \sum_{u=1}^{Q_s} \|J(x, x + u)\|_{\ell^2(x)}^2. \end{aligned}$$

Since supports of $\eta_s(\cdot - a/q)$ are disjoint while a/q varies over $\mathcal{R}_s$, by (6) we get

$$\begin{aligned} \|J(x, x + u)\|_{\ell^2(x)}^2 &= \int_0^1 \left| \sum_{a/q \in \mathcal{R}_s^\beta} G(\chi_q, a) e^{2\pi i \xi u a/q} \eta_s(\xi - a/q) \right|^2 |\hat{f}(\xi)|^2 d\xi \\ &\lesssim 2^{-s(1-\epsilon)} \|f\|_{\ell^2}^2. \end{aligned}$$

Therefore,

$$\sum_{u=1}^{Q_s} \|I(x, x + u)\|_{\ell^2(x)}^2 \lesssim 2^{-s(1-\epsilon)} Q_s \|f\|_{\ell^2}^2,$$

which together with (24) imply (23) and the theorem follows. $\square$

Given $t > 0$ and $n > t$, we define the multiplier

$$\begin{aligned} \Pi_n^t(\xi) &= \sum_{0 \leq s \leq \sqrt{t}} v_n^s(\xi) \\ &= \sum_{0 \leq s \leq \sqrt{t}} \sum_{a/q \in \mathcal{R}_s} \widehat{L_{2^n}^{a,q}}(\xi - a/q) \eta_s(\xi - a/q). \end{aligned}$$

Corollary 5.4 *There are $C, c > 0$ such that for each $t > 0$, and any finitely supported function $f \in \mathbb{Z} \rightarrow \mathbb{C}$,*

$$\left\| \sup_{t \leq n} \left| \mathcal{M}_{2^n} f - \mathcal{F}^{-1}(\Pi_n^t \hat{f}) \right| \right\|_{\ell^2} \leq C \exp(-c\sqrt{t}) \|f\|_{\ell^2}.$$

Proof Since

$$\mathfrak{m}_{2^n} - \Pi_n^t = (\mathfrak{m}_{2^n} - v_n) + \sum_{s > \sqrt{t}} v_n^s,$$

our assertion follows from Theorems 3.2 and 5.1. Indeed, by the Plancherel's theorem and Theorem 3.2 we get

$$\left\| \sup_{t \leq n} \left| \mathcal{F}^{-1}((m_{2^n} - v_n)\hat{f}) \right| \right\|_{\ell^2} \lesssim \left(\sum_{n \geq t} \exp(-2c\sqrt{n}) \right)^{\frac{1}{2}} \|f\|_{\ell^2}.$$

On the other hand, by Theorem 5.1,

$$\left\| \sup_{t \leq n} \left| \sum_{s > \sqrt{t}} \mathcal{F}^{-1}(v_n^s \hat{f}) \right| \right\|_{\ell^2} \lesssim \sum_{s > \sqrt{t}} 2^{-\frac{s}{4}} \|f\|_{\ell^2},$$

which concludes the proof. $\square$

6 Weak Type Estimates

In this section we investigate the weak type estimates for the multipliers $(\Pi_n^t : n \geq t)$. Then together with results from Sect. 5 we deduce Theorem C.

Theorem 6.1 *There is $C > 0$ such that for all $t > 0$ and any finitely supported function $f : \mathbb{Z} \rightarrow \mathbb{C}$,*

$$\sup_{\lambda > 0} \lambda \cdot \left| \left\{ x \in \mathbb{Z} : \sup_{t \leq n} |\mathcal{F}^{-1}(\Pi_n^t \hat{f})(x)| > \lambda \right\} \right| \leq Ct \|f\|_{\ell^1}.$$

Proof Let us fix $2^s \leq q < 2^{s+1}$ for some $1 \leq s \leq \sqrt{t}$. Let $\frac{1}{2} \leq \beta \leq 1$. Suppose that χ is a quadratic Dirichlet character modulo q induced by χ^* having the conductor q_0 . We claim that the following holds true.

