The Fourier Transform on the Group $GL_2(\mathbb{R})$ and the Action of the Overalgebra $\mathfrak{gl}_4$

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Received: 18 April 2017 / Published online: 4 December 2017
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Abstract We define a kind of ‘operational calculus’ for the Fourier transform on the group $GL_2(\mathbb{R})$. Namely, $GL_2(\mathbb{R})$ can be regarded as an open dense chart in the Grassmannian of 2-dimensional subspaces in $\mathbb{R}^4$. Therefore the group $GL_4(\mathbb{R})$ acts in L^2 on $GL_2(\mathbb{R})$. We transfer the corresponding action of the Lie algebra $\mathfrak{gl}_4$ to the Plancherel decomposition of $GL_2(\mathbb{R})$, the algebra acts by differential-difference operators with shifts in an imaginary direction. We also write similar formulas for the action of $\mathfrak{gl}_4 \oplus \mathfrak{gl}_4$ in the Plancherel decomposition of $GL_2(\mathbb{C})$.

Keywords Unitary representations · Plancherel formula · Differential-difference operators · Grassmannian · Principal series

1 The Statement of the Paper

1.1 The Group $GL_n(\mathbb{R})$

Let $GL_n(\mathbb{R})$ be the group of invertible real matrices of order n . We denote elements of $GL_2(\mathbb{R})$ by

Communicated by Hartmut Führ.

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$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$

The Haar measure on $GL_2(\mathbb{R})$ is given by

$$(\det X)^{-2} dX = (\det X)^{-2} dx_{11} dx_{12} dx_{21} dx_{22}.$$

Recall some basic facts on representations of the group $GL_2(\mathbb{R})$, for systematic exposition of the representation theory of $SL_2(\mathbb{R})$, see, e.g., [4, 6]. Denote by $\mathbb{R}^\times$ the multiplicative group of $\mathbb{R}$. Let $\mu \in \mathbb{C}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. We define the function $x^{\mu//\varepsilon}$ on $\mathbb{R}^\times$ by

$$x^{\mu//\varepsilon} := |x|^\mu \operatorname{sgn}(x)^\varepsilon,$$

these functions are precisely all homomorphisms from $\mathbb{R}^\times$ to the multiplicative group of $\mathbb{C}$. Denote by Λ the set of all collections

$$(\mu_1, \varepsilon_1; \mu_2, \varepsilon_2),$$

i.e.,

$$\Lambda \simeq \mathbb{C} \times \mathbb{Z}_2 \times \mathbb{C} \times \mathbb{Z}_2.$$

For each element of Λ we define a representation $T_{\mu, \varepsilon}$ of $GL_2(\mathbb{R})$ in the space of functions on $\mathbb{R}$ by

$$\begin{aligned} & T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \varphi(t) \\ &= \varphi \left(\frac{x_{12} + tx_{22}}{x_{11} + tx_{21}} \right) \cdot (x_{11} + tx_{21})^{-1 + \mu_1 - \mu_2//\varepsilon_1 - \varepsilon_2} \det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}^{1/2 + \mu_2//\varepsilon_2} \end{aligned} \quad (1.1)$$

A space of functions can be specified in various ways, it is convenient to consider the space $C_{\mu_1 - \mu_2, \varepsilon_1 - \varepsilon_2}^\infty$ of C^∞ -functions on $\mathbb{R}$ such that¹

$$\varphi(-1/t)(-t)^{-1 + \mu_1 - \mu_2//\varepsilon_1 - \varepsilon_2}$$

also is C^∞ -smooth.

Thus, for any fixed $X \in GL_2(\mathbb{R})$ we get an operator-valued function $(\mu_1, \varepsilon_1; \mu_2, \varepsilon_2) \mapsto T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}$ on Λ holomorphic in the variables μ_1, μ_2 .

Generators of the Lie algebra $\mathfrak{gl}_2(\mathbb{R})$ act by formulas

$$L_{11} = -t \frac{d}{dt} + (-1/2 + \mu_1), \quad L_{12} = \frac{d}{dt}, \quad (1.2)$$

$$L_{21} = -t^2 \frac{d}{dt} + t(-1 + \mu_1 - \mu_2), \quad L_{22} = t \frac{d}{dt} + (1/2 + \mu_2). \quad (1.3)$$

¹ This condition means that functions φ are smooth as sections of line bundles on the projective line $\mathbb{R} \cup \infty$.

The expressions for the generators do not depend on $\varepsilon_1, \varepsilon_2$. However the space of the representation depends on $\varepsilon_1 - \varepsilon_2$.

Consider integral operators

$$A_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} : C_{\mu_1 - \mu_2, \varepsilon_1 - \varepsilon_2}^\infty \rightarrow C_{\mu_2 - \mu_1, \varepsilon_1 - \varepsilon_2}^\infty$$

defined by

$$A_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} f(t) := \int_{\mathbb{R}} (t - s)^{-1 - \mu_1 + \mu_2 // \varepsilon_1 - \varepsilon_2} f(s) ds.$$

The integral is convergent if $\operatorname{Re}(\mu_1 - \mu_2) < 0$ and determines a function holomorphic in μ_1, μ_2 . As usual (see, e.g., [3], §I.3), the integral admits a meromorphic continuation to the whole plane $(\mu_1, \mu_2) \in \mathbb{C}^2$ with poles at $\mu_1 - \mu_2 = 0, 1, 2, \dots$. The operators $A_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}$ are intertwining,

$$A_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} = T_{\mu_2, \varepsilon_2; \mu_1, \varepsilon_1} A_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2} \quad (1.4)$$

(parameters (μ_1, ε_1) and (μ_2, ε_2) of T are transposed). If $\mu_1 - \mu_2 \neq \pm 1, \pm 2, \dots$, then representations $T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}$ are irreducible and operators $T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}$ are invertible.

Representations $T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}$ form a so-called (nonunitary) *principal series of representations*. Recall description of some unitary irreducible representations of $\mathrm{GL}_2(\mathbb{R})$.

Unitary principal series. If $\mu_1, \mu_2 \in i\mathbb{R}$, then the representation is unitary in $L^2(\mathbb{R})$. Denote by $\Lambda_{\text{principal}} \subset \Lambda$ the subset consisting of tuples $(is_1, \varepsilon_1; is_2, \varepsilon_2)$ with $s_1 \geq s_2$.

