❚ GAFA Geometric And Functional Analysis

VOLUME AND MACROSCOPIC SCALAR CURVATURE

SABINE BRAUN AND ROMAN SAUER

Abstract. We prove the macroscopic cousins of three conjectures: (1) a conjectural bound of the simplicial volume of a Riemannian manifold in the presence of a lower scalar curvature bound, (2) the conjecture that rationally essential manifolds do not admit metrics of positive scalar curvature, (3) a conjectural bound of ℓ^2-Betti numbers of aspherical Riemannian manifolds in the presence of a lower scalar curvature bound. The macroscopic cousin is the statement one obtains by replacing a lower scalar curvature bound by an upper bound on the volumes of 1-balls in the universal cover.

1 Introduction

1.1 Results. Scalar curvature is a microscopic concept. The scalar curvature at a point p of a Riemannian manifold M can be read off from the volumes of balls around p whose radii approach zero. A lower scalar curvature bound for M corresponds to an upper bound on the volumes of sufficiently small balls in M or, equivalently, in $\widetilde{M}$ as the universal cover projection $\widetilde{M} \to M$ is locally isometric.

How can we replace scalar curvature by a macroscopic concept? For instance, by replacing a lower scalar curvature bound by an upper bound on the volumes of balls of a fixed radius, say radius 1, in the universal cover. We form the *macroscopic cousin* of a mathematical statement involving a lower scalar curvature bound for a Riemannian manifold M by the replacing it with an upper bound on the volumes of 1-balls in $\widetilde{M}$. Guth's ICM report [Gut10] describes the analogies and connections that emerge from the macroscopic point of view.

Our first main theorem is the macroscopic cousin of a conjecture by Gromov [Gro86, Conjecture 3A] for which it is assumed that the scalar curvature is bounded from below by -1. Via the Bishop-Gromov inequality one sees that it also generalizes

Gromov's main inequality [Gro82, Section 0.5] for which it is assumed that the Ricci curvature is bounded from below by -1.

Theorem 1.1.   *For every $V_1 > 0$ and $d \in \mathbb{N}$ there is constant $\mathrm{const}(d, V_1) > 0$ with the following property. If M is a d-dimensional closed Riemannian manifold such that the volume of every 1-ball in the universal cover of M is at most V_1, then*

$$\|M\| \leq \mathrm{const}(d, V_1) \cdot \mathrm{vol}(M),$$

where $\|M\|$ denotes the simplicial volume of M.

Theorem 1.1 generalizes another theorem, Guth's volume theorem, as the simplicial volume of a hyperbolic manifold coincides with its volume up to a dimensional constant by a result of Gromov and Thurston. Other generalizations of Guth's theorem can be found in [BK19, AF17, Theorem 1.3].

We adopt the following convention. Within a statement P about a manifold, a *dimensional constant* just means a positive real constant that only depends on the dimension of the manifold. In other words, if the dimensional constant is denoted by $c(d)$ the statement P should be read with the preface: *For every dimension d there is a constant $c(d) > 0$ such that the following holds true.*

Let us denote the supremal volume of an r-ball in a Riemannian manifold (M, g) by $V_{(M,g)}(r)$. The induced metric on the universal cover is denoted by $\tilde{g}$.

Theorem 1.2. (Guth's Volume theorem [Gut11]).   *Let M be a d-dimensional closed hyperbolic manifold, and let g be another metric on M. Suppose that*

$$V_{(\widetilde{M}, \tilde{g})}(1) \leq V_{\mathbb{H}^d}(1),$$

where $\mathbb{H}^d$ is d-dimensional hyperbolic space. Then

$$\mathrm{vol}(M, g_{\mathrm{hyp}}) \leq \mathrm{const}(d) \cdot \mathrm{vol}(M, g)$$

for a dimensional constant $\mathrm{const}(d) > 0$.

Guth's volume theorem is the macroscopic cousin of Schoen's conjecture [Sch89, p. 127] which says that $\mathrm{scal}_{M,g} \geq \mathrm{scal}_{M,g_{\mathrm{hyp}}}$ implies $\mathrm{vol}(M, g_{\mathrm{hyp}}) \leq \mathrm{vol}(M, g)$. More precisely, it is the *non-sharp* macroscopic cousin because of the dimensional constant $\mathrm{const}(d)$.

Our second main theorem for $R = 1$ is the non-sharp macroscopic cousin of the conjecture that rationally essential manifolds do not admit a metric of positive scalar curvature (the statement for $R = 1$ readily implies the one for all $R > 0$ by scaling the metric).

Theorem 1.3.   *There is a dimensional constant $\epsilon(d) > 0$ with the following property. For every rationally essential Riemannian manifold (M, g) of dimension d and every $R > 0$ we have*

$$V_{(\widetilde{M}, \tilde{g})}(R) > \epsilon(d) \cdot R^d.$$

If $\epsilon(d)$ could be chosen to be the volume of a Euclidean d-ball, then the above conjecture would follow. A closed oriented manifold is *rationally essential* if its classifying map sends the fundamental class to a non-zero class in rational homology. Guth proves the volume estimate in Theorem 1.3 for Riemannian manifolds whose universal cover have infinite filling radius [Gut11, Theorem 1]. Not every rationally essential manifold has a universal cover with infinite filling radius according to [BH10, Theorem 1.4 and Proposition 2.8].

Our third main theorem is the macroscopic cousin of the combined conjectures [Gro86, Conjecture 3A; 21, 3.1 (e) on p. 769] by Gromov (see also [Gro93, p. 232]). It generalizes the main result in [Sau09] where a lower Ricci curvature bound was assumed. See also [Sau16] for related results in the residually finite case.

Theorem 1.4.   *For every $V_1 > 0$ and $d \in \mathbb{N}$ there is a constant* $\mathrm{const}(d, V_1) > 0$ *with the following properties.*

(1) *If (M, g) is a d-dimensional connected closed oriented Riemannian manifold with classifying map $c\colon M \to B\Gamma$ such that $V_{(\widetilde{M}, \widetilde{g})}(1) \leq V_1$, then the von Neumann rank of $c_*([M]) \in H_d(B\Gamma)$, where $[M]$ is the fundamental class of M, is bounded from above by $\mathrm{const}(d, V_1) \cdot \mathrm{vol}(M)$.*

(2) *If, in addition, the manifold M is aspherical, then its ℓ^2-Betti numbers satisfy*

$$\beta_i^{(2)}(M) \leq \mathrm{const}(d, V_1) \cdot \mathrm{vol}(M, g)$$

for every $i \in \mathbb{N}$, and the Euler characteristic satisfies

$$|\chi(M)| \leq \mathrm{const}(d, V_1) \cdot \mathrm{vol}(M, g).$$

Section 3.3 contains an overview of what we need from the theory of ℓ^2-Betti numbers, including the definition of von Neumann rank.

We present a result which is an outcome of our methods but is of independent interest. See Remark 3.6 for the relevant notions.

Theorem 1.5.   *For every $V_1 > 0$ and $d \in \mathbb{N}$ there is constant* $\mathrm{const}(d, V_1) > 0$ *with the following properties:*

The integral foliated simplicial volume of every d-dimensional closed aspherical Riemannian manifold (M, g) that satisfies $V_{(\widetilde{M}, \widetilde{g})}(1) \leq V_1$ is bounded from above by $\mathrm{const}(d, V_1) \cdot \mathrm{vol}(M, g)$. *More precisely, the inequality $|M|^{\alpha} \leq \mathrm{const}(d, V_1) \cdot \mathrm{vol}(M, g)$ holds for any free measurable pmp action α on a standard probability space.*

Theorem 1.5 generalizes Corollary 1.2 of [Fau21] since the assumption in [Fau21, Corollary 1.2] implies the vanishing of the minimal volume [CG86, Theorem 3.1]. Theorem 1.5 shows the vanishing of the integral foliated simplicial volume (and its variants above) for 3-dimensional graph manifolds since their minimal volume vanishes [CG86, Example 0.2 and Theorem 3.1]. This vanishing result is a special case of [FFL19, Theorems 1.2 and 1.6].

1.2   Comment on the proof.   The proof of Theorem 1.1 involves an action of the fundamental group on the Cantor set. We will explain why.

A close reading of Guth's proof of the volume theorem yields a proof of Theorem 1.1 for closed aspherical manifolds whose smallest non-contractible loop (*systole*) is of length at least 1 (see the discussion [AF17, Section 4]). If the fundamental group is residually finite, then we can pass to a finite cover whose systole is of length at least 1. Since the stated inequality between simplicial volume and Riemannian volume may be verified on every finite cover, Theorem 1.1 follows for closed aspherical manifolds with residually finite fundamental groups.

We have to get rid of the assumptions of asphericity and residual finiteness. Residual finiteness is hard to verify beyond locally symmetric spaces; we do not even know whether fundamental groups of closed negatively curved manifolds are residually finite. The attempt to get rid of residual finiteness leads to actions on the Cantor set.

If the fundamental group Γ of our manifold M is not residually finite, we may not have enough finite covers to enforce a large systole. Let us consider the inverse limit

$$\varprojlim_{i \in I} M_i$$

of the directed system of all connected finite regular covers of M – even if there are none except M itself in the most extreme case. By covering theory the system M_i, $i \in I$, corresponds to a directed system of finite index normal subgroups $\Gamma_i < \Gamma$, $i \in I$. Each M_i is just the quotient $\Gamma_i \backslash \widetilde{M}$. The inverse limit $\widehat{\Gamma} = \varprojlim \Gamma / \Gamma_i$ is the *profinite completion* of Γ. We have

$$\varprojlim_{i \in I} M_i \cong \varprojlim_{i \in I} \Gamma_i \backslash \widetilde{M} \cong \varprojlim_{i \in I} \Gamma / \Gamma_i \times_\Gamma \widetilde{M} \cong \widehat{\Gamma} \times_\Gamma \widetilde{M}.$$

The Γ-quotient on the right is the quotient by the diagonal action. The profinite completion has an obvious action by translations. The Γ-space $\widehat{\Gamma}$ has three important properties:

  (1) It is homeomorphic to the Cantor set.
  (2) It possesses an invariant probability measure (the Haar measure).
  (3) If Γ is residually finite then the Γ-action is free.

Let X be the Cantor set. In Theorem 2.1 we reprove an observation of Hjorth-Molberg that every countable group Γ admits a free, continuous action on X having a Γ-invariant probability measure. With regard to such an action we form the space

$$X \times_\Gamma \widetilde{M},$$

which acts as a replacement for $\varprojlim M_i$ if the fundamental group is not residually finite. Guth's methods need the finite covers M_i. Our contribution is to generalize them so that we can work with the global object $X \times_\Gamma \widetilde{M}$ instead.

Recent results of Liokumovich–Lishak–Nabutovsky–Rotman [LLNR] and Papasoglu [Pap20] generalize Guth's theorem in [Gut17, Theorem 0.1] on Uryson width. Papasoglu's proof is simpler than Guth's proof which underlies our work. Can one combine the method in [Pap20] with our ideas to obtain shorter proofs for our main results or to obtain explicit constants as in Nabutovsky's paper [Nab]? We do not think this is possible in the case of Theorems 1.1 and 1.4, but it might be possible in the case of Theorem 1.3.

1.3   Structure of the proof.   We establish a framework of equivariant bundles over X (*Cantor bundles*). After Section 2 on preliminaries we introduce the notion of Cantor bundle in Section 4. The space $X \times \widetilde{M}$ with its diagonal action of $\Gamma = \pi_1(M)$ is a trivial example. A more interesting toy example is Example 4.9. Cantor bundles can be regarded as spaces with a groupoid action, namely the groupoid given by the orbit equivalence relation on X, endowed with additional geometric data. Spaces with groupoid actions are considered in many contexts. Our main inspiration came from Gaboriau's $\mathcal{R}$-simplicial complexes in the measurable world [Gab02]. Another influence is Gromov's paper [Gro91].

After discussing transverse Hausdorff measures on Cantor bundles in Section 5 we introduce the rectangular nerve construction in the framework of Cantor bundles (Section 6). The *rectangular Cantor nerve* of an equivariant cover on $X \times \widetilde{M}$ as described above is a non-trivial Cantor bundle. The toy example gives a good impression how such a rectangular Cantor nerve might look like.

In Section 7 we establish the existence of good covers in our framework and prove the analog of Guth's result on the exponential decay of the volume of the high multiplicity set in our framework. This is the main point about the auxiliary space X: We cannot obtain a good, equivariant cover on $\widetilde{M}$, only on the Cantor bundle $X \times \widetilde{M}$.

We then bound the transverse volume of the image of the map to the rectangular Cantor nerve. In Section 8 this map is homotoped as a Cantor bundle map to the d-skeleton where d is the dimension of M. In Section 9 we relate what we have done so far to the simplicial volume of M. Here we use tools from homological algebra and equivariant topology.

2   Topological Preliminaries

In Section 2.1 we present a short proof of the existence of suitable actions on the Cantor set which is a result of Hjorth-Molberg. In Section 2.2 we review the notion of an equivariant CW-complex and of a classifying space. In Sections 2.3 and 2.4 we give a detailed review of rectangular complexes and Guth's rectangular nerve since special care is needed in our equivariant context.

We adhere to the following notation. Let M be a closed d-dimensional Riemannian manifold with fundamental group Γ. Its universal cover is denoted by $\widetilde{M}$ and endowed with the Riemannian metric induced by M. The Cantor set is denoted

by X. We fix a free continuous Γ-action on X and a Γ-invariant Borel probability measure μ on X whose existence is stated in Theorem 2.1. If $B = B(p, r) \subset \widetilde{M}$ is the open ball of radius r around p, then aB is the concentric ball of radius $a \cdot r$ around p.

2.1 Free actions on the Cantor set. The following observation is due to Hjorth and Molberg [HM06, Theorem 0.1]Based on the notion of co-induction we formulate a shorter proof here for the convenience of the reader. A stronger statement, which we only need in the proof of Theorem 1.5, was obtained by Elek [Ele21].

Theorem 2.1. *Let Γ be a countable discrete group and let X be the Cantor set. Then there is a free, continuous Γ-action on X having a Γ-invariant probability measure.*

Proof. The case of finite groups is easy. We may and will assume that Γ is infinite. For every element $\gamma \in \Gamma$ let X_γ be the profinite completion of the cyclic subgroup $\langle \gamma \rangle$ endowed with the left translation action by $\langle \gamma \rangle$ and the normalized Haar measure ν_γ. Depending on the order of γ, X_γ is either a finite set or homeomorphic to the profinite completion $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$, which is a Cantor set. Let Y_γ be the co-induction of the $\langle \gamma \rangle$-space X_γ, that is

$$Y_\gamma := \mathrm{map}(\Gamma, X_\gamma)^{\langle \gamma \rangle} = \left\{ f \colon \Gamma \to X_\gamma \mid \forall_{x \in \Gamma} f(\gamma x) = \gamma \cdot f(x) \right\}$$

endowed with the compact-open topology and the left Γ-action $(\lambda \cdot f)(x) = f(x\lambda)$ for $x \in \Gamma$ and $\lambda \in \Gamma$. Non-equivariantly, Y_γ is homeomorphic to the product $\prod_{\langle \gamma \rangle \backslash \Gamma} X_\gamma$, which is a Cantor set. One easily verifies that the product measure μ_γ of the ν_γ is invariant under the Γ-action on Y_γ. Finally, we define X to be the product

$$X := \prod_{\gamma \in \Gamma} Y_\gamma$$

endowed with diagonal Γ-action and the product measure of the measures μ_γ. The product measure is clearly Γ-invariant. As a countable product of Cantor sets, X is a Cantor set. It remains to show that the Γ-action on X is free. Let $x = (y_\gamma) \in X$ and $\gamma_0 \in \Gamma$. Assume that $\gamma_0 \cdot x = (\gamma_0 \cdot y_\gamma)_{\gamma \in \Gamma} = (y_\gamma)_{\gamma \in \Gamma}$. Since the $\langle \gamma_0 \rangle$-action on Y_{γ_0} is free, it implies that $\gamma_0 = e$. $\square$

2.2 Equivariant CW-complexes. We recall some terminology concerning equivariant CW-complexes and classifying spaces. For the notion of an *(equivariant) Γ-CW-complex* we refer to [Tom87, Section II.1]. The skeleta $N^{(n)}$ of a Γ-CW-complex N are built inductively via Γ-pushouts of the form

$$
\begin{array}{ccc}
\coprod_{i \in I_n} \Gamma/H_i \times S^{n-1} & \longrightarrow & N^{(n-1)} \\
\downarrow & & \downarrow \\
\coprod_{i \in I_n} \Gamma/H_i \times D^n & \longrightarrow & N^{(n)}
\end{array}
$$

The conjugates of the subgroups H_i, $i \in I_n$, $n \geq 0$, are precisely the isotropy groups of the Γ-space N. If all subgroups H_i are trivial, then N is a *free* Γ-CW-complex. If all subgroups H_i are finite, then N is a *proper* Γ-CW complex. The universal cover of a CW-complex with fundamental group Γ has a natural structure of a free Γ-CW-complex.

A *cellular action* of a discrete group Γ on a CW-complex W is a continuous action of Γ on W such that

(1) for every open cell e and $\gamma \in \Gamma$ the translate γe is an open cell and
(2) if $\gamma \in \Gamma$ fixes an open cell set-wise then it does so point-wise.

A CW-complex with a cellular Γ-action is a Γ-CW-complex in the sense of [Tom87, p. 98] (see [Tom87, Proposition (1.15) on p. 101]), which means that is obtained from glueing equivariant cells $\Gamma/H \times D^k$ along their boundaries $\Gamma/H \times S^{k-1}$ where $H < \Gamma$ is a subgroup.

The equivariant homotopy category of free Γ-CW complexes possesses a terminal object which is denoted by $E\Gamma$. The space $E\Gamma$ is unique up to equivariant homotopy and called the *classifying space of* Γ. The quotient of $E\Gamma$ is commonly denoted by $B\Gamma$ and also called *classifying space*. Each free Γ-CW-complex admits an equivariant map to the classifying space of Γ. Any such map—they are unique up to homotopy— is called *classifying map*.

2.3 Rectangular complexes. A *rectangular complex* is a M_κ-polyhedral complex with $\kappa = 0$ in the sense of Bridson–Haefliger [BH99, Definition 7.37 on p. 114] such that each cell is isometric to a Euclidean d-cuboid $[0, a_1] \times [0, a_2] \times \cdots \times [0, a_d] \subset \mathbb{R}^d$ and the intersection of two cells is either empty or a single face. We recall some terminology and basic facts from the book of Bridson–Haefliger [BH99, Chapter I.7].

The faces of $[0, a]$ are just $\{0\}$, $\{a\}$ and $[0, a]$. The *faces* of a Euclidean d-cuboid $[0, a_1] \times \cdots \times [0, a_d]$ are the subsets given by $F_1 \times \cdots \times F_d$ where each F_i is a face of $[0, a_i]$. Faces of dimensions 0 and 1 are also called *vertices* and *edges*, respectively. The *barycenter* of a Euclidean d-cuboid $C = [0, a_1] \times [0, a_2] \times \cdots \times [0, a_d]$ with $d > 0$ is the point $(\frac{1}{2}a_1, \ldots, \frac{1}{2}a_d)$. It lies in the interior of C and is fixed by any isometry of C. The barycenter of a vertex is the vertex itself.

A rectangular complex has the structure of a CW-complex with the cells corresponding to the Euclidean cuboids. Depending on the context, we refer to the latter as cells or (Euclidean) cuboids or faces. A rectangular complex is endowed with the path metric that is induced by the Euclidean metric on each Euclidean cuboid.

The second barycentric subdivision of a rectangular complex is simplicial complex, even a M_0-simplicial complex [BH99, Proposition 7.49 on p. 118].

Let J be a, possibly countably infinite, index set. The real vector space with basis J will be denoted by $\mathbb{E}^J$. We regard $\mathbb{E}^J$ as the vector space of real sequences indexed over J that have only finitely many non-zero components. We endow $\mathbb{E}^J$ with the Euclidean norm and metric.

For a family $(a_j)_{j \in J}$ of positive real numbers we will define a rectangular complex

$$N\big((a_j)_{j \in J}\big) \subset \mathbb{E}^J$$

as a subset of $\mathbb{E}^J$ in the following way. The vertices of N are the sequences of $\mathbb{E}^J \setminus \{0\}$ whose j-component, $j \in J$, is either 0 or a_j. Two vertices are *adjacent* if they differ in exactly one component. A family of 2^k vertices span a k-face (or k-cell, or k-cuboid) given by their convex hull if each vertex is adjacent to exactly k vertices. We call $N\big((a_j)_{j \in J}\big)$ the *rectangular complex associated to the family* $(a_j)_{j \in J}$.

To see that the previous definition yields a rectangular complex we have to verify that the intersection of two faces is empty or a single face. To this end, we start with following remark.

REMARK 2.2. A face F in $N((a_j)_{j \in J})$ is a subset of $\mathbb{E}^J$ of the following type: There is a finite subset $J' \subset J$ and there are $c_j \in \{0, a_j\}$ for every $j \in J \setminus J'$ with $(c_j)_{j \in J \setminus J'} \neq 0$ such that

$$F = \big\{(b_j)_{j \in J} \mid \forall_{j \in J'} \ b_j \in [0, a_j] \wedge \forall_{j \in J \setminus J'} \ b_j = c_j\big\}. \tag{2.1}$$

Vice versa, every such subset is a face in $N((a_j)_{j \in J})$, namely the convex hull of the following set of vertices of cardinality $2^{\#J'}$

$$\big\{(b_j)_{j \in J} \mid \forall_{j \in J'} \ b_j \in \{0, a_j\} \wedge \forall_{j \in J \setminus J'} \ b_j = c_j\big\}.$$

Depending on F, we define the following four subsets of the index set J:

$$J_0(F) := \big\{j \in J \setminus J' \mid c_j = 0\big\}, \qquad J_{\frac{1}{2}}(F) := J',$$

$$J_1(F) := \big\{j \in J \setminus J' \mid c_j = a_j\big\}, \qquad J_+(F) := J_1(F) \cup J_{\frac{1}{2}}(F).$$

Equivalently we could define $J_0(F)$, $J_{\frac{1}{2}}(F)$, and $J_1(F)$ as the subset of indices $j \in J$ for which the j-component of the barycenter of F is 0, $\frac{1}{2}a_j$, and 1, respectively.

Let us consider two faces

$$F = \big\{(b_j)_{j \in J} \mid \forall_{j \in J_{\frac{1}{2}}(F)} \ b_j \in [0, a_j] \wedge \forall_{j \in J \setminus J_{\frac{1}{2}}(F)} \ b_j = c_j\big\},$$

$$\tilde{F} = \big\{(b_j)_{j \in J} \mid \forall_{j \in J_{\frac{1}{2}}(\tilde{F})} \ b_j \in [0, a_j] \wedge \forall_{j \in J \setminus J_{\frac{1}{2}}(\tilde{F})} \ b_j = \tilde{c}_j\big\}.$$

The intersection

$$F \cap \tilde{F} = \big\{(b_j)_{j \in J} \mid \forall_{j \in J_{\frac{1}{2}}(F) \cap J_{\frac{1}{2}}(\tilde{F})} \ b_j \in [0, a_j] \wedge \forall_{j \notin J_{\frac{1}{2}}(F)} \ b_j = c_j \wedge \forall_{j \notin J_{\frac{1}{2}}(\tilde{F})} \ b_j = \tilde{c}_j\big\}$$

is empty or again of the type (2.1) and thus a single face of $N((a_j)_{j \in J})$. We conclude that $N((a_j)_{j \in J})$ is indeed a rectangular complex.

