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# A METRIC FIXED POINT THEOREM AND SOME OF ITS APPLICATIONS

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**Abstract.** A general fixed point theorem for isometries in terms of metric functionals is proved under the assumption of the existence of a conical bicombing. It is new for isometries of convex sets of Banach spaces as well as for non-locally compact CAT(0)-spaces and injective spaces. Examples of actions on non-proper CAT(0)-spaces come from the study of diffeomorphism groups, birational transformations, and compact Kähler manifolds. A special case of the fixed point theorem provides a novel mean ergodic theorem that in the Hilbert space case implies von Neumann's theorem. The theorem accommodates classically fixed-point-free isometric maps such as those of Kakutani, Edelstein, Alspach and Prus. Moreover, from the main theorem together with some geometric arguments of independent interest, one can deduce that every bounded invertible operator of a Hilbert space admits a nontrivial invariant metric functional on the space of positive operators. This is a result in the direction of the invariant subspace problem although its full meaning is dependent on a future determination of such metric functionals.

## 1 Introduction

Brouwer's fixed point theorem from 1912 as well as the infinite dimensional extension that Schauder proved around 1930 have many applications. These theorems require compactness, as Kakutani's 1943 examples of fixed point free $(1+\epsilon)$-Lipschitz maps of the closed unit ball in Hilbert spaces illustrated. For isometries there is an influential paper [BM48] by Brodskii and Milman proving conditions for the existence of fixed points. For 1-Lipschitz maps, also called nonexpansive maps, there are fixed point theorems of Browder, Göhde and Kirk [Bro651, Bro652, Goh65, Kir65] without compactness assumption but instead requiring certain good properties of the Banach spaces. When the maps are uniformly strict contractions, there is the simple but fundamental contraction mapping principle, abstractly formulated in Banach's thesis from 1920, applicable in any complete metric space and providing a unique fixed point. On the other hand, for isometric maps Kakutani gave examples of the closed unit ball in certain Banach spaces without fixed points (see below). An example of Alspach [Dal81] showed moreover that for isometries even with weak compactness of the convex set, there may not be any fixed point. See the handbook [KS01] for more details on these topics.

The main purpose of this paper is to establish an entirely new type of fixed point theorem. It could have been discovered decades ago and is in several ways more general than existing fixed point theorems for individual isometries. In particular it is new for isometries of convex sets of Banach spaces. It has wide applicability as is testified below with corollaries in ergodic theory, operator theory, geometric group theory, dynamical systems, complex analysis, and other topics. In some cases working out the exact consequences is beyond the scope of this paper and may require further study.

Let $(X, d)$ be a metric space. A metric space is *proper* if every closed bounded set is compact. We call a map $T : X \to X$ *isometric*, or *isometric embedding*, if it preserves distances. It is an *isometry* in case it has an (isometric) inverse, which amounts to $T$ being a surjective isometric map.

Let $x_0$ be a point of $X$. We map $X$ into the space $\mathrm{Hom}(X, \mathbb{R})$ of nonexpansive functions $X \to \mathbb{R}$ with the topology of pointwise convergence,

$$\Phi : X \to \mathrm{Hom}(X, \mathbb{R})$$

defined via

$$x \mapsto h_x(\cdot) := d(\cdot, x) - d(x_0, x).$$

This is a continuous injective map and the closure $\overline{\Phi(X)}$ is compact. We call $\overline{X} := \overline{\Phi(X)}$ the *metric compactification of* $X$ (following the terminology of [Rie02] for proper spaces) and we call its elements *metric functionals*. (The choice of topology here is different from the one taken in [Gro78] where it is uniform convergence on bounded sets. The limit functions in this other topology are here called *horofunctions*.)

The action of isometries on $X$ extends to an action on $\overline{X}$ by homeomorphisms, via

$$(Th)(x) = h(T^{-1}x) - h(T^{-1}x_0).$$

Alternatively one can consider everything up to additive constants. This is related to the fact that the compactification is independent of $x_0$.

A bicombing is a selection of a geodesic between every two points (*distinguished geodesics*). More precisely, a *weak conical bicombing* is a map $\sigma : X \times X \times [0, 1] \to X$ such that $\sigma(x, y, \cdot)$ is a constant speed geodesic from $x$ to $y$ such that

$$d(\sigma_{xy}(t), \sigma_{xy'}(t)) \le td(y, y')$$

for all $x, y, y' \in X$ and $0 \le t \le 1$.

Examples of spaces admitting such bicombings include Banach spaces, and convex subsets thereof, complete CAT(0)-spaces and injective metric spaces. In addition, a particularly interesting specific example is Pos, the space of positive bounded linear operators of a Hilbert space, explained more below, and on which every invertible

bounded linear operator acts by isometry. The 1- Kantorovich-Wasserstein distance on spaces of probability measures and the Weil-Petersson metric on Teichmüller spaces are two further examples.

**Theorem 1.** *Let $X$ be a metric space with a weak conical bicombing. Let $T : X \to X$ be an isometric embedding. Then there is an element $h \in \overline{X}$ such that*

$$h(Tx) = h(x) - d$$

*for all $x$ and where $d = \inf_x d(x, Tx)$. In the case $h = h_y$ for some point $y \in X$ then $Ty = y$. If $T$ is an isometry, then*

$$Th = h.$$

An inequality for nonexpansive maps was previously known from Gaubert and Vigeral's paper [GV12], and even earlier from [Kar01] with a weaker but more general statement. For Banach spaces and in terms of linear functionals an inequality is also proved in [KN81], but the use of linear functionals instead of metric functionals gives a strictly weaker result as can be inferred from an example by Mertens in that same paper. The simple case of the theorem when $d = 0$ was observed by Gromov in [Gro78], see also Proposition 13 below. We call $h$ *invariant* under $T$ when the first equality in Theorem 1, $h(Tx) = h(x) - d$, holds for all $x \in X$, and we say that $h$ is a *fixed point* of $T$ when $T$ acts on the compactification $\overline{X}$ and $Th = h$. Note that some convexity or bicombing property is necessary (as is the case for Brouwer, Schauder et al) since a nontrivial rigid rotation of a circle has no fixed point and the circle is a compact space, it means that there are also no fixed metric functional. In contrast:

COROLLARY 2. *Any isometry $T$ of a metric space as in the theorem and with $\inf_x d(x, Tx) > 0$ must have infinite order, in fact $d(x, T^n x) \to \infty$ for any $x$.*

For comparison, Basso proved a fixed point theorem for groups of isometries of spaces with conical bicombings with a midpoint property, in a more traditional sense and assuming compactness [Bas18].

*Banach spaces.*    Here is a first application. Let $U$ be a norm-preserving linear operator, that is $\|Uv\| = \|v\|$ for all $v$ in a Banach space $X$, where as usual all line segments provide the required bicombing. In this setting the metric fixed point theorem proves a new mean ergodic theorem:

COROLLARY 3. (Mean ergodic theorem in Banach spaces) *Let $U$ be a norm-preserving linear operator of a Banach space $X$ and $v \in X$. Then there exists a metric functional $h$ such that*

$$\frac{1}{n} h \left( \sum_{k=0}^{n-1} U^k v \right) = - \inf_x \|Ux + v - x\|$$

*for all $n \geq 1$.*

In case of a Hilbert space one can deduce the mean ergodic theorem proved by von Neumann using spectral theory, in a very different way:

COROLLARY 4. (von Neumann, 1931) *Let $U$ be a unitary operator of a Hilbert space $H$ and $v \in H$. Then the strong limit*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k v = \pi_U(v),$$

*where $\pi_U$ is the projection onto the $U$-invariant vectors.*

(See Sect. 4 for the proofs.) The mean ergodic theorem can thus be viewed as a fixed point theorem. The classical formulation (the convergence part in Corollary 4) is known to fail for general Banach spaces, such as $\ell^\infty(\mathbb{N})$ and $\ell^1(\mathbb{N})$ for shift operators, while Corollary 3 always holds true. An abstract characterization for when the classical version holds is the content of the *mean ergodic theorem in Banach spaces* of Yosida-Kakutani in [YK41]. There are many papers on the topic of which operators are mean ergodic, see [Eis10] for a good exposition. See Sect. 4 for a new mean ergodic theorems for power bounded operators.

