Sharp Multiplier Theorem for Multidimensional Bessel Operators

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Abstract

Consider the multidimensional Bessel operator

$$Bf(x) = - \sum_{j=1}^N \left(\partial_j^2 f(x) + \frac{\alpha_j}{x_j} \partial_j f(x) \right), \quad x \in (0, \infty)^N.$$

Let $d = \sum_{j=1}^N \max(1, \alpha_j + 1)$ be the dimension of the space $(0, \infty)^N$ equipped with the measure $x_1^{\alpha_1} \dots x_N^{\alpha_N} dx_1 \dots dx_N$. In the general case $\alpha_1, \dots, \alpha_N > -1$ we prove multiplier theorems for spectral multipliers $m(B)$ on $L^{1,\infty}$ and the Hardy space H^1 . We assume that m satisfies the classical Hörmander condition

$$\sup_{t>0} \|\eta(\cdot)m(t\cdot)\|_{W^{2,\beta}(\mathbb{R})} < \infty$$

with $\beta > d/2$. Furthermore, we investigate imaginary powers B^{ib} , $b \in \mathbb{R}$, and prove some lower estimates on $L^{1,\infty}$ and L^p , $1 < p < 2$. As a consequence, we deduce that our multiplier theorem is sharp.

Keywords Spectral multiplier · Hardy space · Bessel operator · space of homogeneous type

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1 Introduction and Main Results

1.1 The Bessel Operator

Let $N \in \mathbb{N}$ and $\alpha = (\alpha_1, \dots, \alpha_N)$, where $\alpha_j > -1$ for $j = 1, \dots, N$. Consider the space $X = (0, \infty)^N$ equipped with the Euclidean metric and the measure $d\nu(x) = x^\alpha dx = x_1^{\alpha_1} \dots x_N^{\alpha_N} dx_1 \dots dx_N$. It is well-known that X satisfies the doubling property, i.e.

$$\nu(B(x, 2r)) \leq C\nu(B(x, r)), \quad x \in X, r > 0,$$

where $B(x, r) = \{y \in X : |x - y| < r\}$. In other words, there exist $d, C_d > 0$ such that

$$\nu(B(x, \gamma r)) \leq C_d(1 + \gamma)^d \nu(B(x, r)), \quad x \in X, r, \gamma > 0. \quad (\text{D})$$

We choose the constant d (dimension of X) as small as possible. In this case

$$d = \sum_{j=1}^N \max(1, \alpha_j + 1). \quad (1.1)$$

The multidimensional Bessel operator is given by $B = B_1 + \dots + B_N$, where

$$B_j f(x) = -\partial_j^2 f(x) - \frac{\alpha_j}{x_j} \partial_j f(x), \quad x \in X.$$

The operator B , initially defined on, say, $(C_c^2((0, \infty)^N))^n$, extends to a self-adjoint operator on $L^2(X)$. Slightly abusing notation, we shall denote this extension by the same symbol B . For a precise definition of B we refer the reader to e.g. [6, Sec. 2] (see also [32]). Also, B is the infinitesimal generator of the Bessel semigroup $\mathbf{T}_t f(x) = \int_X T_t(x, y) f(y) d\nu(y)$, where $T_t(x, y) = T_t^{[1]}(x_1, y_1) \dots T_t^{[N]}(x_N, y_N)$ and

$$T_t^{[j]}(x_j, y_j) = \frac{1}{2t} (x_j y_j)^{-(\alpha_j-1)/2} I_{(\alpha_j-1)/2} \left(\frac{x_j y_j}{2t} \right) \exp \left(-\frac{x_j^2 + y_j^2}{4t} \right), \quad x_j, y_j, t > 0. \quad (1.2)$$

Here $I_\tau(x) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m+\tau+1)} \left(\frac{x}{2}\right)^{2m+\tau}$ is the modified Bessel function of the first kind. The kernel $T_t(x, y)$ satisfies the upper and lower gaussian bounds, i.e. there exist constants $c_1, c_2, C_1, C_2 > 0$, such that

$$\begin{aligned} C_1 v(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{|x-y|^2}{c_1 t}\right) &\leq T_t(x, y) \\ &\leq C_2 v(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{|x-y|^2}{c_2 t}\right). \end{aligned} \quad (\text{G})$$

This fact is well known and follows from the asymptotics for $v(B(x, \sqrt{t}))$ and I_t . For details see e.g. [20, Lem. 4.2].

Since B is self-adjoint and nonnegative, for a Borel function $m : (0, \infty) \rightarrow \mathbb{C}$ the spectral theorem defines the operator

$$m(B) = \int_0^\infty m(\lambda) dE_B(\lambda),$$

where E_B is the spectral resolution of B .

1.2 Multiplier Theorems for B

Multiplier theorems for B and other operators are one of the main topics in harmonic analysis. Many authors investigated assumptions on m that guarantee boundedness of $m(B)$ on various function spaces, such as $L^p(X)$, $H^p(X)$, $L^{p,q}(X)$ and others.

For example, in [23] the authors proved weak type $(1, 1)$ estimates on $m(B)$ assuming $N = 1$, $\alpha > 0$ and

$$\left(\int_{R/2}^R |m^{(s)}(\lambda)|^2 d\nu(\lambda) \right)^{1/2} \leq C R^{(\alpha+1)/2-s}, \quad R > 0,$$

where $s = 0, \dots, K$ and K is the least even integer greater than $(\alpha + 1)/2 = d/2$ (see also [27]). In [18], assuming still $N = 1$ and $\alpha > 0$, it is proved that if

$$\sup_{t>0} \|\eta(\cdot) m(t \cdot)\|_{W^{2,\beta}(\mathbb{R})} < \infty \quad (\text{S})$$

with some $\beta > d/2$, then $m(B)$ is bounded on the Hardy space $H^1(B)$ related to B . Here and thereafter $W^{2,\beta}(\mathbb{R})$ is the L^2 -Sobolev space on $\mathbb{R}$ and η is a fixed nonnegative smooth cut-off function such that $\text{supp } \eta \subseteq (2^{-1}, 2)$.

In the multidimensional case $N \geq 1$ in [5] the authors prove weak type $(1, 1)$ estimates for $m(B)$, where m is of Laplace transform type, i.e. there exists $\phi \in L^\infty(0, \infty)$, such that

$$m(x) = |x|^2 \int_0^\infty e^{-t|x|^2} \phi(t) dt, \quad x \in (0, \infty)^N.$$

Notice, that if m is of Laplace transform type, then m is radial and (as a function on $(0, \infty)$) satisfies (S) with any $\beta > 0$. Another multidimensional result can be found in [20], where it is proved that $m(B)$ is weak type $(1, 1)$ and bounded on the Hardy space

$H^1(X)$ provided that $\alpha_j > 1$ for $j = 1, \dots, N$ and m satisfies (S) with $\beta > d/2$. See also e.g. [21,22,38] for other multiplier results for the Bessel operator.

Our first main goal is to obtain multiplier theorem for B in the most general case $N \geq 1$ and $\alpha_j > -1$, $j = 1, \dots, N$. Let us notice that many of the results before assumed that $\alpha_j > 0$ and the case $\alpha_j < 0$ is more difficult and less known. One reason for that is the singularity at zero of the measure $x^{\alpha_j} dx_j$ when $\alpha_j < 0$. Also, so-called “generalized translation” operators and convolution structure for B (see, e.g. [7, Sec. 2]), does not help when $\alpha_j < 0$. This is strictly related to the fact, that the generalized eigenfunctions of B are no longer bounded if $\alpha_j < 0$ for some j and, therefore, the generalized translation is not even bounded on L^2 . Let us also notice that, we are interested in multiplier results that are sharp in the sense that we assume (S) with β as small as possible. In this case this is expected to be $\beta > d/2$ (we shall discuss this in Sect. 1.3 below).

To state the multiplier result let us recall that the weak L^1 space is given by the semi-norm

$$\|f\|_{L^{1,\infty}(X)} = \sup_{\lambda>0} \lambda \nu \{x \in X : |f(x)| > \lambda\},$$

and the Hardy space $H^1(B)$ related to B can be defined by the norm

$$\|f\|_{H^1(B)} = \left\| \sup_{t>0} |\mathbf{T}_t f| \right\|_{L^1(X)}.$$

In the case $N = 1$ the space $H^1(B)$ was studied in [7], where $H^1(B)$ was characterized by means of atomic decompositions and the Riesz transforms. In the general case $N \geq 1$ and $\alpha_j > -1$, $j = 1, \dots, N$ the atomic characterization of $H^1(B)$ can be found in [19] (see also [16,20]). We shall recall this characterization in Sect. 2.3 below.

Theorem A *Let $N \geq 1$ and $\alpha_j > -1$ for $j = 1, \dots, N$. Assume that $m : (0, \infty) \rightarrow \mathbb{C}$ satisfies (S) with $\beta > d/2$, see (1.1). Then:*

1. $m(B)$ is bounded from $L^1(X)$ to $L^{1,\infty}(X)$,
2. $m(B)$ is bounded from $H^1(B)$ to $H^1(B)$,
3. $m(B)$ is bounded from $L^p(X)$ to $L^p(X)$, $1 < p < \infty$.

Part 1. of Theorem A will be proved by using results of [36]. More precisely, we shall check the assumptions of [36, Theorem 3.1]. The proof of 2. will be given in Sect. 2. In fact, in the proof we shall only use general properties of B , such as e.g. (D), (G), and (P_q) below. Thus, the multiplier result in Sect. 2 will be formulated in a more general context. This section can be read independently of the rest of the paper and we shall use different notation. As usual, 3. is a consequence of either 1. or 2. by duality and interpolation, see e.g. [4].