Let F be a face in $N((a_j)_{j \in J})$. We denote the dimension of F by $d(F)$. By definition of the rectangular complex we have $J_1(F) \neq \emptyset$. One has $d(F) = \#J_{\frac{1}{2}}(F)$. Further, F is a cuboid with side lengths a_j, $j \in J_{\frac{1}{2}}(F)$. For every face F we enumerate these side lengths by

$$r_1(F), \ldots, r_{d(F)}(F) \quad \text{such that} \quad r_1(F) \leq \cdots \leq r_{d(F)}(F). \tag{2.2}$$

2.4   Rectangular nerves of covers.   We recall the definition of the rectangular nerve of a cover by balls which was introduced by Guth.

Definition 2.3. Let $\mathcal{V} = \{B_j \mid j \in J\}$ be a cover of a Riemannian manifold W by open balls such that the balls $\frac{1}{2}B_j$ still cover W. Let r_j be the radius of the ball B_j. The *rectangular nerve* $N(\mathcal{V})$ of $\mathcal{V}$ is the subcomplex of $N((r_j)_{j \in J})$ whose faces F are precisely the ones for which

$$\bigcap_{j \in J_+(F)} B_j \neq \emptyset$$

and $J_1(F) \neq \emptyset$.

We turn to the equivariant setting with regard to the action of the fundamental group $\Gamma = \pi_1(M)$ on the universal cover $\widetilde{M}$.

LEMMA 2.4. *Let J be a free cofinite Γ-set, and $\mathcal{V} = \{B_j \mid j \in J\}$ be an equivariant cover of $\widetilde{M}$ by balls B_j of radius r_j in the sense that $\gamma B_j = B_{\gamma j}$ for every $j \in J$ and $\gamma \in \Gamma$. Then $N(\mathcal{V})$ is a locally finite rectangular complex. The left shift action*

$$\Gamma \curvearrowright \prod_{j \in J} [0, r_j], \ \gamma \cdot (x_j)_{j \in J} = (x_{\gamma^{-1}j})_{j \in J}$$

restricts to a proper Γ-action on $N(\mathcal{V})$ that permutes cells. Further, the barycentric subdivision of $N(\mathcal{V})$ is a proper Γ-CW-complex.

Proof. Since Γ is cofinite and and the Γ-action on $\widetilde{M}$ by deck transformations is proper, the cover $\mathcal{V}$ is locally finite. Hence $N(\mathcal{V})$ is a locally finite CW-complex. Clearly, the action permutes cells, thus the action satisfies property (1) of a cellular action. Each stabilizer of a cell is contained in the set-stabilizer of a finite subset of J, thus is a finite group. The CW-structure of $N(\mathcal{V})$ does not necessarily satisfy property (2) of a cellular action. Next we show that its barycentric subdivision does. A k-face of the barycentric subdivision is given by the convex hull of the barycenters of a strictly ascending chain $F_0 \subset F_1 \subset \cdots \subset F_k$ where F_i is an i-face of $N(\mathcal{V})$. Let $\gamma \in \Gamma$ fix the k-face C associated with $F_0 \subset F_1 \subset \cdots \subset F_k$ as a set. Then γ fixes each face F_i as a set. Since F_0 is a vertex, γ fixes F_0 pointwise. By induction we may assume that γ fixes F_i pointwise for $i < k$. Since γ fixes the $i + 1$-dimensional face F_{i+1} as a set and its i-dimensional subface F_i pointwise, it must fix F_{i+1} pointwise. Hence γ fixes C pointwise. So the Γ-action on the barycentric subdivision is cellular. The stabilizers of cells are finite as discussed above, which means that the barycentric subdivision of $N(\mathcal{V})$ is a proper Γ-CW-complex. $\square$

3 Homological Preliminaries

In Section 3.1 we review Thurston's measure homology which is isomorphic and isometric to real singular homology on CW-complexes but has better functorial properties with regard to Cantor bundles which are considered later. In Section 3.2 we discuss normed abelian groups and chain complexes. There we review integral variants (X-parametrised integral simplicial volume, integral foliated simplicial volume) of the simplicial volume that take an action of the fundamental group on a Cantor set or a probability space into account. In Section 3.3 we collect what we need from Lück's approach to ℓ^2-Betti numbers. We prove a bound on the von Neumann rank (Definition 3.8) via the X-parametrised integral simplicial norm which slightly generalizes a bound of ℓ^2-Betti numbers of a closed manifold by the foliated simplicial volume due to Schmidt [Sch05].

We collect some notation. The space of real-valued and integer-valued continuous functions on X is denoted by $C(X)$ and $C(X; \mathbb{Z})$, respectively. The action of Γ on X induces a (left) action on $C(X)$. Tensor products $M \otimes_{\mathbb{Z}} N$ over the ring $\mathbb{Z}$ are denoted by $M \otimes N$. The integral group ring of Γ is denoted by $\mathbb{Z}[\Gamma]$. Modules over a (non-commutative) ring are assumed to be left modules unless said otherwise. Since $\mathbb{Z}[\Gamma]$ is a ring with involution—induced by $\gamma \mapsto \gamma^{-1}$—we can turn any left $\mathbb{Z}[\Gamma]$-module into a right $\mathbb{Z}[\Gamma]$-module. We do implicitly so if we write $M \otimes_{\mathbb{Z}[\Gamma]} N$ for two left $\mathbb{Z}[\Gamma]$-modules. The singular chain complex of a space Y is denoted by $C_*(Y)$. We write $C_*(Y; \mathbb{R})$ for the singular chain complex with real coefficients. Similar for singular homology. If the group Γ is acting continuously on Y and M is a $\mathbb{Z}[\Gamma]$-module, then we write the equivariant homology and cohomology as

$$H_p^{\Gamma}(Y; M) := H_p\big(M \otimes_{\mathbb{Z}[\Gamma]} C_*(Y)\big) \text{ and } H_{\Gamma}^p(Y; M) := H^p\big(\hom_{\mathbb{Z}[\Gamma]}(C_*(Y), M)\big).$$

The projection $\widetilde{M} \to M$ yields a canonical isomorphism $H_*^{\Gamma}(\widetilde{M}; \mathbb{Z}) \xrightarrow{\cong} H_*(M; \mathbb{Z})$.

3.1 Measure homology. Measure homology replaces the finite linear combinations in singular homology by signed measures on the space of singular simplices. It was invented by Thurston. We recall its basic notions. For more details we refer to [Loh06].

For a topological space N we endow the space of continuous maps $\mathrm{map}(\Delta^n, N)$ from the standard n-simplex to N with the compact-open topology. We define $\mathcal{C}_n(N)$ as the $\mathbb{R}$-vector space of signed Borel measures on $\mathrm{map}(\Delta^n, N)$ that have compact support and finite variation. The elements of $\mathcal{C}_n(N)$ are called *measure chains*. The alternating sum of pushforwards of face maps turn $\mathcal{C}_*(N)$ into a chain complex whose homology is called the *measure homology* of N.

The variation of measures induces a seminorm on the measure homology. The map from the singular chain complex to the chain complex of measure chains $C_*(N) \to \mathcal{C}_*(N)$ that sends a singular n-simplex to the point measure on that simplex is a natural chain homomorphism, which induces an isometric isomorphism in homology [Loh06].

3.2 Norms on abelian groups and chain complexes. We consider norms and seminorms on $\mathbb{R}$-modules, i.e. real vector spaces, and on $\mathbb{Z}$-modules, i.e. abelian groups. The defining properties of a (semi-)norm on an $\mathbb{R}$-module make sense for a $\mathbb{Z}$-module A and a function $|\_|\colon A \to \mathbb{R}^{\geq 0}$ with the slight modification that $|r \cdot a|_A = |r| \cdot |a|_A$ is only required for $r \in \mathbb{Z}$ and $a \in A$.

Theorem 3.1. ([Ste85]). *An abelian group endowed with a norm that induces the discrete topology is free.*

The supremum norm on the abelian group $C(X;\mathbb{Z})$ of integer valued continuous functions on X induces the discrete topology. We record the following consequence for later use.

COROLLARY 3.2. *The abelian group $C(X;\mathbb{Z})$ is free.*

A chain complex of $\mathbb{Z}$- or $\mathbb{R}$-modules equipped with a (semi-)norm on each chain group is called a *(semi-)normed chain complex* provided the boundary maps are continuous. We endow the quotient of a (semi-)normed module with the quotient semi-norm. In general, a norm does not induce a norm on the quotient but only a semi-norm. In the context of semi-norms being *isometric* does not imply being *injective*.

The singular chain complexes $C_*(N)$ and $C_*(N;\mathbb{R})$ of a topological space N with integer or real coefficients, respectively, are normed via the ℓ^1-norm with respect to the basis by singular simplices. They induce semi-norms on $H_*(N)$ and $H_*(N;\mathbb{R})$, respectively. The latter is denoted by $\|\_\|$ and called *simplicial norm*. The induced chain homomorphism and homology homomorphism of a map of spaces do not increase the simplicial norms. Gromov and Thurston defined the *simplicial volume* of a closed manifold M as the simplicial norm of its fundamental class $[M]$. We denote it by $\|M\|$.

The ℓ^1-norm on $C(X;\mathbb{Z})$ with respect to the measure μ and the simplicial norm induce the following norm on each abelian group $C(X;\mathbb{Z}) \otimes C_p(N)$ which we call the *X-parametrised integral simplicial norm* and denote by $\|\_\|_{\mathbb{Z}}^X$: For functions $f_1, \ldots, f_k \in C(X;\mathbb{Z})$ and distinct singular p-simplices $\sigma_1, \ldots, \sigma_k$ we set

$$\|f_1 \otimes \sigma_1 + \cdots + f_k \otimes \sigma_k\|_{\mathbb{Z}}^X := \int_X |f_1| d\mu + \cdots + \int_X |f_k| d\mu.$$

Let us now consider the situation where N is a topological space endowed with the action of a group Γ. Then $C_*(N)$ is a chain complex over the group ring $\mathbb{Z}[\Gamma]$. We obtain an induced semi-norm on the quotient $C(X;\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C_p(N)$ of $C(X;\mathbb{Z}) \otimes C_p(N)$ which we call by the same name and denote by the same symbol.

Definition 3.3. Let Y be a connected space with fundamental group Γ and universal cover $\widetilde{Y}$. The composition of chain maps

$$C_*(Y) \overset{\cong}{\longleftarrow} \mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} C_*(\widetilde{Y}) \hookrightarrow C(X;\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C_*(\widetilde{Y})$$

is denoted by j_*^Y. Here the right hand map is induced by the inclusion of constant functions.

REMARK 3.4. Let $i_*^{\mathbb{R}}$ be the change of coefficients $C_*(Y) \to C_*(Y; \mathbb{R})$. We have

$$\left\| j_*^Y(z) \right\|_{\mathbb{Z}}^X \leq \left\| i_*^{\mathbb{R}}(z) \right\|$$

for every chain z in $C_*(Y)$ and thus a similar statement for every homology class. This follows from the fact that invariant measure μ on X yields by integration a chain map

$$C(X; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C_*(\widetilde{Y}) \to \mathbb{R} \otimes_{\mathbb{Z}[\Gamma]} C_*(\widetilde{Y}) \xrightarrow{\cong} C_*(Y; \mathbb{R})$$

that does not increase norms.

Definition 3.5. The *X-parametrised integral simplicial volume* of a connected closed oriented manifold M with fundamental group Γ is defined as

$$\|M\|_{\mathbb{Z}}^X := \left\| j_*^M([M]) \right\|,$$

where $[M] \in H_d(M)$ is the fundamental class.

Note that we take the liberty to skip the dependency on the measure μ in the notation of the *X*-parametrised integral simplicial volume.

REMARK 3.6 (*Relation to integral foliated simplicial volume*). Let us denote the free and probability measure preserving (*pmp*) action of Γ on (X, μ) by α. Then $\|M\|_{\mathbb{Z}}^X$ only depends on the measure isomorphism class of α, and $\|M\|_{\mathbb{Z}}^X$ coincides with the α-parametrised simplicial volume $|M|^\alpha$ as defined in [FLPS16, Definition 2.2]. The *integral foliated simplicial volume* is defined as the infimum of α-parametrised simplicial volumes over all free measurable pmp actions α of Γ, and is thus bounded from above by the *X*-parametrised integral simplicial volume. We refer to [FLPS16, Sch05] for more details.

3.3 ℓ^2-Betti numbers.   We use Lück's approach to ℓ^2-Betti numbers which is based on the dimension function for modules over finite von Neumann algebras. This is not just a matter of taste as it is important in our context to work with singular chains and to be able to read off ℓ^2-Betti numbers from the singular chain complex instead of the simplicial chain complex.

Lück [Luc98] defines a dimension function $\dim_A$ taking values in $[0, \infty]$ for arbitrary modules over a von Neumann algebra A with a finite trace, where A is regarded just as a ring, not as functional-analytic object. Our most important example is the *group von Neumann algebra* $L(\Gamma)$ with its canonical trace. The complex group ring $\mathbb{C}[\Gamma]$ is a subring of $L(\Gamma)$. The trace of an element in $\mathbb{C}[\Gamma]$ is the coefficient of 1_Γ. The involution of $\mathbb{C}[\Gamma]$ induced by complex conjugation and taking inverses extends to an involution of $L(\Gamma)$ which corresponds to taking adjoint operators. In particular,

we can turn any left $L(\Gamma)$-module into a right $L(\Gamma)$-module via this involution. The p-th ℓ^2-*Betti number of a* Γ-*space* Y is then defined as

$$\beta_p^{(2)}(Y;\Gamma) := \dim_{L(\Gamma)} H_p^\Gamma\big(Y;L(\Gamma)\big).$$

In the case of the universal covering $\widetilde{M} \to M$ and $\Gamma = \pi_1(M)$ we simply write $\beta_p^{(2)}(M)$ instead of $\beta_p^{(2)}(\widetilde{M};\Gamma)$ and call it the p-*th* ℓ^2-*Betti* number of M. In the case of Riemannian manifolds and simplicial complexes the above definitions coincide with those by Atiyah and Dodziuk, respectively. For more information and proofs we refer to Lück's book [Luc02].

Next we describe another von Neumann algebra whose relevance to ℓ^2-Betti numbers became clear in the work of Gaboriau [Gab02]. The probability space (X,μ) from Theorem 2.1 gives rise to the abelian von Neumann algebra $L^\infty(\mu)$ of complex-valued measurable functions on X with the integral as finite trace. The measure preserving action of Γ induces a unitary Γ-action on $L^\infty(\mu)$. One can then form the *crossed product von Neumann algebra* $L^\infty(\mu)\bar{\rtimes}\Gamma$ which contains $L(\Gamma)$ and $L^\infty(\mu)$ as subalgebras and which possesses a (unique) finite trace that extends those of $L(\Gamma)$ and $L^\infty(\mu)$. For $\gamma \in \Gamma \subset \mathbb{C}[\Gamma] \subset L(\Gamma)$ and $f \in L^\infty(\mu)$ we have

$$\gamma \cdot f = f\big(\gamma^{-1}{}_-\big) \cdot \gamma \in L^\infty(\mu)\bar{\rtimes}\Gamma.$$

The involution on $L^\infty(\mu)\bar{\rtimes}\Gamma$ extends the one of $L(\Gamma)$ and the complex conjugation on $L^\infty(\mu)$. We indicate the involution in all cases with a bar. We refer for more information to [Gab02, Sau05].

The following theorem was suggested by ideas of Connes and Gromov and was proved in the PhD thesis of Schmidt [Sch05].

Theorem 3.7. *Every* ℓ^2-*Betti number of a closed oriented manifold is bounded from above by its* X-*parametrised integral simplicial volume.*

We formulate a slightly more general version (Theorem 3.10) based on the notion of *von Neumann rank* which is defined below. The proof of Theorem 3.10 can be extracted from Schmidt's proof of Theorem 3.7. To make it easier for the reader we present a proof of Theorem 3.10 which is a streamlined version of Schmidt's method.

Some preparations are in order. Let C_* be a chain complex of left $\mathbb{Z}[\Gamma]$-modules. We denote by C^{-*} the chain complex whose p-th chain module is $\hom_{\mathbb{Z}[\Gamma]}\big(C_{-p},\mathbb{Z}[\Gamma]\big)$ with the induced differential. We may extend chain complexes that are indexed over non-negative degrees like the singular chain complex to all degrees in $\mathbb{Z}$ by setting them zero in negative degrees. Since the group ring is a ring with involution we may regard te module C^{-*} which is naturally a right $\mathbb{Z}[\Gamma]$-module as a left $\mathbb{Z}[\Gamma]$-module. Let D_* be another $\mathbb{Z}[\Gamma]$-chain complex. We consider the following commutative diagram of $\mathbb{Z}$-chain complexes:

$$\begin{array}{ccc}
\mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} \left(C_* \otimes_{\mathbb{Z}} D_* \right) & \longrightarrow & \hom_{\mathbb{Z}[\Gamma]}\left(C^{-*}, D_* \right) \\
\downarrow & & \downarrow \\
L^\infty(\mu) \otimes_{\mathbb{Z}[\Gamma]} \left(C_* \otimes_{\mathbb{Z}} D_* \right) & \longrightarrow & \hom_{L^\infty(\mu) \bar{\rtimes} \Gamma}\left(L^\infty(\mu) \bar{\rtimes} \Gamma \otimes_{\mathbb{Z}[\Gamma]} C^{-*}, L^\infty(\mu) \bar{\rtimes} \Gamma \otimes_{\mathbb{Z}[\Gamma]} D_* \right)
\end{array} \quad (3.1)$$

The tensor product of chain complexes $C_* \otimes D_*$ is itself a $\mathbb{Z}[\Gamma]$-chain complex via the diagonal Γ-action. The complex on the upper right is the hom-complex; its p-th chain group consists of chain maps $C^{-*} \to D_*$ of degree p; its p-th homology consists of the group of chain homotopy classes of degree p chain maps, which we denote by $[C^{-*}, D_*]$. We refer to [Bro94, I.0] for a detailed description of these standard constructions of chain complexes. The left vertical map comes from the inclusion of constant functions. The right vertical map is the induction from $\mathbb{Z}[\Gamma]$ to $L^\infty(\mu)\bar{\rtimes}\Gamma$. The upper horizontal arrow sends $1 \otimes x \otimes y$ to the map $g \mapsto \overline{g(x)} \cdot y$ for $g \in C^{-*}$. The lower horizontal arrow is the map

$$f \otimes x \otimes y \mapsto \left(a \otimes g \mapsto a \cdot \overline{f \cdot g(x)} \otimes y \right). \quad (3.2)$$

To verify that this map is well defined we check that $f(\gamma_{\text{-}}) \otimes x \otimes y$ and $f \otimes \gamma x \otimes \gamma y$ have the same image. This follows from

$$a\overline{f(\gamma_{\text{-}})g(x)} \otimes y = a\overline{f(\gamma_{\text{-}})g(x)}\gamma^{-1} \otimes \gamma y = a\overline{\gamma f(\gamma_{\text{-}})g(x)} \otimes \gamma y = a\overline{f(\text{-})\gamma g(x)} \otimes \gamma y$$
$$= a\overline{f(\text{-})g(\gamma x)} \otimes \gamma y.$$

We leave the verification of the property of being a chain map to the reader.

Next let Y be a topological space with a free Γ-action. Set $C_* = D_* = C_*(Y)$. Let $A_* \colon C_*(Y \times Y) \to C_*(Y) \otimes_{\mathbb{Z}} C_*(Y)$ be the Alexander-Whitney map, and let $\Delta_* \colon C_*(Y) \to C_*(Y \times Y)$ be the map induced by the diagonal embedding. If we compose the horizontal maps in the commutative square above with the chain maps $\mathrm{id}_{\mathbb{Z}} \otimes_{\mathbb{Z}[\Gamma]} A_* \circ \Delta_*$ and $\mathrm{id}_{L^\infty(\mu)} \otimes_{\mathbb{Z}[\Gamma]} A_* \circ \Delta_*$, respectively, and take homology, we obtain the following commutative square. The left vertical map is induced by the inclusion of constant functions.

$$\begin{array}{ccc}
H_d(\Gamma \backslash Y) \cong H_d^\Gamma(Y; \mathbb{Z}) & \xrightarrow{\cap_{\text{-}}} & \left[C^{-*}(Y), C_{d+*}(Y) \right] \\
\downarrow & & \downarrow \\
H_d^\Gamma\left(Y; L^\infty(\mu) \right) & \longrightarrow & \left[L^\infty(\mu) \bar{\rtimes} \Gamma \otimes_{\mathbb{Z}[\Gamma]} C^{-*}(Y), L^\infty(\mu) \bar{\rtimes} \Gamma \otimes_{\mathbb{Z}[\Gamma]} C_{d+*}(Y) \right]
\end{array}$$
$$(3.3)$$

The upper and lower horizontal maps are variants of the cap product.

Definition 3.8. Let Z be a connected space with fundamental group Γ. Let $x \in H_d(Z)$. The *von Neumann rank* of x is defined as

$$\dim_{L(\Gamma)}\left(\bigoplus_{n \geq 0} \mathrm{im}\left(H_\Gamma^n\left(\widetilde{Z}; L(\Gamma) \right) \xrightarrow{x \cap_{\text{-}}} H_{d-n}^\Gamma\left(\widetilde{Z}; L(\Gamma) \right) \right) \right).$$

REMARK 3.9. Let M be a closed oriented d-manifold. The sum of the ℓ^2-Betti numbers of M is the von Neumann rank of its fundamental class. This is direct consequence of (equivariant) Poincare duality which says that the image of the fundamental class under the upper horizontal map in (3.3) is chain homotopy equivalence.

