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L^p-Estimates for Singular Oscillatory Integral Operators

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Abstract In this paper we study singular oscillatory integrals with a nonlinear phase function. We prove estimates of $L^2 \to L^2$ and $L^p \to L^p$ type.

Keywords Singular integral · Oscillatory integral · Nonlinear phase function

Mathematics Subject Classification 42B20

1 Introduction

Let K denote a singular kernel in $\mathbb{R}^n$. Singular integral operators T, defined by $Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy$, $x \in \mathbb{R}^n$, $f \in C_0^\infty(\mathbb{R}^n)$, have been studied for a very long time. Since approximately 1970 there has also been a lot of interest in oscillatory integral operators. The following theorem describes a typical result.

Theorem 1.1 (see Stein [6], p. 377) *Let* $\psi_1 \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ *and* $\lambda > 0$ *and let* Φ *be real-valued and smooth. Set*

$$\mathcal{U}_\lambda f(x) = \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,\xi)}\psi_1(x,\xi)f(x)dx, \;\; \xi \in \mathbb{R}^n,$$

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and assume that $\det\left(\frac{\partial^2\Phi}{\partial x_i \partial \xi_j}\right) \neq 0$ *on* $\operatorname{supp}\psi_1$. *Then one has*

$$||\mathcal{U}_\lambda f||_{L^2(\mathbb{R}^n)} \le C\lambda^{-n/2}||f||_{L^2(\mathbb{R}^n)}.$$

We shall here consider singular oscillatory integral operators, that is operators defined by integrals containing both a singular kernel and an oscillating factor. Operators of this type have been much studied in the theory of convergence of Fourier series and also in for instance Phong and Stein [4]. We shall continue this study.

Let $\psi_0 \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$ and $n \ge 2$. For $f \in L^2(\mathbb{R}^{n-1})$ set

$$T_\lambda f(x) = \int\limits_{\mathbb{R}^{n-1}} e^{i\lambda|x-(y',0)|^\gamma}\psi_0(x,y')K\big(x-(y',0)\big)f(y')dy'$$

for $x \in \mathbb{R}^n$, $\gamma > 0$, and $\lambda \ge 2$. Here for $\gamma > 1$ we set

$$K(z) = |z|^{-(n-m-1)},\ z \in \mathbb{R}^n \setminus \{0\},$$

and for $0 < \gamma \le 1$ we set

$$K(z) = |z|^{-(n-m-1)}\omega(z),\ z \in \mathbb{R}^n \setminus \{0\},$$

where $\omega \in C^\infty(\mathbb{R}^n \setminus \{0\})$, ω is homogeneous of degree 0, and $\omega(z) = 0$ for all z with $|z| = 1$ and $|z_n| \le \varepsilon_0$ for some given $\varepsilon_0 > 0$. We also assume that $0 < m < n-1$.

We shall study the norm of T_λ as an operator from $L^p(\mathbb{R}^{n-1})$ to $L^p(\mathbb{R}^n)$ and denote this norm by $||T_\lambda||_p$. In Aleksanyan et al. [1] the following theorem was proved.

Theorem 1.2 *Set* $\alpha = (n-1)/2$ *and assume* $\gamma \ge 1$. *Then one has*

$$||T_\lambda||_2 \le \begin{cases} C\lambda^{-(m+1/2)/\gamma}, & m < \gamma\alpha - 1/2, \\ C\lambda^{-\alpha}\log\lambda, & m = \gamma\alpha - 1/2, \\ C\lambda^{-\alpha}, & m > \gamma\alpha - 1/2. \end{cases}$$

The above choice of phase function is partially motivated by an application to an inhomogeneous Helmholtz equation where we give estimates for solutions. In this case we take $\gamma = 1$ (see [1], p. 544). It is also possible to use T_λ to give L^p-estimates for convolution operators. This will be studied in a forthcoming paper.

In [1] it is also proved that $||T_\lambda||_2 \ge c\lambda^{-(m+1/2)/\gamma}$ for $\gamma > 1$, where c denotes a positive constant. We shall here prove that this also holds for $\gamma = 1$ and that $||T_\lambda||_2 \ge c\lambda^{-\alpha}$ for $\gamma \ge 1$. It follows that the results in Theorem 1.2 are essentially sharp.

In this paper we shall first study the case $n = 2$ and $1 < p < \infty$. We have the following theorem.

Theorem 1.3 *Assume* $n = 2$ *and* $0 < \gamma \le 1$. *Then* $||T_\lambda||_2 \le C\lambda^{-1/2}$, *and for* $2 < p \le 4$ *one has*

$$||T_\lambda||_p \le \begin{cases} C\lambda^{-(1/p+m)/\gamma}, & 1/p + m < \gamma/2, \\ C_\varepsilon \lambda^{\varepsilon - 1/2}, & 1/p + m \ge \gamma/2, \end{cases}$$

where ε *denotes an arbitrary positive number. Also set* $\beta(p) = 1 - 1/p$ *for* $1 < p < 2$, *and* $\beta(p) = 2/p$ *for* $4 < p < \infty$. *For* $1 < p < 2$ *and* $4 < p < \infty$ *one has*

$$||T_\lambda||_p \le \begin{cases} C\lambda^{-(1/p+m)/\gamma}, & 1/p + m < \gamma\beta(p), \\ C\lambda^{-\beta(p)} \log \lambda, & 1/p + m = \gamma\beta(p), \\ C\lambda^{-\beta(p)}, & 1/p + m > \gamma\beta(p). \end{cases}$$

We shall also study the sharpness of the estimates in Theorem 1.3. We shall then estimate the operator S_λ given by

$$S_\lambda f(x) = \int_{\mathbb{R}^{n-1}} e^{i\lambda|x-y|^\gamma} \psi_0(x, y) K(x - y) f(y) dy, \ x \in \mathbb{R}^{n-1},$$

where $n \ge 2$, $\psi_0 \in C_0^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, and $K(z) = |z|^{-(n-m-1)}$, $z \in \mathbb{R}^{n-1} \setminus \{0\}$. We let $||S_\lambda||_p$ denote the norm of S_λ as an operator from $L^p(\mathbb{R}^{n-1})$ to $L^p(\mathbb{R}^{n-1})$. We shall prove the following theorem.