1.3 Imaginary Powers of B

Another goal of this paper is to study the imaginary powers B^{ib} , $b \in \mathbb{R}$, of the Bessel operator and establish lower bounds of these operators on some function spaces. We

shall concentrate our attention on the dependence of the lower estimates on b for large b . This is related with sharpness of multiplier theorems and may be of independent interest. To state these estimates let us restrict ourselves to the one-dimensional case $N = 1$ ($X = (0, \infty)$, $d\nu(x) = x^\alpha dx$, $\alpha > -1$). Motivated by the identity

$$B^{ib} = \Gamma(-ib)^{-1} \int_0^\infty t^{-ib} e^{-tB} \frac{dt}{t} \quad (1.3)$$

let us define for $x \neq y$ the integral kernel

$$K_b(x, y) = \Gamma(-ib)^{-1} \int_0^\infty t^{-ib} T_t(x, y) \frac{dt}{t}. \quad (1.4)$$

Notice, that the integral in (1.3) is not absolutely convergent, thus we have to explain how the kernel $K_b(x, y)$ is related to the operators B^{ib} . Indeed, in Sect. 3.3 we shall prove that for $f \in L^\infty(X)$ with compact support we have

$$B^{ib} f(x) = \int_X K_b(x, y) f(y) d\nu(y), \quad x \notin \text{supp } f \quad (1.5)$$

One of our goals is to provide lower estimates for B^{ib} .

Theorem B Assume that $\alpha > -1$. Then there exist a constant $C > 0$ and a function f such that $\|f\|_{L^1(X)} = 1$ and for $|b|$ large enough we have

$$\|B^{ib} f\|_{L^{1,\infty}(X)} \geq C|b|^{d/2}.$$

Theorem C Assume that $\alpha > 0$ and $p \in (1, 2)$. Then there exist $C_p > 0$ and f such that $\|f\|_{L^p(X)} = 1$ and for $|b|$ large enough we have

$$\|B^{ib} f\|_{L^p(X)} \geq C_p |b|^{\frac{d}{2} \frac{(2-p)}{p}}.$$

The proofs of Theorems B and C are presented in Sect. 3.3. To prove Theorem B we shall carefully analyze the kernels $K_b(x, y)$. More precisely, we prove the following lemma.

Lemma 1.1 Assume that $\alpha > -1$ and $b \in \mathbb{R}$. Then

$$\begin{aligned} K_b(x, y) = & c_1(b) (x^2 + y^2)^{-ib-(\alpha+1)/2} \\ & + c_2(b) (xy)^{-\alpha/2} |x - y|^{-2bi-1} \chi_{\{y/2 < x < 2y\}}(x, y) \\ & + c_3(b) R_b(x, y), \end{aligned} \quad (1.6)$$

where

$$c_1(b) = \frac{2^{2ib+1}}{\Gamma((\alpha+1)/4)} \frac{\Gamma(ib + (\alpha+1)/2)}{\Gamma(-ib)},$$

$$c_2(b) = \frac{2^{2ib}}{\sqrt{\pi}} \frac{\Gamma(ib + 1/2)}{\Gamma(-ib)}, \quad c_3(b) = \Gamma(-ib)^{-1}.$$

Moreover, there exists $C > 0$ that does not depend on b , such that

$$|R_b(x, y)| \leq Cxy(x + y)^{-\alpha-3}.$$

Notice that the kernel $R_b(x, y)$ is related to an operator that is bounded on every $L^p(X)$, $1 \leq p \leq \infty$, uniformly in $b \in \mathbb{R}$. Thus we may think of $R_b(x, y)$ as of some kind of “error term”. However, for $|b| > 1$ the sizes of the constants are the following:

$$|c_1(b)| \simeq |b|^{(\alpha+1)/2}, \quad |c_2(b)| \simeq |b|^{1/2}, \quad |c_3(b)| \simeq |b|^{1/2} \exp\left(\frac{\pi|b|}{2}\right), \quad (1.7)$$

c.f. Lemma 4.1. Thus, $c_3(b)$ grows exponentially when $|b| \rightarrow \infty$, while the constants $c_1(b)$ and $c_2(b)$ are much smaller. It appears that the growth of the constant $c_3(b)$ will lead to a problem in deriving lower estimates for B^{ib} (since our goal is to find the exact dependence on b). However, we can overcome this difficulty when analyzing weak $(1, 1)$ norm as in Theorem B. The same trick seems not to work in other function spaces (such as $H^1(B)$, $L^p(X)$ and $L^{p,\infty}(X)$ with $p > 1$), thus the proof of Theorem C is different and uses the integral representation of the Bessel function I_τ instead of Lemma 1.1.

As a corollary of Theorems B and C we obtain that Theorem A is sharp (at least for $N = 1$) in the sense that $d/2$ cannot be replaced by a smaller number. The argument is standard, but we shall present it now for the convenience of the reader. One can check that for $m_b(\lambda) = \lambda^{ib}$ we have

$$M_b := \sup_{t>0} \|\eta(\cdot)m_b(t\cdot)\|_{W^{2,\beta}(\mathbb{R})} \leq |b|^\beta.$$

Also, Theorem A actually gives that $\|m_b(B)f\|_{L^{1,\infty}(X)} \leq CM_b \|f\|_{L^1(X)}$, where C does not depend on b . Combining these estimates with Theorem B for $|b|$ large enough we have

$$|b|^{d/2} \leq C \|m_b(B)\|_{L^1(X) \rightarrow L^{1,\infty}(X)} \leq C|b|^\beta.$$

Therefore $\beta \geq d/2$. Actually, one expects that $\beta \neq d/2$, but this question is beyond the scope of this paper.

Similarly, the constant $d/2$ cannot be improved for the Hardy spaces. If $\alpha < 0$ then $d/2 = 1/2$ and (S) with $\beta < 1/2$ would not even guarantee that m is bounded. On the other hand, for $\alpha > 0$ if we could prove multiplier theorem on $H^1(B)$ with a constant lower than $d/2$, then by interpolation we would have better upper bounds for $m_b(B)$ on $L^p(X)$ for $1 < p < 2$, which contradicts Theorem C by an argument similar to the one above.

1.4 Organization of the Paper and Notation

In Sect. 2 we state and prove a “sharp” multiplier theorem on Hardy spaces for self-adjoint operators on spaces of homogeneous type with certain assumptions (Theorem D). This is a slight generalization of Theorem A 2. in the spirit of [36, Theorem 3.1]. In Sect. 2 we shall use different notation, so that it can be read independently of the rest of the paper. In Sect. 3 we prove the results stated above. More precisely, first we check that B satisfies assumption (P_2) (see Sect. 2 below) in the full generality $N \geq 1, \alpha_j > -1$ for $j = 1, \dots, N$. Thus Theorem D can be applied for B . Then we prove Lemma 1.1 and Theorems B and C. We shall use standard notations, i.e. C and c denote positive constants that may change from line to line.

2 Sharp Multiplier Theorem on Hardy Spaces

2.1 Background and General Assumptions

In this section we consider a space Y with a metric ρ and a nonnegative measure μ . We shall assume that the triple (Y, ρ, μ) is a space of homogeneous type, i.e. there exists $C > 0$ such that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$, for all $x \in Y$ and $r > 0$, where $B(x, r) = \{y \in Y : \rho(x, y) < r\}$, c.f. [14]. It is well-known that this implies the existence of $d, C_d > 0$ such that

$$\mu(B(x, \gamma r)) \leq C_d(1 + \gamma)^d \mu(B(x, r)), \quad x \in Y, \quad r, \gamma > 0. \quad (\text{D})$$

As usual, we choose d as small as possible, even at the cost of enlarging C_d .

Let A denote a self-adjoint positive operator and let E_A be its spectral measure, i.e. $A = \int_0^\infty \lambda dE_A(\lambda)$. Denote by $\mathbf{P}_t = \exp(-tA)$ the semigroup generated by A . Assume that there exists an integral kernel $P_t(x, y)$ such that $\mathbf{P}_t f(x) = \int_Y P_t(x, y) f(y) d\mu(y)$ and that satisfies the upper gaussian bounds, i.e. there exist $c_2, C_2 > 0$ such that

$$P_t(x, y) \leq C_2 \mu(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{\rho(x, y)^2}{c_2 t}\right), \quad t > 0, \quad x, y \in Y. \quad (\text{UG})$$

2.2 Multiplier Theorems

By the spectral theorem, for a Borel function m on $(0, \infty)$, we have the operator

$$m(A) = \int_0^\infty m(\lambda) dE_A(\lambda).$$

In the classical case $A = -\Delta$, $Y = \mathbb{R}^D$, the Hörmander multiplier theorem states that if m satisfies (S) with $\beta > D/2$, then $m(-\Delta)$ is weak type $(1, 1)$ and bounded on $L^p(\mathbb{R}^D)$ for $1 < p < \infty$. It is well-known that the constant $D/2$ is sharp in the sense that it cannot be replaced by a smaller constant, see e.g. [34].

At this point let us recall one of many multiplier theorems on spaces of homogeneous type. Suppose Y and A are as in Sect. 2.1. Following [36] we introduce additional assumption. Suppose that there exists $C > 0$ and $q \in [2, \infty]$, such that for $R > 0$ and every Borel function m on $\mathbb{R}$ satisfying $\text{supp } m \subseteq [R/2, 2R]$ we have

$$\int_Y |K_{m(A)}(x, y)|^2 d\mu(x) \leq C \mu \left(B \left(y, R^{-1/2} \right) \right)^{-1} \|m(R \cdot)\|_{L^q(\mathbb{R})}^2. \quad (P_q)$$

Theorem 2.1 [36, Theorem 3.1] *Assume that on a space of homogeneous type (Y, ρ, μ) there is a self-adjoint positive operator A that satisfies (UG). Moreover, assume that (P_q) holds with some $q \in [2, \infty]$ and m satisfies*

$$\sup_{t>0} \|\eta(\cdot)m(t \cdot)\|_{W^{q,\beta}(\mathbb{R})} < \infty \quad (S_q)$$

with some $\beta > d/2$. Then $m(A)$ is of weak type $(1, 1)$ and bounded on $L^p(Y)$ for $p \in (1, \infty)$.

At this point let us make a few comments.