Theorem 3.10. *Let Z be a connected space with fundamental group Γ. Then the von Neumann rank of a homology class $[x] \in H_d(Z)$ is bounded from above by $d \cdot \big\| [j_*^Z(x)] \big\|_{\mathbb{Z}}^X$.*

Proof. Suppose the image of x under the map induced by inclusion of constant functions is homologous to a cycle $\sum_{k=1}^m a_k \otimes \sigma_k$ where $a_k \in C(X; \mathbb{Z})$ and σ_k is a singular d-simplex in $\widetilde{Z}$. Via the embedding $C(X; \mathbb{Z}) \hookrightarrow L^\infty(\mu)$ we obtain a homology class $[\sum_{k=1}^m a_k \otimes \sigma_k] \in H_d^\Gamma(\widetilde{Z}; L^\infty(\mu))$. The cap product with $\sum_{k=1}^m a_k \otimes \sigma_k$, that is the image of $\sum_{k=1}^m a_k \otimes \sigma_k$ under the lower horizontal map of (3.3), is represented by the $L^\infty(\mu) \bar{\rtimes} \Gamma$-chain homomorphism whose degree i part is

$$L^\infty(\mu) \bar{\rtimes} \Gamma \otimes_{\mathbb{Z}[\Gamma]} C^i(\widetilde{Z}) \to L^\infty(\mu) \bar{\rtimes} \Gamma \otimes_{\mathbb{Z}[\Gamma]} C_{d-i}(\widetilde{Z}), \quad 1 \otimes g$$

$$\mapsto \sum_{k=1}^m \overline{a_k \cdot g(\sigma_k \rfloor_i)} \otimes \sigma_k \lfloor_{d-i}.$$

Here $\sigma \rfloor_i$ and $\sigma \lfloor_{d-i}$ denote the front i-face and the back $(d-i)$-face of σ respectively. It clearly factorizes over the $L^\infty(\mu) \bar{\rtimes} \Gamma$-homomorphism

$$L^\infty(\mu) \bar{\rtimes} \Gamma \otimes_{\mathbb{Z}[\Gamma]} C^i(\widetilde{Z}) \to \bigoplus_{k=1}^m L^\infty(\mu) \bar{\rtimes} \Gamma \cdot \chi_{\mathrm{supp}(a_k)},$$

$$y \otimes g \mapsto \big(y \overline{a_k g(\sigma_k \rfloor_i)} \big)_k = \big(y \overline{g(\sigma_k \rfloor_i)} a_k \big)_k.$$

Further, we have

$$\dim_{L^\infty(\mu) \bar{\rtimes} \Gamma} \Big(\bigoplus_{k=1}^m L^\infty(\mu) \bar{\rtimes} \Gamma \cdot \chi_{\mathrm{supp}(a_k)} \Big) = \sum_{k=1}^m \mu(\mathrm{supp}(a_k)) \leq \big\| \sum_{k=1}^m a_k \otimes \sigma_k \big\|_{\mathbb{Z}}^X.$$

The last inequality uses the fact that a_k is integer-valued. Hence the $L^\infty(\mu) \bar{\rtimes} \Gamma$-dimension of the image of the cap product with $[\sum_{k=1}^m a_k \otimes \sigma_k]$

$$\bigoplus_{n \geq 0} \mathrm{im} \Big(H_\Gamma^n(\widetilde{Z}; L^\infty(\mu) \bar{\rtimes} \Gamma) \to H_{d+n}^\Gamma(\widetilde{Z}; L^\infty(\mu) \bar{\rtimes} \Gamma) \Big) \tag{3.4}$$

is bounded by $\big\| [j_*^Z(x)] \big\|_{\mathbb{Z}}^X$. It remains to verify that the $L^\infty(\mu) \bar{\rtimes} \Gamma$-dimension of (3.4) is the von Neumann rank of $[x]$. Since $L(\Gamma) \subset L^\infty(\mu) \bar{\rtimes} \Gamma$ is a flat ring extension [Sau05, Theorem 4.3], we obtain that

$$\bigoplus_{n \geq 0} \mathrm{im} \Big(H_\Gamma^n(\widetilde{Z}; L^\infty(\mu) \bar{\rtimes} \Gamma) \to H_{d+n}^\Gamma(\widetilde{Z}; L^\infty(\mu) \bar{\rtimes} \Gamma) \Big)$$

$$\cong L^\infty(\mu) \bar{\rtimes} \Gamma \otimes_{L(\Gamma)} \bigoplus_{n \geq 0} \mathrm{im} \Big(H_\Gamma^n(\widetilde{Z}; L(\Gamma)) \to H_{d+n}^\Gamma(\widetilde{Z}; L(\Gamma)) \Big)$$

where the maps on the right hand side are induced by the cap product with $[x]$. Since the von Neumann dimension is compatible with induction [Sau05, Theorem 2.6], the proof is finished. □

4 The Category of Cantor Bundles

In 4.1 we introduce the central notion of a Cantor bundle. A Cantor bundle comes with a map to the Cantor set X. In general, a Cantor bundle is not a locally trivial bundle over X. See Example 4.9. It is, however, locally trivial, when restricted to compacta (see Lemma 4.3). Metric Cantor bundles are also introduced which are Cantor bundle whose fibers over X come with the structure of a metric space. In 4.2 we define Cantor bundle maps. We also consider pushouts of Cantor bundles. In 4.3 we study the functoriality of certain chain complexes attached to Cantor bundles.

4.1 Cantor bundles. We define the data of a product atlas for a space over X.

Definition 4.1. Let W be a topological space and $\mathrm{pr}\colon W \to X$ a continuous map to the Cantor set. We denote the fiber over $x \in X$ by W_x. For $A \subset X$ we write $W|_A$ for $\mathrm{pr}^{-1}(A) \subset W$. We introduce the notion of a *product atlas* for W:

▷ A *product chart* for W consists of a clopen subset $A \subset X$, an open subset $U \subset W$, a space F, and a homeomorphism $U \to A \times F$ over A.

▷ Two product charts $c_i\colon U_i \to A_i \times F_i$, $i \in \{1,2\}$ are *compatible* if there are subspaces $F_i' \subset F_i$ such that $c_i(U_1 \cap U_2) = (A_1 \cap A_2) \times F_i'$ and the transition map

$$c_2 \circ c_1^{-1}\colon (A_1 \cap A_2) \times F_1' \to (A_1 \cap A_2) \times F_2'$$

is a product of $\mathrm{id}_{A_1 \cap A_2}$ and a homeomorphism $g\colon F_1' \to F_2'$.

▷ A *product atlas* for W consists of a family of compatible product charts whose domains cover W; it is *maximal* if it contains every product chart that is compatible with the product charts of the atlas.

If in the definition of a product atlas we would replace the Cantor set by a connected space then the existence of a product atlas for $W \to X$ would imply that $W \to X$ is trivial. This in stark contrast to our situation.

Definition 4.2. Let $\mathrm{pr}\colon W \to X$ be a topological space over X endowed with a maximal product atlas.

▷ A relatively compact subset $K \subset W$ is called a *box* if there is a product chart $c\colon U \to A \times F$ such that $K \subset U$ and $c(K) = A \times F'$ for a subspace $F' \subset F$.

▷ For a box K and for all $x, y \in A := \mathrm{pr}(K)$ the map

$$\tau_{x,y}\colon K_x \xrightarrow{\cong} K_y$$

defined by

$$\tau_{x,y}(p) = c^{-1}\big(y, \mathrm{pr}_2(c(p))\big)$$

for a choice of product chart $K \subset U \xrightarrow{c} A \times F$ is independent of the choice of chart. We say that $\tau_{x,y}$ is the *parallel transport* inside the box K.

LEMMA 4.3. *Let* $\mathrm{pr}\colon W \to X$ *be a locally compact Hausdorff space over X endowed with a maximal product atlas. For every compact subset $K \subset W$ there is a relatively compact, open subset $L \subset W$ containing K and a clopen partition $X = A_1 \cup \cdots \cup A_n$ such that $L|_{A_i}$ is a box for each $i \in \{1, \ldots, n\}$. Further, if K is a finite union of open boxes, we may choose $L = K$.*

Proof. Since W is locally compact, each point lies in an open box. Since K is compact it is covered by finitely many open boxes $B_1, \ldots, B_n$. Let L be the union of these boxes. Every box is relatively compact, and so is L. Since the clopen subsets of X form a set algebra, there is a clopen partition $A_1, \ldots, A_m$ of $\mathrm{pr}(L)$ that is subordinate to $\mathrm{pr}(B_1), \ldots, \mathrm{pr}(B_n)$.

We claim that $L|_{A_i}$ is box: Pick $x_0 \in A_i$. We construct a product chart

$$f_i \colon L|_{A_i} \to A_i \times (W_{x_0} \cap L)$$

as follows. Every $p \in L_x \subset L|_{A_i}$ lies in a box B_k with $A_i \subset \mathrm{pr}(B_k)$. We set $f_i(p) = (x, \tau_{x,x_0}^{B_k}(p))$ where $\tau_{x,x_0}^{B_k}$ is the parallel transport inside B_k. We have

$$\tau_{x,x_0}^{B_k}(p) = \tau_{x,x_0}^{B_l}(p) \text{ for } p \in W_x \cap B_k \cap B_l \text{ and } \{x, x_0\} \subset \mathrm{pr}(B_k \cap B_l),$$

hence f_i is well defined. The map f_i is a homeomorphism, and its inverse maps (x, q) to $\tau_{x_0,x}^{B_k}(q)$ for any box B_k with $q \in B_k$.

Since f_i is compatible with all the boxes B_k and its domain is covered by them, f_i lies in the maximal product atlas. So $L|_{A_i}$ is box. Moreover, we can add the complement of $\mathrm{pr}(L)$ to the clopen partition above to get the statement of the lemma. $\qquad\square$

Definition 4.4 (*Cantor bundle*). A *Cantor bundle* is a locally compact Hausdorff space W endowed with a continuous proper Γ-action and a continuous Γ-equivariant map $\mathrm{pr}\colon W \to X$ and a maximal product atlas such that the Γ-action on W has a Borel fundamental domain that is a union of finitely many boxes.

Note that the action on a Cantor bundle is automatically free since it lies over the free action on X.

Definition 4.5. Let $\mathrm{pr}\colon W \to X$ be a Cantor bundle, and let $V \subset W$ be a Γ-invariant subspace so that for every $p \in V$ there is a product chart $U \to A \times F$ such that $p \in U$ and $U \cap V$ is a box. Then we call $\mathrm{pr}|_V\colon V \to X$ a *Cantor subbundle* of $\mathrm{pr}\colon W \to X$.

LEMMA 4.6. *Let $\mathcal{A}$ be a maximal product atlas of a Cantor bundle $W \to X$. A Cantor subbundle $V \subset W$ is a Cantor bundle with respect to the product atlas*

$$\mathcal{A}_V := \left\{ U \xrightarrow{c} A \times F \mid c \in \mathcal{A}, U \cap V \text{ is a box} \right\}.$$

Proof. The only non-obvious statement is the existence of a fundamental domain for V consisting of finitely many boxes. Let $D \subset W$ be a fundamental domain for W which is a union of boxes $B_1, \ldots, B_n$. Note that D is relatively compact as every B_i is relatively compact. Then $V \cap D$ is a fundamental domain for V. At every point $p \in V \cap D$ we can choose a product chart with a domain $U_p \ni p$ such that $U_p \cap V$ is a box. By relative compactness we can cover $V \cap D$ with finitely many such product chart domains $U_{p_1}, \ldots, U_{p_m}$. Every set

$$U_{p_i} \cap V \cap D = \bigcup_{j=1}^{n} U_{p_i} \cap V \cap B_j$$

is a box, hence $V \cap D$ is a union of such. $\qquad\square$

EXAMPLE 4.7. Let $A \subset X$ be a clopen subset and Y any compact space. We consider the trivial Cantor bundle $X \times (\Gamma \times Y)$ with the projection to the first factor and endowed with the Γ-action

$$\gamma \cdot (x, \gamma', y) = (\gamma x, \gamma \gamma', y).$$

Then $A \times \Gamma \times Y$ endowed with the projection

$$A \times \Gamma \times Y \to X, \ (a, \gamma, y) \mapsto \gamma a$$

and the left translation Γ-action on the second factor is a Cantor subbundle via the embedding

$$A \times \Gamma \times Y \hookrightarrow X \times \Gamma \times Y, \ (a, \gamma, y) \mapsto (\gamma a, \gamma, y).$$

REMARK 4.8 (*Finite isotropy disappears*). Let $H < \Gamma$ be a finite subgroup. Then $X \times \Gamma/H$ is a Cantor bundle with the projection to X and the diagonal Γ-action. Let $A \subset X$ be a Borel fundamental domain for the H-action on X. Then

$$X \times \Gamma/H \cong A \times \Gamma$$

are isomorphic as Cantor bundles where the latter is the one from the previous example. An isomorphism is given by

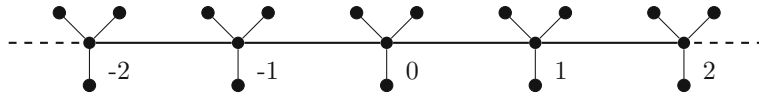$$A \times \Gamma \to X \times \Gamma/H, \ (a, \gamma) \mapsto (\gamma a, \gamma H).$$
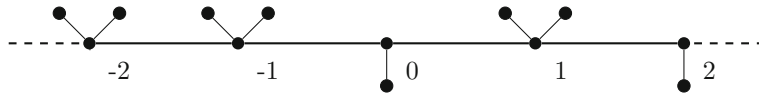
Figure 1: The simplicial complex L.

Figure 2: L_x for $x = (\ldots, 1, 1, \mathbf{0}, 1, 0, \ldots) \in \{0, 1\}^{\mathbb{Z}}$.

EXAMPLE 4.9 (*Non-trivial Cantor bundle*). We describe an example of a Cantor bundle whose fibers exhibit uncountably many homeomorphism types. In particular, it is not a trivial Cantor bundle. Let $\Gamma = \mathbb{Z}$ and let X be a minimal subshift of the shift action of $\mathbb{Z}$ on $\{0, 1\}^{\mathbb{Z}}$ such that the $\mathbb{Z}$-action on the Cantor set X is free. Such a minimal subshift exists due to [GU09, Theorem 4.2]. Let L be the following infinite 1-dimensional simplicial complex: (Fig. 1) We have an obvious $\mathbb{Z}$-action on L by translation. For $x = (x_i) \in X \subset \{0, 1\}^{\mathbb{Z}}$ let $L_x \subset L$ be the subcomplex that consists of the horizontal line and of an upward caret at each n with $x_n = 1$ and a downward segment at each n with $x_n = 0$ (Fig. 2).

Then

$$W = \big\{ (x, p) \mid p \in L_x \big\} \subset X \times L$$

is a Cantor subbundle of the trivial Cantor bundle $X \times L$: The diagonal $\mathbb{Z}$-action on $X \times L$ restricts to W. For every $x = (x_i) \in X$ we consider the clopen neighborhood of x

$$A_x(n) = \big\{ (y_i) \in X \mid y_i = x_i \text{ for } i \in \{-n, \ldots, n\} \big\},$$

and let $L_x(n) \subset L_x$ be the finite subgraph obtained from L_x by cutting off the horizontal line at $-n$ and n. We then have

$$W \cap \big(A_x(n) \times L_x(n) \big) = A_x(n) \times L_x(n) \subset X \times L.$$

Running through $x \in X$ and $n \in \mathbb{N}$ we cover all of W. So W is a Cantor subbundle.

Since the valency of L_x at each vertex is at least 3, two fibers L_x and L_y are homeomorphic if and only if they are simplicially isomorphic. Since X is uncountable, W has uncountably many homeomorphism types of fibers.

Definition 4.10 (*Metric and Riemannian Cantor bundles*). A Cantor bundle $\mathrm{pr}\colon W \to X$ is *metric* if

   $\triangleright$ each fiber W_x is endowed with a metric inducing the topology of W_x, and
   $\triangleright$ the maps $W_x \to W_{\gamma \cdot x}$ induced by multiplication are isometries for every $x \in X$ and every $\gamma \in \Gamma$.
   $\triangleright$ for each product chart $c\colon U \to A \times F$ there is a metric on F such that c is fiberwise an isometry.

If, in addition, each fiber W_x is a d-dimensional Riemannian manifold with the induced Riemannian metric, we say that $\mathrm{pr}\colon W \to X$ is a *d-dimensional Riemannian Cantor bundle*.

Finally, *metric Cantor subbundles* are defined similarly to Cantor subbundles.

EXAMPLE 4.11 The product space $X \times \widetilde{M}$ with the diagonal Γ-action is a Riemannian Cantor bundle. Each fiber $\{x\} \times \widetilde{M} \cong \widetilde{M}$ carries the Riemannian metric lifted from the Riemannian metric from M. The maximal product atlas is defined to be the set of all product charts that are compatible with $\mathrm{id}\colon X \times \widetilde{M} \to X \times \widetilde{M}$. The Γ-action possesses a relatively compact Borel fundamental domain $F \subseteq \widetilde{M}$. Then the box $X \times F$ is a Borel fundamental domain of the Γ-action on $X \times \widetilde{M}$.

4.2 Cantor bundle maps.   To obtain a category of Cantor bundles we define the morphisms next.

Definition 4.12 Let V and W be topological spaces over X endowed with maximal product atlases. Let $\Phi\colon V \to W$ be a continuous map over X.

Let $c\colon U_V \to A_V \times F_V$ be a product chart of V. We say that $\Phi|_U$ is a *product map* if there is a product chart $d\colon U_W \to A_W \times F_W$ of W such that $d \circ \Phi \circ c^{-1}\colon A_V \times F_V \to A_W \times F_W$ is a product of the identity on X and a continuous map.

We say that Φ is *locally product-like* if every point of V lies in the domain of a product chart on which Φ is a product map.

Definition 4.13 A continuous map over X between Cantor bundles is called a *Cantor bundle map* if it is Γ-equivariant and locally product-like. A Cantor bundle map between metric Cantor bundles is called *Lipschitz* if there is some $L > 0$ such that it is L-Lipschitz on each fiber.

REMARK 4.14 A Cantor bundle map $V \to W$ is automatically proper since the Γ-actions on V and W are proper and both actions possess relatively compact fundamental domains.

The composition of (Lipschitz) Cantor bundle maps is a (Lipschitz) Cantor bundle map. So we obtain a category of Cantor bundles with Cantor bundle maps as morphisms.

The notion of product map does not depend on the choices of product charts as we show next.

LEMMA 4.15 *Let V and W be locally compact Hausdorff spaces over X equipped with maximal product atlases. Let $\Phi\colon V \to W$ be locally product-like and proper. Then every compact subset $K \subset W$ is contained in a relatively compact, open subset $L \subset W$ such that there is a clopen partition $X = A_1 \cup \cdots \cup A_n$ with the following properties.*

  ▷ *$L|_{A_i}$ is a box for each $i \in \{1, \ldots, n\}$.*
  ▷ *$\Phi^{-1}(L)|_{A_i}$ is a box for each $i \in \{1, \ldots, n\}$.*

▷ *The restriction of* Φ *to*

$$\Phi^{-1}(L)|_{A_i} \to L|_{A_i}$$

is a product map for each $i \in \{1, \dots, n\}$.

If K *is an open box, we may choose* L *to be* K.

Proof Every compact subset of W is contained in a relatively compact, open subset $L \subset W$ with the properties as in Lemma 4.3. We may assume that L itself is a box. Since Φ is proper, $\Phi^{-1}(L)$ is relatively compact as well. We cover $\Phi^{-1}(L)$ by finitely many open boxes B_i, $i \in I$, such that Φ is a product of maps on each box. The intersection of two boxes is a box. The preimage of a box under $\Phi|_{B_i}$ is a box. Hence $\Phi^{-1}(L)$ is the union of boxes $B_i' := \Phi^{-1}(L) \cap B_i$, $i \in I$. On each B_i' the map Φ is a product. Let $X = A_1 \cup \dots \cup A_n$ be a clopen partition subordinate to $\mathrm{pr}_V(B_i')$, $i \in I$. It exists since the clopen sets of X form a set algebra. By the same argument as in the proof of Lemma 4.3 each $\Phi^{-1}(L)|_{A_j}$, $j \in \{1, \dots, n\}$, is a box. As in the proof of Lemma 4.3 one sees that the parallel transport on the boxes B_i' and the choice of some $x_0 \in A_j$ yields a product chart

$$\Phi^{-1}(L)|_{A_j} \xrightarrow{\cong} A_j \times \left(\Phi^{-1}(L) \cap V_{x_0}\right).$$

Similarly for $L|_{A_j}$. Since Φ is a product map on each B_i', it is compatible with the parallel transport within each B_i', and so the restriction of Φ to $\Phi^{-1}(L)|_{A_j} \to L|_{A_j}$ is a product map. □

LEMMA 4.16 *Let* V *and* W *be locally compact Hausdorff spaces over* X *equipped with maximal product atlases. Let* $\Phi \colon V \to W$ *be locally product-like and proper. Then every compact subset* $K \subset V$ *is contained in a relatively compact, open subset* $L \subset V$ *such that there is a clopen partition* $X = A_1 \cup \dots \cup A_n$ *with the following properties.*

▷ $L|_{A_i}$ *is a box for each* $i \in \{1, \dots, n\}$.
▷ *The restriction of* Φ *to* $L|_{A_i}$ *is a product map for each* $i \in \{1, \dots, n\}$.

If K *is an open box, we may choose* L *to be* K.

Proof This follows from applying Lemma 4.15 to the relatively compact subset $\Phi(K)$. The last statement follows from the fact that the intersection of two boxes is a box. □

Next we discuss categorical pushouts in the category of Cantor bundles.

LEMMA 4.17 *A commutative square of Cantor bundles and Cantor bundle maps that is a pushout of topological spaces is a pushout in the category of Cantor bundles.*

Proof Let the following diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & C \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\
B & \xrightarrow{\ g\ } & D
\end{array}
$$

be commutative and consist of Cantor bundles and Cantor bundle maps. We further assume that it is a pushout of topological spaces. Let Z be a Cantor bundle, and let $r\colon B \to Z$ and $s\colon C \to Z$ be Cantor bundle maps compatible with A. By the pushout property there is unique map $t\colon D \to Z$ that makes everything commute. It is obvious that t is equivariant and lies over X. It remains to show that t is locally product-like. Let $p \in D$ be a point over $x \in X$. Let $U \subset D$ be an open box around p. By Lemma 4.16 there are clopen subsets X_C and X_B of X that contain x such that $j^{-1}(U)|_{X_C}$ and $g^{-1}(U)|_{X_B}$ are boxes and the restrictions of j and g to $j^{-1}(U)|_{X_C}$ and $g^{-1}(U)|_{X_B}$ are product maps. By the same lemma we obtain a clopen neighborhood $X_A \subset X_B \cap X_C$ of x such that $i^{-1}(g^{-1}(U))|_{X_A} = f^{-1}(j^{-1}(U))|_{X_A}$ is a box and the restrictions of i and f to $i^{-1}(g^{-1}(U))|_{X_A}$ are product maps. By applying Lemma 4.16 to the boxes $g^{-1}(U)|_{X_A}$ and $j^{-1}(U)|_{X_A}$ we find a smaller clopen neighborhood $X_Z \subset X_A$ of x such that r and s are product maps on these boxes. Now the box $U|_{X_Z}$ is a pushout of the boxes $g^{-1}(U)|_{X_Z}$ and $j^{-1}(U)|_{X_Z}$ along the box $i^{-1}(g^{-1}(U))|_{X_Z}$. All maps in the pushout square are product maps as well as the restrictions of r and s to the corners. Hence t restricted to $U|_{X_Z}$ is a product map. □

EXAMPLE 4.18 (*Continuing Example* 4.9). The Cantor bundle W in Example 4.9 can be written as a pushout. Let

$$
A = \big\{ x \in X \subset \{0,1\}^{\mathbb{Z}} \mid x_0 = 1 \big\}
$$

and $A^c \subset X$ its complement. Let $X \times \mathbb{R}$ be the trivial Cantor bundle with the diagonal $\mathbb{Z}$-action. On $\mathbb{R}$ we consider the usual translation action of $\mathbb{Z}$. The pushout for W can be written semi-formally as

Here we regard a space of the type $A \times \mathbb{Z} \times M$ as a Cantor bundle as in Example 4.7.