Claim 6.2 *There is $C > 0$ such that for any finitely supported function $f : \mathbb{Z} \rightarrow \mathbb{C}$,*

$$\begin{aligned} & \sup_{\lambda > 0} \lambda \cdot \left| \left\{ \sup_{t \leq n} \left| \sum_{a \in A_q} G(\chi, a) \mathcal{F}^{-1}(\widehat{M_{2^n}^\beta}(\cdot - a/q) \eta_s(\cdot - a/q) \hat{f})) \right| > \lambda \right\} \right| \\ & \leq C \frac{1}{\varphi(q)} \|f\|_{\ell^1}. \end{aligned} \quad (25)$$

The constant C is independent of q , β and χ .

Let us first see that from Claim 6.2, we can deduce the theorem. Indeed, from (25) we easily get

$$\left| \left\{ \sup_{t \leq n} \left| \sum_{a \in A_q} \mathcal{F}^{-1}(\widehat{L_{2^n}^{a,q}}(\cdot - a/q) \eta_s(\cdot - a/q) \hat{f})) \right| > \lambda \right\} \right| \leq C \frac{1}{\lambda \varphi(q)} \|f\|_{\ell^1}.$$

Recall that (see e.g. [18]),

$$\sum_{1 \leq q < 2^{\sqrt{t}}} \frac{1}{\varphi(q)} \simeq \sqrt{t},$$

thus

$$\Phi(t) = \sum_{1 \leq q < 2^{\sqrt{t}}} \frac{1 + \log q}{\varphi(q)} \lesssim t.$$

Hence, by log-convexity of $\ell^{1,\infty}(\mathbb{Z})$, (see [11, 19]) we obtain

$$\begin{aligned} & \left| \left\{ x \in \mathbb{Z} : \sup_{t \leq n} |\mathcal{F}^{-1}(\Pi_n^t \hat{f})(x)| > \lambda \right\} \right| \\ &= \left| \left\{ \sup_{t \leq n} \left| \sum_{0 \leq s \leq \sqrt{t}} \sum_{q=2^s}^{2^{s+1}-1} \sum_{a \in A_q} \mathcal{F}^{-1}(\widehat{L_{2^n}^{a,q}}(\cdot - a/q) \eta_s(\cdot - a/q) \hat{f})) > \lambda \right. \right\} \right| \\ &\leq \left| \left\{ \sum_{0 \leq s \leq \sqrt{t}} \sum_{q=2^s}^{2^{s+1}-1} \frac{1}{\varphi(q)} \sup_{t \leq n} \left| \sum_{a \in A_q} \mathcal{F}^{-1}(\widehat{L_{2^n}^{a,q}}(\cdot - a/q) \eta_s(\cdot - a/q) \hat{f})) > \lambda \right. \right\} \right| \\ &\lesssim \lambda^{-1} \Phi(t) \|f\|_{\ell^1}, \end{aligned}$$

which is bounded by $C\lambda^{-1}t\|f\|_{\ell^1}$.

What is left now is to prove Claim 6.2. Let $r \in \{1, \dots, q\}$. For $x \equiv r \pmod{q}$, we have

$$\begin{aligned} & \sum_{a \in A_q} G(\chi, a) \mathcal{F}^{-1}(\widehat{M_{2^n}^\beta}(\cdot - a/q) \eta_s(\cdot - a/q) \hat{f})(x) \\ &= \sum_{a \in A_q} G(\chi, a) e^{2\pi i r a/q} \mathcal{F}^{-1}(\widehat{M_{2^n}^\beta} \eta_s \hat{f}(\cdot + a/q))(x) \\ &= \mathcal{F}^{-1}(\widehat{M_{2^n}^\beta} \eta_s F_q(\cdot; r))(x), \end{aligned}$$

where

$$F_q(\xi; r) = \sum_{a \in A_q} G(\chi, a) \hat{f}(\xi + a/q) e^{2\pi i r a/q}.$$