1. Assuming (UG) the operators $m(A)$ appearing in (P_q) always have integral kernels $K_{m(A)}(x, y)$, c.f. [36, Lemma 2.2].
2. For the Bessel operator we are interested in (S_q) and (P_q) for $q = 2$ only, $(S) = (S_2)$. However, in Sect. 2 the results are stated and proved with an arbitrary $q \in [2, \infty]$.
3. The assumption (P_q) in some sense plays a role of Plancherel theorem in the proof of Theorem 2.1. It is a key to obtain the sharp range $\beta > d/2$. For example, if we would allow m to satisfy (S_q) with $\beta > d/2 + 1/2$, then (P_q) would be superfluous.
4. The assumption (P_q) is written in [36] for m having support in $[0, R]$ not in $[R/2, 2R]$. However, a simple inspection of the proof shows that (P_q) is needed only for m with $\text{supp } m \subseteq [R/2, 2R]$. This makes no difference for many operators. However, it matters e.g. when considering the Bessel operator with negative parameters α_j .
5. Assumption (P_q) in [36] is written for $m(\sqrt{A})$, but we use equivalent version with $m(A)$ (therefore we replace $B(y, R^{-1})$ by $B(y, R^{-1/2})$).

One of the main goals of this paper is to establish a multiplier theorem on Hardy spaces. We shall use the definition of the Hardy space $H^1(A)$ associated with A by means of the maximal operator of the semigroup $\mathbf{P}_t$, namely

$$H^1(A) = \left\{ f \in L^1(Y) : \|f\|_{H^1(A)} := \left\| \sup_{t>0} |\mathbf{P}_t f| \right\|_{L^1(Y)} < \infty \right\}.$$

To state our result we shall assume additionally that $P_t(x, y)$ satisfies also the lower Gaussian bounds, namely there exist $c_1, C_1 > 0$, such that

$$P_t(x, y) \geq C_1 \mu(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{\rho(x, y)^2}{c_1 t}\right), \quad t > 0, x, y \in Y, \quad (\text{LG})$$

and that the space (Y, ρ, μ) satisfies the following assumption:

$$\text{for all } x \in Y \text{ the function } r \mapsto \mu(B(x, r)) \text{ is a bijection on } (0, \infty). \quad (\text{Y})$$

Notice that (Y) implies that $\mu(Y) = \infty$ and that μ is non-atomic. Now we are ready to state the theorem.

Theorem D *Assume that (Y, ρ, μ) is a space of homogeneous type, d is as in (D), and (Y) is satisfied. Suppose that there is a self-adjoint positive operator A such that (UG), (LG), and (P_q) hold with some $q \in [2, \infty]$. If m satisfies (S_q) and $\beta > d/2$, then $m(A)$ is bounded from $H^1(A)$ to $H^1(A)$, i.e. there exists $C > 0$, such that*

$$\|m(A)f\|_{H^1(A)} \leq C \|f\|_{H^1(A)}.$$

The history of multiplier theorems for spaces of homogeneous type is long and wide. The interested reader is referred to [2,9–11,13,17,20,21,24,26,31,33,35,36] and references therein. Let us concentrate for a moment on the range of parameters β in Theorem D. Obviously, in general, the range $\beta > d/2$ is optimal. However, it may happen that for some particular operators one may obtain multiplier results assuming that $\beta > \tilde{d}/2$ with $\tilde{d} < d$, see e.g. [29,30,33]. On the other hand, there are known families of operators for which the constant $d/2$ cannot be lower. One of the methods to prove this is to derive lower estimates for A^{ib} in terms of $b \in \mathbb{R}$, see [12,30,34,36]. Lastly, let us mention that some multiplier results hold also in the non-doubling case, see e.g. [15].

Boundedness of operators on the Hardy space H^1 is a natural counterpart of weak type $(1, 1)$ bound. For example, it is a good end point for the interpolation, see e.g. [4]. However, the Hardy spaces are strictly related to some cancellation conditions and it is usually more involving to study properties of operators on the Hardy space, than on L^p or $L^{p,\infty}$ spaces. Let us also mention that boundedness from H^1 to H^1 obviously implies boundedness from H^1 to L^1 , which is usually much easier to prove.

2.3 Hardy Spaces

The Hardy spaces on spaces of homogeneous type are studied extensively from the 60's, see e.g. [14]. In particular, now we have many atomic decompositions for H^p on various spaces and operators acting on this spaces. We refer the reader to e.g. [3,7,19,25] and references therein.

In this subsection we recall some results on Hardy spaces related to A , assuming that (D), (UG), (LG) and (Y) are satisfied. For the proofs and more details we refer

the reader to [19]. Firstly, there exists the unique (up to a multiplicative constant) A -harmonic function $\omega : Y \rightarrow \mathbb{R}$ such that

$$C^{-1} \leq \omega(x) \leq C, \quad x \in Y.$$

The function ω plays a special role in the analysis of A and $\mathbf{P}_t$. In particular we have the following Hölder-type estimate.

Theorem 2.2 *Suppose that the semigroup $\mathbf{P}_t$ satisfies (UG), (LG). Then there exist positive constants γ, c, C , such that if $\rho(y, z) \leq \sqrt{t}$, then*

$$\left| \frac{P_t(x, y)}{\omega(y)} - \frac{P_t(x, z)}{\omega(z)} \right| \leq C \mu(B(x, \sqrt{t}))^{-1} \left(\frac{\rho(y, z)}{\sqrt{t}} \right)^\gamma \exp \left(-\frac{\rho(x, y)^2}{ct} \right).$$

Theorem 2.2 is quite well-known and follows from a general theory. For a short and independent proof see [19, Sec. 4].

Corollary 2.3 *There exist $\gamma, C > 0$ such that if $\rho(y, z) \leq \sqrt{t}$, then*

$$\int_Y \left| \frac{P_t(x, y)}{\omega(y)} - \frac{P_t(x, z)}{\omega(z)} \right| d\mu(x) \leq C \left(\frac{\rho(y, z)}{\sqrt{t}} \right)^\gamma.$$

Using Theorem 2.2 the authors of [19] obtained the following atomic decomposition for the elements of $H^1(A)$. Let us call a function $a : Y \rightarrow \mathbb{C}$ an (μ, ω) -atom, if there exists a ball B in Y , such that:

$$\text{supp } a \subseteq B, \quad \|a\|_\infty \leq \mu(B)^{-1}, \quad \int_B a(x) \omega(x) d\mu(x) = 0.$$

Theorem 2.4 [19, Theorem 1] *There exists a constant $C > 0$ such that for each $f \in H^1(A)$ there exist $\lambda_k \in \mathbb{C}$ and (μ, ω) -atoms a_k ($k \in \mathbb{N}$), such that*

$$f(x) = \sum_{k \in \mathbb{N}} \lambda_k a_k(x), \quad \text{and} \quad C^{-1} \|f\|_{H^1(A)} \leq \sum_{k \in \mathbb{N}} |\lambda_k| \leq C \|f\|_{H^1(A)}.$$

Let us start by recalling a few consequences of (D) and (UG).

Lemma 2.5 [36, Lemma 2.1] *Suppose that (D) and (UG) hold. Then*

$$\int_{B(y, r)^c} |P_t(x, y)|^2 d\mu(x) \leq C \mu(B(y, \sqrt{t}))^{-1} \exp \left(-\frac{r^2}{c_2 t} \right).$$

In particular

$$\|P_t(x, \cdot)\|_{L^2(Y)}^2 \leq C \mu(B(x, \sqrt{t}))^{-1}.$$

Lemma 2.6 [36, Lemma 4.1] *For $\kappa \geq 0$ there exists a constant $C = C(\kappa) > 0$ such that*

$$\int_Y |P_{(1+i\tau)R^{-1}}(x, y)|^2 (1 + R^{1/2}\rho(x, y))^\kappa d\mu(x) \leq C\mu\left(B\left(y, R^{-1/2}\right)\right)^{-1} (1 + |\tau|)^\kappa.$$

Lemma 2.7 [36, Lemma 4.4] *Suppose that (D) holds and $\delta > 0$. Then*

$$\int_{B(y, r)^c} (1 + R^{1/2}\rho(x, y))^{-d-2\delta} d\mu(x) \leq C\mu\left(B\left(y, R^{-1/2}\right)\right) (1 + rR^{1/2})^{-2\delta}.$$

2.4 Key Kernel Estimates

This subsection is devoted to obtain key estimates needed for the proof of Theorem D. We shall assume (temporarily) that m satisfies $\text{supp } m \subseteq [R/2, 2R]$ with some $R > 0$. Later we shall use a partition of unity for general m . Denote $m_R(\lambda) = m(R\lambda)$, so that $\text{supp } m_R \subseteq [2^{-1}, 2]$. Let us notice that below the letter $q \in [2, \infty]$ is always the exponent related to (P_q) and (S_q) . Moreover, all the spectral operators below admit related integral kernels, which can be seen by using an argument identical as in [36, Lem. 2.2]. Let us denote $\tilde{m}_t(\lambda) = \exp(-t\lambda)m(\lambda)$ and let $M_t(x, y)$ be the kernel associated with $\tilde{m}_t(A) = \mathbf{P}_t m(A)$.

Proposition 2.8 *Assume that $\text{supp } m \subseteq [R/2, 2R]$ and $m_R \in W^{q, \beta}(\mathbb{R})$ with $\beta > d/2$. Then, there exist $\delta, \gamma, C > 0$ such that for $y, z \in Y$ and $r > 0$ we have*

$$\int_{B(y, r)^c} \sup_{t>0} |M_t(x, y)| d\mu(x) \leq C \left(1 + rR^{1/2}\right)^{-\delta} \|m_R\|_{W^{q, \beta}(\mathbb{R})}, \quad (2.1)$$

and, for $\rho(y, z) < R^{-1/2}$,

$$\int_{B(y, r)^c} \sup_{t>0} \left| \frac{M_t(x, y)}{\omega(y)} - \frac{M_t(x, z)}{\omega(z)} \right| d\mu(x) \leq C \left(R^{1/2}\rho(y, z)\right)^\gamma \|m_R\|_{W^{q, \beta}(\mathbb{R})}. \quad (2.2)$$

Let us start by showing the following lemma.