Theorem 1.4 *Assume* $n \ge 2$, $0 < m < n - 1$, $\gamma > 0$, *and* $\gamma \ne 1$. *Then*

$$||S_\lambda||_2 \le \begin{cases} C\lambda^{-m/\gamma}, & m < \gamma\alpha, \\ C\lambda^{-\alpha} \log \lambda, & m = \gamma\alpha, \\ C\lambda^{-\alpha}, & m > \gamma\alpha, \end{cases}$$

where $\alpha = (n - 1)/2$. *Here the constant* C *depends on* n, m, *and* γ.

We shall point out a relation between the operators T_λ and S_λ. We choose $\gamma > 1$ and take $K(z) = |z|^{-(n-m-1)}$, $z \in \mathbb{R}^n \setminus \{0\}$, and let T_λ be defined as above. Then setting $x = (x', x_n)$, where $x' = (x_1, x_2, \ldots, x_{n-1})$ we obtain

$$T_\lambda f(x', 0) = \int_{\mathbb{R}^{n-1}} e^{i\lambda|x'-y'|^\gamma} \psi_0(x', 0, y') K(x' - y', 0) f(y') dy',$$

that is we obtain an operator of type S_λ. The reason for introducing the homogeneous function ω in the above definition of T_λ for $0 < \gamma \le 1$ is that we want certain determinant conditions to be satisfied. This is discussed in [1, p. 539], and in this paper after the proof of Lemma 2.2.

We shall also make some remarks on an operator which is somewhat similar to S_λ. Set

$$L(x) = \frac{e^{i|x|^a}}{|x|^\alpha}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

where $a > 0$, $a \neq 1$, and $\alpha < n$. Then L belongs to the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions and we set

$$Tf = L \star f, \quad f \in C_0^\infty(\mathbb{R}^n).$$

We say that the operator T is bounded on $L^p(\mathbb{R}^n)$ if

$$||Tf||_p \leq C_p ||f||_p, \quad f \in C_0^\infty(\mathbb{R}^n).$$

In Sjölin [5] the following theorem is proved.

Theorem 1.5 *If $\alpha \geq n(1 - a/2)$ set $p_0 = na/(na - n + \alpha)$. Then T is bounded on $L^p(\mathbb{R}^n)$ if and only if $p_0 \leq p \leq p_0'$. If $\alpha < n(1 - a/2)$ then T is not bounded on any $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$.*

We finally remark that Theorem 1.1 is due to Hörmander.

In Sect. 2 we shall give the proofs of Theorems 1.3 and 1.4. In Sect. 3 we shall discuss the sharpness of the results in these theorems.

2 Proofs of Theorems 1.3 and 1.4

We shall apply the following theorem.

Theorem 2.1 (see Hörmander [3], p. 3) *Let $\psi_1 \in C_0^\infty(\mathbb{R}^3)$, let $\varphi \in C^\infty(\mathbb{R}^3)$ be real-valued, and assume that the determinant*

$$\mathcal{J} = \begin{vmatrix} \varphi_{xt} & \varphi_{yt} \\ \varphi_{xtt} & \varphi_{ytt} \end{vmatrix} \neq 0$$

on suppψ_1. *Here $\varphi = \varphi(x, y, t)$ and $\varphi_{xt} = \frac{\partial^2 \varphi}{\partial x \partial t}$ etc. Set*

$$\mathcal{U}_N f(x, y) = \int_{\mathbb{R}} e^{iN\varphi(x,y,t)} \psi_1(x, y, t) f(t) dt, \ \ N \geq 1,$$

for $f \in L^1(\mathbb{R})$ and $(x, y) \in \mathbb{R}^2$. It follows that

$$||\mathcal{U}_N f||_{L^q(\mathbb{R}^2)} \leq C N^{-2/q} (q/(q-4))^{1/4} ||f||_{L^r(\mathbb{R})}$$

if $q > 4$ and $3/q + 1/r = 1$.

We shall need an estimate of the norm of $\mathcal{U}_N$ as an operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R}^2)$. We denote this norm by $||\mathcal{U}_N||_p$. An application of Theorem 2.1 will give the inequalities in the following lemma.

Lemma 2.2 *Let $\mathcal{U}_N$ be defined as in Theorem 2.1. Then one has*

$$||\mathcal{U}_N||_p \leq CN^{-\beta(p)},\ 1 < p < \infty,$$

where

$$\beta(p) = \begin{cases} 1 - 1/p, & 1 < p \leq 2, \\ 1/2 - \varepsilon, & 2 < p \leq 4, \\ 2/p, & 4 < p < \infty. \end{cases}$$

Here ε is an arbitrary positive number and C depends on φ and p, and in the case $2 < p \leq 4$, also on ε.

Proof Assume that $\text{supp}\psi_1 \subset B_2 \times B_1$, where B_1 is a ball in $\mathbb{R}$ and B_2 a ball in $\mathbb{R}^2$. We then have $\mathcal{U}_N f = \mathcal{U}_N(\mu f)$ if $\mu \in C_0^\infty(\mathbb{R})$ and $\mu(t) = 1$ for $t \in B_1$. Now take $q > 4$ and assume that $3/q + 1/r = 1$. It follows that $1 < r < 4$ and using Hölder's inequality twice and Theorem 2.1 we obtain

$$||\mathcal{U}_N f||_4 \leq C||\mathcal{U}_N f||_q = C||\mathcal{U}_N(\mu f)||_q \leq \\ CN^{-2/q}||\mu f||_r \leq CN^{-2/q}||\mu f||_4 \leq CN^{-2/q}||f||_4.$$

Hence

$$||\mathcal{U}_N f||_4 \leq CN^{\varepsilon - 1/2}||f||_4 \tag{2.1}$$

for every $\varepsilon > 0$, where the constant depends on ε. Then we shall obtain an L^2-estimate for the operator $\mathcal{U}_N$. From the condition on $\mathcal{J}$ in Theorem 2.1 it follows that there exists a number $\delta_0 > 0$ such that

$$\delta_0 \leq |\mathcal{J}| \leq C_0(|\varphi_{xt}| + |\varphi_{yt}|)$$

on $\text{supp}\psi_1$, where C_0 depends on φ.