Hence, by Theorem 4.3, we obtain

$$\begin{aligned} & \left| \left\{ x \in \mathbb{Z} : \sup_{t \leq n} \left| \sum_{a \in A_q} G(\chi, a) \mathcal{F}^{-1}(\widehat{M_{2^n}^\beta(\cdot - a/q) \eta_s(\cdot - a/q) \hat{f}})(x) \right| > \lambda \right\} \right| \\ &= \sum_{r=1}^q \left| \left\{ x \in \mathbb{Z} : \sup_{t \leq n} \left| \mathcal{F}^{-1}(\widehat{M_{2^n}^\beta \eta_s F_q(\cdot; r)})(qx + r) \right| > \lambda \right\} \right| \\ &\lesssim \sum_{r=1}^q \lambda^{-1} \|\mathcal{F}^{-1}(\eta_s F_q(\cdot; r))(qx + r)\|_{\ell^1(x)}. \end{aligned}$$

Next, by Young's convolution inequality we get

$$\begin{aligned} \sum_{r=1}^q \|\mathcal{F}^{-1}(\eta_s F_q(\cdot; r))(qx + r)\|_{\ell^1(x)} &= \left\| \sum_{a \in A_q} G(\chi, a) \mathcal{F}^{-1}(\eta_s(\cdot - a/q) \hat{f}) \right\|_{\ell^1} \\ &\leq \left\| \sum_{a \in A_q} G(\chi, a) \mathcal{F}^{-1}(\eta_s(\cdot - a/q)) \right\|_{\ell^1} \|f\|_{\ell^1}, \end{aligned}$$

and

$$\left\| \sum_{a \in A_q} G(\chi, a) \mathcal{F}^{-1}(\eta_s(\cdot - a/q)) \right\|_{\ell^1} = \left\| \sum_{a \in A_q} G(\chi, a) e^{2\pi i x a/q} \mathcal{F}^{-1}(\eta_s)(x) \right\|_{\ell^1(x)}.$$

Now, by Theorem 2.1, we can compute

$$\begin{aligned} & \left\| \sum_{a \in A_q} G(\chi, a) e^{2\pi i x a/q} \mathcal{F}^{-1}(\eta_s)(x) \right\|_{\ell^1(x)} \\ &= \sum_{r|q/q_0} \left\| \sum_{a \in A_q} G(\chi, a) e^{2\pi i r a/q} \mathcal{F}^{-1}(\eta_s)(qx + r) \right\|_{\ell^1(x)} \\ &\leq q_0 \sum_{r|q/q_0} \frac{\varphi(r)}{\varphi(q)} \|\mathcal{F}^{-1}(\eta_s)(qx + r)\|_{\ell^1(x)} \\ &\lesssim \frac{q_0}{q} \sum_{r|q/q_0} \frac{\varphi(r)}{\varphi(q)}, \end{aligned}$$

where in the last inequality we have used Lemma 4.2 together with Lemma 4.1. Since (see e.g. [18])

$$\sum_{r|q/q_0} \varphi(r) = \frac{q}{q_0},$$

we conclude that

$$\left\| \sum_{a \in A_q} G(\chi, a) e^{2\pi i x a / q} \mathcal{F}^{-1}(\eta_s)(x) \right\|_{\ell^1(x)} \lesssim \frac{1}{\varphi(q)},$$

proving the claim and the theorem follows. $\square$

Theorem 6.3 *There is $C > 0$ such that for any subset $F \subset \mathbb{Z}$ of a finite cardinality and all $0 < \lambda < 1$,*

$$\left| \left\{ x \in \mathbb{Z} : \sup_{n \in \mathbb{N}} \mathcal{M}_{2^n}(\mathbb{1}_F)(x) > \lambda \right\} \right| \leq C \lambda^{-1} \log^2(e/\lambda) |F|.$$

Proof We start by proving the following statement.