Lemma 2.9 *For $\varepsilon > 0, \kappa \geq 0$ there exists a constant $C = C(\kappa, \varepsilon)$ such that*

$$\begin{aligned} & \int_Y \sup_{t>0} |M_t(x, y)|^2 \left(1 + R^{1/2}\rho(x, y)\right)^\kappa d\mu(x) \\ & \leq C\mu\left(B\left(y, R^{-1/2}\right)\right)^{-1} \|m_R\|_{W^{q, \kappa/2+\varepsilon}(\mathbb{R})}^2. \end{aligned}$$

Proof Fix a cut-off function $\psi \in C_c^\infty(4^{-1}, 4)$, such that $\psi \equiv 1$ on $[2^{-1}, 2]$. Set

$$n_{t, R}(\lambda) = m_R(\lambda) \underbrace{e^{-tR\lambda} e^\lambda \psi(\lambda)}_{\lambda_{t, R}(\lambda)}.$$

By the Fourier inversion formula,

$$\tilde{m}_t(A) = n_{t,R}(AR^{-1})e^{-AR^{-1}} = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{n}_{t,R}(\tau) \exp\left((i\tau - 1)AR^{-1}\right) d\tau$$

and

$$M_t(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{n}_{t,R}(\tau) P_{(1-i\tau)R^{-1}}(x, y) d\tau. \quad (2.3)$$

Notice that $\text{supp } \lambda_{t,R}^{(N)} \subseteq (4^{-1}, 4)$ for arbitrary $N \in \mathbb{N}$. By simple calculus we can find a constant C_N such that

$$\sup_{R>0, t>0} |\widehat{\lambda}_{t,R}(\tau)| \leq C_N (1 + |\tau|)^{-N}.$$

Since $\widehat{n}_{t,R} = \widehat{m}_R * \widehat{\lambda}_{t,R}$ and $(1 + |\tau|) \leq (1 + |\theta|)(1 + |\tau - \theta|)$, for $\kappa \geq 0$ and $\varepsilon > 0$ we use the Cauchy-Schwarz inequality, getting

$$\begin{aligned} \int_{\mathbb{R}} \sup_{t>0} |\widehat{n}_{t,R}(\tau)| (1 + |\tau|)^{\kappa/2} d\tau &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \sup_{t>0} |\widehat{m}_R(\theta)| |\widehat{\lambda}_{t,R}(\tau - \theta)| (1 + |\tau|)^{\kappa/2} d\theta d\tau \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \sup_{t>0} |\widehat{m}_R(\theta)| |\widehat{\lambda}_{t,R}(\tau - \theta)| \\ &\quad (1 + |\theta|)^{\kappa/2} (1 + |\tau - \theta|)^{\kappa/2} d\tau d\theta \\ &\leq C \int_{\mathbb{R}} |\widehat{m}_R(\theta)| (1 + |\theta|)^{(\kappa+1)/2+\varepsilon} (1 + |\theta|)^{-1/2-\varepsilon} d\theta \\ &\leq C \|m_R\|_{W^{2,(1+\kappa)/2+\varepsilon}(\mathbb{R})} \left(\int_{-\infty}^{\infty} (1 + |\theta|)^{-1-\varepsilon} d\theta \right)^{1/2} \\ &\leq C \|m_R\|_{W^{2,(1+\kappa)/2+\varepsilon}(\mathbb{R})}. \end{aligned}$$

Hence, by (2.3), the Minkowski inequality, and Lemma 2.6 we obtain

$$\begin{aligned} &\left(\int_Y \sup_{t>0} |M_t(x, y)|^2 (1 + R^{1/2} \rho(x, y))^{\kappa} d\mu(x) \right)^{1/2} \\ &\leq \int_{\mathbb{R}} \sup_{t>0} |\widehat{n}_{t,R}(\tau)| \left(\int_Y |P_{(1-i\tau)R^{-1}}(x, y)|^2 (1 + R^{1/2} \rho(x, y))^{\kappa} d\mu(x) \right)^{1/2} d\tau \\ &\leq C \mu \left(B \left(y, R^{-1/2} \right) \right)^{-1/2} \int_{\mathbb{R}} \sup_{t>0} |\widehat{n}_{t,R}(\tau)| (1 + |\tau|)^{\kappa/2} d\tau \quad (2.4) \\ &\leq C \mu \left(B \left(y, R^{-1/2} \right) \right)^{-1/2} \|m_R\|_{W^{2,(1+\kappa)/2+\varepsilon}(\mathbb{R})} \\ &\leq C \mu \left(B \left(y, R^{-1/2} \right) \right)^{-1/2} \|m_R\|_{W^{q,(1+\kappa)/2+\varepsilon}(\mathbb{R})}. \end{aligned}$$

In the last inequality we have used that $\text{supp } m_R \subseteq [2^{-1}, 2]$ and $q \geq 2$.

Observe that (2.4) is exactly the estimate we look for, but the Sobolev parameter is higher by $1/2$ than we want. To sharpen this estimate, we make use of known interpolation method. Notice, that $M_t(x, y) = \mathbf{P}_t(K_{m(A)}(\cdot, y))(x)$. It is well-known that (UG) implies boundedness on $L^2(Y)$ of the maximal operator $\mathcal{M}f = \sup_{t>0} |\mathbf{P}_t f|$. A second estimate needed for an interpolation is the following

$$\begin{aligned} \left(\int_Y \sup_{t>0} |M_t(x, y)|^2 d\mu(x) \right)^{1/2} &= \|\mathcal{M}K_{m(A)}(\cdot, y)\|_{L^2(Y)} \\ &\leq C \|K_{m(A)}(\cdot, y)\|_{L^2(Y)} \\ &\leq C \mu \left(B \left(y, R^{-1/2} \right) \right)^{-1/2} \|m_R\|_{L^q(\mathbb{R})}. \end{aligned} \quad (2.5)$$

In the last inequality we have used (P_q) . Now, Lemma 2.9 follows by interpolating (2.4) and (2.5), see e.g. proofs of [36, Lem. 4.3(a)] and [20, Lem. 2.2] for details. $\square$

Proof of (2.1) By the Cauchy-Schwarz inequality and Lemmas 2.7 and 2.9,

$$\begin{aligned} &\int_{B(y,r)^c} \sup_{t>0} |M_t(x, y)| d\mu(x) \\ &\leq \left(\int_Y \sup_{t>0} |M_t(x, y)|^2 (1 + R^{1/2} \rho(x, y))^{d+2\delta} d\mu(x) \right)^{1/2} \\ &\quad \left(\int_{B(y,r)^c} (1 + R^{1/2} \rho(x, y))^{-d-2\delta} d\mu(x) \right)^{1/2} \\ &\leq C \mu \left(B \left(y, R^{-1/2} \right) \right)^{-1/2} \|m_R\|_{W^{q,d/2+\delta+\varepsilon}(\mathbb{R})} \mu \left(B \left(y, R^{-1/2} \right) \right)^{1/2} (1 + rR^{1/2})^{-\delta} \\ &\leq C (1 + rR^{1/2})^{-\delta} \|m_R\|_{W^{q,\beta}(\mathbb{R})}, \end{aligned}$$

where $\delta, \varepsilon > 0$ are such that $d/2 + \delta + \varepsilon \leq \beta$. $\square$

Consider for a moment the operator $\mathbf{P}_t m(A) \exp(AR^{-1})$ and let $\tilde{M}_{t,R}(x, y)$ be its kernel. By almost identical arguments as in the proofs of Lemma 2.9 and (2.1), we can show that for $\beta > d/2$ we also have

$$\int_{B(y,r)^c} \sup_{t>0} |\tilde{M}_{t,R}(x, y)| d\mu(x) \leq C \|m_R\|_{W^{q,\beta}(\mathbb{R})}. \quad (2.6)$$

Proof of (2.2) Notice, that $M_t(x, y) = \int_Y \tilde{M}_{t,R}(x, u) P_{R^{-1}}(u, y) d\mu(u)$. For $\rho(y, z) < R^{-1/2}$, by Corollary 2.3 and (2.6),

$$\begin{aligned} & \int_{B(y,r)^c} \sup_{t>0} \left| \frac{M_t(x, y)}{\omega(y)} - \frac{M_t(x, z)}{\omega(z)} \right| d\mu(x) \\ &= \int_{B(y,r)^c} \sup_{t>0} \left| \int_Y \tilde{M}_{t,R}(x, u) \left(\frac{P_{R^{-1}}(u, y)}{\omega(y)} - \frac{P_{R^{-1}}(u, z)}{\omega(z)} \right) d\mu(u) \right| d\mu(x) \\ &\leq \int_Y \left| \frac{P_{R^{-1}}(u, y)}{\omega(y)} - \frac{P_{R^{-1}}(u, z)}{\omega(z)} \right| \int_{B(y,r)^c} \sup_{t>0} |\tilde{M}_{t,R}(x, u)| d\mu(x) d\mu(u) \\ &\leq C \left(R^{1/2} \rho(y, z) \right)^\gamma \|m_R\|_{W^{q,\beta}(\mathbb{R})}. \end{aligned}$$

□

2.5 Proof of Theorem D

Theorem D follows from Proposition 2.8 by a quite standard argument. We present the details for completeness and convenience of the reader. As usual, by a continuity argument, in order to prove boundedness of the operator $m(A)$ on $H^1(A)$ it is enough to show that there exists $C > 0$ such that

$$\|m(A)a\|_{H^1(A)} = \|\mathcal{M}m(A)a\|_{L^1(Y)} \leq C$$

holds for every (μ, ω) -atom a , see Theorem 2.4. Assume then that: $\text{supp } a \subseteq B(y_0, r) =: B$, $\|a\|_\infty \leq \mu(B)^{-1}$, and $\int a \omega d\mu = 0$. As always, the analysis on $2B = B(y_0, 2r)$ follows by the Cauchy-Schwarz inequality and boundedness of $\mathcal{M}$ and $m(A)$ on $L^2(Y)$. More precisely,

$$\begin{aligned} \|\mathcal{M}m(A)a\|_{L^1(2B)} &\leq \mu(2B)^{1/2} \|\mathcal{M}m(A)a\|_{L^2(Y)} \\ &\leq C \mu(B)^{1/2} \|a(x)\|_{L^2(Y)} \leq C. \end{aligned}$$