Choose $\mu_j \in C_0^\infty(\mathbb{R}^3)$, $j = 2, 3, \ldots, M$, such that $\sum_2^M \mu_j(x, y, t) = 1$ for $(x, y, t) \in Q$ and each μ_j has support in a small cube. Here Q is a cube in $\mathbb{R}^3$ with center at the origin and $\text{supp}\psi_1 \subset Q$. It follows that

$$\psi_1 = \sum_2^M \psi_1 \mu_j = \sum_2^M \psi_j,$$

where $\psi_j = \psi_1\mu_j$. Setting

$$\mathcal{U}_N^{(j)} f(x, y) = \int_{\mathbb{R}} e^{iN\varphi(x,y,t)}\psi_j(x, y, t) f(t)dt$$

we have

$$\mathcal{U}_N = \sum_{j=2}^{M} \mathcal{U}_N^{(j)}$$

and shall estimate each $\mathcal{U}_N^{(j)}$.

If $(x_0, y_0, t_0) \in \text{supp}\psi_j$ then $(x_0, y_0, t_0) \in \text{supp}\psi_1$ and $|\varphi_{xt}| \geq \delta/2$ or $|\varphi_{yt}| \geq \delta/2$ at (x_0, y_0, t_0), where $\delta = \delta_0/C_0$. Say that $|\varphi_{xt}| \geq \delta/2$. Then $|\varphi_{xt}| \geq \delta/4$ on $\text{supp}\psi_j$ since $\text{supp}\psi_j$ is contained in a small cube.

Invoking Theorem 1.1 we get

$$\left(\int |\mathcal{U}_N^{(j)} f(x, y)|^2 dx\right)^{1/2} \leq CN^{-1/2}\left(\int |f(t)|^2 dt\right)^{1/2}$$

for every y. Integrating in y and summing over j we then obtain

$$||U_N f||_{L^2(\mathbb{R}^2)} \leq CN^{-1/2}||f||_{L^2(\mathbb{R})}. \tag{2.2}$$

Interpolating between the inequalities (2.1) and (2.2) one has

$$||\mathcal{U}_N f||_{L^p(\mathbb{R}^2)} \leq CN^{\varepsilon-1/2}||f||_{L^p(\mathbb{R})},\ 2 < p \leq 4 \tag{2.3}$$

for every $\varepsilon > 0$.

We then assume $q > 4$. Choosing μ as above we have $\mathcal{U}_N(f) = \mathcal{U}_N(\mu f)$ and it follows that

$$||\mathcal{U}_n f||_q \leq CN^{-2/q}||\mu f||_r \leq CN^{-2/q}||\mu f||_q \leq CN^{-2/q}||f||_q, \tag{2.4}$$

where we have used Hölder's inequality. It remains to study the case $1 < p < 2$. Interpolating between (2.2) and the trivial estimate $||\mathcal{U}_N f||_1 \leq C||f||_1$ one obtains

$$||\mathcal{U}_n f||_p \leq CN^{-(1-1/p)}||f||_p,\ 1 < p < 2, \tag{2.5}$$

and Lemma 2.2 follows from (2.2), (2.3), (2.4), and (2.5). □

Now let $\varphi(x, y, t) = d^\gamma$, where $d = ((x - t)^2 + y^2)^{1/2}$ and $0 < \gamma \leq 1$. A computation shows that

$$\mathcal{J} = \gamma^2(\gamma - 2)y\big((\gamma - 1)(x - t)^2 - y^2\big)$$

for $d = 1$. Since $\mathcal{J}$ is a homogeneous function of degree $2\gamma - 5$ of (x_0, y) where $x_0 = x - t$, we conclude that if $1/2 \le d \le 2$ and $|y| \ge c > 0$ on $\text{supp}\psi_1$, then $|\mathcal{J}| \ge c_1 > 0$ on $\text{supp}\psi_1$. Hence (2.2)–(2.5) hold in this case.

We remark that in the case $\gamma = 1$ $\mathcal{J}$ was computed in Carleson and Sjölin [2], and that in the case $\gamma = 1$ (2.2) and (2.3) are proved in [2] in the case $\psi_1(x, y, t) = \chi_1(t)\chi_2(x, y)$, where χ_1 is the characteristic function for the interval $[0, 1]$ and χ_2 is the characteristic function for the square $[0, 1] \times [2, 3]$. We shall now prove Theorem 1.3.

Proof of Theorem 1.3. We shall estimate the norm of T_λ where

$$T_\lambda f(x) = \int_{\mathbb{R}} e^{i\lambda|x-(y',0)|^\gamma} \psi_0(x, y') K\big(x - (y', 0)\big) f(y')dy',$$

where $x \in \mathbb{R}^2$. Here $\lambda \ge 2$, $0 < \gamma \le 1$, and $\psi_0 \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R})$. Also $K(z) = |z|^{m-1}\omega(z)$, $z \in \mathbb{R}^2 \setminus \{0\}$, where $0 < m < 1$ and ω is described in the introduction.