Claim 6.4 *There are $C, c > 0$ such that for each $t > 0$, there are two sequences of operators $(A_n^t : n \in \mathbb{N})$ and $(B_n^t : n \in \mathbb{N})$ such that $\mathcal{M}_{2^n} = A_n^t + B_n^t$, and for any finitely supported function $f : \mathbb{Z} \rightarrow \mathbb{C}$,*

$$\sup_{\lambda > 0} \lambda \cdot \left| \left\{ x \in \mathbb{Z} : \sup_{n \in \mathbb{N}} |A_n^t f(x)| > \lambda \right\} \right| \leq C t \|f\|_{\ell^1}, \quad (26)$$

and

$$\left\| \sup_{n \in \mathbb{N}} |B_n^t f| \right\|_{\ell^2} \leq C \exp(-c\sqrt{t}) \|f\|_{\ell^2}. \quad (27)$$

Without loss of generality, we may assume that f is non-negative finitely supported function on $\mathbb{Z}$. For $1 \leq n < t$, we set

$$A_n^t f = \mathcal{M}_{2^n} f, \quad \text{and} \quad B_n^t f \equiv 0.$$

Since by the prime number theorem,

$$\frac{2^n}{C} \leq \vartheta(2^n),$$

we have

$$\mathcal{M}_{2^n} f(x) \leq C n M_{2^n} f(x).$$

Hence, by the Hardy–Littlewood theorem,

$$\left| \left\{ x \in \mathbb{Z} : \sup_{1 \leq n < t} \mathcal{M}_{2^n} f(x) > \lambda \right\} \right| \leq \left| \left\{ x \in \mathbb{Z} : \sup_{1 \leq n < t} M_{2^n} f(x) > \frac{\lambda}{Ct} \right\} \right| \\ \lesssim \lambda^{-1} t \|f\|_{\ell^1}.$$

For $t \leq n$, we set

$$A_n^t f = \mathcal{F}^{-1}(\Pi_n^t \hat{f}), \quad \text{and} \quad B_n^t f = \mathcal{M}_{2^n} f - A_n^t f.$$

In view of Corollary 5.4 and Theorem 6.1, we obtain (27) and (26), respectively, and the claim follows.

Now, the theorem is an easy consequence of Claim 6.4. Indeed, given a subset $F \subset \mathbb{Z}$ of a finite cardinality, for any $t > 0$, we can write

$$\left| \left\{ \sup_{n \in \mathbb{N}} \mathcal{M}_{2^n}(\mathbb{1}_F) > \lambda \right\} \right| \lesssim \left| \left\{ \sup_{n \in \mathbb{N}} |A_n^t(\mathbb{1}_F)| > \frac{1}{2}\lambda \right\} \right| + \left| \left\{ \sup_{n \in \mathbb{N}} |B_n^t(\mathbb{1}_F)| > \frac{1}{2}\lambda \right\} \right| \\ \lesssim \lambda^{-1} t |F| + \lambda^{-2} \exp(-2c\sqrt{t}) |F|.$$

Thus, taking

$$t = (2c)^{-2} \log^2(e/\lambda),$$

we get the desired conclusion. $\square$

In view of (7), Theorem 6.3 entails the following corollary, which is precisely Theorem C.

Corollary 6.5 *There is $C > 0$ such that for any subset $F \subset \mathbb{Z}$ of a finite cardinality and all $0 < \lambda < 1$,*

$$\left| \left\{ x \in \mathbb{Z} : \sup_{n \in \mathbb{N}} \mathcal{A}_{2^n}(\mathbb{1}_F)(x) > \lambda \right\} \right| \leq C \lambda^{-1} \log^2(e/\lambda) |F|.$$

7 Applications

In this section we show two applications of Theorem 6.3 and Corollary 6.5. First, we prove that the restricted weak Orlicz estimates together with strong ℓ^2 bounds are sufficient to get ℓ^p maximal inequalities for all $1 < p \leq 2$. Next, we conclude almost everywhere convergence of ergodic averages for functions in some Orlicz space close to L^1 .

7.1 ℓ^p theory

Theorem 7.1 *For each $p \in (1, 2]$ there is $C > 0$ such that for any function $f \in \ell^p(\mathbb{Z})$,*

$$\left\| \sup_{N \in \mathbb{N}} |\mathcal{M}_N f| \right\|_{\ell^p} \leq C(p-1)^{-4} \|f\|_{\ell^p}.$$

Proof With loss of generality, we may restrict the supremum to dyadic numbers. We claim the following holds true.