Therefore, it is enough to prove that

$$\|\mathcal{M}m(A)a\|_{L^1((2B)^c)} \leq C. \quad (2.7)$$

Let $\eta \in C_c^\infty(2^{-1}, 2)$ be a fixed function such that $\sum_{j \in \mathbb{Z}} \eta(2^{-j}\lambda) = 1$ for all $\lambda \in (0, \infty)$. By using this partition of unity, we decompose m as

$$m(\lambda) = \sum_{j \in \mathbb{Z}} \eta(2^{-j}\lambda) m(\lambda) = \sum_{j \in \mathbb{Z}} m_j(\lambda).$$

Fix $N \in \mathbb{Z}$ such that $2^{-N} \leq r^2 < 2^{-N+1}$. Then

$$\|\mathcal{M}m(A)a\|_{L^1((2B)^c)} \leq \sum_{j \in \mathbb{Z}} \|\mathcal{M}m_j(A)a\|_{L^1((2B)^c)} = \sum_{j \geq N} \cdots + \sum_{j < N} \cdots = S_1 + S_2.$$

Denote $m_{j,t}(\lambda) = \exp(-t\lambda)m_j(\lambda)$ and let $M_{j,t}(x, y)$ be the kernel of $m_{j,t}(A) = \mathbf{P}_t m_j(A)$. Obviously, $\text{supp } m_{j,t} \subseteq [2^{j-1}, 2^{j+1}]$ and applying (2.1) we obtain that

$$\begin{aligned} S_1 &\leq \sum_{j \geq N} \int_{(2B)^c} \int_B \sup_{t>0} |M_{j,t}(x, y)| |a(y)| d\mu(y) d\mu(x) \\ &\leq \sum_{j \geq N} \int_B |a(y)| \int_{B^c} \sup_{t>0} |M_{j,t}(x, y)| d\mu(x) d\mu(y) \\ &\leq C \|a\|_{L^1(Y)} \sum_{j \geq N} (1 + 2^{j/2}r)^{-\delta} \left\| \eta(\cdot) m(2^j \cdot) \right\|_{W^{q,\beta}(\mathbb{R})} \\ &\leq C \sup_{t>0} \|\eta(\cdot) m(t \cdot)\|_{W^{q,\beta}(\mathbb{R})} \leq C. \end{aligned}$$

If $y \in B$ and $j < N$, then $\rho(y, y_0) < r < 2^{-j/2}$ and we can apply (2.2) for the kernel $M_{j,t}$ with $R = 2^j$. Using the cancellation condition of a ,

$$\begin{aligned} S_2 &\leq \sum_{j < N} \int_{(2B)^c} \sup_{t>0} \left| \int_B M_{j,t}(x, y) a(y) d\mu(y) \right| d\mu(x) \\ &= \sum_{j < N} \int_{(2B)^c} \sup_{t>0} \left| \int_B \left(\frac{M_{j,t}(x, y)}{\omega(y)} - \frac{M_{j,t}(x, y_0)}{\omega(y_0)} \right) a(y) \omega(y) d\mu(y) \right| d\mu(x) \\ &\leq \sum_{j < N} \int_B |a(y)| \int_{B(y,r)^c} \sup_{t>0} \left| \frac{M_{j,t}(x, y)}{\omega(y)} - \frac{M_{j,t}(x, y_0)}{\omega(y_0)} \right| d\mu(x) \omega(y) d\mu(y) \\ &\leq C \sum_{j < N} 2^{\frac{j\gamma}{2}} \int_B |a(y)| \rho(y, y_0)^\gamma d\mu(y) \left\| \eta(\cdot) m(2^j \cdot) \right\|_{W^{q,\beta}(\mathbb{R})} \\ &\leq C \sup_{t>0} \|\eta(\cdot) m(t \cdot)\|_{W^{q,\beta}(\mathbb{R})} r^\gamma \sum_{j < N} 2^{\frac{j\gamma}{2}} \leq C. \end{aligned}$$

This finishes the proof of (2.7) and Theorem D.

3 The Multidimensional Bessel Operator

In this Section we turn back to the analysis related to B and prove the results stated in Sect. 1.

3.1 The Hankel Transform

Recall that $N \in \mathbb{N}$ and $\alpha_j > -1$ for $j = 1, \dots, N$. For $x, \xi \in X = (0, \infty)^N$ denote $\varphi_\alpha(x\xi) = \varphi_1(x_1\xi_1) \cdots \varphi_N(x_N\xi_N)$, where

$$\varphi_j(z) = 2^{(\alpha_j-1)/2} \Gamma((\alpha_j+1)/2) z^{-(\alpha_j-1)/2} J_{(\alpha_j-1)/2}(z), \quad z > 0.$$

Here J_τ denotes the Bessel function of the first kind. By the asymptotics of J_τ one has

$$|\varphi_j(z)| \leq C(1+z)^{-\alpha_j/2}, \quad z > 0. \quad (3.1)$$

The Hankel transform is defined by

$$H_\alpha f(\xi) = \int_X f(x) \varphi_\alpha(x\xi) d\nu(x), \quad \xi \in X, \quad (3.2)$$

As we have already mentioned, $\varphi_j \in L^\infty$ if and only if $\alpha_j \geq 0$. Nevertheless, it is known that H_α always extends uniquely to an isometric isomorphism on $L^2(X)$, see [5] and [8, Lem. 2.7]. The multipliers $m(B)$ and H_α are related in the same way, as $m(-\Delta)$ and the Fourier transform on $\mathbb{R}^D$. In particular, if

$$n(\lambda) = m(|\lambda|^2), \quad \lambda \in X,$$

then m is radial and

$$m(B)f = H_\alpha(n \cdot H_\alpha). \quad (3.3)$$

3.2 (P_2) for Multidimensional Bessel Operator

Let us first recall that $\mathbf{T}_t$ satisfies (G) and, obviously, X satisfies (Y). Therefore, Theorem A follows from Theorems 2.1 and D provided that (P_q) holds with $q = 2$, which we now prove. The case $N = 1$ follows by similar and simpler argument, thus we shall concentrate on $N \geq 2$. Let $k \in \{1, \dots, N-1\}$ and $c_j < 2^{-N}$ for $j = 1, \dots, k$. Define the sets

$$S_{c_1, \dots, c_k} = \left\{ x \in X : 1/2 < |x|^2 < 2 \text{ and } x_j < c_j \text{ for } j = 1, \dots, k \right\}.$$

Lemma 3.1 *Suppose that $\text{supp } m \subset [R/2, 2R]$, $N \geq 2$, $k \leq N-1$, and $c_j < 2^{-N}$ for $j = 1, 2, \dots, k$. Then there exists $C > 0$ such that*

$$\int_{S_{c_1, \dots, c_k}} \left| m(R|x|^2) \right|^2 x_1^{\alpha_1} \dots x_k^{\alpha_k} dx_1 \dots dx_N \leq C c_1^{\alpha_1+1} \dots c_k^{\alpha_k+1} \|m(R \cdot)\|_{L^2(\mathbb{R})}^2.$$

Proof Introduce the spherical coordinates $(r, \theta_1, \dots, \theta_{N-1})$ on $\mathbb{R}^N$, namely

$$\begin{cases} x_1 = r \sin(\theta_1), \\ x_i = r \sin(\theta_i) \prod_{j=1}^{i-1} \cos(\theta_j), & \text{for } i = 2, 3, \dots, N-1, \\ x_N = r \prod_{j=1}^{N-1} \cos(\theta_j) \\ dx_1 \dots dx_N = r^{N-1} \prod_{j=1}^{N-2} \cos^{N-1-j}(\theta_j) dr d\theta_1 \dots d\theta_{N-1}. \end{cases}$$

Since $x \in X$, then $\theta_j \in (0, \pi/2)$ for $j = 1, \dots, N-1$. We claim that if $x \in S_{c_1, \dots, c_k}$, then

$$\sin(\theta_j) < 2^{-N/2} \leq 2^{-1/2} \quad (3.4)$$

for $j = 1, \dots, k$. Obviously, if $\sin(\beta) < 2^{-1/2}$ for $\beta \in (0, \pi/2)$, then $(\cos \beta)^{-1} < 2^{1/2}$. Observe that $2^{-1/2} < r < 2^{1/2}$, since $\text{supp } m \subseteq [R/2, 2R]$. Therefore, (3.4) follows easily by induction, i.e. for $i = 1, \dots, k$,

$$\sin \theta_i = x_i r^{-1} (\cos \theta_1)^{-1} \dots (\cos \theta_{i-1})^{-1} \leq 2^{-N} 2^{i/2} \leq 2^{-N/2}.$$

Denote $S = S_{c_1, \dots, c_k}$. As a consequence of (3.4) we have that $\sin \theta_i \simeq \theta_i$ and $\cos \theta_i \simeq C$ for $i = 1, \dots, k$. Using this, $x_i^{\alpha_i} \simeq r^{\alpha_i} \theta_i^{\alpha_i}$ for $i = 1, \dots, k$ and

$$\begin{aligned} & \int_S \left| m(R|x|^2) \right|^2 x_1^{\alpha_1} \dots x_k^{\alpha_k} dx_1 \dots dx_N \\ & \leq C \int_S \left| m(Rr^2) \right|^2 r^{N-1+\alpha_1+\dots+\alpha_k} \theta_1^{\alpha_1} \dots \theta_k^{\alpha_k} dr d\theta_1 \dots d\theta_{N-1} \\ & \leq C \int_0^{cc_1} \theta_1^{\alpha_1} d\theta_1 \dots \int_0^{cc_k} \theta_k^{\alpha_k} d\theta_k \cdot \int_{1/2 < r^2 < 2} \left| m(Rr^2) \right|^2 dr \\ & \leq C c_1^{\alpha_1+1} \dots c_k^{\alpha_k+1} \cdot \|m(R \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

□

Proposition 3.2 Assume that $N \in \mathbb{N}$ and $\alpha_j > -1$ for $j = 1, \dots, N$. Then (P_q) holds for B with $q = 2$.