Discrete series. Notice that the group $\mathrm{GL}_2(\mathbb{R})$ acts on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ by linear-fractional transformations

$$X : z \mapsto \frac{x_{12} + zx_{22}}{x_{11} + zx_{21}}, \quad (1.5)$$

these transformations preserve the real projective line $\mathbb{R} \cup \infty$ and therefore preserves its complement $\mathbb{C} \setminus \mathbb{R}$. If $\det X > 0$, then (1.5) leave upper and lower half-planes invariant, if $\det X < 0$, then (1.5) permutes the half-planes.

Let $n = 1, 2, 3, \dots$. Consider the space H_n of holomorphic functions φ on $\mathbb{C} \setminus \mathbb{R}$ satisfying

$$\int_{\mathbb{C} \setminus \mathbb{R}} |\varphi(z)|^2 |\operatorname{Im} z|^{-n-1} d \operatorname{Re} z d \operatorname{Im} z < \infty.$$

In fact, φ is a pair of holomorphic functions φ_+ and φ_- determined on half-planes $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$. These functions have boundary values on $\mathbb{R}$ in distributional sense (we omit a precise discussion since it is not necessary for our purposes). The space H_n is a Hilbert space with respect to the inner product

$$\langle \varphi_1, \varphi_2 \rangle = \int_{\mathbb{C} \setminus \mathbb{R}} \varphi_1(z) \overline{\varphi_2(z)} |\operatorname{Im} z|^{-n-1} d \operatorname{Re} z d \operatorname{Im} z.$$

For $s \in \mathbb{R}$, $\delta \in \mathbb{Z}_2$ we define a unitary representation $D_{n,s}$ of $\mathrm{GL}_2(\mathbb{R})$ in H_n by

$$D_{n,s} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \varphi(z) = \varphi \left(\frac{x_{12} + zx_{22}}{x_{11} + zx_{21}} \right) (x_{11} + zx_{21})^{-1-n} \\ \times \det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}^{1/2+n/2+is//\delta}.$$

In fact, we have operators (1.1) for

$$\mu_1 = -n/2 - is, \varepsilon_1 = -1 + n + \delta; \mu_2 = n/2 + is, \varepsilon_2 = \delta$$

restricted to the subspace generated by boundary values of functions f_+ and f_- .

We denote by $\Lambda_{\text{discrete}}$ the set of all parameters of discrete series.

1.2 The Fourier Transform

Let F be contained in the space $C_0^\infty(\mathrm{GL}_2(\mathbb{R}))$ of compactly supported function on $\mathrm{GL}_2(\mathbb{R})$. We consider a function that sent each $\tilde{\mu} = (\mu_1, \varepsilon_1; \mu_2, \varepsilon_2)$ to an operator in $C_{\mu_1 - \mu_2, \varepsilon_1 - \varepsilon_2}^\infty$ given by

$$T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}(F) = \int_{\mathrm{GL}_2(\mathbb{R})} F(X) T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}(X) \frac{dX}{\det(X)^2}.$$

By definition the *Fourier transform* is the map $F \mapsto T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}(F)$, for each F we get a function on the space of parameters $(\mu_1, \varepsilon_1; \mu_2, \varepsilon_2)$.

Next, we define a subset $\Lambda_{\text{tempered}} \subset \Lambda$ by

$$\Lambda_{\text{tempered}} := \Lambda_{\text{principal}} \cup \Lambda_{\text{discrete}}.$$

Let us define *Plancherel measure* $d\mathcal{P}(\tilde{\mu})$ on $\Lambda_{\text{tempered}}$. On the piece $\Lambda_{\text{principal}}$ it is given by

$$d\mathcal{P}(is_1, 0; is_2, \varepsilon_2) = \frac{1}{16\pi^3} (s_1 - s_2) \tanh \pi(s_1 - s_2)/2 ds_1 ds_2; \\ d\mathcal{P}(is_1, 1; is_2, \varepsilon_2) = \frac{1}{16\pi^3} (s_1 - s_2) \coth \pi(s_1 - s_2)/2 ds_1 ds_2.$$

On n -th piece of $\Lambda_{\text{discrete}}$ it is given by

$$d\mathcal{P} = \frac{n}{8\pi^3} ds.$$

Consider the space $\mathcal{L}^2$ of functions Q on $\Lambda_{\text{tempered}}$ taking values in the space of Hilbert–Schmidt operators in the corresponding Hilbert spaces and satisfying the condition

$$\int_{\Lambda_{\text{tempered}}} \text{tr} \left(Q(\tilde{\mu}) Q^*(\tilde{\mu}) \right) d\mathcal{P}(\tilde{\mu}) < \infty.$$

This is a Hilbert space with respect to the inner product

$$\langle Q_1, Q_2 \rangle_{\mathcal{L}^2} := \int_{\Lambda_{\text{tempered}}} \text{tr} \left(Q_1(\tilde{\mu}) Q_2^*(\tilde{\mu}) \right) d\mathcal{P}(\tilde{\mu}).$$

According the Plancherel theorem, for any $F_1, F_2 \in C_0^\infty(\text{GL}_2(\mathbb{R}))$ we have

$$\langle F_1, F_2 \rangle_{L^2(\text{GL}_2(\mathbb{R}))} = \langle T(F_1), T(F_2) \rangle_{\mathcal{L}^2(\Lambda_{\text{tempered}})}.$$

Moreover, the map $F \rightarrow T(F)$ extends to a unitary operator $L^2(\text{GL}_2(\mathbb{R})) \rightarrow \mathcal{L}^2$.

1.3 Overgroup

Let $\text{Mat}_2(\mathbb{R})$ be the space of all real matrices of order 2. By $\text{Gr}_{4,2}(\mathbb{R})$ we denote the Grassmannian of 2-dimensional subspaces in $\mathbb{R}^2 \oplus \mathbb{R}^2$. For any operator $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ its graph is an element $\text{Gr}_{4,2}(\mathbb{R})$. The set $\text{Mat}_2(\mathbb{R})$ of such operators is an open dense chart in $\text{Gr}_{4,2}(\mathbb{R})$.

The group $\text{GL}_4(\mathbb{R})$ acts in a natural way in $\mathbb{R}^4$ and therefore on the Grassmannian. In the chart $\text{Mat}_2(\mathbb{R})$ the action is given by the formula (see, e.g., [18], Theorem 2.3.2)

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : X \mapsto (A + XC)^{-1}(B + XD),$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is an element of $\text{GL}_4(\mathbb{R})$ written as a block matrix of size $2 + 2$. The Jacobian of this transformation is (see, e.g., [18], Theorem 2.3.2)

$$\det(A + XC)^{-4} \det \begin{pmatrix} A & B \\ C & D \end{pmatrix}^2.$$

For $\sigma \in i\mathbb{R}$ we define a unitary representation of $\text{GL}_4(\mathbb{R})$ in $L^2(\text{Mat}_2(\mathbb{R}))$ by

$$\begin{aligned} R_\sigma \begin{pmatrix} A & B \\ C & D \end{pmatrix} F(X) &= F((A + XC)^{-1}(B + XD)) \\ &\quad \times |\det(A + XC)|^{-2+2\sigma} \left| \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right|^{1-\sigma} \end{aligned}$$

(these representations are contained in degenerate principal series).