Claim 7.2 *There is $C > 0$ such that for any subset $F \subset \mathbb{Z}$ of finite cardinality, and any $p_0 \in (1, 5)$,*

$$\sup_{\lambda > 0} \lambda \cdot \left| \left\{ x \in \mathbb{Z} : \sup_{n \in \mathbb{N}} \mathcal{M}_{2^n}(\mathbb{1}_F)(x) > \lambda \right\} \right|^{\frac{1}{p_0}} \leq C(p_0 - 1)^{-\frac{2}{p_0}} |F|^{\frac{1}{p_0}}.$$

Since $\mathcal{M}_N$ are averaging operators, we may assume that $0 < \lambda < 1$. Observe that the function

$$(0, 1) \ni \lambda \mapsto \lambda^{p_0-1} \log^2(e/\lambda)$$

attains its maximum at

$$\lambda = \exp \left(1 - \frac{2e}{p_0 - 1} \right).$$

The maximal value equals $4e^{p_0-2e+1}(p_0-1)^{-2}$, thus

$$\lambda^{-1} \log^2(e/\lambda) \leq 4e^{p_0-2e+1}(p_0-1)^{-2} \lambda^{-p_0}.$$

Hence, by Theorem 6.3, we get

$$\begin{aligned} \left| \left\{ x \in \mathbb{Z} : \sup_{n \in \mathbb{N}} \mathcal{M}_{2^n}(\mathbb{1}_F)(x) > \lambda \right\} \right| &\leq C \lambda^{-1} \log^2(e/\lambda) |F| \\ &\leq 4C e^{p_0-2e+1} (p_0-1)^{-2} \lambda^{-p_0} |F|, \end{aligned}$$

which is what we claimed.

Next, we notice that by Theorems 3.2 and 5.1, we have

$$\left\| \sup_{n \in \mathbb{N}} |\mathcal{M}_{2^n} f| \right\|_{\ell^2} \leq C \|f\|_{\ell^2}. \quad (28)$$

Let us consider $p \in (1, 2)$. Set $p_0 = (1 + p)/2$. Since $p_0 > 1$ at the cost of the additional factor of $(p-1)^{-1}$, we get

$$\left\| \sup_{n \in \mathbb{N}} |\mathcal{M}_{2^n} f| \right\|_{\ell^{p_0, \infty}} \leq C(p-1)^{-1-\frac{2}{p_0}} \|f\|_{\ell^{p_0, 1}} \quad (29)$$

for any $f \in \ell^{p_0,1}(\mathbb{Z})$. Now, by the Marcinkiewicz interpolation theorem, [8, Theorem 11.9], based on (28) and (29) we obtain

$$\left\| \sup_{n \in \mathbb{N}} |\mathcal{M}_{2^n} f| \right\|_{\ell^p} \leq C \frac{p(2-p_0)}{(p-p_0)(2-p)} (p-1)^{-\frac{p_0+2}{p_0}\theta} \|f\|_{\ell^p}$$

where $\theta \in (0, 1)$ satisfies

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{2}.$$

Since

$$\frac{p(2-p_0)}{(p-p_0)(2-p)} (p-1)^{-\frac{p_0+2}{p_0}\theta} = \frac{p(3-p)}{(p-1)(2-p)} (p-1)^{-\frac{p+5}{p+1}, \frac{p+2-p^2}{p(3-p)}} \lesssim (p-1)^{-3},$$

the theorem follows. $\square$

7.2 Pointwise Convergence

Let $(X, \mathcal{B}, \mu)$ be a probability space with a measurable and measure preserving transformation $T : X \rightarrow X$. We consider the following averages

$$\mathcal{A}_N f(x) = \frac{1}{\pi(N)} \sum_{p \in \mathbb{P}_N} f(T^p x), \quad x \in X.$$

With a help of the Calderón transference principle from [7] applied to Corollary 6.5, we deduce the following proposition.