We first observe that there exists $\psi \in C_0^\infty(\mathbb{R}^2)$, with support in $\{x \in \mathbb{R}^2 : 1/2 \le |x| \le 2\}$ such that $K(z) = \sum_{k=-\infty}^{\infty} 2^{k(1-m)}\psi(2^k z)\omega(z)$ (see Stein [6, p. 393]). Since $\text{supp}\psi_0$ is bounded it follows that there exists an integer k_0 such that $K(z) = \sum_{k=k_0}^{\infty} 2^{k(1-m)}\psi(2^k z)\omega(z)$ for all $z = x - (y', 0)$ with $(x, y') \in \text{supp}\psi_0$. We shall assume that $k_0 = 0$. The proof in the general case is the same as for $k_0 = 0$. Also choose $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp}\chi \subset [-1/2 - 1/10, 1/2 + 1/10]$ and $\sum_{j=-\infty}^{\infty} \chi(t - j) = 1$.

We have $T_\lambda f = \sum_{k=0}^{\infty} T_{\lambda,k} f$ where

$$T_{\lambda,k} f(x) = \int_{\mathbb{R}} e^{i\lambda|x-(y',0)|^\gamma} \psi_0(x, y')2^{k(1-m)}\psi\big(2^k(x - (y', 0))\big)\omega(x - (y', 0)) f(y')dy',$$

Also $T_{\lambda,k} f = \sum_j T_{\lambda,k} f_j$ where $f_j(t) = f(t)\chi\big(2^k(t - 2^{-k}j)\big)$. Assuming $1 < p < \infty$ and invoking Hölder's inequality we obtain

$$|T_{\lambda,k} f(x)|^p \le C \sum_j |T_{\lambda,k} f_j(x)|^p,$$

since the number of terms in the above sum is bounded.

Setting $y' = 2^{-k}z'$ we get

$$\begin{aligned} &T_{\lambda,k} f_j(x) \\ &= \int_{\mathbb{R}} e^{i\lambda|x-(y',0)|^\gamma} 2^{k(1-m)}\psi_0(x, y')\psi\big(2^k(x - (y', 0))\big)\omega\big(x - (y', 0)\big) f_j(y')dy' \end{aligned}$$

$$= 2^{-mk} \int_{\mathbb{R}} e^{i\lambda|x-2^{-k}(z',0)|^{\gamma}} \psi_0(x, 2^{-k}z')\psi\big(2^k x - (z', 0)\big)\omega\big(x - 2^{-k}(z', 0)\big) f_j(2^{-k}z')dz'$$

$$= 2^{-mk} \int_{\mathbb{R}} e^{i\lambda 2^{-k\gamma}|2^k x-(z',0)|^{\gamma}} \psi_0(x, 2^{-k}z')\psi\big(2^k x - (z', 0)\big)\omega\big(2^k x-(z', 0)\big) f(2^{-k}z')\chi(z'-j)dz'$$

$$= [\text{with } y' = z' - j]2^{-mk} \int_{\mathbb{R}} e^{i\lambda 2^{-k\gamma}|2^k x-(y'+j,0)|^{\gamma}} \psi_0(x, 2^{-k}(y' + j))\psi(2^k x - (y' + j, 0))$$

$$\times\omega\big(2^k x - (y' + j, 0)\big) f(2^{-k}(y' + j))\chi(y')dy' = 2^{-mk} \int_{\mathbb{R}} e^{i\lambda 2^{-k\gamma}|2^k(x-(2^{-k}j,0))-(y',0)|^{\gamma}}$$

$$\times\psi_0(x, 2^{-k}j + 2^{-k}y')\psi\big(2^k(x - (2^{-k}j, 0)) - (y', 0)\big)\omega\big(2^k(x - (2^{-k}j, 0)) - (y', 0)\big)$$

$$\times f(2^{-k}j + 2^{-k}y')\chi(y')dy'.$$

We also have

$$\int_{\mathbb{R}^2} |T_{\lambda,k} f_j(x)|^p dx = [\text{with } x = u + (2^{-k}j, 0)]$$

$$\int_{\mathbb{R}^2} \left|T_{\lambda,k} f_j\big(u + (2^{-k}j, 0)\big)\right|^p du = [\text{with } \xi = 2^k u]$$

$$2^{-2k} \int_{\mathbb{R}^2} \left|T_{\lambda,k} f_j\big(2^{-k}\xi + (2^{-k}j, 0)\big)\right|^p d\xi. \tag{2.6}$$

Now let $\widetilde{\chi} \in C_0^{\infty}(\mathbb{R})$ be so that $\widetilde{\chi} = 1$ on suppχ and supp$\widetilde{\chi} \subset [-1, 1]$. We then have

$$T_{\lambda,k} f_j\big(2^{-k}\xi + (2^{-k}j, 0)\big) = 2^{-mk} \int_{\mathbb{R}} e^{i\lambda 2^{-k\gamma}|\xi-(y',0)|^{\gamma}} \psi_0(2^{-k}\xi$$

$$+(2^{-k}j, 0), 2^{-k}j + 2^{-k}y')\psi\big(\xi - (y', 0)\big)$$

$$\times\omega\big(\xi - (y', 0)\big) f(2^{-k}j + 2^{-k}y')\chi(y')\widetilde{\chi}(y')dy'$$

$$= 2^{-mk} \int_{\mathbb{R}} e^{i\lambda 2^{-k\gamma}\Phi(y',\xi)}\psi_1(y', \xi)g(y')dy'$$

$$= 2^{-mk}\mathcal{U}_{\lambda 2^{-k\gamma}} g(\xi),$$

where

$$\Phi(y', \xi) = |\xi - (y', 0)|^{\gamma} = (|\xi' - y'|^2 + \xi_2^2)^{\gamma/2},$$

$$\psi_1(y', \xi)=\psi\big(\xi - (y', 0)\big)\omega\big(\xi - (y', 0)\big)\psi_0(2^{-k}\xi+(2^{-k}j, 0), 2^{-k}j + 2^{-k}y')\widetilde{\chi}(y'),$$

and

$$g(y') = f(2^{-k}j + 2^{-k}y')\chi(y').$$

Here $\xi = (\xi_1, \xi_2) = (\xi', \xi_2)$.