The group $\text{GL}_2(\mathbb{R})$ is an open dense subset in $\text{Mat}_2(\mathbb{R})$. Therefore, we can identify the spaces L^2 on $\text{GL}_2(\mathbb{R})$ and $\text{Mat}_2(\mathbb{R})$. For this we consider a unitary operator

$$J : L^2(\text{Mat}_2(\mathbb{R})) \rightarrow L^2(\text{GL}_2(\mathbb{R}))$$

given by

$$J_\sigma F(X) = F(X) \cdot |\det X|^{1-\sigma}$$

This determines a unitary representation $U_\sigma := J_\sigma R_\sigma J_\sigma^{-1}$ of $\mathrm{GL}_4(\mathbb{R})$ in $L^2(\mathrm{GL}_2(\mathbb{R}))$, the explicit expression is

$$\begin{aligned} U_\sigma \begin{pmatrix} A & B \\ C & D \end{pmatrix} F(X) &= F((A + XC)^{-1}(B + XD)) \\ &\times \left| \frac{\det(A+XC) \det(B+XD)}{\det X \det \begin{pmatrix} A & B \\ C & D \end{pmatrix}} \right|^{-1+\sigma}. \end{aligned} \quad (1.6)$$

Consider a subgroup $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R}) \subset \mathrm{GL}_4(\mathbb{R})$ consisting of matrices $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$. For this subgroup we get the usual left-right action in $L^2(\mathrm{GL}_2(\mathbb{R}))$,

$$U_\sigma \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} F(X) = F(A^{-1}XD).$$

Formula (1.6) extends this formula to the whole group $\mathrm{GL}_4(\mathbb{R})$. The Lie algebra $\mathfrak{gl}_4$ acts in the space of functions on $\mathrm{Mat}_2(\mathbb{R})$ by first order differential operators, which can be easily written; a list of formulas for all generators e_{kl} , where $k, l = 1, 2, 3, 4$, is given below in Sect. 2.5. We restrict this action to the space of smooth compactly supported functions on $\mathrm{GL}_2(\mathbb{R})$. Notice that the operators $i \cdot e_{kl}$ are symmetric on this domain, but some of them are not essentially self-adjoint.

Our purpose is to write explicitly the images E_{kl} of operators e_{kl} under the Fourier transform.

1.4 Formulas

Operators $T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}(F)$ have the form

$$T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}(F)\varphi(t) = \int_{-\infty}^{\infty} K(t, s | \mu_1, \varepsilon_1; \mu_2, \varepsilon_2) \varphi(s) ds. \quad (1.7)$$

Recall that functions K are holomorphic in μ_1, μ_2 .

We wish to write operators E_{kl} on kernels K . The complete list is contained below in Sect. 2.5, here we present two basic expressions.

The algebra $\mathfrak{gl}_4(\mathbb{R})$ can be decomposed as a linear space into a direct sum of four subalgebras $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ consisting of matrices of the form

$$\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}, \quad \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$$

The subalgebras $\mathfrak{a}$ and $\mathfrak{d}$ are isomorphic to $\mathfrak{gl}_2(\mathbb{R})$, subalgebras $\mathfrak{b}$ and $\mathfrak{c}$ are Abelian.

Formulas for the action of $\mathfrak{a}$ and $\mathfrak{d}$ immediately follow from the definition of the Fourier transform. To obtain formulas for the whole $\mathfrak{gl}_4$, it is sufficient to write expressions for one generator of $\mathfrak{b}$, say E_{14} , and one generator of $\mathfrak{c}$, say E_{32} . After this other generators can be obtained by evaluation of commutators.

We define shift operators V_1^+ , V_1^- , V_2^+ , V_2^- by

$$V_1^\pm K(t, s | \mu_1, \varepsilon_1; \mu_2, \varepsilon_2) = K(t, s | \mu_1 \pm 1, \varepsilon_1 + 1; \mu_2, \varepsilon_2); \quad (1.8)$$

$$V_2^\pm K(t, s | \mu_1, \varepsilon_1; \mu_2, \varepsilon_2) = K(t, s | \mu_1, \varepsilon_1; \mu_2 \pm 1, \varepsilon_2 + 1). \quad (1.9)$$

Theorem 1.1 *The operators E_{14} , E_{32} are given by the formulas*

$$E_{14} = \frac{-1/2 - \sigma + \mu_1}{\mu_1 - \mu_2} \frac{\partial}{\partial s} V_1^- + \frac{-1/2 - \sigma + \mu_2}{\mu_1 - \mu_2} \frac{\partial}{\partial t} V_2^-;$$

$$E_{32} = \frac{1/2 + \mu_1 + \sigma}{\mu_1 - \mu_2} \frac{\partial}{\partial t} V_1^+ + \frac{1/2 + \mu_2 + \sigma}{\mu_1 - \mu_2} \frac{\partial}{\partial s} V_2^+.$$

Remark We emphasize that these formulas determining unbounded skew-symmetric operators in $\mathcal{L}$ include shifts transversal to the contour of integration in the Plancherel formula (the contour corresponds to pure imaginary μ_1, μ_2 and (1.8), (1.9) are a shift operators in real directions). $\square$

1.5 Remarks on a General Problem

In [17] the author formulated the following question: *Assume that we know the explicit Plancherel formula for the restriction of a unitary representation ρ of a group G to a subgroup H . Is it possible to write the action of the Lie algebra of G in the direct integral of representations of H ?*

Now it seems that an answer to this question is affirmative.

The initial paper [17] contains a solution for a tensor product² of a highest and lowest weight representations of $\mathrm{SL}_2(\mathbb{R})$. In this case the overalgebra acts by differential-difference operators in the space $L^2(S^1 \times \mathbb{R}_+)$ having the form

$$Lf(\varphi, s) = D_1 f(\varphi, s + i) + D_2 f(\varphi, s) + D_3 f(\varphi, s - i),$$

where D_1, D_2, D_3 are differential operators in the variable φ of orders 0, 1, 2 respectively.