Proposition 7.3 *There is $C > 0$ such that for any subset $A \in \mathcal{B}$, and all $0 < \lambda < 1$,*

$$\mu \left\{ x \in X : \sup_{N \in \mathbb{N}} \mathcal{A}_N(\mathbb{1}_A)(x) > \lambda \right\} \leq C \lambda^{-1} \log^2(e/\lambda) \mu(A).$$

Proof Fix $A \in \mathcal{B}$ and $x \in X$. For $R > L > 0$, we define a finite subset of $F \subset \mathbb{Z}$ by setting

$$F = \{0 \leq n \leq R : T^n x \in A\}.$$

Then for $0 \leq n \leq R - N$, $N \leq L$,

$$\begin{aligned} \mathcal{A}_N(\mathbb{1}_A)(T^n x) &= \frac{1}{\pi(N)} \sum_{p \in \mathbb{P}_N} \mathbb{1}_A(T^{n+p} x) \\ &= \frac{1}{\pi(N)} \sum_{p \in \mathbb{P}_N} \mathbb{1}_F(n+p) = \mathcal{A}_N(\mathbb{1}_F)(n). \end{aligned}$$

Hence,

$$\begin{aligned} & \left| \left\{ 0 \leq n \leq R - L : \max_{1 \leq N \leq L} \mathcal{A}_N(\mathbb{1}_A)(T^n x) > \lambda \right\} \right| \\ & \leq \left| \left\{ n \in \mathbb{Z} : \max_{1 \leq N \leq L} \mathcal{A}_N(\mathbb{1}_F)(n) > \lambda \right\} \right|. \end{aligned}$$

By Corollary 6.5,

$$\begin{aligned} \left| \left\{ n \in \mathbb{Z} : \max_{1 \leq N \leq L} \mathcal{A}_N(\mathbb{1}_F)(n) > \lambda \right\} \right| & \leq C \lambda^{-1} \log^2(e/\lambda) \sum_{n \in \mathbb{Z}} \mathbb{1}_F(n) \\ & = C \lambda^{-1} \log^2(e/\lambda) \sum_{n=0}^R \mathbb{1}_A(T^n x). \end{aligned}$$

Since T preserves the measure μ , by integrating with respect to $x \in X$ we obtain

$$\begin{aligned} & (R - L + 1) \cdot \mu \left(x \in X : \max_{1 \leq N \leq L} \mathcal{A}_N(\mathbb{1}_A)(x) > \lambda \right) \\ & = \sum_{n=0}^{R-L} \mu \left(x \in X : \max_{1 \leq N \leq L} \mathcal{A}_N(\mathbb{1}_A)(T^n x) > \lambda \right) \\ & = \int_X \left| \left\{ 0 \leq n \leq R - L : \max_{1 \leq N \leq L} \mathcal{A}_N(\mathbb{1}_A)(T^n x) > \lambda \right\} \right| d\mu(x) \\ & \leq C \lambda^{-1} \log^2(e/\lambda) \sum_{n=0}^R \int_X \mathbb{1}_A(T^n x) d\mu(x) \\ & = C(R + 1) \lambda^{-1} \log^2(e/\lambda) \mu(A). \end{aligned}$$

We now divide by R and take R approaching infinity to get

$$\mu \left(x \in X : \max_{1 \leq N \leq L} \mathcal{A}_N(\mathbb{1}_A)(T^n x) > \lambda \right) \leq C \lambda^{-1} \log^2(e/\lambda) \mu(A).$$

Finally, taking L tending to infinity by the monotone convergence theorem we conclude the proof. $\square$