Theorem 1 applies in particular to any closed convex set $X$ of a Banach space, and provides fixed points in a generalized sense. Note that it is known that the Mazur-Ulam theorem fails in general, that is, not every isometry of a convex set is affine, see the examples of Schechtman in [Sch]. We now relate Theorem 1 to well-known conventionally fixed-point-free isometric maps.

EXAMPLE (Kakutani). The map $T(x_1, x_2, \ldots) = (1, x_1, x_2, \ldots)$ is an isometric fixed-point-free map of the closed unit ball in the sequence space $c_0$ with the supremum norm and all sequences tending to 0. The metric functional of the theorem for this map is $h(x_1, x_2, \ldots) = \sup_k |x_k - 1| - 1$.

EXAMPLE The isometric map $T(x_1, x_2, \ldots) = (1, x_1, x_2, \ldots)$ of either $\ell^1(\mathbb{N})$ or $\ell^2(\mathbb{N})$, clearly has no fixed point in the usual sense. On the other hand, in the $\ell^1(\mathbb{N})$ case it leaves the metric functional associated to $(1, 1, 1, \ldots)$ invariant ([GK21])

$$h(x_1, x_2, \ldots) = \sum_{k=1}^{\infty} |x_k - 1| - 1,$$

and in the $\ell^2(\mathbb{N})$ case the trivial metric functional $h \equiv 0$ is invariant.

EXAMPLE (Edelstein [Ede64]). A more sophisticated example in the Hilbert space setting was given by Edelstein. It is an isometry with unbounded but recurrent orbits. The trivial metric functional is fixed. This map is an isometry also of $\ell^1(\mathbb{N})$ in which case the metric functional in the previous example is fixed also for this map, as shown in [Pen21]. In this context, note that by [Gut19], the linear functional $f(x) \equiv 0$ is not a metric functional of $\ell^1(\mathbb{N})$.

EXAMPLE (Alspach [Dal81]). This example is essentially what is called the Baker's transformation in ergodic theory. The convex set $X$ in $L^1([0,1])$ consists of the integrable functions taking values in $[0,2]$ of integral 1. The isometric map is $Tf(t)$ is $\min\{2, 2f(2t)\}$ for $0 \leq t \leq 1/2$ and $\max\{0, 2f(2t-1)-2\}$ for $1/2 < t \leq 1$. Gutièrrez noticed an invariant metric functional of this map in [Gut20].

EXAMPLE (Shift in $\ell^1$). Bader, Gelander, and Monod [BGM12] proved an unexpected fixed point theorem for $L^1$ spaces. In the traditional sense there may not be any fixed points of isometries preserving a bounded closed convex set, but they showed that there is a fixed point which may lie outside the set. An example, mentioned in the introduction of [BGM12], is the shift map $T$ on $\ell^1(\mathbb{Z})$ and the convex bounded set being all the non-negative functions whose values sum to 1. The only fixed point of this isometry is 0. From the viewpoint of the present article, given any two points $x$ and $y$, applying the shift enough times on one of them, the $\ell^1$-distance $d(x, T^n y)$ is approaching $\|x\| + \|T^n y\| = \|x\| + \|y\|$. This shows that any orbit converges to the metric functional $h(x) = \|x\| = h_0(x)$, which is the metric functional of Theorem 1. Thus $T0 = 0$ recovering the obvious fixed point in this case.

EXAMPLE (Basso [Bas18]). This is an example of an isometry of a bounded complete convex subset of a strictly convex Banach space without a fixed point. The map is the shift map $T$ on $\ell^1(\mathbb{Z})$ which is renormed with an equivalent norm that is strictly convex and $T$ still an isometry. The closed convex hull $C$ of the vectors which has 1 in one coordinate and 0 elsewhere has T as an isometry. The only fixed point in the usual sense is clearly the zero-vector which lies outside of $C$. Let $x_0$ be the vector that is 1 in the 0th coordinate and 0 elsewhere. The metric functional that is the limit of $h_{T^n x_0}$ as $n \to \infty$,

$$h(x) = \sqrt{(\|x\|_1 + 1)^2 + \|x\|_2^2 + 1} - 1$$

is fixed by $T$.

*CAT(0)-spaces.* When $X$ is a complete CAT(0)-space, geodesics are unique and provide a conical bicombing almost directly from definitions. Such spaces have a standard bordification adding equivalence classes of geodesic rays, sometimes called the *visual bordification,* or the *visual compactification* in case $X$ is proper, see [BH99]. In Gromov's choice of topology [Gro78] on $\mathrm{Hom}(X, \mathbb{R})$, the closure of $\Phi(X)$ above is the visual bordification [Gro78, BH99, Theorem 8.13]. In particular, for nonproper $X$ the visual boundary may be empty, but when proper it coincides with the metric compactification. Immediately from Theorem 1 we therefore get:

COROLLARY 5. *Let $X$ be a complete $CAT(0)$-space. Then any isometry of $X$ fixes a point in $\overline{X}$. In case $X$ is proper, any isometry fixes a point in the visual compactification.*

In the proper case this is a standard fact based on the classification of isometries, the elliptic part going back to E. Cartan. This is of fundamental importance for the

theory [BGS85]. In the special case of finite dimensional Cartan-Hadamard manifolds, the compactification is a closed ball in this case and one could instead appeal to Brouwer's fixed point theorem. The present paper provides a different proof of this, and in fact the first equality in Theorem 1 gives more precise information.

In the nonproper case, as explained in [BH99] there are isometries that have no fixed points in the visual bordification. For example take $\ell^2(\mathbb{Z})$ and consider the shift and adding 1 to the 0th coordinate. Edelstein's isometry mentioned above has $d = 0$ (for example since the orbit does not tend to infinity, thus $\tau(f) = 0$, see below for the definition). This is no contradiction since for Hilbert spaces the function $h \equiv 0$ is a metric functional [Kar22, Gut19]. On the other hand, if $d > 0$ the fixed metric functional is nontrivial. See also Theorem 21 below for more information. This is also the case for CAT(-1)-spaces in general since it is Gromov hyperbolic and for such spaces it follows from Maher-Tiozzo [MT18] that $h \equiv 0$ is *not* a metric functional even without local compactness. For isometries of CAT(-1)-spaces previous results can be found in [Kar01, KN04, Theorem 5]. Claassens [Cla20] has completely determined the metric functionals for the infinite dimensional real hyperbolic space.

Important examples of isometric actions on nonproper CAT(0)-space are provided by the Cremona groups of birational transformations in algebraic geometry [Can11, LU21].

*Injective metric space.*    Metric space that are hyperconvex, or equivalently injective, have properties which generalizes $L^\infty$ and trees. One important feature of such spaces is that they admit a conical bicombing as shown in [Lan13, Prop. 3.8]. A result from 1979 by Sine and Soardi shows that every nonexpansive self-map of a bounded injective metric space has a fixed point, see [KS01, Ch. 13]. Baillon asked whether the boundedness could be relaxed to just assuming that the orbit is bounded, but a counterexample was found by Prus [KS01].