In [9–13] Molchanov solved several rank 1 problems of this type, expressions are similar, but there appear differential operators of order 4. In [20] there was obtained the action of the overalgebra in restrictions from $\mathrm{GL}_{n+1}(\mathbb{C})$ to $\mathrm{GL}_n(\mathbb{C})$, in this case

² The problem of a decomposition of a tensor product $\rho_1 \otimes \rho_2$ of representations of a group G is a special case of a decomposition of restrictions. Indeed, $\rho_1 \otimes \rho_2$ is a representation of the group $G \times G$, and we must restrict this representation to the diagonal G .

differential operators have order n . In all the cases examined by now formulas include shift operators in the imaginary direction.

In the present paper, we write the action of the overalgebra in the restriction of a degenerate principal series of the group $\mathrm{GL}_4(\mathbb{R})$ to $\mathrm{GL}_2(\mathbb{R})$. Notice that canonical overgroups exist for all 10 series of real classical groups.³ Moreover overgroups exist for all 52 series of classical semisimple symmetric spaces G/H , see [8, 15], see also [18], Addendum D.6. So the problem makes sense for all classical symmetric spaces.

Sturm–Liouville problems for difference operators in $L^2(\mathbb{R})$ in the imaginary direction

$$\lambda f(s) = a(s)f(s+i) + b(s)f(s) + c(s)f(s-i)$$

arise in a natural way in the theory of hypergeometric orthogonal polynomials, see, e.g., [1, 7], apparently a first example (the Meixner–Pollaczek system) was discovered by J. Meixner in 1930s. On such operators with continuous spectra see [5, 16, 19]. See also a multi-rank work of Cherednik [2] on Harish-Chandra spherical transforms.

1.6 The Fourier Transform on $\mathrm{GL}_2(\mathbb{C})$

For a detailed exposition of representations of the Lorentz group, see [14]. For $\nu, \nu' \in \mathbb{C}$ satisfying $\nu - \nu' \in \mathbb{Z}$ we define the function $z^{\nu\| \nu'}$ on the multiplicative group of $\mathbb{C}$ by

$$z^{\nu\| \nu'} := z^\nu z^{\nu'} := |z|^{2\nu} \bar{z}^{\nu'-\nu}.$$

Denote by Δ the set of all $(\mu_1, \mu'_1; \mu_2, \mu'_2) \in \mathbb{C}^4$ such that $\mu_1 - \mu'_1 \in \mathbb{Z}, \mu_2 - \mu'_2 \in \mathbb{Z}$. For $(\mu_1, \mu'_1; \mu_2, \mu'_2) \in \Delta$ we define a representation of $\mathrm{GL}_2(\mathbb{C})$ in the space of functions on $\mathbb{C}$ by

$$\begin{aligned} & T_{\mu_1, \mu'_1; \mu_2, \mu'_2} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \varphi(t) \\ &= \varphi\left(\frac{x_{12} + tx_{22}}{x_{11} + tx_{21}}\right) (x_{11} + tx_{21})^{-1+\mu_1-\mu_2\| -1+\mu'_1-\mu'_2} \det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}^{1/2+\mu_2\| 1/2+\mu'_2}. \end{aligned}$$

We consider the space of C^∞ -smooth functions φ on $\mathbb{C}$ such that

$$\varphi(-t^{-1})t^{-1+\mu_1-\mu_2\| -1+\mu'_1-\mu'_2}$$

is C^∞ -smooth at 0. These representations form the (nonunitary) *principal series*.

It is convenient to complexify the Lie algebra of $\mathrm{GL}_2(\mathbb{C})$,

$$\mathfrak{gl}_2(\mathbb{C})_{\mathbb{C}} \simeq \mathfrak{gl}_2(\mathbb{C}) \oplus \mathfrak{gl}_2(\mathbb{C}).$$

³ Emphasize that in the present paper we consider groups GL and not SL, also we must consider groups $U(p, q)$ and not $SU(p, q)$.

Under this isomorphism, the operators of the Lie algebra act in our representation by

$$L_{11} = -t \frac{d}{dt} + (-1/2 + \mu_1), \quad L_{12} = \frac{d}{dt}, \quad (1.10)$$

$$L_{21} = -t^2 \frac{d}{dt} + t(-1 + \mu_1 - \mu_2), \quad L_{22} = t \frac{d}{dt} + (1/2 + \mu_2), \quad (1.11)$$

$$\bar{L}_{11} = -\bar{t} \frac{d}{d\bar{t}} + (-1/2 + \mu'_1), \quad \bar{L}_{12} = \frac{d}{d\bar{t}}, \quad (1.12)$$

$$\bar{L}_{21} = -\bar{t}^2 \frac{d}{d\bar{t}} + \bar{t}(-1 + \mu'_1 - \mu'_2), \quad \bar{L}_{22} = \bar{t} \frac{d}{d\bar{t}} + (1/2 + \mu'_2). \quad (1.13)$$

Formally, we have duplicated expressions (1.2)–(1.3).

If

$$\operatorname{Re}(\mu_1 + \mu'_1) = 0, \quad \operatorname{Re}(\mu_2 + \mu'_2) = 0, \quad (1.14)$$

then the representation $T_{\mu_1, \mu'_1; \mu_2, \mu'_2}$ is unitary in L^2 . Denote by Δ_{tempered} the set of such tuples $(\mu_1, \mu'_1; \mu_2, \mu'_2)$, it is a union of a countable family of parallel 2-dimensional real planes in $\mathbb{C}^4$, we equip it by a natural Lebesgue measure $d\lambda(\mu)$.

For any compactly supported smooth function F on $\operatorname{GL}_2(\mathbb{C})$ we define its *Fourier transform* as an operator-valued function on Δ given by

$$T_{\mu_1, \mu'_1; \mu_2, \mu'_2}(F) = \int_{\operatorname{GL}_2(\mathbb{C})} F(X) |\det X|^{-4} \prod_{k,l=1,2} d \operatorname{Re} x_{kl} d \operatorname{Im} x_{kl}.$$

The Plancherel formula is the following identity

$$\begin{aligned} \langle F_1, F_2 \rangle_{L^2(\operatorname{GL}_2(\mathbb{C}))} &= -C \cdot \int_{\Delta_{\text{tempered}}} \operatorname{tr} \left(T_{\mu_1, \mu'_1; \mu_2, \mu'_2}(F_1) T_{\mu_1, \mu'_1; \mu_2, \mu'_2}(F_2) \right)^* \\ &\quad \times (\mu_1 - \mu_2)(\mu'_1 - \mu'_2) d\lambda(\mu), \end{aligned}$$

where C is an explicit constant.