Proof In the proof we consider only the case $N \geq 2$. Let $q = 2$ and suppose that m is supported in $[R/2, 2R]$ for some $R > 0$. Notice that by (3.3) and (3.2) we have

$$\begin{aligned} m(B)f(x) &= \int_X f(y) \int_X n(\xi) \varphi_\alpha(y\xi) \varphi_\alpha(x\xi) d\nu(\xi) d\nu(y) \\ &= \int_X f(y) H_\alpha(n(\cdot) \varphi_\alpha(y \cdot))(x) d\nu(y) \end{aligned}$$

and the kernel associated with $m(B)$ has the form

$$K_{m(B)}(x, y) = H_\alpha(n(\cdot) \varphi_\alpha(y \cdot))(x).$$

Therefore, by the Plancherel identity for H_α , (P_2) is equivalent to

$$\int_{R/2 < |x|^2 < 2R} \left| m(|x|^2) \right|^2 |\varphi_\alpha(xy)|^2 d\nu(x) \leq C \nu\left(B\left(y, R^{-1/2}\right)\right)^{-1} \|m(R \cdot)\|_{L^2(\mathbb{R})}^2. \quad (3.5)$$

For each $i = 1, \dots, N$ we consider four cases:

C1. $x_i < 2^{-N}$, $\sqrt{R}y_i > 2^N$, $\sqrt{R}y_i x_i < 1$,

- C2.** $x_i < 2^{-N}$, $\sqrt{R}y_i \leq 2^N$,
C3. $x_i < 2^{-N}$, $\sqrt{R}y_i > 2^N$, $\sqrt{R}y_i x_i \geq 1$,
C4. $2^{-N} \leq x_i \leq \sqrt{2}$.

Divide the set $\{x \in X : 1/2 < |x|^2 < 2\}$ into several disjoint regions using the cases above. Without loss of generality we may consider the set S of points $x \in X$ such that:

- x_i satisfies **C1.** for $i = 1, \dots, k_1$,
- x_i satisfies **C2.** for $i = k_1 + 1, \dots, k_2$,
- x_i satisfies **C3.** for $i = k_2 + 1, \dots, k_3$,
- x_i satisfies **C4.** for $i = k_3 + 1, \dots, N$,

where $0 \leq k_1 \leq k_2 \leq k_3 < N$. The fact that $k_3 < N$ is implied by $|x|^2 > 1/2$. Notice that it may happen that S is empty. Recall that $\nu(B(y, r)) \simeq \prod_{j=1}^N \nu_j(B(y_j, r))$, where $d\nu_j(x_j) = x_j^{\alpha_j} dx_j$ is the one-dimensional measure, and

$$\nu_j \left(B \left(y_j, R^{-1/2} \right) \right)^{-1} \simeq R^{(\alpha_j+1)/2} \left(1 + \sqrt{R}y_j \right)^{-\alpha_j}.$$

Denote $d_{gl} = N + \alpha_1 + \dots + \alpha_N$. Using (3.1) and Lemma 3.1 with $k = k_2$, we have

$$\begin{aligned} & \int_{R/2 < |x|^2 < 2R} \left| m \left(|x|^2 \right) \right|^2 |\varphi_\alpha(xy)|^2 d\nu(x) \\ & \leq C \sum_S R^{\frac{d_{gl}}{2}} \int_S \left| m \left(R|x|^2 \right) \right|^2 \prod_{j=1}^N \left(x_j^{-1} + \sqrt{R}y_j \right)^{-\alpha_j} dx \\ & \leq C \sum_S R^{\frac{d_{gl}}{2}} \int_S \left| m \left(R|x|^2 \right) \right|^2 x_1^{\alpha_1} \dots x_{k_2}^{\alpha_{k_2}} \prod_{j=k_2+1}^N \left(1 + \sqrt{R}y_j \right)^{-\alpha_j} dx_1 \dots dx_N \\ & \leq C \sum_S R^{\frac{d_{gl}}{2}} \prod_{i=1}^{k_1} \left(\sqrt{R}y_i \right)^{-\alpha_i-1} \prod_{j=k_2+1}^N \left(1 + \sqrt{R}y_j \right)^{-\alpha_j} \|m(R \cdot)\|_{L^2(\mathbb{R})}^2 \\ & \leq C \prod_{i=1}^{k_1} R^{(\alpha_i+1)/2} \left(\sqrt{R}y_i \right)^{-\alpha_i} \prod_{k=k_1+1}^{k_2} R^{(\alpha_k+1)/2} \prod_{j=k_2+1}^N R^{(\alpha_j+1)/2} \\ & \quad \left(1 + \sqrt{R}y_j \right)^{-\alpha_j} \|m(R \cdot)\|_{L^2(\mathbb{R})}^2 \\ & \leq C \nu \left(B \left(y, R^{-1/2} \right) \right)^{-1} \|m(R \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

□

3.3 Imaginary Powers of B

In this subsection we prove Lemma 1.1 and Theorems B and C. From now on we consider one-dimensional Bessel operator, i.e. $N = 1$, $X = (0, \infty)$, $\alpha > -1$, and $d\nu(x) = x^\alpha dx$.

Let us start this section by recalling well-known asymptotics of the Bessel function I_τ [28,37], i.e.

$$I_\tau(x) = \Gamma\left(\frac{\tau+1}{2}\right)^{-1} \left(\frac{x}{2}\right)^\tau + O\left(x^{\tau+1}\right), \quad x \sim 0, \quad (3.6)$$

$$I_\tau(x) = (2\pi x)^{-1/2} e^x \left(1 + O(x^{-1})\right), \quad x \sim \infty. \quad (3.7)$$

Now we provide a short argument for (1.5). In [6, Sec. 4.3] it is proved that B^{ib} is associated with the kernel

$$-\Gamma(-ib+1)^{-1} \int_0^\infty t^{ib} \partial_t T_t(x, y) dt$$

in the sense as in (1.5) (let us notice that in [6] only positive values of α 's are considered, but the proof works for $\alpha_j > -1$ as well). By integrating by parts,

$$\begin{aligned} & -\Gamma(-ib+1)^{-1} \int_0^\infty t^{-ib} \partial_t T_t(x, y) dt \\ &= -\Gamma(-ib+1)^{-1} \lim_{\varepsilon \rightarrow 0} \left(\varepsilon^{ib} T_{\varepsilon^{-1}}(x, y) - \varepsilon^{-ib} T_\varepsilon(x, y) \right) \\ &+ \Gamma(-ib)^{-1} \int_0^\infty t^{-ib} T_t(x, y) \frac{dt}{t} = K_b(x, y). \end{aligned}$$

Proof of Lemma 1.1 Let us first notice that for $\kappa \in \mathbb{R}$ and $c, M > 0$, there exists $C = C(\kappa, c, M)$ such that

$$\int_{cz}^\infty t^\kappa \exp\left(-\frac{t}{4}\right) \frac{dt}{t} \leq Cz^{-M}, \quad z \geq 1. \quad (3.8)$$

Using (1.4) and (1.2) one obtains

$$\begin{aligned} 2\Gamma(-ib)K_b(x, y) &= \int_0^\infty t^{-ib-1} (xy)^{-(\alpha-1)/2} I_{(\alpha-1)/2}\left(\frac{xy}{2t}\right) \exp\left(-\frac{x^2+y^2}{4t}\right) \frac{dt}{t} \\ &= \int_0^{xy} \dots + \int_{xy}^\infty \dots = A_1 + A_2. \end{aligned}$$

Denote $\chi_{loc}(x, y) = \chi_{\{y/2 < x < 2y\}}(x, y)$ and $\chi_{glob}(x, y) = 1 - \chi_{loc}(x, y)$, $x, y \in X$. In the proof below all expressions denoted by R_k shall be parts of the kernel $R_b(x, y)$.

Using (3.7), we write $A_1 = A_{1,1} + R_1$, where

$$A_{1,1} = \pi^{-1/2} \int_0^{xy} t^{-ib-1/2} (xy)^{-\alpha/2} \exp\left(-\frac{|x-y|^2}{4t}\right) \frac{dt}{t}$$

and

$$\begin{aligned} |R_1| &= \left| \int_0^{xy} t^{-ib-1} (xy)^{-(\alpha-1)/2} \exp\left(-\frac{x^2+y^2}{4t}\right) \right. \\ &\quad \left. \left(I_{(\alpha-1)/2}\left(\frac{xy}{2t}\right) - \left(\frac{\pi xy}{t}\right)^{-1/2} \exp\left(\frac{xy}{2t}\right) \right) \frac{dt}{t} \right| \\ &\leq C \int_0^{xy} t^{1/2} (xy)^{-\alpha/2-1} \exp\left(-\frac{|x-y|^2}{4t}\right) \frac{dt}{t} \\ &= C|x-y|(xy)^{-\alpha/2-1} \int_{\frac{|x-y|^2}{xy}}^{\infty} t^{-1/2} e^{-t/4} \frac{dt}{t} \\ &\leq Cxy(x+y)^{-\alpha-3}. \end{aligned}$$