LEMMA 4.19 *Let N be a cocompact proper Γ-CW complex. Then $X \times N$ with the diagonal Γ-action is a Cantor bundle. Further, there is a locally product-like map $X \times N \to X \times E\Gamma$ over X to the classifying space of Γ which is Γ-equivariant with respect to the diagonal actions.*

Proof The Γ-CW complex N is built via equivariant pushouts where we successively attach finitely many equivariant cells of the form $\Gamma/H \times D^n$ with finite $H < \Gamma$:

$$
\begin{array}{ccc}
\coprod_{i \in I_n} \Gamma/H_i \times S^{n-1} & \longrightarrow & N^{(n-1)} \\
\downarrow & & \downarrow \\
\coprod_{i \in I_n} \Gamma/H_i \times D^n & \longrightarrow & N^{(n)}
\end{array}
$$

Taking a product with the compact Hausdorff space X preserves pushouts. With Remark 4.8 we see that $X \times N$ is inductively built via finitely many pushouts of the form

$$
\begin{array}{ccc}
\coprod_{i \in I_n} A_i \times \Gamma \times S^{n-1} & \longrightarrow & X \times N^{(n-1)} \\
\downarrow & & \downarrow \\
\coprod_{i \in I_n} A_i \times \Gamma \times D^n & \longrightarrow & X \times N^{(n)}.
\end{array}
$$

There is a Borel fundamental domain of $X \times N$ consisting of the finite union of the products of A_i and the open n-cell associated with $i \in I_n$ over all n and $i \in I_n$. Hence $X \times N$ is a Cantor bundle.

Next we apply repeatedly Lemma 4.17 to the above pushout for $n = 1, 2, \ldots, \dim(N)$ and to the target $X \times E\Gamma$ to construct equivariant, locally product-like maps $X \times N^{(n)} \to X \times E\Gamma$. Strictly speaking, the target $X \times E\Gamma$ is not necessarily a Cantor bundle as required in the lemma since $E\Gamma$ might not be a finite Γ-CW complex. However, the image of the maps below lie in the product of X and a finite Γ-CW subcomplex which allows us to use Lemma 4.17.

On the 0-skeleton $\coprod_{i \in I_0} A_i \times \Gamma$ we define an equivariant and locally product-like map to $X \times E\Gamma$ as the equivariant extension that maps $(a, 1)$ to (a, p) for some chosen point $p \in E\Gamma$. To proceed inductively via the pushout property, we need to extend a continuous locally product-like equivariant map $A_i \times \Gamma \times S^{n-1} \to X \times E\Gamma$ to $A_i \times \Gamma \times D^n$. By decomposing each A_i into a suitable clopen partition we may assume that the restriction $A_i \times \{1\} \times S^{n-1} \to X \times E\Gamma$ is a product of the inclusion $A \hookrightarrow X$ and a continuous map $S^{n-1} \to E\Gamma$. Since $E\Gamma$ is contractible, we can extend the map $A \times \{1\} \times S^{n-1} \to E\Gamma$ to $A \times \{1\} \times D^n$ and then extend further to $A \times \Gamma \times D^n$ by equivariance. The resulting map is locally product-like. $\qquad\square$

4.3 Chains and norms of chains in the context of Cantor bundles.

We consider various chain complexes involving singular chains, locally finite chains and measure chains for Cantor bundles or spaces over X.

The abelian group $C(X; \mathbb{Z})$ carries a left $\mathbb{Z}[\Gamma]$-module structure and via the involution on the group ring also a right $\mathbb{Z}[\Gamma]$-module structure. Let N be a Γ-space. For every $x \in X$ the map

$$\mathrm{ev}_x \colon C(X; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C_*(N) \to C_*^{\mathrm{lf}}(N)$$
$$\sum_i f_i \otimes \sigma_i \mapsto \sum_{\gamma \in \Gamma} \sum_i f_i(\gamma^{-1} x) \gamma \sigma_i$$

to locally finite chains is a chain map [FLPS16, Lemma 2.5]. We consider the map

$$\mathrm{map}(\Delta^n, N) \to \mathcal{C}_n(X \times N), \quad \sigma \mapsto \sigma^X$$

that sends a singular n-simplex σ in N to the finite, compactly supported Borel measure σ^X on $\mathrm{map}(\Delta^n, X \times N)$ characterized by the property

$$\int_{\mathrm{map}(\Delta^n, X \times N)} g \, d\sigma^X = \int_X g\big(\Delta^n \xrightarrow{\sigma} \{x\} \times N \hookrightarrow X \times N\big) d\mu(x)$$

for every compactly supported continuous function g on $\mathrm{map}(\Delta^n, X \times N)$. If $f \in C(X)$, then $f * \sigma^X$ denotes the measure with

$$\int_{\mathrm{map}(\Delta^n, X \times N)} g \, d(f * \sigma^X) = \int_X f(x) g\big(\Delta^n \xrightarrow{\sigma} \{x\} \times N \hookrightarrow X \times N\big) d\mu(x).$$

The map

$$C(X; \mathbb{Z}) \otimes C_*(N) \hookrightarrow \mathcal{C}_*(X \times N)$$
$$\sum_i f_i \otimes \sigma_i \mapsto \sum_i f_i * \sigma_i^X$$

is a Γ-equivariant injective chain map. Here Γ acts diagonally on the left hand side, and the left action on the right hand side is induced by the diagonal action on $X \times N$.

Definition 4.20 The chain map $C(X; \mathbb{Z}) \otimes C_*(N) \hookrightarrow \mathcal{C}_*(X \times N)$ is called the *diffusion embedding*.

LEMMA 4.21 *Let U and V be topological Hausdorff spaces. We endow $X \times U$ and $X \times V$ with the obvious maximal product atlasses. Let $\Phi \colon X \times U \to X \times V$ be a locally product-like map over X. Then there is a chain map, indicated by the dashed arrow, such that the following diagram commutes. Furthermore, this chain is non-increasing with respect to the X-parametrised integral simplicial norm.*

$$
\begin{array}{ccc}
\mathcal{C}_*(X \times U) & \xrightarrow{\ \Phi_* \ } & \mathcal{C}_*(X \times V) \\
\uparrow & & \uparrow \\
C(X; \mathbb{Z}) \otimes C_*(U) & \dashrightarrow & C(X; \mathbb{Z}) \otimes C_*(V)
\end{array}
$$

Here the upper horizontal map is the induced map in measure chains. The vertical maps are the diffusion embeddings. We will also denote the dashed arrow by Φ_*.

Proof Let $\sigma\colon \Delta_p \to U$ and $f \in C(X;\mathbb{Z})$. We apply Lemma 4.15 to the map

$$X \times \Delta^n \xrightarrow{\mathrm{id}_X \times \sigma} X \times U \xrightarrow{\Phi} X \times V.$$

The image of this map is a compact subspace of the Hausdorff space $X \times V$, hence we can apply Lemma 4.15 even if V is not locally compact. As a result there is a finite Borel partition $X = A_1 \cup \cdots \cup A_n$ and there are continuous maps $g_i\colon U \to V$ for $i \in \{1, \ldots, n\}$ such that

$$\left(\Delta_p \xrightarrow{\sigma} \{x\} \times U \xrightarrow{\Phi_x} X \times V \xrightarrow{\mathrm{pr}} V\right) = g_i \circ \sigma \ \text{ for } x \in A_i.$$

Hence Φ_* maps $f \otimes \sigma$ to the measure that is the image of $\sum_{i=1}^n f \cdot \chi_{A_i} \otimes g_i \circ \sigma$ under the diffusion embedding. The statement about the X-parametrised integral simplicial norm follows directly from this description. $\qquad\square$

REMARK 4.22 The advantage of measure homology is that Φ obviously induces a chain map. It automatically follows that its restriction to the complex $C(X;\mathbb{Z}) \otimes C_*(\_)$ is a Γ-equivariant chain map provided Φ is Γ-equivariant, for example a Cantor bundle map. A direct verification of functoriality that avoids measure homology would be more cumbersome.

5   Transverse Measure Theory on Cantor Bundles

In this short section we define the notion of transverse measure which gives, in particular, a finite measure on Γ-invariant Borel subsets of $X \times \widetilde{M}$. As before, μ denotes the Γ-invariant probability measure on the Cantor set X.

Definition 5.1 Let W be a standard Γ-space endowed with a Γ-invariant Borel measure λ. For any choice of a measurable Γ-fundamental domain $F \subset W$ we define a Borel measure λ^{tr} on the σ-algebra of Γ-invariant Borel subsets of W by

$$\lambda^{\mathrm{tr}}(A) = \lambda(A \cap F).$$

We call λ^{tr} the *transverse measure* induced by λ.

REMARK 5.2 The definition of the transverse measure does not depend on the choice of the Borel fundamental domain. This is proved similarly to the situation of a lattice in a locally compact group. Let $F' \subset W$ another Borel Γ-fundamental domain. Let $A \subset W$ be a Γ-invariant Borel subset. Then

$$\lambda(A \cap F) = \lambda\left(A \cap \bigcup_{\gamma \in \Gamma} \gamma F' \cap F\right) = \lambda\left(\bigcup_{\gamma \in \Gamma} A \cap \gamma F' \cap F\right)$$

$$= \lambda\left(\bigcup_{\gamma \in \Gamma} \gamma^{-1} A \cap F' \cap \gamma^{-1} F\right)$$

$$= \lambda\left(A \cap F' \cap \bigcup_{\gamma \in \Gamma} \gamma^{-1} F'\right)$$

$$= \lambda(A \cap F').$$

Definition 5.3 Retaining the setting of the previous definition, the pushforward of λ^{tr} under the restriction of the quotient map $F \to \Gamma \backslash W$ is a Borel measure on $\Gamma \backslash W$ that we also call the *transverse measure* induced by λ and denote by the same symbol.

Definition 5.4 Let $\mathrm{pr} \colon W \to X$ be a metric Cantor bundle. Let $\mathcal{H}_x^d$ be the d-dimensional Hausdorff measure on W_x. Then

$$\int_X \mathcal{H}_x^d d\mu(x)$$

defines a Γ-invariant Borel measure on W that we call the *d-dimensional Hausdorff measure* of the Cantor bundle pr. We denote it generically by vol_d. The induced transverse measure is denoted by $\mathrm{vol}_d^{\mathrm{tr}}$.

The fact that $A \mapsto \int_X \mathcal{H}_x^d(A \cap W_x) d\mu(x)$ is Borel measurable for a Borel subset $A \subset W$ follows from the existence of a product atlas. The Γ-invariance follows from the fact that μ is Γ-invariant and the multiplication with an element $\gamma \in \Gamma$ is fiberwise an isometry. So the above definition is justified.

LEMMA 5.5 *Let $\Phi \colon V \to W$ be a Lipschitz Cantor bundle map, and let ϕ be the induced map on Γ-quotients. Then the map*

$$\Gamma \backslash W \to \mathbb{N} \cup \{\infty\}, \ w \mapsto \#\phi^{-1}(\{w\})$$

is $\mathrm{vol}_d^{\mathrm{tr}}$-measurable, and we define the transverse *d-volume of Φ as*

$$\mathrm{vol}_d^{\mathrm{tr}}(\Phi) = \int_{\Gamma \backslash W} \#\phi^{-1}(\{w\}) d\,\mathrm{vol}_d^{\mathrm{tr}}(w).$$

Proof Let $F_W \subset W$ be a Borel fundamental domain of $\Gamma \curvearrowright W$ that is a finite union of boxes. Then $F_V = \Phi^{-1}(F_W)$ is a fundamental domain of $\Gamma \curvearrowright V$. By Lemma 4.15 there is a clopen partition $X = A_1 \cup \cdots \cup A_n$ such that each $V|_{A_i} \cap F_V$ is a box, each $W|_{A_i} \cap F_W$ is a box and the restriction

$$\Phi \colon V|_{A_i} \cap F_V \to W|_{A_i} \cap F_W$$

is a product $\mathrm{id}_{A_i} \times h_i$ (in product charts) with h_i being Lipschitz.

The statement is equivalent to the measurability of the map

$$f \colon F_W \to \mathbb{N} \cup \{\infty\}, \ \ w \mapsto \#\big(\Phi^{-1}(\{w\}) \cap F_V\big).$$

In product chart coordinates $(x, p) \in A_i \times F \cong (F_W)|_{A_i}$ we have $f(x, p) = \#h_i^{-1}(\{p\})$. So the measurability over A_i and hence everywhere follows from [Fed69, 2.10.9]. $\square$

Let $f\colon M \to N$ be a Lipschitz map between Riemannian manifolds. By Rademacher's theorem f is differentiable almost everywhere. Let $J_d f$ be the almost everywhere defined d-*dimensional Jacobian* of Df. If f is smooth and M and N are d-dimensional, then $J_d f(m)$ is the absolute value of the determinant of the differential at $m \in M$ with respect to orthonormal bases. Let $\Phi\colon V \to W$ be a Lipschitz Cantor bundle map between metric Cantor bundles for which every fiber is a Riemannian manifold. We consider the quotient map $\phi\colon \Gamma\backslash V \to \Gamma\backslash W$. By equivariance of Φ the d-dimensional Jacobian of ϕ is well defined on a conull set by

$$J_d\phi([p]) = J_d\Phi(p).$$

We prove the following version of the area formula for Cantor bundles.

Theorem 5.6 (Area formula).   *Let* $\Phi\colon V \to W$ *be a Lipschitz Cantor bundle maps between Riemannian Cantor bundles, where each fiber of* V *is* d-*dimensional. Then the* d-*dimensional Jacobian* $J_d\phi$ *is* $\mathrm{vol}_d^{\mathrm{tr}}$-*measurable and*

$$\mathrm{vol}_d^{\mathrm{tr}}(\Phi) = \int_{\Gamma\backslash W} \#\phi^{-1}(\{w\})d\,\mathrm{vol}_d^{\mathrm{tr}}(w) = \int_{\Gamma\backslash V} J_d\phi(v)d\,\mathrm{vol}_d^{\mathrm{tr}}(v).$$

Proof Let $F_W \subset W$ be a Borel fundamental domain consisting of finitely many boxes. Let F_V be its Φ-preimage. The statement is equivalent to

$$\int_{F_W} \#\Phi^{-1}(\{w\})d\,\mathrm{vol}_d(w) = \int_{F_V} J_d\Phi(v)d\,\mathrm{vol}_d(v). \tag{5.1}$$

Let $X = A_1 \cup \cdots \cup A_n$ be a clopen partition as in Lemma 4.15 such that $(F_V)|_{A_i}$ and $(F_W)|_{A_i}$ are boxes. Choose homeomorphisms $A_i \times F_i \cong (F_V)|_{A_i}$ and $A_i \times F_i' \cong (F_W)|_{A_i}$ such that under these homeomorphisms the map Φ is of the form $\mathrm{id}_{A_i} \times h_i$. The left hand side of (5.1) becomes

$$\sum_{i=1}^{n} \mu(A_i) \int_{F_i'} \#h_i^{-1}(\{w\})d\mathcal{H}^d(w).$$

The right hand side of (5.1) becomes

$$\sum_{i=1}^{n} \mu(A_i) \int_{F_i} J_d h_i(v)d\mathcal{H}^d(v).$$

The i-th summand of left and right hand side coincide by the classical area formula [Fed69, 3.2.3 on p. 243]. $\qquad\square$

6   Rectangular Cantor Nerves and Cantor Covers

In Section 6.1 we introduce the notion of Cantor cover on $X \times \widetilde{M}$ which is an analog of a Γ-equivariant cover by balls on $\widetilde{M}$. Then we define a good Cantor cover. The proof of its existence is postponed to Section 7.1. In Section 6.2 we introduce the analog of Guth's rectangular nerve in our setting. In Section 6.3 we introduce the nerve map as a Cantor bundle map.

6.1 Cantor covers.

A *Cantor cover* of $X \times \widetilde{M}$ is an open cover

$$\mathcal{U} = \{A_j \times B_j \mid j \in J\}$$

of $X \times \widetilde{M}$ by product sets of clopen sets $A_j \subset X$ and open balls $B_j \subset \widetilde{M}$ indexed over a free cofinite Γ-set J such that $A_{\gamma j} = \gamma A_j$ and $B_{\gamma j} = \gamma B_j$ for all $\gamma \in \Gamma$ and $j \in J$. We further require that $\{A_j \times \frac{1}{2}B_j \mid j \in J\}$ still covers $X \times \widetilde{M}$. If we replace the property of being a cover by requiring that the elements of $\mathcal{U}$ are pairwise disjoint then we call $\mathcal{U}$ a *Cantor packing*.

Since the index set is cofinite, i.e. consists of finitely many orbits, and the Γ-action on $\widetilde{M}$ is proper, a Cantor cover is always locally finite. Let $\mathcal{U} = \{A_j \times B_j \mid j \in J\}$ be a Cantor cover or Cantor packing of $X \times \widetilde{M}$. Let $\mathcal{V}$ be an arbitrary family of subsets of a space.

$\triangleright$ We denote the union of the elements of $\mathcal{V}$ by $\bigcup \mathcal{V}$.

$\triangleright$ For $x \in X$ we denote by

$$\mathcal{U}_x = \{B_j \mid j \in J, x \in A_j\}$$

the induced open cover (packing, respectively) of $\widetilde{M} \cong \{x\} \times \widetilde{M}$.

$\triangleright$ We say that $\mathcal{U}$ has *no self intersections* if $(\gamma A_j \times \gamma B_j) \cap (A_j \times B_j) \neq \emptyset$ implies $\gamma = 1$ for every $j \in J$ and $\gamma \in \Gamma$.

$\triangleright$ For $a > 0$ we write $a\mathcal{U} := \{A_j \times aB_j \mid j \in J\}$.

We produce a suitable Cantor cover of $X \times \widetilde{M}$ that consists of good balls in every fiber. The notion of goodness goes back to Gromov. We refer to [Gut11, Section 1] for this notion. A cover by good balls will be called a good cover which is a bit unfortunate since this terminology is also used for covers with contractible sets and intersections.

Let N be a d-dimensional Riemannian manifold and $V_N(1)$ be the supremal volume of 1-balls in N. The ball $B(p, r) \subseteq N$ of radius r around a point $p \in M$ is called a *good ball* if the following conditions are satisfied.

(1) Reasonable growth: $\mathrm{vol}(B(p, 100r)) \leqslant 10^{4(d+3)} \, \mathrm{vol}(B(p, \frac{1}{100}r))$.

(2) Volume bound: $\mathrm{vol}(B(p, r)) \leqslant 10^{2(d+3)} V_N(1) r^{d+3}$.

(3) Small radius: $r \leqslant \frac{1}{100}$.

A *good cover* of a Riemannian manifold is an open cover by good balls where the concentric $\frac{1}{6}$-balls are disjoint and the $\frac{1}{2}$-balls provide a cover of the manifold as well. A Cantor cover $\mathcal{U}$ of $X \times \widetilde{M}$ is called *good* if $\mathcal{U}_x$ is a good cover of $\widetilde{M}$ for every $x \in X$.

Guth showed that any closed Riemannian manifold has a good cover [Gut11, Lemma 2]. At the end of Section 7.1 we will be able to give the proof of the equivariant statement:

Theorem 6.1 *There exists a good Cantor cover on $X \times \widetilde{M}$ that has no self intersections.*

6.2   The rectangular Cantor nerve of a Cantor cover.   In the sequel we consider a Cantor cover

$$\mathcal{U} = \{A_j \times B_j \mid j \in J\}$$

of $X \times \widetilde{M}$. We adhere to the following notation.

  ▷ By picking a set of Γ-representatives we write the Γ-set J as $\Gamma \times I$ with finite I.
  ▷ Let r_j denote the radius of the ball B_j and m_j the center of B_j.
  ▷ Let $\mathcal{V} := \{B_j \mid j \in J\}$. This is a locally finite cover of $\widetilde{M}$ since Γ acts freely and properly on $\widetilde{M}$.

The nerve $N(\mathcal{V})$ satisfies the requirements of Lemma 2.4. In particular, its barycentric subdivision is a proper Γ-CW-complex. By properness and cofiniteness of J the maximal multiplicity of $\mathcal{V}$ is finite, hence $N(\mathcal{V})$ is cocompact. Since $\mathcal{U}_x$ is a subcover of $\mathcal{V}$, $N(\mathcal{U}_x)$ is a subcomplex of $N(\mathcal{V})$ for every $x \in X$.

Definition 6.2 The *rectangular Cantor nerve* $N^{\mathrm{Ca}}(\mathcal{U})$ of the Cantor cover $\mathcal{U}$ is the subset

$$\big\{(x,p) \mid p \in N(\mathcal{U}_x), x \in X\big\} \subset X \times N(\mathcal{V}).$$

Clearly, $N^{\mathrm{Ca}}(\mathcal{U})$ is a Γ-invariant subset of $N(\mathcal{V})$. We restrict the metric of $N((r_j)_{j \in J})$ to $N(\mathcal{V})$ and, further, to each $N(\mathcal{U}_x)$.

LEMMA 6.3 *The rectangular Cantor nerve $N^{\mathrm{Ca}}(\mathcal{U})$ is a metric Cantor subbundle of the trivial metric Cantor bundle $X \times N(\mathcal{V})$ endowed with its diagonal Γ-action.*

Proof It suffices to construct an open box neighborhood around each point

$$(x,p) \in N^{\mathrm{Ca}}(\mathcal{U}) \subset X \times N(\mathcal{V}) \subset X \times N\big((r_j)_{j \in J}\big).$$

Let F be an open face in $N(\mathcal{U}_x)$ that contains p. The star of F within $N\big((r_j)_{j \in J}\big)$ consists of all open faces in $N\big((r_j)_{j \in J}\big)$ that contain F in their closure. A face E of $N\big((r_j)_{j \in J}\big)$ is in the star of F if and only if $J_+(F) \subset J_+(E)$. For a face E which lies in the star of F and in $N(\mathcal{V})$ the subset $J_+(E)$ lies in the subset

$$J_F := \big\{j \in J \mid B_j \cap \bigcap_{i \in J_+(F)} B_i \neq \emptyset\big\},$$

which is finite since $\mathcal{V}$ is locally finite. As a finite intersection of clopen sets the set

$$C := \bigcap_{\substack{j \in J_F \\ x \in A_j}} A_j \cap \bigcap_{\substack{j \in J_F \\ x \notin A_j}} X \backslash A_j$$

is a clopen neighborhood of x. Let S be the star of F within $N(\mathcal{V})$. Let S' be the star of F within $N(\mathcal{U}_x)$.

We claim that every face E of S' lies in $N(\mathcal{U}_y)$ for all $y \in C$. For such E we have $J_1(E) \neq \emptyset$ and $\bigcap_{j \in J_+(E)} B_j \neq \emptyset$. Since E lies in $N(\mathcal{U}_x)$ we have $x \in \bigcap_{j \in J_+(E)} A_j$. The inclusion $J_+(E) \subset J_F$ implies that $C \subset \bigcap_{j \in J_+(E)} A_j$. Thus E lies in $N(\mathcal{U}_y)$ for every $y \in C$. Therefore

$$N^{\mathrm{Ca}}(\mathcal{U}) \cap \big(C \times S' \big) = C \times S',$$

where the left hand intersection is taken within $X \times N(\mathcal{V})$.

Next we show that $C \times S'$ is open in $N^{\mathrm{Ca}}(\mathcal{U})$. To this end, we show that $S' = N(\mathcal{U}_y) \cap S$ for all $y \in C$. Since S is open in $N(\mathcal{V})$ this proves that $C \times S'$ is open. Let E be a face in $N(\mathcal{U}_y) \cap S$. In particular, $y \in \bigcap_{j \in J_+(E)} A_j$. Then E lies in S' if $x \in \bigcap_{j \in J_+(E)} A_j$. Suppose there is $j_0 \in J_+(E)$ with $x \notin A_{j_0}$. Since $y \in C$, this would imply $y \in X \backslash A_{j_0}$ and contradict $y \in \bigcap_{j \in J_+(E)} A_j$. Hence E lies in S'.