EXAMPLE (Prus). Consider the hyperconvex Banach space $X = \ell^\infty(\mathbb{N})$ and the map $T : X \to X$ defined by $T((y_n)) = (1 + \lim y_n, y_0, y_1, \ldots)$ where the limit is an ultrafilter or Banach limit. This map is isometric with bounded orbits but no fixed point in $X$. This is in contrast with Theorem 1 that provides an invariant metric functional for $T$. Moreover it must be nontrivial, since as shown below in Proposition 22, no metric functional for $\ell^\infty(\mathbb{N})$ is trivial, that is, the linear functional $f(x) \equiv 0$ is not a metric functional.

Once again, Theorem 1 provides the missing fixed point as it were:

COROLLARY 6. *Let $X$ be an injective metric space. Then every isometry of $X$ fixes a point in $\overline{X}$. In case $X$ is proper, any isometry fixes a point in the visual compactification.*

Details for how to deduce the second assertion of the corollary, which is very easy, will be given below. A new topic in geometric group theory is the subject of Helly groups, which are groups admitting geometric isometric actions on injective metric

spaces [HO21]. Note that, although similar to the CAT(0)-case above, the boundaries (metric vs visual) are now different even for proper spaces, the case of $\ell^\infty$ illustrates this as is shown below. As Basso informed me there is a fixed point theorem by U. Lang for isometry groups of bounded orbits in [Lan13]. For Busemann spaces one should mention Navas paper [Nav13] in this context.

Every metric space $X$ has an (essentially) unique injective hull $E(X)$ into which it is isometrically embedded [Isb64], and every isometry $f$ of $X$ extends to an isometry of $E(X)$. This is immediate from the defining properties, see [Lan13, Prop. 3.7]. Therefore we may conclude:

**Theorem 7** (Isometry fixed point theorem). *Let $f$ be an isometry of a metric space $X$. Then $f$ has a fixed point in $\overline{E(X)}$.*

In this way, even a nontrivial rigid rotation of the circle has an associated fixed point.

*Diffeomorphims and biholomorphisms.* It is well-known that the Weil-Petersson metric of the Teichmüller space of a closed orientable surface has nonpositive curvature, is not complete but is geodesically convex. Therefore:

COROLLARY 8. *Any element in the mapping class group of a genus g surface fixes a metric functional of the corresponding Teichmüller space equipped with the Weil-Petersson metric.*

Fixed points of mapping classes were already well-understood from Thurston's compactification of Teichmüller spaces, using Brouwer's theorem.

The space of all Riemannian metrics Met($M$) of a manifold $M$ is a space first introduced by DeWitt in general relativity and studied early on by Ebin in the 1960s. The diffeomorphisms group acts on it by isometry.

COROLLARY 9. *Let $f$ be a diffeomorphism of a closed orientable compact manifold $M$. Then $f$ fixes a metric functional of Met($M$).*

*Proof.* It is known since some time that this space has features of nonpositive curvature with a good description of what the geodesics are. The version that applies best for us here is the theorem in [Cla13] that asserts that the metric completion is a CAT(0)-space. From this the main fixed point theorem applies directly taking into account Lemma 15 below. □

A certain space of Kähler potentials of a connected compact Kähler manifold was introduced by Calabi in the 1950s. It has a Weil-Petersson type metric and the following holds for the same reasons as the previous corollary:

COROLLARY 10. *Any complex automorphism of a compact Kähler manifold fixes a metric functional of the associated space of Kähler potentials.*

The completed space which is a CAT(0)-space was used for progress on a conjecture of Donaldson on the asymptotic behavior of the Calabi flow, see [Bac] for a

discussion and references on this topic. The Calabi flow is in fact nonexpansive in this metric and when there is no constant scalar curvature metric on the underlying Kähler manifold, the flow may diverge. Note that the theorems in [KM99, Kar01] (compare with Theorem 21 below) apply to (the time-one map of) the Calabi flow. These results provide a geodesic ray or metric functional and may thus be relevant for Donaldson's conjecture.

Holomorphic maps are always nonexpansive in the Kobayashi pseudo-metric and thus biholomorphisms are isometries. The fixed point theorem in this case will be treated in a forthcoming paper since some additional arguments are needed in this case.

In these examples, from algebraic geometry, topology and complex geometry, the metric fixed point theorem provides something new and nontrivial, at least in the case that $d > 0$, and the metric compactification in these cases deserves to be studied.

*The invariant subspace problem.* The invariant subspace problem, open since the 1950s, asks whether every bounded linear operator of a separable infinite dimensional complex Hilbert space has a nontrivial closed invariant subspace. In finite dimensions the statement follows for example from the fundamental theorem of algebra, and von Neumman, who is one of the first to have considered this problem, proved it in the case of compact operators. Without loss of generality one may assume that the operator is invertible otherwise one can add a multiple of the identity map, which does not change the invariant subspaces.

The set of bounded positive operators Pos of a Hilbert space $H$ has a natural metric which admits a weak conical bicombing. Moreover every invertible bounded linear operator $g : H \to H$ acts by isometry on Pos via $p \mapsto gpg^*$. Applying Theorem 1 we get:

**Theorem 11.** *Every bounded, invertible linear operator $g$ of a Hilbert space has an invariant nonconstant metric functional $h : \mathrm{Pos} \to \mathbb{R}$,*

$$gh = h.$$

*Moreover, the following equality holds*:

$$\inf_{p \in \mathrm{Pos}} \left| \log \sup_{v \neq 0} \frac{(pg^*v, g^*v)}{(pv, v)} \right| = \lim_{n \to \infty} \frac{1}{n} \left| \log \|g^n g^{*n}\| \right|.$$

This presents a new approach to the invariant subspace problem and a novel viewpoint of operator theory more generally. In finite dimensions the statement above implies the existence of a nontrivial invariant subspace as follows, either the operator is unitary with respect to some scalar product or from the explicit determination of the metric functionals in Lemmens' paper [Lem] there is an associated closed linear subspace (the kernel of a semipositive operator), which is nontrivial in case the dimension is at least two and that is invariant.

The fact that the metric functional is nontrivial comes from the following statement of independent interest. This is in contrast with linear functionals and weak limits that can be just 0.

PROPOSITION 12. *Every metric functional of the space* Pos *is an unbounded function.*

In conclusion, Theorem 11 shows that a general bounded linear operator does have some structure. Carleson once speculated that the invariant subspace problem in the Hilbert space case perhaps is a fixed point theorem on some Grassmannian [CJ02]. Even though this is not achieved in the present paper, the author believes that the metric perspective on operator theory is of future interest more generally.

One suggestion could be to investigate the validity of an extension of the Ryll-Nardzewski fixed point theorem for affine isometry groups and see how it relates to Dixmier's unitarization problem. As Monod pointed out to me years ago, a uniformly bounded representation of a group corresponds to an isometric action with bounded orbits in Pos and the equivalence to a unitary representation is precisely the existence of a fixed point in Pos (that fixed points may not exist, in contrast to CAT(0)-spaces, is also coherent with Basso's example mentioned above). It follows therefore from [dN47] that if $g$ is an invertible, bounded linear operator with bounded orbits in Pos, then it fixes a point in Pos. Some groups are non-unitarizable, so once again the metric functionals are potentially an appropriate device. This topics connects to the power-bounded mean ergodic theorem in Sect. 4, since the latter corresponds to uniformly bounded representations of $\mathbb{Z}$ (or $\mathbb{N}$).