It is clear that ψ_1 has a support which is uniformly bounded in j and k, and the derivatives of ψ_1 can be bounded uniformly in j and k. Here we use the fact that $k \geq 0$.

Invoking (2.6) we conclude that

$$\left(\int_{\mathbb{R}^2} |T_{\lambda,k} f_j(x)|^p dx\right)^{1/p} = 2^{-2k/p} 2^{-mk} \left(\int_{\mathbb{R}^2} |\mathcal{U}_{\lambda 2^{-k\gamma}} g(\xi)|^p d\xi\right)^{1/p}.$$

We set $d = (|\xi' - y'|^2 + \xi_2^2)^{1/2}$. It follows from the definitions of ψ and ω that $1/2 \leq d \leq 2$ and $|\xi_2| \geq c > 0$ on suppψ_1. Hence the determinant $\mathcal{J}$ for the phase function Φ satisfies $|\mathcal{J}| \geq c > 0$ on suppψ_1, as we remarked after the proof of Lemma 2.2. We can therefore apply Lemma 2.2 and one obtains

$$\left(\int_{\mathbb{R}^2} |\mathcal{U}_{\lambda 2^{-k\gamma}} g(\xi)|^p d\xi\right)^{1/p} \leq C(\lambda 2^{-k\gamma})^{-\beta(p)} ||g||_{L^p(\mathbb{R})}.$$

We have $g = g_{j,k}$ and

$$\int_{\mathbb{R}} |g_{j,k}|^p dy' \leq \int_{-1}^{1} |f(2^{-k}j + 2^{-k}y')|^p dy' = 2^k \int_{|z'| \leq 2^{-k}} |f(2^{-k}j + z')|^p dz'$$

and it follows that

$$\sum_{j=-\infty}^{\infty} \int_{\mathbb{R}} |g_{j,k}|^p dy' \leq C 2^k ||f||_p^p.$$

Hence

$$\int_{\mathbb{R}^2} |T_{\lambda,k} f|^p dx \leq C \sum_j \int_{\mathbb{R}^2} |T_{\lambda,k} f_j|^p dx \leq C 2^{-2k} 2^{-mkp} (\lambda 2^{-k\gamma})^{-\beta(p)p}$$
$$\sum_j \int_{\mathbb{R}} |g_{j,k}|^p dy' \leq C 2^{-k} 2^{-mkp} (\lambda 2^{-k\gamma})^{-p\beta(p)} ||f||_p^p$$

and we obtain the inequality

$$||T_{\lambda,k}||_p \leq C 2^{-k/p} 2^{-mk} (\lambda 2^{-k\gamma})^{-\beta(p)}.$$

Making a trivial estimate we also have

$$||T_{\lambda,k}||_p \leq C 2^{-k/p} 2^{-mk}.$$

Invoking the inequality $||T_\lambda||_p \leq \sum_0^\infty ||T_{\lambda,k}||_p$ we obtain

$$||T_\lambda||_p \leq C\lambda^{-\beta(p)} \sum_{2^k \leq \lambda^{1/\gamma}} 2^{k(-1/p-m+\gamma\beta(p))} + C \sum_{2^k \geq \lambda^{1/\gamma}} 2^{-k(1/p+m)} = A + B.$$

It is clear that $B \leq C\lambda^{-(1/p+m)/\gamma}$ and in the case $1/p + m < \gamma\beta(p)$ we get

$$A \leq C\lambda^{-\beta(p)}\lambda^{(-1/p-m+\gamma\beta(p))/\gamma} = C\lambda^{-(1/p+m)/\gamma}$$

and

$$||T_\lambda||_p \leq C\lambda^{-(1/p+m)/\gamma}.$$

In the case $1/p + m = \gamma\beta(p)$ we get $A \leq C\lambda^{-\beta(p)} \log\lambda$ and $||T_\lambda||_p \leq C\lambda^{-\beta(p)} \log\lambda$.

Finally, in the case $1/p + m > \gamma\beta(p)$ we have $A \leq C\lambda^{-\beta(p)}$ and $||T_\lambda||_p \leq C\lambda^{-\beta(p)}$.

We remark that in the case $p = 2$ only the case $1/p + m > \gamma\beta(p)$ can occur. The proof of Theorem 1.3 is complete. □

Before proving Theorem 1.4 we shall make a preliminary observation. Set $\xi = (\xi', \xi_n)$ where $\xi' = (\xi_1, \xi_2, \ldots, \xi_{n-1})$ and $n \geq 2$. Also set $x' = (x_1, x_2, \ldots, x_{n-1})$ and $\Phi(x', \xi) = d^\gamma$ where $\gamma > 0$ and $d = (|\xi' - x'|^2 + \xi_n^2)^{1/2}$. In [1, Section 4.1], we studied the determinant

$$P(x', \xi', \xi_n) = \det\left(\frac{\partial^2\Phi}{\partial x_i \partial \xi_j}\right)_{i,j=1}^{n-1}$$

for $1/2 \leq d \leq 2$. In [1] it is proved that

$$P(x', \xi', \xi_n) = (-\gamma d^{\gamma-2})^{n-1} \frac{(\gamma - 1)|\xi' - x'|^2 + \xi_n^2}{d^2}. \tag{2.7}$$

Now let $\Phi_1(x', \xi') = |\xi' - x'|^\gamma = d_1^\gamma$ where $d_1 = |\xi' - x'|$. We shall need the determinant

$$P_1(x', \xi') = \det\left(\frac{\partial^2\Phi_1}{\partial x_i \partial \xi_j}\right)_{i,j=1}^{n-1}.$$