Denote by $K(t, s | \mu_1, \mu'_1; \mu_2, \mu'_2)$ the kernel of the operator $T_{\mu_1, \mu'_1; \mu_2, \mu'_2}(F)$,

$$T_{\mu_1, \mu'_1; \mu_2, \mu'_2}(F)\varphi(t) = \int_{\mathbb{C}} K(t, s | \mu_1, \mu'_1; \mu_2, \mu'_2)\varphi(s) ds.$$

1.7 Overgroup for $\operatorname{GL}_2(\mathbb{C})$

Consider the complex Grassmannian $\operatorname{Gr}_{4,2}(\mathbb{C})$ of 2-dimensional planes in $\mathbb{C}^4$, again the set $\operatorname{Mat}_2(\mathbb{C})$ is an open dense set on $\operatorname{Gr}_{4,2}(\mathbb{C})$. For $\sigma, \sigma' \in i\mathbb{R}$ consider a unitary representation $R_{\sigma, \sigma'}$ of $\operatorname{GL}_4(\mathbb{C})$ in $L^2(\operatorname{Mat}_2(\mathbb{C}))$ given by

$$R_{\sigma,\sigma'} F(X) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} F(X) = F((A + XC)^{-1}(B + XD)) \\ \times |\det(A + XC)|^{-2+2\sigma\| - 2+2\sigma'} \left| \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right|^{1-\sigma\| 1-\sigma'}.$$

We define a unitary operator $J : L^2(\text{Mat}_2(\mathbb{C})) \rightarrow L^2(\text{Mat}_2(\mathbb{C}))$ by

$$J_{\sigma,\sigma'} F(X) = F(X)(\det X)^{-1+\sigma\| - 1+\sigma'}.$$

In this way we get a unitary representation $U_{\sigma,\sigma'} = J_{\sigma,\sigma'} R_{\sigma,\sigma'} J_{\sigma,\sigma'}^{-1}$ of $\text{GL}_4(\mathbb{C})$ in $L^2(\text{Mat}_2(\mathbb{C}))$:

$$U_{\sigma\| \sigma'} \begin{pmatrix} A & B \\ C & D \end{pmatrix} F(X) = F((A + XC)^{-1}(B + XD)) \\ \times \left| \frac{\det(A+XC) \det(B+XD)}{\det X \det \begin{pmatrix} A & B \\ C & D \end{pmatrix}} \right|^{-1+\sigma\| - 1+\sigma'}.$$

1.8 Formulas for $\text{GL}_2(\mathbb{C})$

We wish to write the action of the Lie algebra

$$\mathfrak{gl}_4(\mathbb{C})_{\mathbb{C}} \simeq \mathfrak{gl}_4(\mathbb{C}) \oplus \mathfrak{gl}_4(\mathbb{C})$$

in the Plancherel decomposition of $\text{GL}_2(\mathbb{C})$. Denote the standard generators of $\mathfrak{gl}_4(\mathbb{C}) \oplus 0$ and $0 \oplus \mathfrak{gl}_4(\mathbb{C})$ by E_{kl} and $\bar{E}_{kl}$ respectively. Define the following shift operators

$$V_1 K(t, s; \mu_1, \mu'_1, \mu_2, \mu'_2) = K(t, s; \mu_1 + 1, \mu'_1, \mu_2, \mu'_2); \\ V'_1 K(t, s; \mu_1, \mu'_1, \mu_2, \mu'_2) = K(t, s; \mu_1, \mu'_1 + 1, \mu_2, \mu'_2),$$

and similar operators V_2 and V'_2 shifting μ_2 and μ'_2 .

Theorem 1.2 *The operators E_{14} , $\bar{E}_{14}$, E_{23} , $\bar{E}_{23}$ are given by the formulas*

$$E_{14} = \frac{-1/2 - \sigma + \mu_1}{\mu_1 - \mu_2} \frac{\partial}{\partial s} V_1^{-1} - \frac{-1/2 - \sigma + \mu_2}{\mu_1 - \mu_2} \frac{\partial}{\partial t} V_2^{-1}; \\ \bar{E}_{14} = \frac{-1/2 - \sigma' + \mu'_1}{\mu'_1 - \mu'_2} \frac{\partial}{\partial \bar{s}} (V'_1)^{-1} - \frac{-1/2 - \sigma' + \mu'_2}{\mu'_1 - \mu'_2} \frac{\partial}{\partial \bar{t}} (V'_2)^{-1}; \\ E_{32} = \frac{1/2 + \mu_1 + \sigma}{\mu_1 - \mu_2} \frac{\partial}{\partial t} V_1 + \frac{1/2 + \mu_2 + \sigma}{\mu_1 - \mu_2} \frac{\partial}{\partial s} V_2; \\ \bar{E}_{32} = \frac{1/2 + \mu'_1 + \sigma'}{\mu'_1 - \mu'_2} \frac{\partial}{\partial \bar{t}} V'_1 + \frac{1/2 + \mu'_2 + \sigma'}{\mu'_1 - \mu'_2} \frac{\partial}{\partial \bar{s}} V'_2.$$

2 Calculations

2.1 The Expression for Kernel

Lemma 2.1 *The kernel $K(\cdot)$ of an integral operator $T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}(F)$ is given by the formula*

$$\begin{aligned} K(t, s | \mu_1, \varepsilon_1; \mu_2, \varepsilon_2) \\ = \iiint_{\mathbb{R}^3} F(u - tv, su - stv - tw, v, sv + w) u^{-3/2 + \mu_1 \varepsilon_1} w^{-3/2 + \mu_2 \varepsilon_2} du dv dw. \end{aligned} \quad (2.1)$$

For $F \in C_0^\infty(\mathrm{GL}_2(\mathbb{R}))$ the integration is actually taken over a bounded domain.

The integral converges if $\mathrm{Re} \mu_1 > 1/2$, $\mathrm{Re} \mu_2 > 1/2$. For fixed $\varepsilon_1, \varepsilon_2$ it has a meromorphic continuation to the whole complex plane (μ_1, μ_2) with poles on the hyperplanes $\mu_1 = -1/2 - k$, $\mu_2 = -1/2 - k$, where $k = 0, 1, 2, \dots$ (see, e.g., [3], §I.3).