So $C \times S'$ is a box neighborhood containing $(x, p) \in N^{\mathrm{Ca}}(\mathcal{U})$ with respect to the global product chart on $X \times N(\mathcal{V})$. $\square$

Next we try to understand the rectangular Cantor nerve by pushouts.

LEMMA 6.4 *We assume that $\mathcal{U}$ has no self-intersections. Let C_n be a complete set of representatives of the Γ-orbits of the n-dimensional faces of $N(\mathcal{V})$. The n-skeleton $N^{\mathrm{Ca}}(\mathcal{U})^{(n)} = N^{\mathrm{Ca}}(\mathcal{U}) \cap X \times N(\mathcal{V})^{(n)}$ of $N^{\mathrm{Ca}}(\mathcal{U})$ arises from the $(n-1)$-skeleton as a pushout*

$$
\begin{array}{ccc}
\coprod\limits_{F \in C_n} \Big(\bigcap\limits_{j \in J_+(F)} A_j \Big) \times \Gamma \times \partial\Big(\prod\limits_{k=1}^{n} [0, r_k(F)] \Big) & \longrightarrow & N^{\mathrm{Ca}}(\mathcal{U})^{(n-1)} \\
\Big\downarrow & & \Big\downarrow \\
\coprod\limits_{F \in C_n} \Big(\bigcap\limits_{j \in J_+(F)} A_j \Big) \times \Gamma \times \Big(\prod\limits_{k=1}^{n} [0, r_k(F)] \Big) & \longrightarrow & N^{\mathrm{Ca}}(\mathcal{U})^{(n)}.
\end{array}
\tag{6.1}
$$

whose maps are Cantor bundle maps. The lower horizontal map is fiberwise an isometric embedding of $\prod_{k=1}^{n} [0, r_k(F)]$.

Proof The map

$$\coprod_{F \in C_n} \Big(\bigcap_{j \in J_+(F)} A_j \Big) \times \Gamma \times \Big(\prod_{j \in J_{\frac{1}{2}}(F)} [0, r_j] \Big) \xrightarrow{\Psi} X \times N\big((r_j)_{j \in J} \big)$$

$$\big(a, \gamma, (w_j)_{j \in J_{\frac{1}{2}}(F)} \big) \mapsto \big(\gamma a, \gamma \cdot (\bar{w}_j)_{j \in J} \big)$$

where

$$\bar{w}_j = \begin{cases} w_j & \text{if } j \in J_{\frac{1}{2}}(F), \\ 0 & \text{if } j \in J_0(F), \\ r_j & \text{if } j \in J_1(F), \end{cases}$$

lands in $N^{\mathrm{Ca}}(\mathcal{U})^{(n)}$ and is a Cantor bundle map into $N^{\mathrm{Ca}}(\mathcal{U})^{(n)}$. Next we verify that the restriction Ψ_0 of Ψ

$$\coprod_{F \in C_n} \Big(\bigcap_{j \in J_+(F)} A_j \Big) \times \Gamma \times \prod_{j \in J_{\frac{1}{2}}(F)} (0, r_j) \xrightarrow{\Psi_0} N^{\mathrm{Ca}}(\mathcal{U})^{(n)} \backslash N^{\mathrm{Ca}}(\mathcal{U})^{(n-1)}$$

is bijective. Suppose Φ_0 maps two points $(a, \gamma, (w_j))$ and $(a', \gamma', (w'_j))$ in the left hand summands associated with the n-faces F and F' to the same point. By equivariance it suffices to consider the case $\gamma' = 1$. The open faces γF and F' intersect, hence concide as subsets $\gamma F = F'$[1]. Since F, F' are from a complete set of Γ-representatives C_n we obtain that $F' = F$ and $\gamma F = F$ as subsets. Let $x := \gamma a = a'$. The n-faces of $N(\mathcal{U}_x)$ are exactly the n-faces E with $x \in \bigcap_{j \in J_+(E)} A_j$ and $\bigcap_{j \in J_+(E)} B_j \neq \emptyset$ and $J_1(E) \neq \emptyset$. From $\gamma F = F$ we obtain that $\emptyset \neq \bigcap_{j \in J} B_j = \bigcap_{j \in J} \gamma B_j$ and $x \in \bigcap_{j \in J_+(\gamma F)} A_j = \bigcap_{j \in J_+(F)} \gamma A_j$. In particular, $B_j \cap \gamma B_j \neq \emptyset$ and $A_j \cap \gamma A_j \neq \emptyset$ for every $j \in J_+(F)$. Since $\mathcal{U}$ has no self-intersections, this implies $\gamma = 1$ and proves injectivity. By [Tom08, Proposition 8.3.1 on p. 203] and rewriting $\prod_{j \in J_{\frac{1}{2}}(F)} [0, r_j]$ as $\prod_{k=1}^n [0, r_k(F)]$ the pushout property of (6.1) follows from the bijectivity of Ψ_0 and the fact that Ψ is a quotient map onto its closed image. The image of Ψ is the union of closed n-faces which is closed. Let $C \subset \mathrm{im}(\Psi)$ be a subset such that $\Psi^{-1}(C)$ is closed. Let F be a compact subset of the domain of Ψ whose translates cover the domain. Then

$$C = \Phi\Big(\bigcup_{\gamma \in \Gamma} \gamma F \cap \Phi^{-1}(C) \Big) = \bigcup_{\gamma \in \Gamma} \gamma \cdot \Phi\big(F \cap \gamma^{-1} \Phi^{-1}(C) \big).$$

Moreover, each subset $\Phi\big(F \cap \gamma^{-1} \Phi^{-1}(C) \big)$ is compact and lies in the compact subset $\Phi(F)$. The Γ-action on $N(\mathcal{V})$ is proper. Hence C is closed, and Φ is a quotient map onto its image. $\qquad\square$

6.3 The map to the rectangular Cantor nerve.

A Cantor bundle map $\Phi \colon X \times \widetilde{M} \to N^{\mathrm{Ca}}(\mathcal{U}) \subset X \times N((r_j)_{j \in J})$ is *subordinate to* $\mathcal{U}$ if each component $\Phi_j \colon X \times \widetilde{M} \to [0, r_j]$, $j \in J$, of the map Φ is supported in $A_j \times B_j$.

Next we construct a specific Cantor bundle map subordinate to $\mathcal{U}$. We define the continuous component map Φ_j for each $j \in J$ by

$$\Phi_j \colon X \times \widetilde{M} \to [0, r_j]$$

$$\Phi_j(x, p) = \begin{cases} 0 & \text{if } d_{\widetilde{M}}(p, m_j) \geq r_j \text{ or } x \notin A_j, \\ 2(r_j - d_{\widetilde{M}}(p, m_j)) & \text{if } \frac{r_j}{2} \leq d_{\widetilde{M}}(p, m_j) \leq r_j \text{ and } x \in A_j, \\ r_j & \text{if } d_{\widetilde{M}}(p, m_j) < \frac{r_j}{2} \text{ and } x \in A_j. \end{cases}$$

---

[1] Since $N(\mathcal{V})$ is not a Γ-CW complex (only after barycentric subdivision) one might have $F = \gamma F$ as subsets but not pointwise.

Definition 6.5 The *Cantor nerve map* Φ associated with $\mathcal{U}$ is the product of the maps Φ_j

$$\Phi\colon X \times \widetilde{M} \to N^{\mathrm{Ca}}(\mathcal{U}), \ \Phi(x,p) = \big(x, (\Phi_j(x,p))_{j\in J}\big).$$

One sees immediately that Φ is subordinate to $\mathcal{U}$.

LEMMA 6.6 *The Cantor nerve map associated with $\mathcal{U}$ is a Lipschitz Cantor bundle map.*

Proof Since each A_j is clopen and Φ_j is clearly continuous when restricted to $A_j \times \widetilde{M}$ or $(X\backslash A_j) \times \widetilde{M}$, Φ_j is continuous. Thus Φ is continuous. For each $j \in J$ and $\gamma \in \Gamma$ we have

$$\Phi_{\gamma j}(\gamma x, \gamma p) = \Phi_j(x,p),$$

which implies that Φ is Γ-equivariant. Let $(x,p) \in X \times \widetilde{M}$. Let F be an open face in $N^{\mathrm{Ca}}(\mathcal{U}) = N(\mathcal{U}_x)$ that contains $\Phi(x,p)$. Then $x \in \bigcap_{j\in J_+(F)} A_j$ and $p \in \bigcap_{j\in J_+(F)} B_j$. We consider the following sets

$$J_F := \big\{ j \in J \mid B_j \cap \bigcap_{i\in J_+(F)} B_i \neq \emptyset \big\},$$

$$C := \bigcap_{\substack{j\in J_F \\ x\in A_j}} A_j \cap \bigcap_{\substack{j\in J_F \\ x\notin A_j}} X\backslash A_j.$$

Let S be the star of F within $N(\mathcal{U}_x)$. In the proof of Lemma 6.3 we showed that $C \times S$ is an open box (S was denoted S' in the proof), that is,

$$N^{\mathrm{Ca}}(\mathcal{U}) \cap C \times S = C \times S.$$

Let $y \in C$ and $q \in \bigcap_{j\in J_+(F)} B_j$. Next we show that $\Phi_j(y,q) = \Phi_j(x,q)$ for every $j \in J$ which implies that Φ is a product of maps on $C \times \bigcap_{j\in J_+(F)} B_j$.

First assume that $\Phi_j(y,q) = 0$. If $q \notin B_j$, then $\Phi_j(x,q) = 0$. If $q \in B_j$, then $j \in J_F$, thus $y \notin A_j$. This implies that $y \notin C$ or $x \notin A_j$. Because of $y \in C$ we must have $x \notin A_j$. Hence $\Phi_j(y,q) = \Phi_j(x,q) = 0$.

Second assume that $\Phi_j(y,q) > 0$. Then $q \in B_j$ and $j \in J_F$ and $y \in A_j$. If $x \notin A_j$ then $y \in C$ would imply that $y \notin A_j$. Hence $x \in A_j$. Therefore

$$\Phi_j(y,q) = \Phi_j(x,q) = \begin{cases} 2(r_j - d_{\widetilde{M}}(p, m_j)) & \text{if } \frac{r_j}{2} \le d_{\widetilde{M}}(p, m_j) \le r_j, \\ r_j & \text{if } d_{\widetilde{M}}(p, m_j) < \frac{r_j}{2}. \end{cases}$$

It remains to show that Φ is Lipschitz. Each Φ_j has Lipschitz constant 2. Hence Φ_j has local Lipschitz constant at (x,p) bounded by $2m_x(p)^{1/2}$ where $m_x(p)$ is the multiplicity at p of the cover $\mathcal{U}_x$. The multiplicity is uniformly bounded by the multiplicity of the cover $\mathcal{V}$ (albeit not by a dimensional constant). Hence Φ is a Lipschitz Cantor bundle map. $\square$

REMARK 6.7 Restricted to a fiber $x \in X$ the map $\Phi_x\colon \widetilde{M} \to N$ is exactly Guth's nerve map [Gut11, Section 3] associated with the cover $\mathcal{U}_x$.

7 Volume Estimates

The goal of this section is to prove the existence of a good Cantor cover and to prove the analog of Lemma 4 in [Gut11] in the Cantor setting. In Section 7.1 we define layers of an ordinary cover and of a Cantor cover. The most important technical result is the existence of layers for Cantor covers with no self-intersections. After that, the proof of the analog of Guth's Lemma 4 in Sections 7.2 and 7.3 runs similar to Guth's proof.

7.1 Cantor–Vitali layerings of equivariant covers. Let $\mathcal{V}$ be a cover of a Riemannian manifold by balls. A *Vitali layering* of $\mathcal{V}$ consists of a finite sequence of subsets $\mathcal{V}(1), \ldots, \mathcal{V}(n)$ of $\mathcal{V}$, called *layers*, with the following property:

(1) The balls within each layer are pairwise disjoint.
(2) For every pair $i < j$ in $\{1, \ldots, n\}$ and every ball $B \in \mathcal{V}(j)$ there is a ball in $\mathcal{V}(i)$ that meets B and whose radius is greater or equal than the one of B.
(3) Every ball of $\mathcal{V}$ appears in precisely one of the layers.

We say that a layer $\mathcal{V}(j)$ is *lower* than a layer $\mathcal{V}(i)$ if $i < j$. The relation $<$ on $\mathcal{V}(i)$ *associated to the Vitali layering* is defined as the smallest partial order $<$ on the layer $\mathcal{V}(i)$ such that $B < B'$ whenever there is a ball B'' from a lower layer that meets both B and B' and the radii of these balls satisfy

$$2r \leq r'' \leq r'.$$

The *core* of a layer is the union of all balls $\frac{1}{10}B$ where B is maximal with respect to the relation $<$ on that layer.

LEMMA 7.1 *Let $\mathcal{V} = \{B_j \mid j \in J\}$ be a good cover of a Riemannian manifold by balls. Let $\mathcal{V}(1), \mathcal{V}(2), \ldots, \mathcal{V}(n)$ be a Vitali layering of $\mathcal{V}$ with cores $\mathcal{V}^c(1), \ldots, \mathcal{V}^c(n)$. The following holds for every integer $l \in \{1, \ldots, n\}$.*

(1) *Every point in $\bigcup \mathcal{V}^c(l)$ is contained in at most $10^{8(d+3)}$ balls from lower layers.*
(2) *For $l' \geq l$ we have $\bigcup \mathcal{V}(l') \subset \bigcup 3\mathcal{V}(l)$.*
(3) $\bigcup \mathcal{V}(l) \subset \bigcup 10 \mathcal{V}^c(l)$.

Proof Both statements (1) and (3) are extracted from the proof of Lemma 4 in Guth's paper [Gut11, p. 60]. Concerning (1), Guth shows that the radii of balls in layers $\geq l$ that contain a point $p \in \frac{1}{10}B_j$ in the l-th core, where $B_j \in \mathcal{V}(l)$ is a maximal ball of radius r_j, are pinched in the interval $[\frac{1}{15}r_j, 2r_j]$. The number of such balls is bounded by a dimensional constant [Gut11, Lemma 3] which can be taken to be $10^{8(d+3)}$. Ad (2): Let $l' \geq l$. Let $B_j \in \mathcal{V}(l')$. Then there is a ball $B_k \in \mathcal{V}(l)$ that meets B_j and has $r_k \geq r_j$. Hence $B_j \subset 3B_k \subset \bigcup 3\mathcal{V}(l)$.   □

A *Cantor–Vitali layering* of a Cantor cover $\mathcal{U}$ consists of a finite sequence of Cantor packings $\mathcal{U}(1), \ldots, \mathcal{U}(n)$ such that $\mathcal{U}(1)_x, \ldots, \mathcal{U}(n)_x$ is a Vitali layering of $\mathcal{U}_x$ for every $x \in X$. Further, the *core* of the l-th layer $\mathcal{U}(l)$ is defined to be union of the cores of $\mathcal{U}(l)_x$ over all $x \in X$.

LEMMA 7.2 *Let $\mathcal{U} = \{A_j \times B_j \mid j \in J\}$ be a Cantor packing of $X \times \widetilde{M}$ such that each ball B_j is good. Then*

$$\mathrm{vol}_d^{\mathrm{tr}}\left(\bigcup 10\mathcal{U}\right) \le 10^{4(d+3)} \, \mathrm{vol}_d^{\mathrm{tr}}\left(\bigcup \mathcal{U}\right).$$

Proof We write $J = \Gamma \times I$ as Γ-sets. Let $\mathcal{U}_0 = \{A_i \times B_i \mid i \in I\}$. Since the Γ-translates of $A_i \times 10B_i$, $i \in I$, cover the set $\bigcup 10\mathcal{U}$ and $\bigcup\mathcal{U}_0$ is a Γ-fundamental domain of $\bigcup\mathcal{U}$ by the packing property, we obtain for the transverse measure that

$$\mathrm{vol}_d^{\mathrm{tr}}\left(\bigcup 10\mathcal{U}\right) \le \mathrm{vol}_d\left(\bigcup 10\mathcal{U}_0\right) \le \sum_{i \in I} \mu(A_i) \, \mathrm{vol}_d(10B_i)$$

$$\le \sum_{i \in I} 10^{4(d+3)} \mu(A_i) \, \mathrm{vol}_d(B_i) \qquad \text{(since } B_i \text{ is good)}$$

$$= 10^{4(d+3)} \, \mathrm{vol}_d^{\mathrm{tr}}\left(\bigcup \mathcal{U}\right). \qquad \square$$

LEMMA 7.3 *Let $\mathcal{U}$ be a good Cantor cover of $X \times \widetilde{M}$. Let $\mathcal{U}(1), \mathcal{U}(2), \ldots$ be a Cantor–Vitali layering of $\mathcal{U}$. Further, let $\mathcal{U}^c(l)$ denote the core of $\mathcal{U}(l)$. Then the following hold.*

(1) *For every $x \in X$ and $p \in \bigcup\mathcal{U}^c(l)_x$ the number of balls that contain p and lie in $\mathcal{U}(l')_x$ for some $l' \ge l$ is bounded by $10^{8(d+3)}$.*
(2) *For $l' \ge l$ we have $\bigcup\mathcal{U}(l') \subset \bigcup 3\mathcal{U}(l)$.*
(3) *For $l \ge 1$ we have $\mathrm{vol}_d^{\mathrm{tr}}\left(\bigcup\mathcal{U}(l)\right) \le 10^{4(d+3)} \cdot \mathrm{vol}_d^{\mathrm{tr}}\left(\bigcup\mathcal{U}^c(l)\right)$.*

Proof (1) and (2) follow directly from Lemma 7.1. By applying Lemma 7.1 (3) fiberwise to the packings $\mathcal{U}(l)_x$ and then taking unions over $x \in X$ we obtain that

$$\bigcup\mathcal{U}(l) \subset \bigcup 10\,\mathcal{U}^c(l).$$

Statement (3) follows from 7.1 (3) and 7.2. $\square$

Theorem 7.4   *Every Cantor cover of $X \times \widetilde{M}$ without self-intersections possesses a Cantor–Vitali layering.*

Proof Let $\mathcal{U} = \{\gamma \cdot A_j \times \gamma \cdot B_j \mid (\gamma, j) \in \Gamma \times \{1, \ldots, n\}\}$ be a Cantor cover of $X \times \widetilde{M}$ without self-intersections. Let r_j be the radius of B_j. We assume that the enumeration of balls is such that $r_1 \ge r_2 \ge \cdots \ge r_n$. For the purpose of this proof, we call a set of subsets of $X \times \widetilde{M}$ of *product type* if it is of the form $\{\gamma \cdot Y_j \times \gamma \cdot B_j \mid (\gamma, j) \in \Gamma \times \{1, \ldots, n\}\}$ where each Y_j is a (not necessarily non-empty) clopen subset of X.

We construct the layers by a double induction. For every $l \in \mathbb{N}$ and $s \in \{1, \ldots, n\}$ we define Cantor packings $\mathcal{U}^s(l)$ and $\mathcal{U}(l)$ depending on $\mathcal{U}(1), \ldots, \mathcal{U}(l-1)$ and on

$\mathcal{U}^{s-1}(l)$ in the following way.

$$\mathcal{U}(0) := \emptyset,$$

$$\mathcal{U}^0(l) := \emptyset \ \text{ for every } l \in \{1, \ldots, n\},$$

$$Y^s(l) := \Big\{ x \in X \mid B_s \in \mathcal{U}_x, B_s \text{ is disjoint from all balls in } \mathcal{U}^{s-1}(l)_x,$$

$$\text{and } B_s \text{ is not contained in } \mathcal{U}(1)_x, \ldots, \mathcal{U}(l-1)_x \Big\},$$

$$\mathcal{U}^s(l) := \mathcal{U}^{s-1}(l) \cup \big\{ \gamma Y^s(l) \times \gamma B_s \mid \gamma \in \Gamma \big\},$$

$$\mathcal{U}(l) := \mathcal{U}^n(l).$$

Each set $\mathcal{U}^s(l)$ is of product type: It suffices to verify that each set $Y^s(l)$ is clopen. By induction we may assume that $\mathcal{U}^{s-1}(l)$ and $\mathcal{U}(i)$ are of product type for every $i \in \{1, \ldots, l-1\}$, that is,

$$\mathcal{U}^{s-1}(l) = \big\{ \gamma \cdot Y_j^{s-1}(l) \times \gamma \cdot B_j \mid (\gamma, j) \in \Gamma \times \{1, \ldots, n\} \big\},$$

$$\mathcal{U}(i) = \big\{ \gamma \cdot Y_j(i) \times \gamma \cdot B_j \mid (\gamma, j) \in \Gamma \times \{1, \ldots, n\} \big\}$$

for suitable clopen subsets $Y_j^{s-1}(l) \subset X$ and $Y_j(i) \subset X$. For every $j \in \{1, \ldots, n\}$ let

$$F_j^s := \big\{ \gamma \in \Gamma \mid B_s \cap \gamma B_j \neq \emptyset \big\}.$$

The subset F_j^s is finite. Then we have

$$Y_s(l) = A_s \cap \left(X \backslash \left(\bigcup_{\substack{j \in \{1, \ldots, n\} \\ \gamma \in F_j^s}} \gamma Y_j^{s-1}(l) \ \cup \bigcup_{\substack{j \in \{1, \ldots, n\} \\ i \in \{1, \ldots, l-1\} \\ \gamma \in F_j^s}} \gamma Y_j(i) \right) \right),$$

which is clearly a clopen subset.

Each $\mathcal{U}^s(l)$ is a Cantor packing: Equivalently, we may show that $\mathcal{U}^s(l)_x$ is a packing for every $x \in X$. Since $Y^s(l) \subset A_s$ and $\mathcal{U}$ has no self intersections, $\big\{ \gamma Y^s(l) \times \gamma B_s \mid \gamma \in \Gamma \big\}$ is a Cantor packing for every $s \in \{1, \ldots, n\}$ and $l \in \mathbb{N}$. By induction we may assume that $\mathcal{U}^{s-1}(l)$ is a Cantor packing or, equivalently, $\mathcal{U}^{s-1}(l)_x$ is packing for every $x \in X$. Hence if for some $x \in X$ there is a non-empty intersection of two balls in $\mathcal{U}^s(l)_x$ it has to be a ball γB_s in $\mathcal{U}^s(l)_x$ intersecting a ball in $\mathcal{U}^{s-1}(l)_x$. In particular, $x \in \gamma Y^s(l)$. By Γ-equivariance the ball B_s lies in $\mathcal{U}^s(l)_{\gamma^{-1}x}$ and intersects a ball in $\mathcal{U}^{s-1}(l)_{\gamma^{-1}x}$. This contradicts $\gamma^{-1}x \in Y^s(l)$. So $\mathcal{U}^s(l)_x$ is indeed a packing for every $x \in X$.