One area where spaces of positive operators have already been useful is multiplicative ergodic theorems for operators. The result [KM99, Corollary 7.1] is a substantial and surprising strengthening of an important theorem of Ruelle [Rue82] under an extra Hilbert-Schmidt condition that allows using a submanifold of Pos that is CAT(0). A general result is [GK20, Theorem 1.7] and in [BHL21] several new multiplicative ergodic theorems are obtained using finite traces and a study of the corresponding spaces of positive operators (again CAT(0) here).

## 2    Metric preliminaries

We consider the metric category, that is, objects are metric spaces and morphims are nonexpansive maps, which means that

$$d(f(x), f(y)) \leq d(x, y)$$

for all $x$ and $y$. The composition of nonexpansive maps remains nonexpansive. Isometries are the isomorphisms in this category.

One defines the *minimal displacement*

$$d = d_f = \inf_x d(x, f(x))$$

and the *translation number*

$$\tau = \tau_f = \lim_{n \to \infty} d(x, f^n(x))/n.$$

These numbers are analogs of the operator norm and spectral radius, respectively, in particular note that $\tau \leq d$. To see this, since $\tau$ is independent of the point $x$ we may take a point close to the infimal displacement in the sense that $d(x, f(x)) < d + \epsilon$. By the triangle inequality, $d(x, f^n(x)) < n(d + \epsilon)$, so $\tau < d + \epsilon$ for any $\epsilon > 0$ and hence the inequality. Rigid rotations of a circle show that the inequality can be strict. On the other hand, note that when Theorem 1 applies then necessarily $\tau = d$, since by the theorem and in view of that metric functionals are nonexpansive,

$$d(x, T^n x) \geq |h(x) - h(T^n x)| \geq nd.$$

This was already known from [GV12] and in the nonpositive curvature case for isometries from Gromov.

Let $\mathrm{Hom}(X, \mathbb{R})$ be the space of morphisms $X \to \mathbb{R}$ (*fonctionnelles* in [Ban55]) equipped with the topology of pointwise convergence. Given a base point $x_0$ of the metric space $X$, let

$$\Phi : X \to \mathrm{Hom}(X, \mathbb{R})$$

be defined via

$$x \mapsto h_x(\cdot) := d(\cdot, x) - d(x_0, x).$$

This is a continuous injective map and the closure $\overline{\Phi(X)}$ is compact, say the Arzela-Ascoli or Tychonoff's theorems. This is the metric space analog of the Banach–Alaoglu theorem. We call $\overline{X} := \overline{\Phi(X)}$ the *metric compactification of $X$* and its elements *metric functionals.* These are nonexpansive and satisfy a *metric Hahn-Banach theorem* [Kar211]. Note that $h(x_0) = 0$ for any $h \in \overline{X}$. There is a growing list of studied examples [Wal18, KL18, H+, MT18, Gut19, Cla20, LP23, AK22] to cite a few recent papers. It is easy to see that for Banach spaces metric functionals are convex functions. In [B+22] the same topology is used as here for nonproper spaces, and that paper contains a careful discussion of this construction.

Busemann observed in the early part of the 20th century that any geodesic ray $\gamma$ defines a metric functional: $b_\gamma(y) = \lim_{t \to \infty} d(y, \gamma(t)) - t$, called the *Busemann function* of $\gamma$. This is akin to how a vector defines a linear functional via a scalar product. Busemann functions have been useful for a long time in the theories of spaces with nonpositive respectively nonnegative curvature. The usefulness of these metric notions without curvature conditions is illustrated by the work described in [KL11].

Note the effect of changing base point $x_0$ to $y_0$:

$$d(y, x) - d(x_0, x) = d(y, x) - d(y_0, x) + d(y_0, x) - d(x_0, x).$$

It amounts in other words to just adding the constant $h_x(y_0)$.

Isometries act by homeomorphisms of $\overline{X}$ via

$$(Th)(x) = h(T^{-1}x) - h(T^{-1}x_0).$$

Note that if for some metric functional $h$ one has that $h(Tx) = h(x) + c$ (or which is the same $h(T^{-1}x) = h(x) - c$) for all $x \in X$ with a constant $c$, then $Th = h$. Conversely, having a fixed point, $h(x) = (Th)(x) = h(T^{-1}x) - h(T^{-1}x_0)$ for all $x$, then the same equality holds with constant $c = h(T^{-1}x_0)$. As discussed in most references on the topic, from [Gro78] to [B+22], one could consider functionals up to an additive constant, projectively, removing the role of any fixed based point. Some things are more clear from this point of view, but functionals would then just be defined up to an additive constant.

Let $G$ be a group that fixes $h$, note that $T(g) := -h(gx_0) = -d_g$ then defines a group homomorphism $T : G \to \mathbb{R}$.

Bas Lemmens and I observed the following easy fact during a discussion (essentially as recorded in [Kar211]), for isometries this was pointed out by Gromov long time ago [Gro78]:

PROPOSITION 13. *Let $T$ be a nonexpansive map of a metric space $X$. Suppose that $d(T) = 0$. Then there is a metric functional $h \in \overline{X}$ such that*

$$h(Tx) \le h(x)$$

*for all $x \in X$. In case $T$ is an isometric embedding then $h(Tx) = h(x)$ for all $x$, and when $T$ is an isometry $Th = h$.*

*Proof.* For any $\epsilon > 0$, take $y_\epsilon$ such that $d(y_\epsilon, Ty_\epsilon) < \epsilon$. Take $x \in X$ and consider

$$d(Tx, y_\epsilon) - d(x_0, y_\epsilon) \le d(Tx, Ty_\epsilon) + d(Ty_\epsilon, y_\epsilon) - d(x_0, y_\epsilon) \le \epsilon + d(x, y_\epsilon) - d(x_0, y_\epsilon).$$

This shows that any limit point $h$ of $h_{y_\epsilon}$ as $\epsilon$ approaches 0, which exists by compactness, has the property that

$$h(Tx) \le h(x).$$

Now let us assume that $T$ is isometric, then

$$d(x, y_\epsilon) - d(x_0, y_\epsilon) = d(Tx, Ty_\epsilon) - d(x_0, y_\epsilon) \le d(Tx, y_\epsilon) + d(y_\epsilon, Ty_\epsilon) - d(x_0, y_\epsilon)$$

$$\le \epsilon + d(Tx, y_\epsilon) - d(x_0, y_\epsilon),$$

which in the limit shows that $h(x) \le h(Tx)$. Thus $h(Tx) = h(x)$ and $Th = h$ if $T$ is an isometry thus actually acting on $\overline{X}$.  $\square$

Let us spell out a special case of Theorem 1 (cf. [Kar212, Theorem 2.2]):

PROPOSITION 14. *Let $T$ be an affine isometry of a bounded convex set $C$ of a reflexive Banach space. Then there is a point $x \in C$ such that $Tx = x$.*

*Proof.* From Theorem 1 we have $d = \tau = 0$. Consider the sets

$$\{x : \|x - Tx\| < \epsilon\}.$$

They are clearly convex when $T$ is affine and by reflexivity the intersection of all $\epsilon > 0$ is nonempty (Smulian). The points in this intersection must be fixed points of $T$. $\qquad\square$

A referee pointed out that instead of the reflexivity assumption one could merely assume that $C$ is weakly compact. Another extension would be to complete CAT(0) spaces, or more generally, complete uniformly convex metric spaces, which also are well-known to have the required non-empty intersection property, see e.g. [Kar00, Prop. 2.2] or [Kel14]. Incidentally, these two references suggest different notions of reflexivity of metric spaces.