Proof By the definition

$$\begin{aligned} & \int_{\mathbb{R}} T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}(F) \varphi(t) \psi(t) dt \\ &= \int_{\mathbb{R}} \int_{\mathrm{Mat}_2(\mathbb{R})} F(x_{11}, x_{12}, x_{21}, x_{22}) \varphi\left(\frac{x_{12} + tx_{22}}{x_{11} + tx_{21}}\right) (x_{11} + tx_{21})^{-1 + \mu_1 - \mu_2 \varepsilon_1 - \varepsilon_2} \\ & \quad \times (x_{11}x_{22} - x_{12}x_{21})^{1/2 + \mu_2 \varepsilon_2} \frac{dx_{11} dx_{12} dx_{21} dx_{22}}{(x_{11}x_{22} - x_{12}x_{21})^2} dt. \end{aligned}$$

In the interior integral, we pass from the variables $x_{11}, x_{12}, x_{21}, x_{22}$ to new variables u, v, w, s defined by

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ v & w \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \quad (2.2)$$

or

$$x_{11} = u - tv, \quad x_{12} = su - stv - tw, \quad x_{21} = v, \quad x_{22} = sv + w.$$

The Jacobi matrix of this transformation is triangular, and the Jacobian is $|u|$. The inverse transformation is

$$u = x_{11} + tx_{21}, \quad v = x_{21}, \quad w = \frac{x_{11}x_{22} - x_{12}x_{21}}{x_{11} + tx_{21}}, \quad s = \frac{x_{12} + tx_{22}}{x_{11} + tx_{21}}.$$

We also have $x_{11}x_{22} - x_{12}x_{21} = uw$.

After the change of variables we come to

$$\iint_{\mathbb{R}^2} K(t, s | \mu_1, \varepsilon_1; \mu_2, \varepsilon_2) \varphi(s) \psi(t) ds dt,$$

where $K(\cdot)$ is given by (2.1).

A function F has a compact support in $\mathbb{R}^4 \setminus \{x_{11}x_{22} - x_{12}x_{21} = 0\}$. So, actually, $x_{21} = v, x_{11} = u - tv, x_{22} = w + sv$ are contained in a bounded domain. This implies the second claim of the lemma. $\square$

2.2 Preliminary Remarks

Below F denotes

$$F := F(u - tv, su - stv - tw, v, sv + w).$$

Also, $\partial_{11}F, \partial_{12}F$, etc. denote

$$\partial_{11}F := \frac{\partial}{\partial x_{11}} F(x_{11}, x_{12}, x_{21}, x_{22}) \Big|_{\substack{x_{11} = u - tv \quad x_{12} = su - stv - tw, \\ x_{21} = v, \quad x_{22} = sv + w}},$$

etc. Partial derivatives of F are

$$\frac{\partial}{\partial u} F = \partial_{11}F + s \partial_{12}F; \quad (2.3)$$

$$\frac{\partial}{\partial v} F = -t \partial_{11}F - st \partial_{12}F + \partial_{21}F + s \partial_{22}F; \quad (2.4)$$

$$\frac{\partial}{\partial w} F = -t \partial_{12}F + \partial_{22}F, \quad (2.5)$$

and

$$\frac{\partial}{\partial s} F = (u - vt) \partial_{12}F + v \partial_{22}F; \quad (2.6)$$

$$\frac{\partial}{\partial t} F = -v \partial_{11}F - (w + sv) \partial_{12}F. \quad (2.7)$$

Also, notice that

$$(y^{v//\delta})' = vy^{v-1//\delta+1}.$$

2.3 A Verification of the Formula for E_{14}

It is easy to verify that the operator e_{14} in $C_0^\infty(\mathrm{GL}_2(\mathbb{R}))$ is given by

$$e_{14} = \frac{\partial}{\partial x_{12}} - (-1 + \sigma) \frac{x_{21}}{x_{11}x_{22} - x_{12}x_{21}}.$$

Therefore,

$$\begin{aligned} T_{\mu_1, \varepsilon_1; \mu_2, \varepsilon_2}(e_{14}F) \\ = \iiint \partial_{12}F u^{-3/2+\mu_1/\varepsilon_1} w^{-3/2+\mu_2/\varepsilon_2} du dv dw \\ - (-1 + \sigma) \iiint F \frac{v}{uw} u^{-3/2+\mu_1/\varepsilon_1} w^{-3/2+\mu_2/\varepsilon_2} du dv dw, \end{aligned} \quad (2.8)$$

(the integration is taken over $\mathbb{R}^3$ on default).

We must verify that (2.8) coincides with

$$E_{14}K = \left(\frac{-1/2 - \sigma + \mu_1}{\mu_1 - \mu_2} \frac{\partial}{\partial s} V_1^- + \frac{-1/2 - \sigma + \mu_2}{\mu_1 - \mu_2} \frac{\partial}{\partial t} V_2^- \right) K \quad (2.9)$$

Below we establish two formulas

$$\begin{aligned} \iiint \partial_{12}F u^{-3/2+\mu_1/\varepsilon_1} w^{-3/2+\mu_2/\varepsilon_2} du dv dw - \frac{\partial}{\partial s} V_1^- K \\ = (-3/2 + \mu_2) \iiint F \frac{v}{uw} u^{-3/2+\mu_1/\varepsilon_1} w^{-3/2+\mu_2/\varepsilon_2} du dv dw, \end{aligned} \quad (2.10)$$

$$\begin{aligned} \iiint \partial_{12}F u^{-3/2+\mu_1/\varepsilon_1} w^{-3/2+\mu_2/\varepsilon_2} du dv dw + \frac{\partial}{\partial t} V_2^- K \\ = (-3/2 + \mu_1) \iiint F \frac{v}{uw} u^{-3/2+\mu_1/\varepsilon_1} w^{-3/2+\mu_2/\varepsilon_2} du dv dw. \end{aligned} \quad (2.11)$$

Considering the sum of (2.10) and (2.11) with coefficients $\frac{-1/2-\sigma+\mu_1}{\mu_1-\mu_2}$ and $\frac{-1/2-\sigma+\mu_2}{\mu_1-\mu_2}$ we get coincidence of (2.8) and (2.9); for this, we use the identities

$$\begin{aligned} \frac{-1/2 - \sigma + \mu_1}{\mu_1 - \mu_2} - \frac{-1/2 - \sigma + \mu_2}{\mu_1 - \mu_2} &= 1; \\ (-3/2 + \mu_2) \cdot \frac{-1/2 - \sigma + \mu_1}{\mu_1 - \mu_2} - (-3/2 + \mu_1) \cdot \frac{-1/2 - \sigma + \mu_2}{\mu_1 - \mu_2} &= 1 - \sigma. \end{aligned}$$

Now let us check (2.10). The following identity can be verified by a straightforward calculation (with (2.6) and (2.5)):

$$\partial_{12}F - \frac{1}{u} \frac{\partial}{\partial s} F = -\frac{v}{u} \frac{\partial}{\partial w} F.$$

Therefore the left-hand side of (2.10) equals to

$$\iiint \left[-\frac{v}{u} \frac{\partial}{\partial w} F \right] \cdot u^{-3/2+\mu_1\|\varepsilon_1\|} w^{-3/2+\mu_2\|\varepsilon_2\|} du dv dw.$$

We integrate this expression by parts in the variable w and come to (2.10).