The sequence $\mathcal{U}(1), \mathcal{U}(2), \ldots, \mathcal{U}(n)$ is a Cantor–Vitali layering of $\mathcal{U}$: Let $x \in X$. We show that $\mathcal{U}(1)_x, \mathcal{U}(2)_x, \ldots, \mathcal{U}(n)_x$ is a Vitali layering of $\mathcal{U}_x$. Let $i, j \in \{1, \ldots, n\}$ with $i < j$. Consider a ball γB_s in $\mathcal{U}(j)_x$. In particular, $x \in \gamma Y^s(j)$. We have to find a ball in $\mathcal{U}(i)_x$ that meets γB_s and has radius at least r_s. Equivalently, we have to find a ball in $\mathcal{U}(i)_{\gamma^{-1}x}$ that meets $B_s \in \mathcal{U}(j)_{\gamma^{-1}x}$ and has radius at least r_s. If such

a ball did not exist, then B_s would lie in $\mathcal{U}^s(i)_{\gamma^{-1}x} \subset \mathcal{U}(i)_{\gamma^{-1}x}$ or B_s would lie in $\mathcal{U}(1)_{\gamma^{-1}x}, \ldots, \mathcal{U}(i-1)_{\gamma^{-1}x}$. Both possibilities are absurd.

Let $x \in X$. If $s \in \{1, \ldots, n\}$ is the smallest number so that $B_s \in \{B_1, \ldots, B_n\} \cap \mathcal{U}_x$ but B_s is not in one of the layers $\mathcal{U}(1)_x, \ldots, \mathcal{U}(n-1)_x$ then $B_s \in \mathcal{U}^s(n)_x \subset \mathcal{U}(n)_x$. Hence every ball of $\{B_1, \ldots, B_n\} \cap \mathcal{U}_x$ is in one of the layers $\mathcal{U}(1)_x, \ldots, \mathcal{U}(n)_x$. By equivariance each ball of $\mathcal{U}_x$ appears in one of the layers $\mathcal{U}(1)_x, \ldots, \mathcal{U}(n)_x$. It is clear from the construction that each ball also appears in at most one layer. $\qquad\square$

Proof of Theorem 6.1 By [Gut11, Lemma 1] around every point of $\widetilde{M}$ there is a good ball. We choose a Γ-fundamental domain of $\widetilde{M}$ and a good ball for every point in the fundamental domain. Since M is compact we can select a finite subset $B_1, \ldots, B_n$ of these balls such that the projections of $\frac{1}{12}B_1, \ldots, \frac{1}{12}B_n$ cover all of M. Hence the translates of $X \times \frac{1}{12}B_1, \ldots, X \times \frac{1}{12}B_n$ form a Cantor cover of $X \times \widetilde{M}$.

By properness of the Γ-action on $\widetilde{M}$ the set

$$F := \big\{\gamma \in \Gamma \mid \exists_{i \in \{1, \ldots, n\}} B_i \cap \gamma B_i \neq \emptyset\big\}$$

is finite. Next we show that there is clopen partition $X = A_1 \cup \cdots \cup A_r$ of X such that $\gamma A_i \cap A_i = \emptyset$ for every $\gamma \in F$ and every $i \in \{1, \ldots, r\}$. To this end, choose a metric d on X that induces the topology on X. For every $\gamma \in F$ the continuous map

$$X \to [0, \infty), \quad x \mapsto d(\gamma x, x)$$

takes on a minimum ϵ_γ which is strictly positive as the Γ-action on X is free. Let $\epsilon := \min_{\gamma \in F} \epsilon_\gamma > 0$. Now we pick a cover of X by clopen subsets of diameter at most $\epsilon/2$. Then there is a subordinate clopen partition $X = A_1 \cup \cdots \cup A_r$ of X. Since the diameter of each A_i is at most $\epsilon/2$ we have $\gamma A_i \cap A_i = \emptyset$ for every $\gamma \in F$ and every $i \in \{1, \ldots, r\}$. Then the translates of the sets $A_i \times B_j$, $i = 1, \ldots, r$, $j = 1, \ldots, n$, form a Cantor cover $\mathcal{U}'$ indexed over $\Gamma \times \{1, \ldots, r\} \times \{1, \ldots, n\}$ without self-intersections. Also the Cantor cover $6\,\mathcal{U}'$ has no self-intersections.

According to Theorem 7.4 the Cantor cover $\mathcal{U}'$ has a Cantor–Vitali layering. Let $\mathcal{U}'(1)$ be the top layer. We claim that $\mathcal{U} := 6\,\mathcal{U}'(1)$ is a good Cantor cover without self-intersections: Since the top layer is always a Cantor packing, $\frac{1}{6}\mathcal{U} = \mathcal{U}'(1)$ is a Cantor packing. Further, $\frac{1}{2}\mathcal{U} = 3\mathcal{U}'$ is a Cantor cover by Lemma 7.3 (2). Finally, since $6\,\mathcal{U}'$ has no self intersections, $\mathcal{U}$ has no self-intersections either. $\qquad\square$

7.2 Exponential decay of the volume of the high multiplicity set.

Similar remarks as in Guth's paper on the multiplicity are valid here: The (fiberwise) multiplicity of a Cantor cover is bounded but not in terms of a universal constant. Therefore we cannot bound later the Lipschitz constant of the Cantor nerve map universally. However, the volume of the high multiplicity set decays exponentially. The argument for that is basically the same as the one in [Gut11, p. 61/62], only with volume replaced by transverse measure and so on.

Theorem 7.5 *Let $\mathcal{U}$ be a good Cantor cover of $X \times \widetilde{M}$ with no self-intersections. For $(x,p) \in X \times \widetilde{M}$ let $m_x(p)$ be the multiplicity of the point p with respect to the cover $\mathcal{U}_x$ of $\widetilde{M}$. There are dimensional constants $\alpha(d) > 0$ and $\beta(d) > 0$ such that for every $\lambda \geq 1$*

$$\mathrm{vol}_d^{\mathrm{tr}}\Big(\big\{(x,p) \mid m_x(p) \geq \lambda + \beta(d)\big\}\Big) \leq e^{-\alpha(d)\lambda} \cdot \mathrm{vol}(M).$$

REMARK 7.6 In the above statement we can choose $\alpha = -\log(1 - 10^{-16(d+3)})$ and $\beta = 10^{8(d+3)}$. This is a consequence of the proof below.

Proof According to Theorem 7.4 we pick a Cantor–Vitali layering $\mathcal{U}(1), \mathcal{U}(2), \ldots$ of $\mathcal{U}$. Let $\mathcal{U}^c(l) \subset \mathcal{U}(l)$ be the associated core of $\mathcal{U}(l)$. Consider the subsets

$$L^\theta(\lambda) = \big\{(x,p) \in X \times \widetilde{M} \mid (x,p) \in \bigcup \mathcal{U}(l) \text{ for at least}$$
$$\theta \text{ values of } l \text{ in the range } l \geq \lambda\big\}.$$

By Lemma 7.1 (2) and Lemma 7.2 we obtain that

$$\mathrm{vol}_d^{\mathrm{tr}}\big(L^1(\lambda)\big) \leq \mathrm{vol}_d^{\mathrm{tr}}\big(\bigcup 3\mathcal{U}(\lambda)\big) \leq 10^{4(d+3)} \, \mathrm{vol}_d^{\mathrm{tr}}\big(\bigcup \mathcal{U}(\lambda)\big). \tag{7.1}$$

With the constant $\beta(d) = 10^{8(d+3)}$ from Lemma 7.1 (1) we define $T(\lambda)$ as the average volume

$$T(\lambda) := \frac{1}{\beta(d)} \sum_{\theta=1}^{\beta(d)} \mathrm{vol}_d^{\mathrm{tr}}\big(L^\theta(\lambda)\big).$$

An element $(x,p) \in L^\theta(\lambda) \backslash L^\theta(\lambda+1)$ lies in $\mathcal{U}(\lambda)$ and in exactly $\theta - 1$ different layers lower than λ. With Lemma 7.1 (1) this implies that

$$\bigcup \mathcal{U}^c(\lambda) \subset \bigcup_{\theta=1}^{\beta(d)} \big(L^\theta(\lambda) \backslash L^\theta(\lambda+1)\big).$$

Note that $\mathcal{U}^c(\lambda)$ and $L^\theta(\lambda)$ are Γ-invariant subsets to which we can apply the measure $\mathrm{vol}_d^{\mathrm{tr}}$. The above inclusion yields

$$\mathrm{vol}_d^{\mathrm{tr}}\big(\mathcal{U}^c(\lambda)\big) \leq \mathrm{vol}_d^{\mathrm{tr}}\big(\bigcup_{\theta=1}^{\beta(d)} \big(L^\theta(\lambda) \backslash L^\theta(\lambda+1)\big) \leq \sum_{\theta=1}^{\beta(d)} \mathrm{vol}_d^{\mathrm{tr}}\big(L^\theta(\lambda) \backslash L^\theta(\lambda+1)\big)$$

$$= \sum_{\theta=1}^{\beta(d)} \mathrm{vol}_d^{\mathrm{tr}}\big(L^\theta(\lambda)\big) - \mathrm{vol}_d^{\mathrm{tr}}\big(L^\theta(\lambda+1)\big)$$

$$\leq \beta(d)\big(T(\lambda) - T(\lambda+1)\big).$$

We conclude further that

$$T(\lambda) - T(\lambda+1) \geq \frac{1}{\beta(d)} \operatorname{vol}_d^{\mathrm{tr}}\left(\bigcup \mathcal{U}^c(\lambda)\right) \geq \frac{10^{-4(d+3)}}{\beta(d)} \operatorname{vol}_d^{\mathrm{tr}}\left(\bigcup \mathcal{U}(\lambda)\right) \quad (\text{Lemma } 7.1\ (3))$$
$$\geq \frac{10^{-8(d+3)}}{\beta(d)} \operatorname{vol}_d^{\mathrm{tr}}\left(L^1(\lambda)\right) \quad (\text{using } (7.1))$$
$$= 10^{-16(d+3)} \operatorname{vol}_d^{\mathrm{tr}}\left(L^1(\lambda)\right)$$
$$\geq 10^{-16(d+3)} T(\lambda).$$

Hence $T(\lambda+1) \leq (1 - 10^{-16(d+3)})T(\lambda)$. So T decays exponentially. More precisely, we obtain that for $\lambda \geq 1$

$$T(\lambda) \leq e^{-\alpha(d)\lambda} \cdot T(1) \leq e^{-\alpha(d)\lambda} \cdot \operatorname{vol}(M), \tag{7.2}$$

where $\alpha = \alpha(d) = -\log(1 - 10^{-16(d+3)})$. Finally, we relate the function T to the volume of the high multiplicity subset. Let $(x,p) \in X \times \widetilde{M}$ be a point with $m_x(p) \geq \lambda + \beta(d)$. Since the balls in layer $\mathcal{U}(l)_x$ are disjoint, the point (x,p) lies in at most λ many balls from the layers $\bigcup \mathcal{U}(1)_x, \ldots, \bigcup \mathcal{U}(\lambda)_x$. Hence $(x,p) \in L^{\beta(d)}(\lambda)$. We conclude that

$$\operatorname{vol}_d^{\mathrm{tr}}\left(\{(x,p) \mid m_x(p) \geq \lambda + \beta(d)\}\right) \leq \operatorname{vol}_d^{\mathrm{tr}}\left(L^{\beta(d)}(\lambda)\right) \leq T(\lambda) \leq e^{-\alpha(d)\lambda} \operatorname{vol}(M).$$

$\square$

7.3 Bounding the transverse volume of the image of the nerve map.

In the sequel let $\mathcal{U} = \{A_j \times B_j \mid j \in J\}$ be a good Cantor cover of $X \times \widetilde{M}$ with no self-intersections. Let $\Phi \colon X \times \widetilde{M} \to N^{\mathrm{Ca}}(\mathcal{U})$ be the Cantor nerve map. The following (non-equivariant) statement only concerns the fiberwise nerve map $\Phi_x \colon \widetilde{M} \to \mathcal{N}(\mathcal{U}_x)$. In view of Remark 6.7 we can cite the following theorem from Guth's paper. Recall that the constant V_1 denotes an upper bound on the volume of 1-balls of $\widetilde{M}$ (see Theorem 1.1) and that $d(F)$ denotes the dimension of a face F.

Theorem 7.7 ([Gut11, Lemma 5]). *There are dimensional constants $C(d) > 0$ and $\beta(d) > 0$ so that for every $x \in X$ and every open face $F \in N(\mathcal{U}_x)$ we have*

$$\operatorname{vol}_d\left(\Phi|_{\Phi^{-1}\left(\{x\} \times \operatorname{star}(F)\right)}\right) < C(d) \cdot V_1 \cdot r_1(F)^{d+1} \cdot e^{-\beta(d) \cdot d(F)}.$$

We now fix a dimensional constant $\beta(d) > 0$ that satisfies the conclusions of Theorems 7.5 and 7.7.

Theorem 7.8 *There is a dimensional constant $C(d) > 0$ such that*

$$\operatorname{vol}_d^{\mathrm{tr}}\left(\Phi\right) \leq C(d) \cdot \operatorname{vol}(M).$$

Proof Let n be the maximal multiplicity of the cover $\{B_j \mid j \in J\}$ of $\widetilde{M}$. For $i \in \mathbb{N}_0$ we define the Γ-invariant subsets

$$S_i := \big\{(x,p) \in X \times \widetilde{M} \mid i + \beta(d) \le m_x(p) < 1 + i + \beta(d)\big\},$$
$$S := \big\{(x,p) \in X \times \widetilde{M} \mid m_x(p) < \beta(d)\big\}.$$

Restricted to S_i or S the map Φ is fiberwise Lipschitz with Lipschitz constant at most $2(1 + i + \beta(d))^{1/2}$ or $2\beta(d)^{1/2}$, respectively (cf. the proof of Lemma 6.6). Therefore we have

$$\mathrm{vol}_d^{\mathrm{tr}}(\Phi) \le \mathrm{vol}_d^{\mathrm{tr}}\big(\Phi|_S\big) + \sum_{i=0}^n \mathrm{vol}_d^{\mathrm{tr}}\big(\Phi|_{S_i}\big)$$

$$\le \big(2\beta(d)^{1/2}\big)^d \cdot \mathrm{vol}(M) + \sum_{i=0}^n \big(2(1 + i + \beta(d))^{1/2}\big)^d \cdot \mathrm{vol}_d^{\mathrm{tr}}(S_i)$$

$$\le \Big(2^d \beta(d)^{d/2} + \sum_{i=0}^n 2^d (1 + i + \beta(d))^{d/2} e^{-\alpha(d)\cdot i}\Big) \cdot \mathrm{vol}(M) \qquad \text{(Theorem 7.5)}$$

$$\le C(d) \cdot \mathrm{vol}(M),$$

where we set $C(d)$ to be the value of the convergent series

$$C(d) := \Big(2^d \beta(d)^{d/2} + \sum_{i=0}^\infty 2^d (1 + i + \beta(d))^{d/2} e^{-\alpha(d)\cdot i}\Big).$$

Since $\alpha(d), \beta(d)$ are dimensional constants, so is $C(d)$. $\qquad\square$

We now fix a dimensional constant $C(d) > 0$ that satisfies the conclusions of Theorems 7.7 and 7.8.

8 Pushing the Equivariant Nerve Map Down to the d-Skeleton

In this section we deform the Cantor nerve map of a Cantor cover $\mathcal{U}$ to the d-skeleton with $d = \dim(M)$. The non-equivariant counterpart in Guth's paper [Gut11] is the one where tools from geometric measure theory enter. An essential tool is the following result.

Theorem 8.1 (Pushout lemma [Gut17, Lemma 0.6]). *For each dimension $d \ge 2$ there is a constant $\sigma(d) > 0$ so that the following holds. Suppose that N is a compact piecewise smooth d-dimensional manifold with boundary. Suppose that $K \subset \mathbb{R}^n$ is a convex set, and $\phi\colon (N, \partial N) \to (K, \partial K)$ is a piecewise smooth map. Then ϕ may be homotoped into a map ϕ' so that the following holds.*

  $\triangleright$ *The map ϕ' agrees with ϕ on ∂N.*
  $\triangleright$ $\mathrm{vol}_d(\phi') \le \mathrm{vol}_d(\phi)$.
  $\triangleright$ *The image $\phi'(N)$ lies in the $\sigma(d) \cdot \mathrm{vol}_d(\phi)^{1/d}$-neighborhood of ∂K.*

Here is a list of dimensional constants to be used below.

$$\begin{array}{c|l} \beta(d) & \text{defined after Theorem 7.7.} \\ C(d) & \text{defined after Theorem 7.8.} \\ \sigma(d) & \text{see the Pushout lemma above.} \end{array}$$

Next we recall the definition of thin and thick faces from Guth's paper. To this end, we choose $\epsilon > 0$ small enough so that

$$\prod_{k=d+1}^{\infty} \left(1 - 2\big(3 \cdot \epsilon \cdot \sigma(d)^d \cdot e^{-\beta(d) \cdot k}\big)^{1/d}\right)^{-d} < 2 \quad \text{and} \quad 2 \cdot \epsilon \cdot e^{\beta(d) \cdot d} < 1. \quad (8.1)$$

The infinite product converges by the exponential decay in the term $e^{-\beta(d) \cdot k}$. Since the value of ϵ only depends on d and the dimensional constant $\beta(d)$, it is a dimensional constant and we write $\epsilon = \epsilon(d)$. Let F be an open k-face with side lengths $r_1(F) \leq \cdots \leq r_k(F)$. We call the face F *thin* if

$$C(d) \cdot V_1 \cdot r_1(F) < \epsilon(d). \quad (8.2)$$

Otherwise it is called *thick*. Next we play off the framework developed in Sections 4 and 6 to transfer Guth's methods to our setting.

8.1 Compression map. Let $\delta \in (0, \frac{1}{2})$. The δ-*truncation* $N^{\mathrm{Ca}}(\mathcal{U})^{(n)}_{\delta}$ of the n-skeleton $N^{\mathrm{Ca}}(\mathcal{U})^{(n)}$ is obtained from $N^{\mathrm{Ca}}(\mathcal{U})^{(n)}$ by removing a smaller cuboid inside each n-dimensional face. Referring to the pushout (6.1), we obtain $N^{\mathrm{Ca}}(\mathcal{U})^{(n)}_{\delta}$ by removing

$$\coprod_{F \in C_n} \Big(\bigcap_{j \in J_+(F)} A_j \Big) \times \Gamma \times \Big(\prod_{k=1}^{n} [\delta r_k(F), (1-\delta) r_k(F)] \Big).$$

The self map R_{δ} of the cuboid given by F stretches linearly the interval $[\delta r_k(F), (1-\delta) r_k(F)]$ to $[0, r_k(F)]$ and sends $[0, \delta r_k(F)]$ to 0 and $[(1-\delta) r_k(F), r_k(F)]$ to $r_k(F)$ in each coordinate. The δ-*compression map* on the n-skeleton is the map $P_{\delta} \colon N^{\mathrm{Ca}}(\mathcal{U})^{(n)} \to N^{\mathrm{Ca}}(\mathcal{U})^{(n)}$ such that P_{δ} is the identity on the $(n-1)$-skeleton and on every summand of the left lower corner of the pushout (6.1) it is the equivariant extension of

$$\Big(\bigcap_{j \in J_+(F)} A_j \Big) \times \{1\} \times \prod_{k=1}^{n} [0, r_k(F)] \xrightarrow{\mathrm{id} \times R_{\delta}} N^{\mathrm{Ca}}(\mathcal{U})^{(n)}.$$

By Lemma 4.17 the map P_{δ} is a Cantor bundle map.

REMARK 8.2 Obviously, we have

$$P_{\delta}\big(N^{\mathrm{Ca}}(\mathcal{U})^{(n)}_{\delta}\big) \subset N^{\mathrm{Ca}}(\mathcal{U})^{(n-1)}.$$

The map P_{δ} is a Lipschitz Cantor bundle map with Lipschitz constant $(1-2\delta)^{-1}$.

8.2 Federer–Fleming deformation in thick faces.

Let $n > d$. Let

$$\Phi \colon X \times \widetilde{M} \to N^{\mathrm{Ca}}(\mathcal{U})^{(n)}$$

be a Lipschitz Cantor bundle map which is subordinate to $\mathcal{U}$. Referring to the pushout (6.1), we consider the subset

$$L_F := \Big(\bigcap_{j \in J_+(F)} A_j \Big) \times \{1\} \times \prod_{k=1}^{n} [0, r_k(F)]$$

of $N^{\mathrm{Ca}}(\mathcal{U})^{(n)}$. Let $L_F^\circ \subset L_F$ be similarly defined as L_F by taking the interior of the cuboid in the right hand factor. By applying Lemma 4.15 to each box L_F and taking a common refinement we obtain a clopen partition $X = B_1 \cup \cdots \cup B_m$ such that the following holds.

▷ For every $i \in \{1, \ldots, m\}$ and every $F \in C_n$ we have either $B_i \subset \bigcap_{j \in J_+(F)} A_j$ or $B_i \cap \bigcap_{j \in J_+(F)} A_j = \emptyset$.

▷ $\Phi^{-1}(L_F)|_{B_i}$ is a box (possibly empty). So we have $\Phi^{-1}(L_F)|_{B_i} = B_i \times W_{i,F}$ for some subset $W_{i,F} \subset \widetilde{M}$.

▷ If $B_i \subset \bigcap_{j \in J_+(F)} A_j$, then $\Phi|_{B_i \times W_{i,F}} = \mathrm{id}_{B_i} \times h_{i,F}$ for some Lipschitz map $h_{i,F} \colon W_{i,F} \to \prod_{k=1}^{n} [0, r_k(F)]$.

Let us denote the restriction of $h_{i,F}$ to the $h_{i,F}$-preimage of the interior of the cube by $h_{i,F}^\circ$. We apply the Federer–Fleming deformation theorem to $h_{i,F}^\circ$ for each thick $F \in C_n$ in the same way as in [Gut11, p. 70]. It gives us points $p_{i,F}$ in the interior of the cube $\prod_{k=1}^{n} [0, r_k(F)]$ such that for the radial projections $\mathrm{pr}_{i,F}$ from the interior of the cube minus the point $p_{i,F}$ to the boundary of the cube we have

$$\int J_{\mathrm{pr}_{i,F} \circ h_{i,F}^\circ} \, d\mathrm{vol}_d^{\widetilde{M}} \leq G(V_1, d) \cdot \int J_{h_{i,F}^\circ} \, d\mathrm{vol}_d^{\widetilde{M}} \tag{8.3}$$

for a constant $G(V_1, d) \geq 1$ only depending on V_1 and d. Here $\mathrm{vol}_d^{\widetilde{M}}$ is the Riemannian volume measure on $\widetilde{M}$ induced by M.

The same two remarks in [Gut11, p. 70] apply here: First, the stretching factor $G(V_1, d)$ depends on the dimension $d(F)$ of the face. However, the maximal dimension of a thick face only depends on V_1 and d as noted before. Second, the usual Federer–Fleming construction takes place in a cube rather than a cuboid. The fact that the face is thick puts a limit on how distorted it is in comparison to a cube. By properness of Φ the infimal distance ϵ of $p_{i,F}$ to $\mathrm{im}\, h_{i,F}$ over all thick $F \in C_n$ and $i \in \{1, \ldots, m\}$ is strictly positive.

Next we describe two Cantor subbundles $Z_1^{(n)}$ and $Z_2^{(n)}$ of $N^{\mathrm{Ca}}(\mathcal{U})^{(n)}$. The first one $Z_1^{(n)}$ is obtained by removing ϵ-balls around the Γ-translates of the points $p_{i,F}$, more precisely, by removing

$$\coprod_{\substack{F \in C_n \text{ thick} \\ i=1,\ldots,m}} \Big(\bigcap_{j \in J_+(F)} A_j \cap B_i \Big) \times \Gamma \times B(p_{i,F}, \epsilon).$$

The second one $Z_2^{(n)}$ is obtained by removing all thick n-faces, that is, $Z_2^{(n)}$ is given by a similar pushout as in (6.1) with the coproduct in the lower left corner running only over thin $F \in C_n$.