In the last inequality we have used (3.8). Denoting

$$\begin{aligned} R_2 &= \chi_{glob}(x, y) A_{1,1}, \\ R_3 &= \pi^{-1/2} \chi_{loc}(x, y) \int_{xy}^{\infty} t^{-ib-1/2} (xy)^{-\alpha/2} \exp\left(-\frac{|x-y|^2}{4t}\right) \frac{dt}{t} \end{aligned}$$

we have

$$\begin{aligned} A_{1,1} - R_2 + R_3 &= \pi^{-1/2} \chi_{loc}(x, y) \left(\int_0^{\infty} t^{-ib-1/2} (xy)^{-\alpha/2} \exp\left(-\frac{|x-y|^2}{4t}\right) \frac{dt}{t} \right) \\ &= \pi^{-1/2} 2^{2bi+1} \Gamma(ib+1/2) \chi_{loc}(x, y) (xy)^{-\alpha/2} |x-y|^{-2ib-1}. \end{aligned}$$

Notice that $A_{1,1}$ is one of the terms from (1.6). Next, by (3.8),

$$\begin{aligned} |R_2| &\leq C \chi_{glob}(x, y) |x-y|^{-1} (xy)^{-\alpha/2} \int_{\frac{|x-y|^2}{xy}}^{\infty} t^{1/2} e^{-t/4} \frac{dt}{t} \\ &\leq Cxy(x+y)^{-\alpha-3}, \\ |R_3| &\leq C \chi_{loc}(x, y) x^{-\alpha} |x-y|^{-1} \int_0^{\frac{|x-y|^2}{xy}} t^{1/2} \frac{dt}{t} \\ &\simeq C \chi_{loc}(x, y) x^{-\alpha-1}. \end{aligned}$$

Now let us turn to study A_2 . Denote $c_\alpha = 4^{-(\alpha-1)/2} \Gamma((\alpha+1)/4)^{-1}$. Then, by using (3.6),

$$A_2 = c_\alpha \int_{xy}^{\infty} t^{-ib-(\alpha+1)/2} \exp\left(-\frac{x^2+y^2}{4t}\right) \frac{dt}{t} + R_4 = A_{2,1} + R_4$$

where, by (3.8),

$$\begin{aligned} |R_4| &= \left| \int_{xy}^{\infty} t^{-ib-1} (xy)^{-(\alpha-1)/2} \exp\left(-\frac{x^2+y^2}{4t}\right) \right. \\ &\quad \left. \left(I_{(\alpha-1)/2}\left(\frac{xy}{2t}\right) - \Gamma\left(\frac{\alpha+1}{4}\right)^{-1} \left(\frac{xy}{4t}\right)^{(\alpha-1)/2} \right) \frac{dt}{t} \right| \\ &\leq Cxy \int_{xy}^{\infty} t^{-(\alpha+3)/2} \exp\left(-\frac{x^2+y^2}{4t}\right) \frac{dt}{t} \\ &\leq Cxy(x^2+y^2)^{-(\alpha+3)/2} \simeq Cxy(x+y)^{-\alpha-3}. \end{aligned}$$

Moreover,

$$\begin{aligned} A_{2,1} + R_5 &= c_\alpha \int_0^{\infty} t^{-ib-(\alpha+1)/2} \exp\left(-\frac{x^2+y^2}{4t}\right) \frac{dt}{t} \\ &= c_\alpha 4^{ib+(\alpha+1)/2} \Gamma(ib + (\alpha+1)/2) (x^2+y^2)^{-ib-(\alpha+1)/2}, \end{aligned}$$

where

$$\begin{aligned} |R_5| &= \left| c_\alpha \int_0^{xy} t^{-ib-(\alpha+1)/2} \exp\left(-\frac{x^2+y^2}{4t}\right) \frac{dt}{t} \right| \\ &\leq C(x^2+y^2)^{-(\alpha+1)/2} \int_{\frac{x^2+y^2}{xy}}^{\infty} t^{(\alpha+1)/2} e^{-t/4} \frac{dt}{t} \\ &\leq Cxy(x+y)^{-\alpha-3}. \end{aligned}$$

□

Proof of Theorem B for $\alpha < 0$. Let $|b| > 1$ and $\varepsilon \in (0, 10^{-1})$ (to be fixed later on). Denote $I = [1, 1 + \varepsilon]$ and $S = [1 + 3\varepsilon, 2]$. Put $f_\varepsilon(x) = \varepsilon^{-1} \chi_I(x) x^{-\alpha}$, so that $\|f_\varepsilon\|_{L^1(X)} = 1$. If $x \in S$, by Lemma 1.1 and the triangle inequality,

$$\begin{aligned}
\left| B^{ib} f_\varepsilon(x) \right| &\leq |c_2(b)| x^{-\alpha/2} |x-1|^{-1} \\
&+ |c_2(b)| \left| \int_I \left((xy)^{-\alpha/2} |x-y|^{-2ib-1} - x^{-\alpha/2} |x-1|^{-2ib-1} \right) f_\varepsilon(y) dv(y) \right| \\
&+ |c_1(b)| \left| \int_I \left(x^2 + y^2 \right)^{-ib-(\alpha+1)/2} f_\varepsilon(y) dv(y) \right| \\
&+ |c_3(b)| \left| \int_I R_b(x, y) f_\varepsilon(y) dv(y) \right| \\
&= |c_2(b)| x^{-\alpha/2} |x-1|^{-1} + \Lambda_1 + \Lambda_2 + \Lambda_3.
\end{aligned} \tag{3.9}$$

Observe that for $x \in S$ and $y \in I$ we have $|x-y| \simeq |x-1|$ and $x \simeq y \simeq 1$. By using the Mean Value Theorem for the function $y \mapsto y^{-\alpha/2} |x-y|^{-2ib-1}$,

$$\begin{aligned}
\Lambda_1 &\leq C |c_2(b)| \varepsilon^{-1} \int_1^{1+\varepsilon} |b| |y-1| |x-1|^{-2} dy \leq C \varepsilon |bc_2(b)| |x-1|^{-2}, \\
\Lambda_2 &\leq C |c_1(b)| \varepsilon^{-1} \int_1^{1+\varepsilon} (x^2 + y^2)^{-(1+\alpha)/2} dy \simeq C |c_1(b)|, \\
\Lambda_3 &\leq C |c_3(b)| \varepsilon^{-1} \int_1^{1+\varepsilon} xy(x+y)^{-\alpha-3} dy \simeq C |c_3(b)|.
\end{aligned} \tag{3.10}$$

Fix $|b| \geq 1$ and λ such that $\lambda > \max(\Lambda_2, \Lambda_3, |bc_2(b)|)$. Recall that $x^{-\alpha/2} \geq 1$ for $x \in S$, so that for ε small enough

$$\begin{aligned}
\nu \left\{ x \in S : |c_2(b)| x^{-\alpha/2} |x-1|^{-1} > 4\lambda \right\} &\geq \nu \left\{ x \in S : |c_2(b)| |x-1|^{-1} > 4\lambda \right\} \\
&= \int_{1+3\varepsilon}^{1+|c_2(b)|/(4\lambda)} x^\alpha dx \\
&\geq C |c_2(b)| / (4\lambda)
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
\nu \{ x \in S : |\Lambda_1| > \lambda \} &\leq \nu \left\{ x \in S : C \varepsilon |bc_2(b)| |x-1|^{-2} > \lambda \right\} \\
&\leq \int_{1+3\varepsilon}^{1+C(\varepsilon\lambda^{-1}|bc_2(b)|)^{1/2}} x^\alpha dx \\
&\leq C \left(\varepsilon\lambda^{-1} |bc_2(b)| \right)^{1/2} \\
&\leq C \varepsilon^{1/2}.
\end{aligned} \tag{3.12}$$

Hence, using (3.9)–(3.12) and (1.7) we get

$$\begin{aligned}
 \|B^{ib} f_\varepsilon\|_{L^{1,\infty}(X)} &\geq \lambda \nu \left\{ x \in S : |B^{ib} f_\varepsilon(x)| > \lambda \right\} \\
 &\geq \lambda \nu \left\{ x \in S : |c_2(b)| x^{-\alpha/2} |x-1|^{-1} > 4\lambda \right\} \\
 &\quad - \lambda \nu \left\{ x \in S : |\Lambda_1| > \lambda \right\} - \underbrace{\lambda \nu \left\{ x \in S : |\Lambda_2| > \lambda \right\}}_{=0} \\
 &\quad - \underbrace{\lambda \nu \left\{ x \in S : |\Lambda_3| > \lambda \right\}}_{=0} \\
 &\geq C|c_2(b)| - C\lambda\varepsilon^{1/2} \geq C|c_2(b)| \simeq |b|^{1/2} = |b|^{d/2}.
 \end{aligned}$$

□

Turning to the case $\alpha > 0$ we could also use Lemma 1.1. In this case, the summand with $c_1(b)$ would play the first role. An alternative proof that we shall present here uses integral representation of the modified Bessel function. The same will be used in the proof of Theorem C. It is known that for $\alpha > 0$

$$I_{(\alpha-1)/2}(z) = \left(\Gamma(\alpha/2)^{-1} \sqrt{\pi} \right)^{-1} \left(\frac{z}{2} \right)^{(\alpha-1)/2} \int_{-1}^1 e^{-zs} (1-s^2)^{\alpha/2-1} ds, \quad z > 0, \quad (3.13)$$

see [37, Ch. 6]. Therefore, for $\alpha > 0$, using (1.4), (1.2), and (3.13) we obtain

$$\begin{aligned}
 K_b(x, y) &= (2\Gamma(-ib))^{-1} \int_0^\infty t^{-ib-1} (xy)^{-(\alpha-1)/2} I_{(\alpha-1)/2} \left(\frac{xy}{2t} \right) \\
 &\quad \exp \left(-\frac{x^2 + y^2}{4t} \right) \frac{dt}{t} \\
 &= (2^\alpha \Gamma(-ib) \Gamma(\alpha/2) \sqrt{\pi})^{-1} \int_{-1}^1 \int_0^\infty t^{-ib-(\alpha+1)/2} \\
 &\quad \exp \left(-\frac{x^2 + y^2 + 2xys}{4t} \right) \frac{dt}{t} (1-s^2)^{\frac{\alpha}{2}-1} ds \\
 &= \frac{2^{2ib+1} \Gamma(ib + (\alpha+1)/2)}{\Gamma(-ib) \Gamma(\alpha/2) \sqrt{\pi}} \int_{-1}^1 (x^2 + y^2 + 2xys)^{-ib-(\alpha+1)/2} \\
 &\quad (1-s^2)^{\alpha/2-1} ds \\
 &= C_\alpha c_1(b) \int_{-1}^1 (x^2 + y^2 + 2xys)^{-ib-(\alpha+1)/2} (1-s^2)^{\alpha/2-1} ds,
 \end{aligned} \quad (3.14)$$

where $C_\alpha = \pi^{-1/2} \Gamma(\alpha/2)^{-1} \Gamma((\alpha+1)/4)$.