We are now in the position to show μ -almost everywhere convergence of the ergodic averages $(\mathcal{A}_N f : N)$ for a function f from the Orlicz space $L(\log L)^2(\log \log L)(X, \mu)$. Let us recall that $L(\log L)^2(\log \log L)(X, \mu)$ consists of functions such that

$$\int_X |f(x)| (\log^+ |f(x)|)^2 (\log^+ \log^+ |f(x)|) d\mu(x) < \infty$$

where $\log^+ t = \max\{0, \log t\}$. The space $L(\log L)^2(\log \log L)(X, \mu)$ is a Banach space with the norm

$$\|f\|_{L(\log L)^2(\log \log L)} = \int_0^1 f^*(t) \phi(t^{-1}) dt$$

where f^* is the decreasing rearrangement of f , that is

$$f^*(t) = \inf \left\{ s > 0 : \mu \{x \in X : |f(x)| \geq s\} \leq t \right\},$$

and

$$\phi(t) = \log^2(1+t) \log(1+\log t).$$

Theorem 7.4 *There is $C > 0$ such that for each $f \in L(\log L)^2(\log \log L)(X, \mu)$,*

$$\sup_{\lambda > 0} \lambda \cdot \mu \left\{ x \in X : \sup_{N \in \mathbb{N}} |\mathcal{A}_N f(x)| > \lambda \right\} \leq C \|f\|_{L(\log L)^2(\log \log L)}.$$

In particular, for each $f \in L(\log L)^2(\log \log L)(X, \mu)$,

$$\text{the limit } \lim_{N \rightarrow \infty} \mathcal{A}_N f(x) \text{ exists}$$

for μ -almost all $x \in X$.

Proof We first prove the following claim.

Claim 7.5 *There is $C > 0$ such that for each $A \in \mathcal{B}$, and any $0 < \lambda < 1$,*

$$\sup_{\lambda > 0} \lambda \cdot \mu \left\{ x \in X : \sup_{N \in \mathbb{N}} \mathcal{A}_N(\mathbb{1}_A)(x) > \lambda \right\} \leq C \mu(A) \log^2 \left(\frac{e}{\mu(A)} \right). \quad (30)$$

Indeed, by monotonicity, if $\lambda \geq \mu(A)$, then

$$\lambda^{-1} \mu(A) \log^2 \left(\frac{e}{\lambda} \right) \leq \lambda^{-1} \mu(A) \log^2 \left(\frac{e}{\mu(A)} \right). \quad (31)$$

Otherwise, $\lambda \leq \mu(A)$, which entails that

$$1 \leq \lambda^{-1} \mu(A) \leq \lambda^{-1} \mu(A) \log^2 \left(\frac{e}{\mu(A)} \right). \quad (32)$$

In view of Proposition 7.3,

$$\mu \left\{ x \in X : \sup_{N \in \mathbb{N}} \mathcal{A}_N(\mathbb{1}_A)(x) > \lambda \right\} \leq \min \left\{ 1, C \mu(A) \lambda^{-1} \log^2(e/\lambda) \right\},$$

which together with (31) and (32) easily lead to (30).

Now, to show the theorem, let us fix $f \in L(\log L)^2(\log \log L)(X, \mu)$. We set

$$A_j = \left\{ x \in X : f^*(2^{-j+1}) < |f(x)| \leq f^*(2^{-j}) \right\},$$

and

$$a_j = f^*(2^{-j}).$$

Since $|f(x)| \leq a_j$ for $x \in A_j$, we have

$$|f| \leq \sum_{j \geq 1} a_j \mathbb{1}_{A_j}.$$

Moreover, if $j > k$ then for $x \in A_j$ and $y \in A_k$, we have $|f(x)| \geq |f(y)|$. Since $\mu(A_j) = 2^{-j}$, we get

$$f^*(t) \geq \sum_{j \geq 1} a_j \mathbb{1}_{[2^{-j-1}, 2^{-j})}(t). \quad (33)$$

Because the space $L^{1,\infty}(X, \mu)$ is log-convex (see [11, 19]), by Claim 7.5, we get

$$\begin{aligned} & \sup_{\lambda > 0} \lambda \cdot \mu \left\{ x \in X : \sup_{N \in \mathbb{N}} |\mathcal{A}_N f(x)| > \lambda \right\} \\ & \lesssim \sum_{j \geq 1} \log(j+1) \sup_{\lambda > 0} \lambda \cdot \mu \left\{ x \in X : a_j \sup_{N \in \mathbb{N}} \mathcal{A}_N(\mathbb{1}_{A_j})(x) > \lambda \right\} \\ & \lesssim \sum_{j \geq 1} \log(j+1) a_j \mu(A_j) \log^2 \left(\frac{e}{\mu(A_j)} \right). \end{aligned} \quad (34)$$