If one does not assume that the isometry is affine there is a celebrated result of Maurey [Mau81], that states that in a weakly compact, convex set of a superreflexive Banach space, every isometry has a fixed point.

## 3    Proof of the metric fixed point theorem

Fix $x_0 \in X$. We define the following family of contractions $r_s : X \to X$, $0 \le s \le 1$ by $r_s(y) = \sigma_{x_0 y}(s)$. Note that by the bicombing definition

$$d(r_s(x), r_s(y)) \le s d(x, y)$$

and $d(r_s(y), y) = (1 - s)d(x_0, y)$ from the constant speed geodesic requirement.

LEMMA 15. *Let $X$ be a metric space and $Y$ its metric completion. Then $\overline{X}$ is canonically homeomorphic to $\overline{Y}$.*

*Proof.* We can consider $X$ as a subset of $Y$ in the natural way and select a basepoint $x_0 \in X$ for both spaces. First note that in a canonical way $\mathrm{Hom}(X, \mathbb{R}) \hookrightarrow \mathrm{Hom}(Y, \mathbb{R})$ since any element $h \in \mathrm{Hom}(X, \mathbb{R})$ has uniquely defined values on $Y$. Indeed, for any sequence $x_n \to y$,

$$|h(x_n) - h(x_m)| \le d(x_n, x_m)$$

which makes $h(x_n)$ a Cauchy sequence of real numbers, hence converging, which is the thus well-defined value $h(y)$. Similarly, by a $3\epsilon$-argument, it follows that every limit point in $\mathrm{Hom}(X, \mathbb{R})$ is also a limit point in $\mathrm{Hom}(Y, \mathbb{R})$. This shows that $\overline{X} \hookrightarrow \overline{Y}$.

Second, every element of $\mathrm{Hom}(Y, \mathbb{R})$ is also an element of $\mathrm{Hom}(X, \mathbb{R})$ by restriction. This also implies that any limit point $h$ in $\overline{Y}$ is also a metric functional in $X$ by approximation and restriction. Indeed, say that $h$ is a limit point of a set $\{h_y\}$, then we can approximate each $y$ by a Cauchy sequence in $x$. This shows the opposite continuous inclusion and the lemma is established. $\qquad\square$

In view of the lemma, and that we can also extend our maps $r_s$ for the same reasons since they are also nonexpansive maps, we may for simplicity assume that $X$ is complete.

The map $Tr_s = T \circ r_s$ is a uniformly strict contraction if $s < 1$ since $T$ preserves distances. By the contraction mapping principle we thus have a unique fixed point $y_s \in X$ for this map.

Here is a lemma that applies even with $T$ is merely nonexpansive:

LEMMA 16. *Let $y_s$ be the unique fixed point of $T \circ r_s$ for $0 \le s < 1$. Then*

$$d(x_0, y_s) \le \frac{d(x_0, Tx_0)}{1 - s}.$$

*Proof.* Let $D = d(x_0, Tx_0)$. Consider the ball of radius $R$ with center $z$

$$B_R(z) = \{x \in X : d(x, z) \le R\}.$$

Note that from the bicombing

$$r_s(B_R(x_0)) = B_{sR}(x_0).$$

From the isometric property of $T$ obviously $T(B_R(x_0)) \subseteq B_R(Tx_0)$ for any $R$. Hence

$$Tr_s(B_R(x_0)) \subseteq B_{sR}(Tx_0).$$

Note that $B_R(Tx_0) \subseteq B_{R+D}(x_0)$ by the triangle inequality. This means that if we take $R \ge D/(1-s)$ so that $sR + D \le R$, then

$$Tr_s(B_R(x_0)) \subseteq B_{sR}(Tx_0) \subseteq B_{sR+D}(x_0) \subseteq B_R(x_0).$$

From the contraction mapping principle, the iterates of the map $Tr_s$ converge towards the unique fixed point $y_s$. That is, for any $x$, $(Tr_s)^n x \to y_s$ as $n \to \infty$. Since $B_R(x_0)$ is an invariant set for $R = D/(1-s)$ we must have that $y_s \in B_R(x_0)$ as required. $\square$

To get the first inequality we follow the proof of Gaubert-Vigeral [GV12] which even applies to nonexpansive maps $T$. For any $x$

$$
\begin{aligned}
&h_{y_s}(x) - h_{y_s}(Tx) \\
&\quad = d(x, y_s) - d(Tx, y_s) = d(x, y_s) - d(Tx, T(r_s y_s)) \ge d(x, y_s) - d(x, r_s y_s) \\
&\quad \ge d(x, y_s) - d(x, r_s x) - d(r_s x, r_s y_s) \\
&\quad \ge d(x, y_s) - d(x, r_s x) - s d(x, y_s) = (1 - s) d(x, y_s) - d(x, r_s x) \\
&\quad \ge (1 - s)(d(x_0, y_s) - d(x_0, x)) - d(x, r_s x) \\
&\quad = d(r_s y_s, y_s) - (1 - s) d(x_0, x) - d(x, r_s x) \\
&\quad \ge d(Tr_s y_s, T y_s) - (1 - s) d(x_0, x) - d(x, r_s x) \\
&\quad = d(y_s, T y_s) - (1 - s) d(x_0, x) - d(x, r_s x)
\end{aligned}
$$

$$\geq d(T) - (1-s)d(x_0, x) - d(x, r_s x).$$

The last two terms go to 0 as $s \to 1$. By compactness there is a limit point of $h_{y_s}$ as $s$ approaches 1 and for any such limit point $h$ the above inequality shows that

$$h(x) - h(Tx) \geq d.$$

So we have obtained $h$ with the property that $h(Tx) \leq h(x) - d$ for all $x \in X$.

On the other hand, to get the other inequality we use that $T$ is isometric,

$$
\begin{aligned}
h_{y_s}(x) &= d(x, y_s) - d(x_0, y_s) = d(Tx, Ty_s) - d(x_0, y_s) \\
&\leq d(Tx, y_s) + d(y_s, Ty_s) - d(x_0, y_s) = d(Tx, y_s) + d(Tr_s y_s, Ty_s) - d(x_0, y_s) \\
&= h_{y_s}(Tx) + d(r_s y_s, y_s) = h_{y_s}(Tx) + (1-s)d(x_0, y_s) \\
&\leq h_{y_s}(Tx) + (1-s)d(x_0, Tx_0)/(1-s) \\
&= h_{y_s}(Tx) + d(x_0, Tx_0),
\end{aligned}
$$

where the last inequality comes from Lemma 16. This means that for any $\epsilon > 0$ we can find a metric functional $h$ (using $x_0$ such that $d(x_0, Tx_0) < d + \epsilon$) such that

$$h(x) \leq h(Tx) + d + \epsilon$$

together with $h(x) \leq h(Tx) - d$ both inequalities holding for all $x$. This applies to any limit point $h$.