It is clear that

$$P_1(x', \xi') = P(x', \xi', 0) = (-\gamma d_1^{\gamma-2})^{n-1}(\gamma - 1)$$

and for $\gamma > 0$, $\gamma \neq 1$, it follows that

$$|P_1(x', \xi')| \geq c > 0 \text{ for } 1/2 \leq d_1 \leq 2. \tag{2.8}$$

Proof of Theorem 1.4. We shall use the method in the proof of Theorem 1.3 and omit some details. We assume that

$$K(z) = \sum_{k=0}^{\infty} 2^{k(n-1-m)} \psi(2^k z),$$

where $\text{supp}\psi \subset \{x \in \mathbb{R}^{n-1},\ 1/2 \leq |x| \leq 2\}$. One obtains

$$S_\lambda f = \sum_{k=0}^{\infty} S_{\lambda,k} f$$

where

$$S_{\lambda,k} f(x) = \int_{\mathbb{R}^{n-1}} e^{i\lambda|x-y|^\gamma} \psi_0(x, y) 2^{k(n-1-m)} \psi\left(2^k(x-y)\right) f(y) dy.$$

We also have

$$f = \sum_{j \in \mathbb{Z}^{n-1}} f_j,$$

where

$$f_j(t) = f(t)\chi\left(2^k(t - 2^{-k}j)\right),\ j \in \mathbb{Z}^{n-1},\ t \in \mathbb{R}^{n-1},$$

and $\chi \in C_0^\infty(\mathbb{R}^{n-1})$ is like χ in the proof of Theorem 1.3.

The Schwarz inequality gives the estimate

$$|S_{\lambda,k} f(x)|^2 \leq C \sum_j |S_{\lambda,k} f_j(x)|^2$$

and arguing as in the proof of Theorem 1.3 we get

$$S_{\lambda,k} f_j(x) = 2^{-mk} \int_{\mathbb{R}^{n-1}} e^{i\lambda 2^{-k\gamma}|2^k(x-2^{-k}j)-y|^\gamma} \psi_0(x, 2^{-k}j + 2^{-k}y)$$
$$\psi(2^k(x - 2^{-k}j) - y) \times f(2^{-k}j + 2^{-k}y)\chi(y) dy$$

and

$$\int_{\mathbb{R}^{n-1}} |S_{\lambda,k} f_j(x)|^2 dx = 2^{-k(n-1)} \int_{\mathbb{R}^{n-1}} |S_{\lambda,k} f_j(2^{-k}\xi + 2^{-k}j)|^2 d\xi.$$

It follows that

$$\begin{aligned} S_{\lambda,k} f_j(2^{-k}\xi + 2^{-k}j) &= 2^{-mk} \int_{\mathbb{R}^{n-1}} e^{i\lambda 2^{-k\gamma}|\xi - y|^\gamma} \psi_0(2^{-k}\xi + 2^{-k}j, 2^{-k}j + 2^{-k}y) \\ &\quad \times \psi(\xi - y) f(2^{-k}j + 2^{-k}y)\chi(y)\widetilde{\chi}(y)dy \\ &= 2^{-mk}\mathcal{U}_{\lambda 2^{-k\gamma}} g(\xi) \\ &= 2^{-mk} \int_{\mathbb{R}^{n-1}} e^{i\lambda 2^{-k\gamma}\Phi_1(y,\xi)} \psi_1(y, \xi) g(y)dy \end{aligned}$$

where $\Phi_1(y, \xi) = |\xi - y|^\gamma$, $\psi_1(y, \xi) = \psi(\xi - y)\psi_0(2^{-k}\xi + 2^{-k}j, 2^{-k}j + 2^{-k}y)\widetilde{\chi}(y)$, and $g(y) = f(2^{-k}j + 2^{-k}y)\chi(y)$.

Invoking the determinant condition (2.8) and Theorem 1.1 we conclude that

$$||\mathcal{U}_{\lambda 2^{-k\gamma}} g||_{L^2(\mathbb{R}^{n-1})} \leq C(\lambda 2^{-k\gamma})^{-\alpha} ||g||_{L^2(\mathbb{R}^{n-1})}$$

where $\alpha = (n - 1)/2$. Arguing as in the proof of Theorem 1.3 we then obtain

$$||S_{\lambda,k}||_2 \leq C 2^{-mk} \lambda^{-\alpha} 2^{k\gamma\alpha}$$

and $||S_{\lambda,k}||_2 \leq C 2^{-mk}$.

Hence

$$||S_\lambda||_2 \leq C\lambda^{-\alpha} \sum_{2^k \leq \lambda^{1/\gamma}} 2^{(\gamma\alpha - m)k} + \sum_{2^k \geq \lambda^{1/\gamma}} 2^{-mk}$$

and Theorem 1.4 follows easily from this inequality. □

3 Counter-examples

Assume $\gamma > 0$, $1 < p < \infty$, and

$$T_\lambda f(x) = \int_{\mathbb{R}^{n-1}} e^{i\lambda|x-(y',0)|^\gamma} \psi_0(x, y') K\big(x - (y', 0)\big) f(y')dy',$$

where $x \in \mathbb{R}^n$, $n \geq 2$, and $K(z) = |z|^{m-n+1}$ with $0 < m < n - 1$. We shall estimate the norm $||T_\lambda||_p = ||T_\lambda||_{L^p(\mathbb{R}^{n-1}) \to L^p(\mathbb{R}^n)}$ from below. We take $y_0' \in \mathbb{R}^{n-1}$ and set $E = B(y_0'; c_0\lambda^{-\rho})$ where $B(x; R)$ denotes a ball with center x and radius R. Also let F denote a cube in $\mathbb{R}^n$ with center $(y_0', 100c_0\lambda^{-\rho})$ and side length $c_0\lambda^{-\rho}$. We assume that $\psi_0(x, y') = 1$ for $x \in F$ and $y' \in E$.