On the other hand, by (33) we have

$$\begin{aligned} \|f\|_{L(\log L)^2(\log \log L)} &= \int_0^1 f^*(t) \phi(t^{-1}) dt \geq \sum_{j \geq 1} a_j \phi(2^j) 2^{-j-1} \\ &\geq \frac{1}{8} \sum_{j \geq 1} a_j \mu(A_j) \log^2 \left(\frac{e}{\mu(A_j)} \right) \log(j+1), \end{aligned}$$

which together with (34) conclude the proof. $\square$

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Birkhoff, G.D.: Proof of the ergodic theorem. *Proc. Natl. Acad. Sci. USA* **17**, 656–660 (1931)
2. Bourgain, J.: Estimations de certaines fonctions maximales. *C. R. Acad. Sci. Paris I* **301**(10), 499–502 (1985)
3. Bourgain, J.: An Approach to Pointwise Ergodic Theorems. *Geometric Aspects of Functional Analysis*, pp. 204–223. Springer, New York (1988)
4. Bourgain, J.: On the maximal ergodic theorem for certain subsets of the integers. *Israel J. Math.* **61**, 39–72 (1988)
5. Bourgain, J.: Pointwise ergodic theorems for arithmetic sets. With an appendix by the author, Harry Furstenberg, Yitzhak Katznelson and Donald S. Ornstein. *Publ. Math. Paris* **69**(1), 5–45 (1989)
6. Buczolich, Z., Mauldin, R.D.: Divergent square averages. *Ann. Math.* **171**(3), 1479–1530 (2010)
7. Calerón, A.P.: Ergodic theory and translational-invariant operators. *Proc. Natl. Acad. Sci.* **59**(2), 349–353 (1968)
8. de Reyna, J.A.: Pointwise Convergence of Fourier Series. *Lecture Notes in Mathematics*. Springer, New York (2002)
9. Fefferman, Ch.: Inequalities for strongly singular convolution operators. *Acta Math.* **124**, 9–36 (1970)
10. Ionescu, A.D.: An endpoint estimate for the discrete spherical maximal function. *Proc. Am. Math. Soc.* **132**(5), 1411–1417 (2004)
11. Kalton, N.J.: Convexity, type, and the three space problem. *Studia Math.* **69**(3), 247–287 (1980/1981)
12. LaVictoire, P.: Universally L^1 -bad arithmetic sequences. *J. Anal. Math.* **113**(1), 241–263 (2011)
13. Mirek, M., Trojan, B.: Cotlar’s ergodic theorem along the prime numbers. *J. Fourier Anal. Appl.* **21**(4), 822–848 (2015)
14. Mirek, M., Trojan, B.: Discrete maximal functions in higher dimensions and applications to ergodic theory. *Am. J. Math.* **138**(6), 1495–1532 (2016)
15. Mirek, M., Trojan, B., Zorin-Kranich, P.: Variational estimates for averages and truncated singular integrals along the prime numbers. *Trans. Am. Math. Soc.* **369**(8), 5403–5423 (2017)
16. Montgomery, H.L., Vaughan, R.C.: *Multiplicative Number Theory I: Classical Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (2006)
17. Nathanson, M.B.: *Additive Number Theory the Classical Bases*. Graduate Texts in Mathematics. Springer, Princeton (1996)
18. Sitaramachandrarao, R.: On an error term of Landau II. *Rocky Mt. J. Math.* **15**, 579–588 (1985)
19. Stein, E.M., Weiss, N.J.: On the convergence of Poisson integrals. *Trans. Am. Math. Soc.* **140**, 34–54 (1969)
20. Vinogradov, I.M.: *The Method of Trigonometrical Sums in the Theory of Numbers*. Dover Books on Mathematics Series. Dover, Mineola (1954)
21. Wierdl, M.: Pointwise ergodic theorem along the prime numbers. *Israel J. Math.* **64**(3), 315–336 (1988)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.