We can now argue as follows, fix a sequence $s_n \nearrow 1$ as $n \to \infty$. Take a non-principal ultrafilter $\mathfrak{U}$ on the natural numbers. This defines the limit

$$h = \lim_{\mathfrak{U}} h_{y_{s_n}}$$

and for metric functionals with another base point $\widetilde{x_0}$, we have the limit $\tilde{h} = \lim_{\mathfrak{U}} \tilde{h}_{y_{s_n}}$. Recall the above discussion about changing the base point, that it does not influence differences $h(x) - h(Tx)$. Indeed, for any $y$

$$h_y(x) - h_y(Tx) = \tilde{h}_y(x) - \tilde{h}_y(Tx)$$

for any $x$, since on each side the terms with the base points cancel. Therefore

$$h(x) - h(Tx) = \lim_{\mathfrak{U}} \left[ h_{y_{s_n}}(x) - h_{y_{s_n}}(Tx) \right] = \lim_{\mathfrak{U}} \left[ \tilde{h}_{y_{s_n}}(x) - \tilde{h}_{y_{s_n}}(Tx) \right]$$

$$= \tilde{h}(x) - \tilde{h}(Tx) \leq d(\tilde{x}_0, T\tilde{x}_0),$$

where the last inequality comes from the above calculation. Now since the base point $\tilde{x}_0$ is arbitrary, we conclude that $h(x) - h(Tx) \leq d$.

In summary we have for this metric functional that for all $x \in X$

$$h(x) - d \leq h(Tx) \leq h(x) - d,$$

hence the desired equality. In case $T$ is an isometry, previous remarks show that $Th = h$.

Suppose that $h = h_y$ for some point $y \in X$. Then $h_y(Tx) = h_y(x) - d$, that is,

$$d(y, Tx) - d(y, x) = -d$$

for all $x$. Applying it to $x = y$, forces $d = 0$ and $d(y, Ty) = 0$, since distances are positive or 0. By the standard axiom for metric spaces it means that $Ty = y$.

With all this, Theorem 1 is proved.

## 4   Mean ergodic theorems

In 1931 von Neumann proved the first ergodic theorem after being inspired by remarks of Koopman and Weil. Around the same time Carleman had similar ideas, and, as Eisner pointed out to me, von Neumann subsequently published a note analyzing Carleman's work on the topic. For references and a good general discussion on this subject can be found in the book by Eisner [Eis10]. The mean ergodic theorem is proved in virtually every book on ergodic theory. It is also well-known that the operator version of the mean ergodic theorem fails in general Banach spaces such as $\ell^1$ and $\ell^\infty$. From the metric fixed point theorem, we deduce a new result that in contrast holds in general (cf. [Kar212, Theorem 2.1] for another version):

COROLLARY 17 (Mean ergodic theorem in Banach spaces). *Let $U$ be a norm-preserving linear operator of a Banach space $X$ and $v \in X$. Then there exists a metric functional $h$ such that*

$$\frac{1}{n} h \left( \sum_{k=0}^{n-1} U^k v \right) = - \inf_x \| Ux + v - x \|$$

*for all $n \geq 1$.*

*Proof.* The line segments in the Banach space provide the required bicombing. From the assumption it follows that $x \mapsto Ux + v$ is an isometric map and the sum in the statement is the application of this map $n$ times to $x = 0$. The result now follows directly from Theorem 1 by iterating the map.     □

In case of a Hilbert space one can deduce the usual mean ergodic theorem in a way quite different from the standard proofs:

COROLLARY 18. *Let $U$ be a unitary operator of a Hilbert space $H$ and $v \in H$. Then the strong limit*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k v = \pi_U(v),$$

*where $\pi_U$ is the projection onto the $U$-invariant vectors.*

*Proof.* Either the limit exists equalling 0, or $\tau = d > 0$ (see Sect. 2) and Corollary 17 gives a nontrivial metric functional, which in this case for a Hilbert space must be linear, see [Kar22]. Hence there is a unit vector $w$ such that for all $n \geq 1$ the scalar product

$$\left( \frac{1}{n} \sum_{k=0}^{n-1} U^k v, w \right) = d,$$

where $d = \inf_x \|Ux + v - x\|$. Considering that $U$ is norm-preserving, $\tau = d$, and $\|w\| = 1$ it is a simple Hilbert space fact that this implies that the sum in the statement converges strongly to $d \cdot w$ (see [Kar22]). From Theorem 1 we in addition have for any $x$ that $(x - Ux - v, w) = \inf_x \|x - Ux - v\| = d$. Since the closed linear span of vectors $\{x - Ux\}$ is orthogonal to the $U$-invariant vectors, the projection of $v$ onto the latter subspace is $d \cdot w$. $\qquad\square$

A linear operator $A$ of a Banach space is *power bounded* if

$$\sup_{k>0} \left\| A^k \right\| < \infty.$$

Mean ergodic theorems have been considered for such operators for example in the works of Kakutani and Yosida. Parallel to the above discussion, the usual mean ergodic formulation fails in general but there is now instead a metric replacement:

COROLLARY 19. (Power bounded ergodic theorem) *Let $A$ be a power bounded operator of a Banach space $X$. Let*

$$\|x\|_1 := \sup_{k>0} \left\| A^k x \right\|$$

*for $x \in X$. Then for any $v \in X$ there is a metric functional $h$ of $X$ with the norm $\|\cdot\|_1$ such that*

$$h \left( \sum_{k=0}^{n-1} A^k v \right) \leq -n\tau$$

*for all $n > 0$ and where $\tau = \inf_x \|Ax + v - x\|_1$.*

*Proof.* Note that $\|\cdot\|_1$ is a norm and that $x \mapsto Ax + v$ is nonexpansive in the corresponding metric. The statement then follows from the first part of the proof of the metric fixed point theorem. $\qquad\square$

## 5    Some remarks on CAT(0), injective spaces, and $\ell^\infty$

The following remarks are related to the main topic of this paper, and although perhaps not new, some of them seems not to be stated in the literature.

CAT(0)-SPACES Let $X$ be a proper CAT(0)-space. It is shown in [Gro78, BH99, Theorem 8.13] that the metric compactification is topologically equivalent to the

visual compactification. Let $g$ be an isometry of $X$ with fixed point $h$ from Theorem 1. In case h is associated with a bounded sequence, then $h = h_y$ for some point $y \in X$ and then as shown above $gy = y$ (the usual argument uses a circumcenter of the bounded orbit, see [BH99, Corollary 2.8]). In the case $h \in \partial X$ then $h$ corresponds precisely to a equivalence class of geodesic rays, and hence this is the fixed point again in the traditional sense. Although this latter case is a simple fact in view of the following argument: If the orbit of $g$ is unbounded, let $g^{n_k} x_0$ be a convergent sequence to a boundary point $\gamma$. Note then that for any isometry $g'$ commuting with $g$ (for example $g$ itself) fixes this boundary point:

$$g'(\gamma) = g'(\lim_k g^{n_k} x_0) = \lim_k g' g^{n_k}(x_0) = \lim_k g^{n_k}(g'(x_0)) = \gamma,$$

where the last equality comes from the fact that any two sequences staying on bounded distance from each other (in the present case $d(g'x_0, x_0)$) have the same accumulation points. Thus:

PROPOSITION 20. *Let $X$ be a proper $CAT(0)$-space and $g$ an isometry with unbounded orbits. The centralizer of $g$ fixes a point in the visual boundary of $X$.*

As noted previously, without properness, the metric compactification is different from the visual bordification (which can be empty even for unbounded spaces). Therefore Corollary 5 is of interest for groups of isometries of nonproper spaces. Important examples of isometric actions on nonproper $CAT(0)$-space are provided by the fundamental Cremona groups of birational transformations of varieties. In the two variable case this action was found by Manin and Zariski, and greatly exploited for example by Cantat in [Can11]. More recently, actions by the other Cremona groups by isometry on certain $CAT(0)$-cube complexes were constructed by Lonjou and Urech [LU21]. Fixed points of individual isometries play a crucial role for these studies and the translation length are related to algebraic notions.