Setting $f = \chi_E$ and taking $x \in F$ we obtain

$$T_\lambda f(x) = \int_E K\big(x - (y', 0)\big)dy' + \int_E (e^{i\lambda|x-(y',0)|^\gamma} - 1)K\big(x - (y', 0)\big)dy'$$
$$= P(x) + R(x).$$

Setting $\rho = 1/\gamma$ we have

$$|e^{i\lambda|x-(y',0)|^\gamma} - 1| \leq \lambda|x - (y', 0)|^\gamma \leq Cc_0\lambda\lambda^{-\rho\gamma} = Cc_0, \; y' \in E,$$

and

$$|R(x)| \leq Cc_0 \int_E K\big(x - (y', 0)\big)dy'.$$

Now taking c_0 small we obtain

$$|T_\lambda f(x)| \geq c \int_E K\big(x - (y', 0)\big)dy' \geq c \int_E \lambda^{-\rho(m-n+1)}dy' = C\lambda^{-\rho m}$$

and

$$\int_F |T_\lambda f(x)|^p dx \geq c\lambda^{-\rho m}(\lambda^{-\rho n})^{1/p} = c\lambda^{-m/\gamma}\lambda^{-n/\gamma p}.$$

On the other hand

$$||f||_p = \left(\int_E dy'\right)^{1/p} = C\lambda^{-\rho(n-1)/p} = C\lambda^{-(n-1)/\gamma p}$$

and we have

$$||T_\lambda||_p \geq c\frac{\lambda^{-m/\gamma}\lambda^{-n/\gamma p}}{\lambda^{-(n-1)/\gamma p}} = c\lambda^{-m/\gamma}\lambda^{-1/\gamma p} = c\lambda^{-(1/p+m)/\gamma}. \tag{3.1}$$

The same proof works also in the case $K(z) = |z|^{m-n+1}\omega(z)$.

In Theorems 1.2 and 1.3 we proved estimates of the type

$$||T_\lambda||_p \leq C\lambda^{-(1/p+m)/\gamma}$$

and the inequality (3.1) shows that these estimates are sharp.

In Theorem 1.4 we proved the estimate

$$||S_\lambda||_2 \leq C\lambda^{-m/\gamma}. \tag{3.2}$$

We shall now prove that also this estimate is sharp. We shall use the same method as in the above counter-example.

We take x_0 and y_0 in $\mathbb{R}^{n-1}$ with $|x_0 - y_0| = 100c_0\lambda^{-\rho}$ and set $E = B(y_0; c_0\lambda^{-\rho})$ and $F = B(x_0; c_0\lambda^{-\rho})$. Here E and F are balls in $\mathbb{R}^{n-1}$. Setting $f = \chi_E$ and arguing as above one obtains

$$|S_\lambda f(x)| \geq c\lambda^{-\rho m} \text{ for } x \in F.$$

It follows that

$$||S_\lambda f||_2 \geq c\lambda^{-m/\gamma}\lambda^{-(n-1)/2\gamma}$$

and

$$||f||_2 = C\lambda^{-(n-1)/2\gamma}.$$

We conclude that

$$||S_\lambda||_2 \geq c\lambda^{-m/\gamma}$$

and it follows that (3.2) is sharp.

In Theorems 1.2 and 1.3 we have

$$T_\lambda f(x) = \int_{\mathbb{R}^{n-1}} e^{i\lambda\varphi(x,y')}\psi_0(x,y')K\big(x-(y',0)\big)f(y')dy'$$

where $x = (x', x_n)$ and $\varphi(x, y') = (|x' - y'|^2 + x_n^2)^{\gamma/2}$.

We let a denote the point $(0, 1) = (0, 0, \ldots, 0, 1)$ in $\mathbb{R}^n$. We assume that $\psi_0(x, y') = 1$ in a neighbourhood of $(a, 0)$ and let $f = \chi_B$ where $B = B(0; c_0\lambda^{-1})$ is a ball in $\mathbb{R}^{n-1}$. For x in a neighbourhood of a one obtains

$$T_\lambda f(x) = \int_B e^{i\lambda\varphi(x,y')}K\big(x-(y',0)\big)dy'.$$

It follows from the mean value theorem that

$$|\varphi(x, y') - \varphi(x, 0)| \leq Cc_0\lambda^{-1} \text{ for } y' \in B$$

and choosing c_0 small we obtain

$$|\lambda\varphi(x, y') - \lambda\varphi(x, 0)| \leq c_1 \text{ for } y' \in B,$$

where c_1 is small. It follows that there is no cancellation in the above integral and we get

$$|T_\lambda f(x)| \geq c_2\lambda^{-(n-1)}$$

in a neighbourhood of a. Hence

$$||T_\lambda f||_2 \geq c_3 \lambda^{-(n-1)}.$$

We have $||f||_2 = c_4 \lambda^{-(n-1)/2}$ and we obtain

$$\frac{||T_\lambda||_2}{||f||_2} \geq \frac{c_3 \lambda^{-(n-1)}}{c_4 \lambda^{-(n-1)/2}} = c_5 \lambda^{-(n-1)/2}.$$

Hence

$$||T_\lambda||_2 \geq c_5 \lambda^{-(n-1)/2} \tag{3.3}$$

and thus the estimates $||T_\lambda||_2 \leq C\lambda^{-(n-1)/2}$ in Theorems 1.2 and 1.3 are sharp.

We shall then construct a similar counter-example for the operator S_λ in Theorem 1.4. Here we have

$$S_\lambda f(x) = \int_{\mathbb{R}^{n-1}} e^{i\lambda\varphi(x,y)} \psi_0(x, y) K(x - y) f(y) dy, \ x \in \mathbb{R}^{n-1},$$

where $\varphi(x, y) = |x - y|^\gamma$. Take $a = (0, 0, \ldots, 0, 1)$ and assume that $\psi_0(x, y) = 1$ in a neighbourhood of $(a, 0)$. Also let $f = \chi_B$ where B is as in the previous counter-example. The same argument as above then gives the estimate $||S_\lambda||_2 \geq c\lambda^{-(n-1)/2}$ and it follows that the estimate $||S_\lambda||_2 \leq C\lambda^{-(n-1)/2}$ in Theorem 1.4 is sharp.