To check (2.11), we verify the identity

$$\partial_{12}F + \frac{1}{v} \frac{\partial}{\partial t} F = \frac{v}{w} \frac{\partial}{\partial u} F$$

and after this integrate by parts as above.

2.4 A Verification of the Formula for E_{32}

We have

$$e_{32} = -\left(x_{11}x_{21} \frac{\partial}{\partial x_{11}} + x_{11}x_{22} \frac{\partial}{\partial x_{12}} + x_{21}^2 \frac{\partial}{\partial x_{21}} + x_{21}x_{22} \frac{\partial}{\partial x_{22}} \right) + (-1 + \sigma)x_{21}.$$

Therefore,

$$T(e_{32}K) = \iiint (G + (-1 + \sigma)vF) \cdot u^{-3/2+\mu_1\|\varepsilon_1\|} w^{-3/2+\mu_2\|\varepsilon_2\|} du dv dw, \quad (2.12)$$

where

$$G = (u - vt)v \partial_{11}F + (u - vt)(w + vs) \partial_{12}F + v^2 \partial_{21}F + v(w + vs) \partial_{22}F.$$

On the other hand,

$$\begin{aligned} E_{32}K &= \frac{1/2+\mu_1+\sigma}{\mu_1-\mu_2} \iiint u \frac{\partial}{\partial t} F \cdot u^{-3/2+\mu_1\|\varepsilon_1\|} w^{-3/2+\mu_2\|\varepsilon_2\|} du dv dw \\ &\quad + \frac{1/2+\mu_2+\sigma}{\mu_1-\mu_2} \iiint w \frac{\partial}{\partial s} F \cdot u^{-3/2+\mu_1\|\varepsilon_1\|} w^{-3/2+\mu_2\|\varepsilon_2\|} du dv dw. \end{aligned} \quad (2.13)$$

We must verify that (2.12) and (2.13) are equal. As in the previous subsection, this statement is reduced to a pair of identities

$$\begin{aligned} & \iiint (G + (-1 + \sigma)vF - u \frac{\partial}{\partial t} F) \cdot u^{-3/2+\mu_1\|\varepsilon_1} w^{-3/2+\mu_2\|\varepsilon_2} du dv dw \\ &= (1/2 + \mu_2 + \sigma) \iiint vF \cdot u^{-3/2+\mu_1\|\varepsilon_1} w^{-3/2+\mu_2\|\varepsilon_2} du dv dw; \end{aligned} \quad (2.14)$$

$$\begin{aligned} & \iiint (G + (-1 + \sigma)vF + w \frac{\partial}{\partial s} F) \cdot u^{-3/2+\mu_1\|\varepsilon_1} w^{-3/2+\mu_2\|\varepsilon_2} du dv dw \\ &= (1/2 + \mu_1 + \sigma) \iiint vF \cdot u^{-3/2+\mu_1\|\varepsilon_1} w^{-3/2+\mu_2\|\varepsilon_2} du dv dw. \end{aligned} \quad (2.15)$$

Let us verify (2.14). It can be easily checked (with (2.7), (2.4), (2.5)) that

$$G - u \frac{\partial}{\partial t} F = -v^2 \frac{\partial}{\partial v} F - vw \frac{\partial}{\partial w} F.$$

We substitute this to the left-hand side of (2.14) and come to

$$\begin{aligned} & \iiint \left(-v^2 \frac{\partial}{\partial v} F - vw \frac{\partial}{\partial w} F + (-1 + \sigma)vF \right) \\ & \quad \times u^{-3/2+\mu_1\|\varepsilon_1} w^{-3/2+\mu_2\|\varepsilon_2} du dv dw. \end{aligned} \quad (2.16)$$

Integrating by parts, we get

$$\begin{aligned} & \iiint F \cdot \frac{\partial}{\partial v} (v^2) \cdot u^{-3/2+\mu_1\|\varepsilon_1} w^{-3/2+\mu_2\|\varepsilon_2} du dv dw \\ & + \iiint vF \cdot u^{-3/2+\mu_1\|\varepsilon_1} \frac{\partial}{\partial v} (w^{-1/2+\mu_2\|\varepsilon_2+1}) du dv dw \\ & + (-1 + \sigma) \iiint vF \cdot u^{-3/2+\mu_1\|\varepsilon_1} w^{-3/2+\mu_2\|\varepsilon_2} du dv dw. \end{aligned}$$

After a summation we come to the right-hand side of (2.14).

A proof of (2.15) is similar, we use the identity

$$G + v \frac{\partial}{\partial s} F = -v^2 \frac{\partial}{\partial v} F - uv \frac{\partial}{\partial u} F$$

and repeat the same steps.

2.5 Table of Formulas

First, we present formulas for the action of the Lie algebra $\mathfrak{gl}_4$ corresponding to U_σ , see (1.6). Denote generators of $\mathfrak{gl}_4$ by e_{kl} , where $1 \leq k, l \leq 4$. Denote by ∂_{pq} the partial derivatives $\frac{\partial}{\partial x_{pq}}$, where $p, q = 1, 2$. The generators e_{kl} naturally split into 4 groups corresponding to blocks A, B, C, D in (1.6).

(a) Generators corresponding to the block A form a Lie algebra $\mathfrak{gl}_2$:

$$\begin{aligned} e_{11} &= -x_{11}\partial_{11} - x_{12}\partial_{12}, & e_{12} &= -x_{21}\partial_{11} - x_{22}\partial_{12}, \\ e_{21} &= -x_{11}\partial_{21} - x_{12}\partial_{22}, & e_{22} &= -x_{21}\partial_{21} - x_{22}\partial_{22}. \end{aligned}$$

(b) Generators corresponding to the block D also form a Lie algebra $\mathfrak{gl}_2$:

$$\begin{aligned} e_{33} &= x_{11}\partial_{11} + x_{21}\partial_{21}, & e_{34} &= x_{11}\partial_{12} + x_{21}\partial_{22}, \\ e_{43} &= x_{12}\partial_{11} + x_{22}\partial_{21}, & e_{44} &= x_{12}\partial_{12} + x_{22}\partial_{22}. \end{aligned}$$