Proof of Theorem B for $\alpha > 0$. Let $|b| > 1$, $\varepsilon \in (0, 10^{-1})$, and $f_\varepsilon(x) = x^{-\alpha} \varepsilon^{-1} \chi_{[\varepsilon, 2\varepsilon]}(x)$. Similarly as in (3.9), using (3.14) and the Mean Value Theorem, for $x > 3\varepsilon$ we have

$$\begin{aligned}
 |B^{-ib} f_\varepsilon(x)| &\leq \left| \int_\varepsilon^{2\varepsilon} K_b(x, 0) f_\varepsilon(y) dv(y) \right| \\
 &\quad + \left| \int_\varepsilon^{2\varepsilon} (K_b(x, 0) - K_b(x, y)) f_\varepsilon(y) dv(y) \right| \\
 &\leq C|c_1(b)|\varepsilon^{-1} \left\{ \left| \int_\varepsilon^{2\varepsilon} \int_{-1}^1 (1-s^2)^{\alpha/2-1} x^{-2ib-(\alpha+1)} ds dy \right| \right. \\
 &\quad + \left. \int_\varepsilon^{2\varepsilon} \int_{-1}^1 (1-s^2)^{\alpha/2-1} \right. \\
 &\quad \left. \left| (x^2 + y^2 + 2sxy)^{-ib-\frac{\alpha+1}{2}} - x^{-2ib-(\alpha+1)} \right| ds dy \right\} \\
 &\leq C|c_1(b)| \left(x^{-\alpha-1} + \varepsilon^{-1} \int_{-1}^1 (1-s^2)^{\alpha/2-1} \right. \\
 &\quad \left. \int_\varepsilon^{2\varepsilon} |b| |y^2 + 2sxy| x^{-\alpha-3} dy ds \right) \\
 &\leq C|c_1(b)| \left(x^{-\alpha-1} + \varepsilon |b| x^{-\alpha-2} \right).
 \end{aligned} \tag{3.15}$$

Let us fix $|b| > 1$ and $\lambda > |bc_1(b)|$. For all ε small enough, we get

$$\nu \left\{ x > 3\varepsilon : C|c_1(b)|x^{-\alpha-1} > 2\lambda \right\} = \int_{3\varepsilon}^{C(|c_1(b)|/\lambda)^{1/(1+\alpha)}} x^\alpha dx \geq C|c_1(b)|/\lambda,$$

and

$$\begin{aligned}
 \nu \left\{ x > 3\varepsilon : C|bc_1(b)|\varepsilon x^{-\alpha-2} > \lambda \right\} &\leq \int_0^{C(|bc_1(b)|\varepsilon/\lambda)^{1/(\alpha+2)}} x^\alpha dx \\
 &= C(|bc_1(b)|\varepsilon/\lambda)^{(\alpha+1)/(\alpha+2)} \\
 &\leq C\varepsilon^{(1+\alpha)/(\alpha+2)}.
 \end{aligned}$$

Therefore, by choosing a proper ε , we obtain

$$\begin{aligned}
 \|B^{ib} f_\varepsilon\|_{L^{1,\infty}(X)} &\geq \lambda \nu \left\{ x \in X : |B^{ib} f_\varepsilon(x)| > \lambda \right\} \\
 &\geq \lambda \nu \left\{ x > 3\varepsilon : C|c_1(b)|x^{-\alpha-1} > 2\lambda \right\} \\
 &\quad - \lambda \nu \left\{ x > 3\varepsilon : C|c_1(b)|\varepsilon x^{-\alpha-2} > \lambda \right\} \\
 &\geq C|c_1(b)| - C\lambda \varepsilon^{(\alpha+1)/(\alpha+2)} \geq C|c_1(b)| \simeq |b|^{(\alpha+1)/2} = |b|^{d/2}.
 \end{aligned}$$

□

Proof of Theorem C Set $\alpha > 0$, $p \in (1, 2)$, $|b| > 1$, $\varepsilon \in (0, 10^{-1})$, $\delta > 1$, and a function $f \in L^p(X)$ such that $\text{supp } f \subseteq (0, \varepsilon)$, and $f \geq 0$. Similarly as in (3.15), using (3.14) and Corollary 4.2,

$$\begin{aligned} \|B^{ib} f\|_{L^p(X)}^p &\geq \int_{\delta}^{\infty} \left| \int_X (K_b(x, 0) - (K_b(x, 0) - K_b(x, y)) f(y) dv(y) \right|^p dv(x) \\ &\geq C \|f\|_{L^1(X)}^p \int_{\delta}^{\infty} |K_b(x, 0)|^p dv(x) \\ &\quad - C \int_{\delta}^{\infty} \left| \int_X (K_b(x, y) - K_b(x, 0)) f(y) dv(y) \right|^p dv(x) \\ &\geq C \|f\|_{L^1(X)}^p |b|^{p(\alpha+1)/2} \left(\int_{\delta}^{\infty} x^{-p(\alpha+1)+\alpha} dx - \int_{\delta}^{\infty} \varepsilon^p |b|^p x^{-p(\alpha+2)+\alpha} dx \right) \\ &\geq C \|f\|_{L^1(X)}^p |b|^{p(\alpha+1)/2} \delta^{(\alpha+1)(1-p)} (1 - \varepsilon^p |b|^p \delta^{-p}). \end{aligned}$$

Now we take $\delta = |b|$ and fix ε small enough, independent of b , getting

$$\|B^{ib} f\|_{L^p(X)} \geq C_p |b|^{\frac{(\alpha+1)(2-p)}{2p}} \|f\|_{L^1(X)} \geq C_{p,\varepsilon} |b|^{\frac{d}{2} \frac{2-p}{p}} \|f\|_{L^p(X)}.$$

□

4 Appendix: Gamma Function Estimate

Lemma 4.1 Let $a + bi \in \mathbb{C}$. For $a \geq 0$ fixed and all $|b| \geq 1$ we have

$$|\Gamma(a + bi)| \simeq |b|^{a-1/2} \exp\left(-\frac{\pi b}{2}\right).$$

The result above is known. It is a consequence of the Stirling's Formula, see [1, Ch. 6]. For the convenience of the reader we present a short proof.

Proof Using the reflection formula

$$\Gamma(1 - z)\Gamma(z) = \pi / \sin(\pi z), \quad (4.1)$$

and the recursion identity

$$z\Gamma(z) = \Gamma(z + 1), \quad (4.2)$$

we have that $|1 - ib| |\Gamma(ib)|^2 = |\Gamma(ib)\Gamma(1 - ib)| = |\pi / \sin(\pi ib)| \simeq \exp(-\pi |b|)$ for $|b| \geq 1$. Thus,

$$|\Gamma(ib)| \simeq |b|^{-1/2} \exp\left(-\frac{\pi |b|}{2}\right), \quad |b| \geq 1. \quad (4.3)$$

Denote $S = \{z \in \mathbb{C} : 1 \leq \operatorname{Re}(z) \leq 2, |\operatorname{Im}(z)| \geq 1\}$ and define a holomorphic function

$$F(z) = \Gamma(z)z^{-z+1/2}, \quad z \in S.$$

Now, we claim that $|F(z)| \leq C$ if $z \in \partial S$. This is clear for $z = a \pm i$, $a \in [1, 2]$. For $z = 1 + ib$, $|b| \geq 1$, we use (4.2) and (4.3) getting

$$\begin{aligned} |F(1 + ib)| &= |\Gamma(1 + ib)| \left| (1 + ib)^{-1/2 - ib} \right| = |b| |\Gamma(ib)| (1 + b^2)^{-1/4} e^{b \operatorname{arctg} b} \\ &\leq C |b|^{1/2} e^{-\pi |b|/2} |b|^{-1/2} e^{b \operatorname{arctg}(b)} \leq C. \end{aligned}$$

Similarly we show boundedness of F for $z = 2 + bi$, $|b| \geq 1$.

Observe that $|F(z)| \leq |\Gamma(z)| |z|^{-z+1/2} \leq C e^{c|z|^2}$ for $z \in S$. Hence, applying the Phragmén-Lindelöf principle, we obtain that $|F(z)| \leq C$ for $z \in S$. Therefore, for a fixed $a \in [1, 2]$ and $|b| \geq 1$ we have

$$\begin{aligned} |\Gamma(a + bi)| &\leq C \left| (a + bi)^{a-1/2+bi} \right| = \left(a^2 + b^2 \right)^{(2a-1)/4} \cdot e^{-b \operatorname{arctg}(b/a)} \\ &\simeq C |b|^{a-1/2} e^{-\pi |b|/2}. \end{aligned} \quad (4.4)$$

This is the desired estimate from above for $a \in [1, 2]$. We extend this for all $a \in [0, \infty)$ by using (4.2). Then, by (4.1), we get estimate from below for $a \in [0, 1]$, and extend this for $a \in [0, \infty)$ using (4.2) once more. $\square$

Corollary 4.2 For fixed $a_1, a_2 \geq 0$ and $|b| \geq 1$ we have

$$\left| \frac{\Gamma(a_1 + bi)}{\Gamma(a_2 + bi)} \right| \simeq |b|^{a_1 - a_2}.$$

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