By equivariance, $\operatorname{im} \Phi \subset Z_1^{(n)}$, so the map Φ factors as

$$X \times \widetilde{M} \to Z_1^{(n)} \hookrightarrow N^{\mathrm{Ca}}(\mathcal{U})^{(n)}.$$

The maps $\operatorname{pr}_{i,F}$ and the identity on the $(n-1)$-skeleton and thin faces yield by the pushout property (see Lemma 4.17) a Cantor bundle map $Z_1^{(n)} \to Z_2^{(n)}$. It depends on the choice of the points $p_{i,F}$. For every such choice the composition of $\Phi \colon X \times \widetilde{M} \to Z_1^{(n)}$ with the radial projection map $Z_1^{(n)} \to Z_2^{(n)} \hookrightarrow N^{\mathrm{Ca}}(\mathcal{U})^{(n)}$ is called a *Federer–Fleming deformation* of Φ.

Since each $\operatorname{pr}_{i,F}$ is Lipschitz when restricted to the complement of a small ball around the center, a Federer–Fleming deformation of Φ is still Lipschitz. We cannot bound the Lipschitz constant by a dimensional constant, though, as we cannot control the above quantity ϵ.

LEMMA 8.3 *Let $\Phi \colon X \times \widetilde{M} \to N^{\mathrm{Ca}}(\mathcal{U})^{(n)}$ be a Lipschitz Cantor bundle map which is subordinate to $\mathcal{U}$. Let Φ' be a Federer–Fleming deformation of Φ. Then Φ' is a Lipschitz Cantor bundle map subordinate to $\mathcal{U}$ and*

$$\operatorname{vol}_d^{\mathrm{tr}}(\Phi') \leq G(V_1, n) \cdot \operatorname{vol}_d^{\mathrm{tr}}(\Phi).$$

Proof Let $E \subset N^{\mathrm{Ca}}(\mathcal{U})^{(n-1)}$ be a Borel Γ-fundamental domain of the $(n-1)$-skeleton. Then

$$\Phi^{-1}(E) \cup \bigcup_{F \in C_n} \Phi^{-1}\big(L_F^\circ\big)$$

is a Borel Γ-fundamental domain of $X \times \widetilde{M}$. The above union is disjoint. By (5.1) we obtain that

$$\operatorname{vol}_d^{\mathrm{tr}}(\Phi') = \int_{\Phi^{-1}(E)} J_{\Phi'} d\operatorname{vol}_d + \sum_{F \in C_n} \int_{\Phi^{-1}(L_F^\circ)} J_{\Phi'} d\operatorname{vol}_d \, .$$

On the Φ-preimage of the $(n-1)$-skeleton the maps Φ and Φ' coincide. The maps Φ and Φ' also coincide on $\Phi^{-1}(L_F^\circ)$ for each thin $F \in C_n$. Hence

$$\operatorname{vol}_d^{\mathrm{tr}}(\Phi') = \int_{\Phi^{-1}(E)} J_\Phi d\operatorname{vol}_d + \sum_{\substack{F \in C_n \\ F \text{ thin}}} \int_{\Phi^{-1}(L_F^\circ)} J_\Phi d\operatorname{vol}_d + \sum_{\substack{F \in C_n \\ F \text{ thick}}} \int_{\Phi^{-1}(L_F^\circ)} J_{\Phi'} d\operatorname{vol}_d \, .$$

For thin $F \in C_n$ the set $\Phi^{-1}(L_F^\circ)$ is the disjoint union of products of $B_i \cap \bigcap_{j \in J_+(F)} A_j$ and the domain of $h_{i,F}^\circ$ where i runs over $1, \ldots, m$. Recall that each

B_i is either disjoint from or contained in $\bigcap_{j \in J_+(F)} A_j$. For thick $F \in C_n$ we obtain from (8.3) that

$$
\int_{\Phi^{-1}(L_F^\circ)} J_{\Phi'} d \operatorname{vol}_d = \sum_{\substack{i=1,\dots,m \\ B_i \subset \bigcap_{j \in J_+(F)} A_j}} \mu(B_i) \int J_{\operatorname{pr}_{i,F} \circ h_{i,F}^\circ} d \operatorname{vol}_d^{\widetilde{M}}
$$

$$
\leq G(V_1, d) \cdot \sum_{\substack{i=1,\dots,m \\ B_i \subset \bigcap_{j \in J_+(F)} A_j}} \mu(B_i) \int J_{h_{i,F}^\circ} d \operatorname{vol}_d^{\widetilde{M}}
$$

$$
= G(V_1, d) \cdot \int_{\Phi^{-1}(L_F^\circ)} J_\Phi d \operatorname{vol}_d .
$$

The claimed inequality follows from Theorem 5.6. $\hfill\square$

8.3 Guth's pushout lemma for thin faces. While the Federer–Fleming deformation allows us to deform the nerve map away from thick faces, the pushout deformation of this subsection, in combination with the compression map, allows us to deform the nerve map away from thin faces.

We retain the setup at the beginning of Section 8.2. We additionally require that Φ is piecewise smooth on each fiber. We apply exactly the same argument as in [Gut17, p. 206] to each thin face and the maps h_i; we only have to take care that everything fits together to a Cantor bundle map in the end.

For each open thin face $F \in C_n$ and each $i \in \{1, \dots, m\}$ one chooses a convex subset $K_{i,F}$ of F containing almost all of F but in general position with respect to h_i: The h_i-preimage of $K_{i,F}$ is a piecewise smooth submanifold $S_{i,F}$ of $\widetilde{M}$ with boundary $\partial S_{i,F}$ which is the h_i-preimage of $\partial K_{i,F}$. We apply the Pushout Lemma 8.1 to

$$
h_i|_{S_{i,F}} \colon (S_{i,F}, \partial S_{i,F}) \to (K_{i,F}, \partial K_{i,F}).
$$

The result is a map $\tilde{h}_{i,F}$ so that $\tilde{h}_{i,F}$ coincides with h_i on $\partial S_{i,F}$ and

$$
\operatorname{vol}_d(\tilde{h}_{i,F}) \leq \operatorname{vol}_d(h_i|_{S_{i,F}}), \tag{8.4}
$$

and the image of $\tilde{h}_{i,F}$ lies in the $w_{i,F}$-neighborhood of $\partial K_{i,F}$ where

$$
w_{i,F} := \sigma(d) \cdot \operatorname{vol}_d\big(h_i|_{S_{i,F}}\big)^{1/d} .
$$

We modify the map Φ as follows. The Cantor bundle $X \times \widetilde{M}$ contains the Γ-invariant subspace

$$
\bigcup_{\substack{\gamma \in \Gamma \\ F \in C_n \text{ thin} \\ i=1,\dots,m}} \gamma \cdot \Big(\bigcap_{j \in J_+(F)} A_j \cap B_i \Big) \times \gamma \cdot S_{i,F} = \Phi^{-1}\Big(\bigcup_{\substack{\gamma \in \Gamma \\ F \in C_n \text{ thin} \\ i=1,\dots,m}} \Big(\bigcap_{j \in J_+(F)} A_j \cap B_i \Big) \times \Gamma \times K_{i,F} \Big)
$$

$$
\tag{8.5}
$$

which is a disjoint union of subspaces and in each fiber a disjoint union of piecewise smooth d-dimensional submanifolds with boundaries. We make the subspace (8.5) slightly smaller by replacing each $S_{i,F}$ with its interior and then consider the complement, which we denote by $R \subset X \times \widetilde{M}$, of this smaller subspace. Then $X \times \widetilde{M}$ can be expressed as the pushout of Cantor bundles

$$
\begin{array}{ccc}
\coprod_{\substack{\gamma \in \Gamma \\ F \in C_n \text{ thin} \\ i=1,\ldots,m}} \left(\bigcap_{j \in J_+(F)} A_j \cap B_i \right) \times \Gamma \times \partial S_{i,F} & \longrightarrow & R \\
\Big\downarrow & & \Big\downarrow \\
\coprod_{\substack{\gamma \in \Gamma \\ F \in C_n \text{ thin} \\ i=1,\ldots,m}} \left(\bigcap_{j \in J_+(F)} A_j \cap B_i \right) \times \Gamma \times S_{i,F} & \longrightarrow & X \times \widetilde{M}.
\end{array}
$$

By the pushout property (Lemma 4.17) we obtain a new Cantor bundle map $\Phi' \colon X \times \widetilde{M} \to N^{\mathrm{Ca}}(\mathcal{U})^{(n)}$ that coincides with Φ on R and is the equivariant extension of $\mathrm{id} \times \tilde{h}_{i,F}$ on each summand of the left lower corner of the pushout. The map Φ' is still Lipschitz and piecewise smooth on each fiber. We say that the map Φ' is a *pushout deformation* of Φ.

Similarly as in Lemma 8.3, we conclude the following statement from (8.4).

LEMMA 8.4 *If Φ' is a pushout deformation of Φ, then*

$$
\mathrm{vol}_d^{\mathrm{tr}}(\Phi') \le \mathrm{vol}_d^{\mathrm{tr}}(\Phi)
$$

and

$$
\mathrm{vol}_d\left(\Phi'_x|_{\Phi'_x{}^{-1}(\mathrm{star}(F))} \right) \le \mathrm{vol}_d\left(\Phi_x|_{\Phi_x^{-1}(\mathrm{star}(F))} \right)
$$

for every thin face $F \in N(\mathcal{V})$ and every $x \in X$.

8.4 Pushing down the skeleta. We consider the Cantor nerve map Φ of a Cantor cover $\mathcal{U}$ with no self-intersections. Let $\mathcal{V}$ be the locally finite cover of $\widetilde{M}$ obtained by the right factors of elements in $\mathcal{U}$ (see Definition 6.2). Let $N \in \mathbb{N}$ be such that the nerve map $\Phi \colon X \times \widetilde{M} \to N^{\mathrm{Ca}}(\mathcal{U})$ lands in the N-skeleton. We define

$$
\delta(k) := \left(3 \cdot \epsilon(d) \cdot \sigma(d)^d \cdot e^{-\beta(d) \cdot k} \right)^{1/d}.
$$

We set $\Phi_N := \Phi$ and construct, by a finite downward induction, a sequence of Lipschitz Cantor bundle maps subordinate to $\mathcal{U}$

$$
\Phi_i \colon X \times \widetilde{M} \to N^{\mathrm{Ca}}(\mathcal{U})^{(i)}, \ i = N, \ldots, d,
$$

such that for every $i \in \{N, N-1, \ldots, d+1\}$

$$
\mathrm{vol}_d^{\mathrm{tr}}\left(\Phi_{i-1} \right) \le \left(1 - 2 \cdot \delta(i) \right)^{-d} \cdot G(V_1, d) \cdot \mathrm{vol}_d^{\mathrm{tr}}\left(\Phi_i \right), \tag{8.6}
$$

and for every thin face $F \in N(\mathcal{V})$ and every $x \in X$ and $i \in \{N, N-1, \ldots, d+1\}$

$$\mathrm{vol}_d\Big(\Phi_{i-1}\big|_{\Phi_{i-1}^{-1}\left(\{x\}\times\mathrm{star}(F)\right)}\Big) \leq \big(1 - 2\cdot\delta(i)\big)^{-d}\cdot\mathrm{vol}_d\Big(\Phi_i\big|_{\Phi_i^{-1}\left(\{x\}\times\mathrm{star}(F)\right)}\Big) \quad (8.7)$$

and for every thin face $F \in N(\mathcal{V})$ and every $x \in X$ and $i \in \{N, N-1, \ldots, d\}$

$$\mathrm{vol}_d\Big(\Phi_i\big|_{\Phi_i^{-1}\left(\{x\}\times\mathrm{star}(F)\right)}\Big) \leq 2\cdot\epsilon(d)\cdot r_1(F)^d\cdot e^{-\beta(d)\cdot d(F)}. \quad (8.8)$$

We combine the deformation steps in the previous subsections to inductively deform Φ_i with $i > d$ which lands in the i-skeleton to map Φ_{i-1} which lands in the $(i-1)$-skeleton. The application of Guth's pushout lemma requires that the map to the nerve is fiberwise piecewise smooth. This is true for the original Cantor nerve map. All deformation steps preserve that property. The first map $\Phi_N = \Phi$ satisfies (8.8) by Theorem 7.7 and (8.2). Next we construct Φ_{i-1} from Φ_i.

First, let Φ_i' be a Federer–Fleming deformation of Φ_i. The image of Φ_i' does not meet any open thick i-faces; it lies in the Cantor subcomplex $Z_2^{(i)}$. By Lemma 8.3,

$$\mathrm{vol}_d^{\mathrm{tr}}\big(\Phi_i'\big) \leq G(V_1, d)\cdot\mathrm{vol}_d^{\mathrm{tr}}\big(\Phi_i\big). \quad (8.9)$$

Let F be an open thin face in $N^{\mathrm{Ca}}(\mathcal{U})_x \subset \{x\}\times N(\mathcal{V})$. Let $x \in X$. We have

$$\Phi_i'\big|_{\Phi_i'^{-1}\left(\{x\}\times F\right)} = \Phi_i\big|_{\Phi_i^{-1}\left(\{x\}\times F\right)}.$$

Since every face in the open star of a thin face is thin, we also obtain that

$$\Phi_i'\big|_{\Phi_i'^{-1}\left(\{x\}\times\mathrm{star}(F)\right)} = \Phi_i\big|_{\Phi_i^{-1}\left(\{x\}\times\mathrm{star}(F)\right)}. \quad (8.10)$$

Next we assume that F is i-dimensional and we consider a pushout deformation Φ_i'' of Φ_i' which does not increase volumes according to Lemma 8.4. Let

$$w_{i,F} := \sigma(d)\cdot\mathrm{vol}_d\Big(\Phi_i'\big|_{\Phi_i'^{-1}\left(\{x\}\times F\right)}\Big)^{1/d} = \sigma(d)\cdot\mathrm{vol}_d\Big(\Phi_i\big|_{\Phi_i^{-1}\left(\{x\}\times F\right)}\Big)^{1/d}.$$

The image of $(\Phi_i'')_x$ within F lies in the $w_{i,F}$-neighborhood of the boundary of a convex subset of F which we can choose arbitrarily large within F. By (8.8) we have

$$w_{i,F} \leq \sigma(d)\cdot\mathrm{vol}_d\Big(\Phi_i\big|_{\Phi_i^{-1}\left(\{x\}\times\mathrm{star}(F)\right)}\Big)^{1/d} \leq \big(2\cdot\sigma(d)^d\cdot\epsilon(d)\cdot e^{-\beta(d)\cdot i}\big)^{1/d}\cdot r_1(F).$$

Hence we choose the convex subset so that $\mathrm{im}(\Phi_i'')_x \cap F$ lies in the $\delta(i)\cdot r_1(F)$-neighborhood of ∂F. Hence the composition with a suitable compression map

$$\Phi_{i-1} := \mathrm{P}_{\delta(i)} \circ \Phi_i''$$

lands in the $(i-1)$-skeleton. Now (8.6) follows from (8.9), the fact that the pushout deformation does not increase volume, and $\mathrm{P}_{\delta(i)}$ having Lipschitz constant

at most $\left(1 - 2 \cdot \delta(i)\right)^{-1}$. Similarly and because of (8.10) for every thin face F and $\mathrm{P}_{\delta(i)}^{-1}(\mathrm{star}(F)) \subset \mathrm{star}(F)$ for every face F we obtain (8.7). Note that (8.1) says that

$$\prod_{l=i}^{N} \left(1 - 2 \cdot \delta(l)\right)^{-d} < 2.$$

Using the induction hypothesis, we obtain (8.8) from

$$
\begin{aligned}
\mathrm{vol}_d\Big(\Phi_{i-1}\big|_{\Phi_{i-1}^{-1}\left(\{x\}\times\mathrm{star}(F)\right)}\Big) &\leq \mathrm{vol}_d\Big(\Phi\big|_{\Phi^{-1}\left(\{x\}\times\mathrm{star}(F)\right)}\Big) \cdot \prod_{l=i}^{N}\left(1 - 2\cdot\delta(l)\right)^{-d} \\
&\leq 2 \cdot \mathrm{vol}_d\Big(\Phi\big|_{\Phi^{-1}\left(\{x\}\times\mathrm{star}(F)\right)}\Big) \\
&\leq 2 \cdot C(d) \cdot V_1 \cdot r_1(F)^{d+1} \cdot e^{-\beta(d)\cdot d(F)} \quad (\text{Theorem } 7.7) \\
&\leq 2 \cdot \epsilon(d) \cdot r_1(F)^d \cdot e^{-\beta(d)\cdot d(F)} \quad (\text{see } (8.2)).
\end{aligned}
$$

Theorem 8.5   *There is constant $C(V_1, d) > 0$ only depending on V_1 and d such that the map $\Phi_d\colon X \times \widetilde{M} \to N^{\mathrm{Ca}}(\mathcal{U})^{(d)}$ satisfies*

$$\mathrm{vol}_d^{\mathrm{tr}}\big(\Phi_d\big) \leq C(V_1, d) \cdot \mathrm{vol}(M).$$

Furthermore, for every thin d-face F in $N(\mathcal{V})$ and every $x \in X$ we have

$$\mathrm{vol}_d\Big(\Phi_d\big|_{\Phi_d^{-1}\left(\{x\}\times\mathrm{star}(F)\right)}\Big) < r_1(F)^d.$$

Proof By the same argument as in [Gut11, Proof of Lemma 9] based on [Gut11, Lemma 3], there is a constant $D(V_1, d) > 0$ only depending on V_1 and the dimension d such that every thick face in $N^{\mathrm{Ca}}(\mathcal{U})$ is at most $D(V_1, d)$-dimensional. Hence we have to apply the Federer–Fleming deformation step at most $D(V_1, d)$ times. The constant $G(V_1, d)$ in (8.6) only appears if a Federer–Fleming deformation was used when deforming Φ_i to Φ_{i-1}. Therefore we obtain that

$$
\begin{aligned}
\mathrm{vol}_d^{\mathrm{tr}}\big(\Phi_d\big) &\leq \prod_{l=d+1}^{\infty}\left(1 - 2\delta(l)\right)^{-d} \cdot G(V_1, d)^{D(V_1, d)} \cdot \mathrm{vol}_d^{\mathrm{tr}}\big(\Phi\big) \\
&\leq 2 \cdot G(V_1, d)^{D(V_1, d)} \cdot \mathrm{vol}_d^{\mathrm{tr}}\big(\Phi\big).
\end{aligned}
$$

The first assertion now follows from Theorem 7.8. The second assertion follows from (8.1) and (8.8). $\square$

9 From Volume to Simplicial Volume

In this section we complete the proofs of the main results in the introduction.

Let us recall the common setup of the main theorems. Let M be a closed connected d-dimensional Riemannian manifold with fundamental group Γ. Let X be a Cantor space endowed with a free continuous action of Γ and a Γ-invariant Borel probability measure μ (Theorem 2.1). Let $V_1 > 0$ be an upper bound of the volumes of 1-balls in the universal cover $\widetilde{M}$. We choose a good Cantor cover $\mathcal{U}$ on $X \times \widetilde{M}$ with no self-intersections (Theorem 6.1). The transverse volume of the associated Cantor nerve map is universally bounded by the volume of M (Theorem 7.8). According to the previous section, in particular Theorem 8.5, we can deform the Cantor nerve map to the d-skeleton without loosing the control on its transverse volume. More precisely, we obtain a Cantor bundle map $\Phi \colon X \times \widetilde{M} \to N^{\mathrm{Ca}}(\mathcal{U})^{(d)}$ subordinate to $\mathcal{U}$ such that

$$\mathrm{vol}_d^{\mathrm{tr}}(\Phi) \leq C(d, V_1) \cdot \mathrm{vol}(M) \tag{8.1}$$

for a constant $C(d, V_1) > 0$ only depending on V_1 and the dimension d. The second assertion of Theorem 8.5 and the fact that $r_1(F)^d \leq \mathrm{vol}_d(F)$ for an open d-face F implies that for every $x \in X$ and every thin open d-face F in $N(\mathcal{U}_x)$ the image of Φ_x misses at least a point of F. As before, we will denote by $\mathcal{V}$ the locally finite cover of $\widetilde{M}$ by all balls appearing as factors of elements of $\mathcal{U}$. For the rest of this section, let N denote the d-skeleton of the nerve of $\mathcal{V}$. In particular, we have

$$N^{\mathrm{Ca}}(\mathcal{U}) \subset X \times N.$$

The remaining steps to complete the proofs of the main theorems are as follows.

In Section 9.1 we present an auxiliary result on the freeness of certain $\mathbb{Z}[\Gamma]$-modules. This is, for example, needed in the proof of Lemma 9.6 where we invoke the fundamental theorem of homological algebra to show that a homology isomorphism between free $\mathbb{Z}[\Gamma]$-chain complexes is induced by a $\mathbb{Z}[\Gamma]$-chain homotopy equivalence. In Section 9.2 we discuss classifying maps to classifying spaces. In Section 9.3 we see how to read off geometric information from the coefficients of a suitable representative of the image of the fundamental class under Φ_*. We finish the proofs of the main theorems in Section 9.4.

9.1   A result on modules over the group ring.   If $H < \Gamma$ is a finite subgroup, then $\mathbb{Q}[\Gamma] \otimes_{\mathbb{Q}[H]} \mathbb{Q}$ is a projective $\mathbb{Q}[\Gamma]$-module. However, if H is a non-trivial finite subgroup, $\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[H]} \mathbb{Z}$ is not a projective $\mathbb{Z}[\Gamma]$-module. The following lemma shows how to remedy the situation for integral coefficients using the $\mathbb{Z}[\Gamma]$-module $C(X; \mathbb{Z})$.

LEMMA 9.1 *The following statements hold true:*

(1) *Let $H < \Gamma$ be a finite subgroup and $\chi \colon H \to \{\pm 1\}$ a character. Let $\mathbb{Z}^\chi$ denote the $\mathbb{Z}[H]$-module $\mathbb{Z}$ endowed with the action $h \cdot x = \chi(h)x$ for $h \in H$ and $x \in \mathbb{Z}$. Then the $\mathbb{Z}[\Gamma]$-module*

$$C(X; \mathbb{Z}) \otimes \left(\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[H]} \mathbb{Z}^\chi \right)$$

with the module structure induced by the diagonal (left) Γ-action is free.

(2) *Let $H < \Gamma$ be a finite subgroup. The $\mathbb{Z}[\Gamma]$-module $C(X;\mathbb{Z}) \otimes \mathbb{Z}[\Gamma/H]$ with the module structure induced by the diagonal (left) Γ-action is free.*

Proof Ad 1). Since H is finite there is a clopen fundamental domain A of the H-action on X. Consider the $\mathbb{Z}$-homomorphism

$$C(A;\mathbb{Z}) \to C(X;\mathbb{Z}) \otimes \left(\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[H]} \mathbb{Z}^\chi \right)$$

that maps $f \in C(A;\mathbb{Z})$ to $f \otimes 1 \otimes 1$. Here we regard $C(A;\mathbb{Z})$ as a subgroup of $C(X;\mathbb{Z})$ by extending functions by zero. The above homomorphism extends to a $\mathbb{Z}[\Gamma]$-homomorphism

$$g \colon \mathbb{Z}[\Gamma] \otimes C(A;\mathbb{Z}) \to C(X;\mathbb{Z}) \otimes \left(\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[H]} \mathbb{Z}^\chi \right)$$

from the induced module. Since $C(A;\mathbb{Z})$ is a free $\mathbb{Z}$-module by Corollary 3.2 it suffices to show that g is bijective.