(c) Elements corresponding to the block B form a 4-dimensional Abelian Lie algebra:

$$\begin{aligned} e_{13} &= \partial_{11} + (-1 + \sigma) \frac{x_{22}}{\det X}, & e_{14} &= \partial_{12} - (-1 + \sigma) \frac{x_{21}}{\det X}, \\ e_{23} &= \partial_{21} - (-1 + \sigma) \frac{x_{12}}{\det X}, & e_{24} &= \partial_{22} + (-1 + \sigma) \frac{x_{11}}{\det X}. \end{aligned}$$

(d) Elements corresponding to the block C also form a 4-dimensional Abelian Lie algebra:

$$\begin{aligned} e_{31} &= -(x_{11}^2\partial_{11} + x_{11}x_{12}\partial_{12} + x_{11}x_{21}\partial_{21} + x_{12}x_{21}\partial_{22}) + (-1 + \sigma)x_{11}, \\ e_{32} &= -(x_{11}x_{21}\partial_{11} + x_{11}x_{22}\partial_{12} + x_{21}^2\partial_{21} + x_{21}x_{22}\partial_{22}) + (-1 + \sigma)x_{21}, \\ e_{41} &= -(x_{11}x_{12}\partial_{11} + x_{12}^2\partial_{12} + x_{11}x_{22}\partial_{21} + x_{12}x_{22}\partial_{22}) + (-1 + \sigma)x_{12}, \\ e_{42} &= -(x_{12}x_{21}\partial_{11} + x_{12}x_{22}\partial_{12} + x_{21}x_{22}\partial_{21} + x_{22}^2\partial_{22}) + (-1 + \sigma)x_{22}. \end{aligned}$$

Denote by E_{kl} the corresponding operators E_{kl} on kernels K . Formulas for operators of groups (a), (b) immediately follow from the definition of the Fourier transform,

$$\begin{aligned} E_{11} &= -t \frac{\partial}{\partial t} - (1/2 - \mu_1), & E_{12} &= \frac{\partial}{\partial t}, \\ E_{21} &= -t^2 \frac{\partial}{\partial t} + (-1 + \mu_1 - \mu_2)t, & E_{22} &= t \frac{\partial}{\partial t} + (1/2 + \mu_2), \end{aligned}$$

and

$$\begin{aligned} E_{33} &= -s \frac{\partial}{\partial s} - (1/2 + \mu_1), & E_{34} &= \frac{\partial}{\partial s}, \\ E_{43} &= -s^2 \frac{\partial}{\partial s} + (-1 - \mu_1 + \mu_2)s, & E_{44} &= s \frac{\partial}{\partial s} + (1/2 - \mu_2). \end{aligned}$$

Next,

$$\begin{aligned}
 E_{13} &:= \frac{1/2 - \mu_2 + \sigma}{\mu_1 - \mu_2} s \frac{\partial}{\partial t} V_2^- + \frac{1/2 - \mu_1 + \sigma}{\mu_1 - \mu_2} \left(\mu_1 - \mu_2 + s \frac{\partial}{\partial s} \right) V_1^-, \\
 E_{14} &= -\frac{1/2 + \sigma - \mu_1}{\mu_1 - \mu_2} \frac{\partial}{\partial s} V_1^- - \frac{1/2 + \sigma - \mu_2}{\mu_1 - \mu_2} \frac{\partial}{\partial t} V_2^-, \\
 E_{23} &:= \frac{1/2 - \mu_2 + \sigma}{\mu_1 - \mu_2} s \left(-(\mu_1 - \mu_2) + t \frac{\partial}{\partial t} \right) V_2^- \\
 &\quad + \frac{1/2 - \mu_1 + \sigma}{\mu_1 - \mu_2} t \left(\mu_1 - \mu_2 + s \frac{\partial}{\partial s} \right) V_1^-, \\
 E_{24} &= -\frac{1/2 + \sigma - \mu_1}{\mu_1 - \mu_2} t \frac{\partial}{\partial s} V_1^- + \frac{1/2 + \mu_2 - \sigma}{\mu_1 - \mu_2} \left(\mu_1 - \mu_2 + t \frac{\partial}{\partial s} \right) V_2^-.
 \end{aligned}$$

and

$$\begin{aligned}
 E_{31} &:= \frac{1/2 + \mu_1 + \sigma}{\mu_1 - \mu_2} \left(\mu_1 - \mu_2 - t \frac{\partial}{\partial t} \right) V_1^+ - \frac{1/2 + \mu_2 + \sigma}{\mu_1 - \mu_2} t \frac{\partial}{\partial s} V_2^+, \\
 E_{32} &= \frac{1/2 + \mu_1 + \sigma}{\mu_1 - \mu_2} \frac{\partial}{\partial t} V_1^+ + \frac{1/2 + \mu_2 + \sigma}{\mu_1 - \mu_2} \frac{\partial}{\partial s} V_2^+, \\
 E_{41} &= \frac{1/2 + \mu_1 + \sigma}{\mu_1 - \mu_2} s \left(\mu_1 - \mu_2 - t \frac{\partial}{\partial t} \right) V_1^+ \\
 &\quad - \frac{1/2 + \mu_2 + \sigma}{\mu_1 - \mu_2} t \left(\mu_1 - \mu_2 + s \frac{\partial}{\partial s} \right) V_2^+, \\
 E_{42} &= \frac{1/2 + \mu_1 + \sigma}{\mu_1 - \mu_2} s \frac{\partial}{\partial t} V_1^+ + \frac{1/2 + \mu_2 + \sigma}{\mu_1 - \mu_2} \left(\mu_1 - \mu_2 + s \frac{\partial}{\partial s} \right) V_2^+.
 \end{aligned}$$

2.6 The Case of $\mathrm{GL}_2(\mathbb{C})$

Notice that formulas in Sects. 1.1–1.4 for $\mathrm{SL}_2(\mathbb{R})$ and in Sects. 1.6–1.8 are very similar, except the Plancherel formulas.

The analog of formula (2.1) is

$$\begin{aligned}
 &K(t, s | \mu_1, \mu'_1; \mu_2, \mu'_2) \\
 &= \iiint_{\mathbb{C}^3} F(u - tv, su - stv - tw, v, sv + w) u^{-3/2 + \mu_1} w^{-3/2 + \mu_2} \\
 &\quad \times d \operatorname{Re} u d \operatorname{Im} u d \operatorname{Re} v d \operatorname{Im} v d \operatorname{Re} w d \operatorname{Im} w.
 \end{aligned} \tag{2.17}$$

Its derivation is based on the same change of variables (2.2), its real Jacobian is $u\bar{u}$. A further calculation one-to-one follows the calculation for $\mathrm{GL}_2(\mathbb{R})$.

Acknowledgements Open access funding provided by University of Vienna. Supported by the grant FWF, Project P28421.

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