We shall then again consider the operator T_λ in Theorem 1.3. Here we have $n = 2$ and the above counter-example also gives

$$||T_\lambda||_p \geq \frac{||T_\lambda f||_p}{||f||_p} \geq c \frac{\lambda^{-1}}{\lambda^{-1/p}} = c\lambda^{-(1-1/p)}$$

for $1 \leq p < 2$. It follows that the estimate

$$||T_\lambda||_p \leq C\lambda^{-\beta(p)}$$

for $1 < p < 2$ in Theorem 1.3 is sharp (since $\beta(p) = 1 - 1/p$).

In Theorem 1.3 we have

$$T_\lambda f(x, y) = \int_{\mathbb{R}} e^{i\lambda\varphi(x,y,t)} \psi_0(x, y, t) K(x - t, y) f(t) dt, \ (x, y) \in \mathbb{R}^2,$$

where $\varphi(x, y, t) = \big((x - t)^2 + y^2\big)^{\gamma/2}$ and $K(z) = |z|^{m-1}\omega(z)$.

Setting

$$T_\lambda^* g(t) = \int_{\mathbb{R}^2} e^{-i\lambda\varphi(x,y,t)} \overline{\psi_0(x, y, t)} K(x - t, y) g(x, y) dx dy, \ t \in \mathbb{R},$$

we get

$$(T_\lambda f, g)_2 = (f, T_\lambda^* g)_1, \ f \in C_0^\infty(\mathbb{R}), \ g \in C_0^\infty(\mathbb{R}^2),$$

where $(,)_2$ and $(,)_1$ denote the inner products in $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R})$. It follows that

$$||T_\lambda||_p = ||T_\lambda||_{L^p(\mathbb{R})\to L^p(\mathbb{R}^2)} \geq ||T_\lambda^*||_{L^r(\mathbb{R}^2)\to L^r(\mathbb{R})}$$

where $1/p + 1/r = 1$. We shall use this inequality for $4 \leq p < \infty$.

Let B denote a disc in $\mathbb{R}^2$ with center $(0, 1)$ and radius $c_0\lambda^{-1}$. Take $g \in C_0^\infty(\mathbb{R}^2)$ with support in B, $0 \leq g \leq 1$, and $g = 1$ in $\frac{1}{2}B$. Then

$$||g||_r \leq \left(\iint_B dxdy \right)^{1/r} = c\lambda^{-2/r}$$

and choosing ψ_0 such that $\psi_0(x, y, t) = 1$ in a neighbourhood of $(0, 1, 0)$ we get

$$|T_\lambda^* g(t)| \geq c\lambda^{-2}$$

in a neighbourhood of 0. Hence

$$||T_\lambda^* g||_r \geq c\lambda^{-2}$$

and

$$||T_\lambda^*||_r \geq \frac{||T_\lambda^* g||_r}{||g||_r} \geq c\frac{\lambda^{-2}}{\lambda^{-2/r}} = c\lambda^{-2(1-1/r)}.$$

Since $1 - 1/r = 1/p$ we conclude that

$$||T_\lambda||_p \geq c\lambda^{-2/p}, \ 4 \leq p < \infty$$

and it follows that the estimate

$$||T_\lambda||_p \leq C\lambda^{-\beta(p)}, \ 4 < p < \infty,$$

in Theorem 1.3 is sharp (since $\beta(p) = 2/p$).

In Theorem 1.3 we also have an estimate of the type

$$||T_\lambda||_p \leq C\lambda^{-1/2+\varepsilon}$$

for $2 < p < 4$. We shall finally discuss the sharpness of this estimate in the case $\gamma = 1$. We shall study the statement

$$||T_\lambda||_p \leq C\lambda^{-1/2-\delta} \text{ for some } p \text{ with } 2 < p < 4 \text{ and some } \delta > 0. \qquad (3.4)$$

Omitting details we shall describe how (3.4) leads to a contradiction.
Following Stein [6], p. 393, we have

$$\frac{1}{|x|^{3/2}} = u(x) + \sum_{k=1}^{\infty} 2^{-3k/2}\psi\left(\frac{x}{2^k}\right),\ x \in \mathbb{R}^2 \setminus \{0\},$$

where $u \in L^1(\mathbb{R}^2)$, ψ is smooth, and supp$\psi \subset \{x \in \mathbb{R}^2;\ 1/2 \leq |x| \leq 2\}$. We set

$$K_0(x) = \frac{e^{i|x|}}{|x|^{3/2}} = e^{i|x|}u(x) + \sum_{k=1}^{\infty} 2^{-3k/2}e^{i|x|}\psi(x/2^k),\ x \in \mathbb{R}^2 \setminus \{0\},$$

and $S_0 f = K_0 \star f$. We define the operator V_k by setting

$$V_k f = 2^{-3k/2}2^{2k}(e^{i2^k|x|}\psi) \star f = 2^{k/2}(e^{i2^k|x|}\psi) \star f = \lambda^{1/2}(e^{i\lambda|x|}\psi) \star f,$$

where $\lambda = 2^k$. Using (3.4) we can prove that

$$||V_k||_p = ||V_k||_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)} \leq C\lambda^{-\delta} = C2^{-k\delta},$$

and the inequality

$$\sum_{k=1}^{\infty} ||V_k||_p < \infty$$

implies that S_0 is a bounded operator on $L^p(\mathbb{R}^2)$. It follows that the characteristic function of the unit disc is a Fourier multiplier for $L^p(\mathbb{R}^2)$. This contradicts Fefferman's multiplier theorem.

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