In this context, it may be worthwhile to point out the following:

**Theorem 21.** *Let $g$ be an isometry of a complete $CAT(0)$-space. If $\inf_x d(x, gx) > 0$, then $g$ has a fixed point in the visual bordification. The metric functional corresponding to this point is fixed and is a Busemann horofunction. It is moreover the unique horofunction such that*

$$h(gx) = h(x) - d_g,$$

*and for the geodesic ray $\gamma$ corresponding to $h$ it holds that*

$$\frac{1}{n} d(g^n x_0, \gamma(d_g n)) \to 0.$$

*Proof.* First, it is known from [BGS85], but also reproved here in a different way that $\tau_g = d_g$. Therefore $\tau_g > 0$, and a special case of the main result in [KM99] then shows that $g^n x_0$ converges to a visual boundary point $\gamma$. In fact the last assertion of the present theorem is proven and which is a finer notion of convergence to the boundary.

As shown above, we then must have $g(\gamma) = \gamma$. This bordification is equivalent to Gromov's horofunction bordification, thus $g$ fixes $h$, the metric functional (which in this case is a Busemann horofunction $b_\gamma$) associated to the geodesic ray emanating from $x_0$ representing $\gamma$. As shown in [KL11] the last assertion in the theorem is equivalent to

$$-\frac{1}{n} h(g^n x_0) \to \tau_g$$

for $h = b_\gamma$. Comparing Theorem 1 this is the metric functional in that statement so there is an equality (which actually is clear from remarks in Sect. 2). ☐

Note that as alluded to above, shown for example in [BH99], in the case of $\inf_x d(x, gx) = 0$, there may not exist any fixed point in the visual bordification. In case of CAT(0) cube complexes there is however a more precise version (no parabolic isometries) available due to Haglund [Hag07].

Some spaces are not complete but still have a conical bicombing, for example the Weil-Petersson metric on the Teichmüller spaces since it is geodesically convex with nonpositive curvature.

INJECTIVE SPACES A visual compactification was defined in [DL15] for *consistent* bicombings of proper injective spaces (although since the choice of bicombing is not unique it is apparently unclear how unique the visual compactification is, while the metric compactification of course is unique), and another one in [Bas24] which work without the consistency condition. We need to justify the second assertion of Corollary 6: From Theorem 1 we have that either there is a point $y \in X$ that is fixed be the isometry, or the isometry has unbounded orbits in which case the small argument above for CAT(0)-spaces applies, since any sequence staying on bounded distance from a convergent unbounded sequence must converge to the same point as follows from the basic properties of the visual compactification as in [Bas24]. This proves that any isometry of a proper injective metric space has a fixed point in a visual compactification.

THE $\ell^\infty$-SPACES The $\ell^\infty$-space is one of the standard injective spaces. Let us first consider it on a finite set $X := \ell^\infty(\{1, 2, \ldots, N\}) = C(\{1, 2, \ldots, N\})$. In this case the visual compactification (as defined from the bicombing of lines) is not equivalent to the metric one. To see this, first notice that the rays can be represented by a unit vector $v$ and $\gamma(t) = tv$. Now for the metric functionals, which are already known from papers by Walsh, Gutiérrez and others. Take $x_0 = 0$. Since we can argue with sequences we take a sequence $y_n$ such that $\|y_n\| \to \infty$. By taking subsequences we may divide the index set into $A$ and $B$ such that the coordinates $y_n^i \geq 0$ for all $n$ and $i \in A$ and $y_n^j < 0$ for all $n$ and $j \in B$. Moreover we can assume that the following limits exist

$$y_n^i - \|y_n\| \to a^i \in [-\infty, 0]$$

for $i \in A$ and for $j \in B$

$$-y_n^j - \|y_n\| \to b^j \in [-\infty, 0].$$

Denote by $A_f$ and $B_f$ those indices with a finite limiting value. Since the norm needs to be realized on some coordinate we have that $A_f \cup B_f$ is non-empty, indeed at least one of these values is 0. The corresponding metric functional which is the limit of $h_{y_n}$ is now

$$h(x) = \max \left\{ \sup_{i \in A_f} a^i - x^i, \sup_{j \in B_f} b^j + x^j \right\}.$$

These are the metric functionals of $X$ in addition to $h_y$. For a ray $y_n = nv$ defined by a unit vector $v$ notice that $A_f = \{i : v^i = 1\}$ and $B_f = \{j : v^j = -1\}$. This means that the other coordinates of $v$ play no role for its Busemann function. This shows the difference between the two compactifications.

Determining the metric compactification of $\ell^\infty$ in infinite dimensions seems not yet have been done, although there is an interesting paper by Walsh [Wal18] determining the so-called Busemann points for Banach spaces (which lead him to a direct alternative proof of the Mazur-Ulam theorem). Maybe one should keep in mind that the determination of the linear dual space of $\ell^\infty$ in functional analysis leads to signed finitely additive measures.

Here is one interesting observation, which is in contrast to Hilbert spaces but similar to a phenomenon Gutiérrez discovered for $\ell^1$ spaces [Gut19] (also true for hyperbolic spaces [MT18, Cla20, B+22]). Actually, as Lytchak pointed out to me, this feature was established for CAT(0)-spaces with finite telescoping dimension already in [CL10, Lemma 4.9].

PROPOSITION 22. *For $\ell^\infty$-spaces the linear functional $f \equiv 0$ is not a metric functional. In fact all metric functionals are unbounded functions.*

*Proof.* Let $S$ be a set and consider $X = \ell^\infty(S)$. The finite dimensional case was treated already. Any accumulation point of a bounded set cannot be identically 0 by evaluating on a point $x$ which lies outside a ball centered at 0 containing the bounded set, showing that they all are unbounded functions.

For the case of accumulation points of unbounded sets of metric functionals, consider the set $C$ of $y \in X$ with $\|y\| > 2$. Take the subset $A$ of $X$ for which there is a coordinate $y^{s_0}$ such that $y^{s_0} > \|y\| - 1/2 > 0$. In the complement of $A$ in $C$ there is instead a coordinate such that $-y^{s_0} > \|y\| - 1/2 > 0$.

Look at $x \in X$ such that $-c$ for every $s \in S$, and some $c > 1$. Then for any $y \in A$

$$h_y(x) = \sup_s |y^s - x^s| - \|y\| \geq y^{s_0} - (-c) - \|y\| > -1/2 + c > 1/2.$$

For $y$ in the complement of $A$ in $C$ we instead look at $z \in X$ such that $z^s = c$ for all $s$, some $c > 1$, and have similarly

$$h_y(z) = \sup_s |y^s - z^s| - \|y\| \geq -y^{s_0} + c - \|y\| > -1/2 + c > 1/2.$$

Note that for any metric functional $|h(z)| \leq d(0, z) = 1$. The two sets $\{f : f(x) < 1/2\}$ and $\{f : f(z) < 1/2\}$ are neighborhoods of $f \equiv 0$ in $\mathrm{Hom}(X, \mathbb{R})$. The above two in-

equalities show that every cluster point of $\Phi(C)$ stays outside the intersection of these two neighborhoods, thus the functional identically equal to 0 is not in $\overline{X}$. Indeed, we see that each $h$ takes values larger than $c - 1/2$ but $c$ is arbitrarily large, hence all metric functionals are unbounded.                                                    □

## 6   The space Pos

Let $H$ be a Hilbert space and $\mathrm{Pos} = \mathrm{Pos}(H)$ the set of positive definite symmetric (self-adjoint) bounded linear operators on $H$. This is an open cone in the Banach space of all symmetric operators equipped with the operator norm (which by the spectral theorem equals the spectral radius). This is the prototype space for spaces of metrics discussed above concerning diffeomorphisms and biholomorphisms. Therefore I expect that some things discussed below will be relevant for the other contexts.

One can put a complete metric on the space Pos, *the Thompson metric*, for $p, q \in Pos$

$$d(p,q) = \sup_{v \in H, \|v\|=1} \left| \log \frac{(qv,v)}{(pv,v)} \right|,$$

which is also a Finsler metric [CPR94]. It is known from this reference and subsequent papers that it has a weak conical bicombing which follows from Segal's inequality:

$$\|\exp(u+v)\| \le \|\exp(u/2)\exp(v)\exp(u/2)\|$$

for self-adjoint bounded operators and the operator norm. The bicombing does not come from the lines in the cone even though these lines are also geodesics, but instead the distinguished geodesics are the image of lines in the tangent space Sym under the exponential map (note that the exponential map in the sense of algebra and analysis, coincides for this space with the notion in differential geometry). We consider Sym with the distance that the operator norm gives. The inequality shows that the exponential map is distance preserving on lines, and metric noncontracting in general [CPR94]. Note that this discussion centered around the point $x = I$ but since Pos is a homogeneous space the same applies to any base point. Instead of considering the full algebra of bounded linear operators one can do the same constructions for $C^*$-algebras, see for example [KM99].

Invertible bounded linear operators $g$ act on Pos by isometry $p \mapsto gpg^*$. This action is transitive. If $g$ fixes a point $p$ it means it is a unitary operator in the corresponding scalar product.

In finite dimensions there are several studies about the metric compactification in the literature, in the standard Riemannian metric (of less interest for us) [BH99] and for various Finsler metrics [KL18, H+, Lem]. For the application we have in mind, we need to go to infinite dimensions, in this setting there is basically only [Wal18].

Here we now establish the following, which remedies a vexing property of weak topologies in general in functional analysis that a weak limit could just be 0:

PROPOSITION 23. *The function identically* 0 *is not a metric functional of* Pos *and thus every metric functional is nonconstant, in fact unbounded.*

*Proof.* First note that since $h(I) = 0$ for any metric functional $h$, being constant is the same as being 0 everywhere. (Here we are taking the identity operator to be the base point of Pos.)

Secondly, any cluster point of a bounded set of $h_p$ is clearly an unbounded function: since taking $q$ with large norm much bigger than the radius of the ball around $I$ that contains the set of $p$ under consideration, then

$$h_p(q) = d(q, p) - d(I, p) \geq d(q, I) - 2d(I, p) \gg 0,$$

and it shows the function is unbounded in $q$. (This applies to any metric space.)

We now consider the set of $p$ outside the ball of a certain sufficiently large radius. We will show that this does not intersect the following neighborhood of 0 in $\mathrm{Hom}(\mathrm{Pos}, \mathbb{R})$:

$$N(0) = \{f : |f(a)| < 1/2 \text{ and } |f(b)| < 1/2\}$$

where $a = \exp(I)$ and $b = \exp(-I)$.

To see this take $p = \exp(Y) \in \mathrm{Pos}$ with $\|Y\| > C > 2$. Consider:

$$h_p(x) = d(x, p) - d(I, p)$$

which is greater or equal to $h_Y(\log x) = \|\log x - Y\| - \|Y\|$ by the metric properties of the exponential map recalled above. For a constant $\epsilon$ with $0 < \epsilon < 1/2$, we take a unit vector $v$ in $H$ such that $|(Yv, v)| > \|Y\| - \epsilon$ in view of a standard fact of symmetric operators. Now, in the case $(Yv, v) > \|Y\| - \epsilon$, we thus have

$$h_p(b) \geq h_Y(-I) = \|-I - Y\| - \|Y\| \geq |(-Iv - Yv, v)| - \|Y\| = |-1 - (Yv, v)| - \|Y\|$$

and assuming $C > 1 + \epsilon$ this expression equals

$$1 + (Yv, v) - \|Y\| \geq 1 - \epsilon > 1/2.$$

In the opposite case $-(Yv, v) > \|Y\| - \epsilon$, we instead look at

$$h_p(a) \geq h_Y(I) = \|I - Y\| - \|Y\| \geq |(Iv - Yv, v)| - \|Y\| = |1 - (Yv, v)| - \|Y\|$$

and assuming $C > 1 + \epsilon$ this expression equals

$$1 - (Yv, v) - \|Y\| \geq 1 - \epsilon > 1/2.$$

In total this shows that the set of all functionals $h_p$ with $\|p\| > \exp(2)$, does not intersect $N(0)$. It follows that $f \equiv 0$ cannot be a cluster point of the set $\Phi(\mathrm{Pos})$ and thus not a metric functional.

Similarly to the case with $\ell^\infty$ one sees that we can conclude that every metric functional of this space is not only nonconstant but in fact unbounded.  $\square$

# 7  Application to bounded linear operators

Recall first that we already noticed one application to norm-preserving linear operators, showing a new result that defies the standard counterexamples, and strong enough to deduce the most classical version of the mean ergodic theorem.

The Invariant Subspace Problem (ISP) asks: does every bounded linear operator $T$ of a separable complex Hilbert space of dimension at least two, admit a nontrivial invariant closed subspace? This is a famous and longstanding open problem, see for example the book [CP11] devoted to it. For the purpose of investigating invariant linear subspaces, we may add a multiple of the identity map or multiply the operator by a scalar, since these operation do not change any invariant subspace. Therefore we may assume that the operator $g = T$ is invertible.

Note that by applying the operator $g$ iteratively on the basepoint $x_0 = Id$ we get

$$\tau(g) = \lim_{n \to \infty} \left| \log \|g^n g^{*n}\|^{1/n} \right|$$

this is semipositive and finite (since $g$ is an invertible bounded operator). It is 0 if and only if the spectrum of $g$ is contained in the unit circle (for example when the operator is unitary). As remarked in Sect. 2 above $d(g) = \tau(g)$ (note that $d$ makes sense in operator theory but is an invariant perhaps not much considered). In view of Proposition 23 we thus get from Theorem 1 a nontrivial metric functional $h$ of Pos which is fixed by $g$

$$g(h) = h.$$

This proves Theorem 11. It is clearly a statement in the direction of the ISP, but how does it actually relate to closed invariant subspaces? As already remarked when $g$ fixes a point inside Pos it is unitary and thus the spectral theorem applies and settles the issue of invariant subspaces. If it fixes a point $s$ in the natural boundary of Pos consisting of bounded semipositive operators, then $g^*$ must preserve its kernel. Indeed, take $v \in \ker s$ then

$$0 = gs(v) = s(g^*v)$$

showing that $g^*(\ker s) \subseteq \ker s$. This is a closed subspace and its closed orthogonal complement is invariant under $g$ and is a nontrivial subspace. In general, one conceivable path forward is to refine the theorem in this special case, and deduce that the metric functional must be a Busemann point, and that as such it has an associated closed linear subspace that must be invariant.

Another discussion of the invariant subspace problem from a metric perspective without the space Pos can be found in [GK21] which establishes some cases when there exist metric or linear functionals $f$ of the Banach space such that $f(g^n(0)) \leq 0$ for all $n > 0$ or $f(g(x)) \leq f(x)$ for all vectors $x$.

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