Let $S \subset \Gamma$ be a set of representatives for the right H-cosets. We obtain a natural isomorphism as $\mathbb{Z}$-modules

$$\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[H]} \mathbb{Z}^\chi \cong \bigoplus_{\gamma \in S} \mathbb{Z}$$

and thus

$$C(X;\mathbb{Z}) \otimes \left(\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[H]} \mathbb{Z}^\chi \right) \cong \bigoplus_{\gamma \in S} C(X;\mathbb{Z}). \tag{8.2}$$

The domain of g is in an obvious way isomorphic to

$$\mathbb{Z}[\Gamma] \otimes C(A;\mathbb{Z}) \cong \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[H]} C(X;\mathbb{Z}) \cong \mathbb{Z}[\Gamma/H] \otimes C(X;\mathbb{Z}) \cong \bigoplus_{\gamma \in S} C(X;\mathbb{Z}). \tag{8.3}$$

Both isomorphisms (8.2) and (8.3) are compatible with g. Thus g is an isomorphism.

Ad 2). This is a special case of (1) for the trivial character. □

9.2 Classifying maps and the simplicial norm.

Upon passing to the barycentric subdivision N becomes a proper Γ-CW complex (Lemma 2.4). We choose a map $\Psi \colon X \times N \to X \times E\Gamma$ as in Lemma 4.19. We consider the following composition of maps of $\mathbb{Z}[\Gamma]$-chain complexes.

$$C_*(\widetilde{M}) \to C(X;\mathbb{Z}) \otimes C_*(\widetilde{M}) \xrightarrow{\Phi_*} C(X;\mathbb{Z}) \otimes C_*(N) \xrightarrow{\Psi_*} C(X;\mathbb{Z}) \otimes C_*(E\Gamma) \tag{8.4}$$

The first map is induced by the inclusion of constant function $\mathbb{Z} \hookrightarrow C(X;\mathbb{Z})$. The next two maps are induced by Φ and Ψ according to Lemma 4.21. Recall that the group Γ acts diagonally on the tensor products. The abelian group $C(X;\mathbb{Z})$ is free, in particular flat, due to Corollary 3.2. Hence we have a resolution of the $\mathbb{Z}[\Gamma]$-module $C(X;\mathbb{Z})$ on the right. Again by Lemma 9.1 it is a free $\mathbb{Z}[\Gamma]$-resolution. The singular

chain complex $C_*(\widetilde{M})$ is a free $\mathbb{Z}[\Gamma]$-chain complex, and on 0-th homology the above composition is the inclusion of constant functions. Let $c\colon \widetilde{M} \to E\Gamma$ be the classifying map. Then

$$C_*(\widetilde{M}) \xrightarrow{c_*} C_*(E\Gamma) \hookrightarrow C(X;\mathbb{Z}) \otimes C_*(E\Gamma)$$

is a $\mathbb{Z}[\Gamma]$-chain map with the same behaviour on 0-th homology. By the fundamental theorem of homological algebra the two chain maps are chain homotopic which we record for later use.

REMARK 9.2 Let $c\colon \widetilde{M} \to E\Gamma$ be the classifying map. The map (8.4) and the chain map $C_*(\widetilde{M}) \xrightarrow{c_*} C_*(E\Gamma) \hookrightarrow C(X;\mathbb{Z}) \otimes C_*(E\Gamma)$ are chain homotopic as $\mathbb{Z}[\Gamma]$-chain maps. Further, the latter map is equal to the composition $C_*(\widetilde{M}) \hookrightarrow C(X;\mathbb{Z}) \otimes C_*(\widetilde{M}) \xrightarrow{\mathrm{id}\,\otimes c_*} C(X;\mathbb{Z}) \otimes C_*(E\Gamma)$.

9.3   Cellular chains and volume in the Cantor nerve.   Let S_n be a complete set of representatives of the Γ-orbits of n-faces of N. For each n-face F let $\Gamma_F < \Gamma$ be the finite subgroup of elements $\gamma \in \Gamma$ with $\gamma F = F$ as subsets of N. After choosing an orientation for an n-face F we obtain a character $\eta_F\colon \Gamma_F \to \{\pm 1\}$ which indicates whether $\gamma \in \Gamma_F$ preserves or reverses the orientation; the character is independent of the choice of orientation. Note that the character would be trivial if the CW-structure of N would be a Γ-CW structure. There is an isomorphism of $\mathbb{Z}[\Gamma]$-modules

$$C_n^{\mathrm{cell}}(N) \cong \bigoplus_{F \in S_n} \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_F]} \mathbb{Z}^{\eta_F}. \tag{8.5}$$

Lemma 9.1 now implies the first statement of the following lemma. The second statement follows similarly by noting that the singular chain groups $C_n(N;\mathbb{Z})$ is a direct sum of $\mathbb{Z}[\Gamma]$-modules of the type $\mathbb{Z}[\Gamma/H] \cong \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[H]} \mathbb{Z}$ with $H < \Gamma$ being finite.

LEMMA 9.3 For every $n \in \mathbb{N}$ the $\mathbb{Z}[\Gamma]$-module $C(X;\mathbb{Z}) \otimes_{\mathbb{Z}} C_n^{\mathrm{cell}}(N)$ endowed with the diagonal Γ-action is free. The same is true when $C_n^{\mathrm{cell}}(N)$ is replaced by $C(X)$.

There is a natural chain map from the cellular chain complex of N to the oriented singular chain complex of N—but not to the singular chain complex of N. Recall that the *oriented singular chain complex* $C_*^o(Y)$ of a space Y is the quotient complex of $C_*(Y)$ obtained by introducing the relation $g\sigma - \mathrm{sign}(g)\sigma$ for $\sigma \in C_p(X)$, $g \in S(p+1)$ and the natural action of the symmetric group $S(p+1)$ on singular p-simplices, and the relation $\sigma = 0$ if there is a transposition t with $t\sigma = \sigma$.

The barycentric subdivision of a (closed) n-face in N consists of $2^n \cdot n!$ many n-simplices. For each oriented n-face F of N the sum of the affine singular n-simplices matching the simplices of the barycentric subdivision with orientation is a chain $c_F \in C_n^o(N)$. We obtain an equivariant chain map

$$s_*\colon C_*^{\mathrm{cell}}(N) \to C_*^o(N)$$

that maps an oriented n-face F to c_F. We endow the oriented singular chain complex with the quotient norm. Since the integral foliated simplicial volume is defined in terms of the norm on singular chains it is important to know that we do not lose too much by passing to oriented singular chains. The following result can be found in [CS19, Theorem 3.3 and Remark 3.4].

Theorem 9.4   *Let Y be a topological space. The projection $\mathrm{pr}_* \colon C_*(Y) \to C_*^o(Y)$ is a natural chain homotopy equivalence. The norm of the map pr_* is at most 1, and the norm of a suitable chain homotopy inverse is at most $(p+1)!$ in degree p.*

REMARK 9.5 The natural chain homotopy inverse constructed in [CS19, Theorem 3.3 and Remark 3.4] takes the equivalence class of a singular simplex σ and maps it to a linear combination of singular simplices with coefficients in $\{1, -1\}$ which corresponds to a barycentric subdivision of σ.

LEMMA 9.6 *The composition*

$$C(X;\mathbb{Z}) \otimes C_*^{\mathrm{cell}}(N) \xrightarrow{\mathrm{id} \otimes s_*} C(X;\mathbb{Z}) \otimes C_*^o(N) \xrightarrow{\mathrm{id} \otimes q_*^N} C(X;\mathbb{Z}) \otimes C_*(N),$$

where q_*^N is any natural chain homotopy inverse as in Theorem 9.4, is a $\mathbb{Z}[\Gamma]$-chain homotopy equivalence (with regard to the diagonal Γ-actions).

Proof Both s_* and q_*^N are homology isomorphisms. Since $C(X;\mathbb{Z})$ is a free abelian group (Corollary 3.2), also $\mathrm{id} \otimes s_*$ and $\mathrm{id} \otimes q_*^N$ are homology isomorphisms. Both the domain and the codomain of the composition are free $\mathbb{Z}[\Gamma]$-modules by Lemma 9.3. By the fundamental theorem of homological algebra the composition is a $\mathbb{Z}[\Gamma]$-homotopy equivalence.   □

In the following we consider the local degree of a map $f \colon \widetilde{M} \to S^d$ which is proper outside a fixed basepoint of S^d [Dol95, VIII.4]. Recall that the *local degree* of f at point $z \in S^d$ different from the basepoint is the integer $\deg_z(f)$ such that the locally finite fundamental class is sent to $\deg_z(f) \cdot [S^d]$ under

$$H_d^{\mathrm{lf}}\big(\widetilde{M}\big) \to H_d\big(\widetilde{M}, \widetilde{M} \backslash f^{-1}(\{z\})\big) \xrightarrow{f_*} H_d\big(S^d, S^d \backslash \{z\}\big) \xleftarrow{\cong} H_d(S^d).$$

The local degree does not depend on the choice of z [Dol95, Proposition 4.4 on p. 267]; thus we denote it by $\deg(f)$. If f is Lipschitz then

$$\deg(f) = \sum_{y \in f^{-1}(\{z\})} \mathrm{sign}\det Df_y \tag{8.6}$$

for almost every $z \in S^d$ by [Fed69, Corollary 4.1.26 on p. 383].

For a d-face F in N we write $S(F)$ for the quotient of the closure of F by its boundary which is homeomorphic to S^d. We take the collapsed boundary as the basepoint of $S(F)$.

Below we refer to the map j_*^M from Definition 3.3.

LEMMA 9.7 *Let S_d and Γ_F for a d-face be as in (8.5). Let $A_F \subset X$ be a fundamental domain for the Γ_F-action on X. Then there are $a_F \in C(X; \mathbb{Z})$ supported on A_F and integral chains $e_F \in C_d(N)$ of ℓ^1-norm at most $2^d \cdot d! \cdot (d+1)!$ such that the image of the fundamental class under*

$$H_d(M) \xrightarrow{j_*^M} H_d^\Gamma\big(\widetilde{M}; C(X; \mathbb{Z})\big) \xrightarrow{\Phi_*} H_d^\Gamma\big(N; C(X; \mathbb{Z})\big)$$

is represented by the cycle $\sum_{F \in S_d} a_F \otimes e_F$. For $x \in A_F$, we have

$$\deg\big(\widetilde{M} \xrightarrow{\Phi_x} N \xrightarrow{\mathrm{pr}_F} S(F)\big) = \pm a_F(x)$$

and

$$|a_F(x)| \le \frac{1}{\mathrm{vol}_d(F)} \int_{\Phi_x^{-1}(F)} J_d \Phi(y) d\, \mathrm{vol}_d^{\widetilde{M}}(y).$$

Proof Every d-cycle in $C(X; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C_*(N)$ is homologous to a cycle coming from $C(X; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C_d^{\mathrm{cell}}(N)$ via the map in Lemma 9.6 (after passing to Γ-coinvariants). Since $C_d^{\mathrm{cell}}(N)$ is an abelian group generated by Γ-translates of $F \in S_d$, it follows that every d-cycle is homologous to a d-cycle of the form

$$\sum_{F \in S_d} b_F \otimes q_*^N\big([c_F]\big) \in C(X; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} C_d(N)$$

for some $b_F \in C(X; \mathbb{Z})$. Set $e_F := q_*^N([c_F])$. The statement about the norm e_F follows from the fact that c_F consists of $2^d \cdot d!$ singular simplices and Theorem 9.4. Next we rewrite the tensor products to obtain functions supported on A_F: Let $b_F' = \chi_{A_F} \cdot b_F$ where χ_{A_F} is the characteristic function of A_F. We have

$$b_F \otimes e_F = \sum_{h \in \Gamma_F} b_F'(h^{-1}{}_-) \otimes e_F = \sum_{h \in \Gamma_F} b_F' \otimes \eta_F(h) \cdot e_F = \Big(\sum_{h \in \Gamma_F} \eta_F(h) b_F' \Big) \otimes e_F$$

where $\eta_F \colon \Gamma_F \to \{\pm 1\}$ is the character as in (8.5). Therefore every homology class is of the form $\big[\sum_{F \in S_d} a_F \otimes e_F\big]$ where a_F are functions supported on A_F and e_F are integral chains of ℓ^1-norm at most $2^d \cdot d! \cdot (d+1)!$. We represent $z_M := \Phi_* \circ j_*^M([M])$ like that with suitable a_F and e_F. The image of z_M under the map ev_x, $x \in X$, from Section 4.3, which is a locally finite homology class of N, is denoted by $(z_M)_x^{\mathrm{lf}}$. We obtain that

$$(\Phi_x)_*\big([\widetilde{M}]^{\mathrm{lf}}\big) = (z_M)_x^{\mathrm{lf}} = \Big[\sum_{\gamma \in \Gamma} \sum_{F \in S_d} a_F(\gamma^{-1}x) \cdot \gamma \cdot e_F\Big]$$

as elements in the locally finite homology of N. Let F_0 be an open d-face in N and z_0 a point in F_0. We consider the image of $(z_M)_x^{\mathrm{lf}}$ under the homomorphism

$$H_d^{\mathrm{lf}}\big(N\big) \to H_d\big(N, N \setminus \mathrm{pr}_{F_0}^{-1}(\{z_0\})\big) \xrightarrow{H_d(\mathrm{pr}_{F_0})} H_d\big(S(F_0), S(F_0) \setminus \{z_0\}\big) \xleftarrow{\cong} H_d\big(S(F_0)\big).$$

For $\gamma \notin \Gamma_{F_0}$ or $F \neq F_0$ the chain $\gamma \cdot e_F$ is mapped to zero under the chain map $C_d^{\mathrm{lf}}(N) \to C_d(N, N \setminus \mathrm{pr}_{F_0}^{-1}(\{z_0\})) \to C_d(S(F_0), S(F_0) \setminus \{z_0\})$. Therefore only the terms $a_F(\gamma^{-1}x)\gamma \cdot e_F$ with $\gamma \in \Gamma_{F_0}$ and $F = F_0$ contribute something potentially non-zero to the image of the homology class. But for $x \in A_{F_0}$ and $\gamma \in \Gamma_{F_0} \setminus \{1\}$ we have $a_F(\gamma^{-1}x) = 0$. Hence $(z_M)_x^{\mathrm{lf}}$ is mapped to $a_{F_0}(x)$ times the generator, which implies the statement about the local degree. The bound for $|a_F(x)|$ is now a direct consequence of the area formula [Fed69, Theorem 3.2.5 on p. 244 and the remark before 3.2.47 on p. 282] and the characterization (8.6) of the local degree. □

9.4 Conclusion of proofs of main results. For the next result we refer the reader to the overview of dimensional constants after Theorem 8.1.

Theorem 9.8 *For every $V_1 > 0$ and $d \in \mathbb{N}$ there are constants $\mathrm{const}(d, V_1) > 0$ and $\epsilon(d) > 0$ with the following properties.*

Let (M, g) be a d-dimensional closed Riemannian manifold with $V_{(\widetilde{M}, \tilde{g})}(1) < V_1$. Let $\Gamma = \pi_1(M)$, and let $c \colon M \to B\Gamma$ be the classifying map. Then

$$\left\| i_*^{\mathbb{R}} \circ c_*([M]) \right\| \leq \left\| j_*^{B\Gamma} \circ c_*([M]) \right\|_{\mathbb{Z}}^X \leq \mathrm{const}(d, V_1) \cdot \mathrm{vol}(M).$$

Furthermore, if $V_{(\widetilde{M}, \tilde{g})}(1) < C(d)^{-1} \cdot \epsilon(d)$, then

$$i_*^{\mathbb{R}} \circ c_*([M]) = 0 \in H_d(B\Gamma; \mathbb{R}).$$

Proof The following diagram contains all the maps we have to consider.

The middle horizontal map is induced by the (equivariant) classifying map $\widetilde{M} \to E\Gamma$ and the identity on $C(X; \mathbb{Z})$. The right-hand horizontal map is induced by integration $C(X; \mathbb{Z}) \to \mathbb{R}$ (see Remark 3.4). The upper triangle is commutative by Remark 9.2. That the lower part commutes is straightforward.

Let z_M be the image of $[M] \in H_d(M)$ in $H_d^{\Gamma}(N; C(X; \mathbb{Z}))$. According to Lemma 9.7 the homology class z_M is represented by a cycle of the form

$$\sum_{F \in S_d} a_F \otimes e_F$$

where the function a_F is supported on A_F and $a_F(x)$ is the local degree of Φ_x followed by the projection to $S(F)$ for $x \in A_F$. If $F \in S_d$ is thin and $x \in X$, then there is

at least one point in the interior of F that is not in the image of Φ_x according to the volume estimate of Theorem 8.5. In this case $a_F(x) = 0$. Under the assumption $V_{(\widetilde{M},\tilde{g})}(1) < C(d)^{-1} \cdot \epsilon(d)$ every d-face is thin. Hence $z_M = 0$. The commutativity of the diagram implies the second statement.

The smallest side length of a thick d-face in N and hence its volume are bounded from below by a constant that only depends on the dimension d and V_1. Let $\mathrm{const}'(d, V_1) > 0$ be such that $1/\mathrm{const}'(d, V_1)$ is a lower volume bound of thick d-faces. We now set $\mathrm{const}(d, V_1) := 2^d \cdot d! \cdot (d+1)! \cdot \mathrm{const}'(d, V_1) \cdot C(d, V_1)$, where $C(d, V_1)$ is the constant in (8.1). Since Ψ_* does not increase the integral foliated norm, the above diagram commutes and because of Remark 3.4, it suffices to show that $\|z_M\|_{\mathbb{Z}}^X \leq \mathrm{const}(d, V_1) \cdot \mathrm{vol}(M)$ to obtain the first statement of the theorem. Let $T_d \subset S_d$ be the subset of thick d-faces. With the norm bound on e_F from Lemma 9.7 and the above argument for thin d-faces we obtain that

$$\|z_M\|_{\mathbb{Z}}^X \leq 2^d \cdot d! \cdot (d+1)! \cdot \sum_{F \in T_d} \int_{A_F} |a_F(x)| d\mu(x).$$

Again with Lemma 9.7 we conclude that

$$\|z_M\|_{\mathbb{Z}}^X \leq 2^d \cdot d! \cdot (d+1)! \cdot \mathrm{const}'(d, V_1) \sum_{F \in T_d} \int_{A_F} \int_{\Phi_x^{-1}(F)} \big|J_d\Phi_x(y)\big| d\,\mathrm{vol}_d^{\widetilde{M}}(y).$$

The subset

$$\left\{ (x, y) \mid F \in T_d, x \in A_F, y \in \Phi_x^{-1}(F) \right\}$$

is contained in a Γ-fundamental domain of $X \times \widetilde{M}$. Hence the Area Formula (Theorem 5.6) and the definition of $\mathrm{const}(d, V_1)$ imply that

$$\|z_M\|_{\mathbb{Z}}^X \leq 2^d \cdot d! \cdot (d+1)! \cdot \mathrm{const}'(d, V_1) \cdot \mathrm{vol}_d(\Phi) \leq \mathrm{const}(d, V_1) \cdot \mathrm{vol}(M). \qquad \square$$

Proof of Theorem 1.1 By Gromov's mapping theorem [Gro82, Section 3.1. on p. 248] we have $\|i_{\mathbb{R},*} \circ c_*([M])\| = \|M\|$. Therefore Theorem 1.1 is implied by Theorem 9.8. $\square$

Proof of Theorem 1.3 By scaling the metric, it is enough to prove the case $R = 1$. The case $R = 1$ is the second statement of Theorem 9.8. $\square$

Proof of Theorem 1.4 According to Theorem 9.8 the X-parametrised integral simplicial norm of $c_*([M]) \in H_d(B\Gamma)$ is bounded from above by $\mathrm{const}(d, V_1) \cdot \mathrm{vol}(M)$. By Theorem 3.10 the von Neumann rank of $c_*([M])$ is bounded by $d \cdot C(d, V_1) \cdot \mathrm{vol}(M)$. If M is, in addition, aspherical, then c is a homotopy equivalence and the von Neumann rank of $c_*([M])$ is the von Neumann rank of $[M]$ which is the sum of the ℓ^2-Betti numbers of M according to Remark 3.9. This implies the second statement of Theorem 1.4. The statement about the Euler characteristic is an immediate consequence since the alternating sum of ℓ^2-Betti numbers equals the Euler characteristic. $\square$

Proof of Theorem 1.5 Let α be a free measurable pmp action of Γ on a standard probability space (Y, μ). By [Ele21, Theorem 2] there is a free continuous action of Γ on the Cantor set X and an equivariant Borel embedding $X \hookrightarrow Y$ such that $\mu(X) = 1$. This means that we can realize every free measurable pmp action by a free continuous action on the Cantor set. Therefore the α-parametrised simplicial volume $|M|^\alpha$ coincides with the X-parametrised simplicial volume (with regard to μ). According to Theorem 9.8 the X-parametrised integral simplicial norm of $c_*([M]) \in H_d(B\Gamma)$ is bounded from above by $\mathrm{const}(d, V_1) \cdot \mathrm{vol}(M)$. Since c is a homotopy equivalence the X-parametrised integral simplicial norm of $c_*([M])$ is the X-parametrised integral simplicial volume of M.   □

References

[AF17]   H. ALPERT and K. FUNANO. Macroscopic scalar curvature and areas of cycles. *Geom. Funct. Anal.*, **27**(4) (2017), 727–743.

[BK19]   F. BALACHEFF and S. KARAM. Macroscopic Schoen conjecture for manifolds with nonzero simplicial volume. *Trans. Am. Math. Soc.*, **372**(10) (2019), 7071–7086.

[Bra18]   S. BRAUN. *Simplicial Volume and Macroscopic Scalar Curvature.* Karlsruher Institut für Technologie (KIT), Ph.D. thesis (2018). https://publikationen.bibliothek.kit.edu/1000086838.

[BH99]   M.R. BRIDSON and A. HAEFLIGER. *Metric Spaces of Non-positive Curvature. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Vol. 319. Springer-Verlag, Berlin (1999).

[Bro94]   K.S. BROWN. *Cohomology of Groups. Graduate Texts in Mathematics*, Vol. 87. Springer-Verlag, New York (1994). Corrected reprint of the 1982 original.

[BH10]   M. BRUNNBAUER and B. HANKE. Large and small group homology. *J. Topol.*, **3**(2) (2010), 463–486.

[CS19]   C. CAMPAGNOLO and R. SAUER. Counting maximally broken Morse trajectories on aspherical manifolds. *Geom. Dedicata*, **202** (2019), 387–399.

[CG86]   J. Cheeger and M. Gromov. Collapsing Riemannian manifolds while keeping their curvature bounded. I. *J. Differ. Geom.*, **23** (1986), 3.

[Dol95]   A. Dold. *Lectures on Algebraic Topology, Classics in Mathematics*. Springer-Verlag, Berlin (1995). Reprint of the 1972 edition.

[Ele21]   G. Elek. Free minimal actions of countable groups with invariant probability measures. *Ergod. Theor. Dyn. Syst.*, **41**(5) (2021), 1369–1389.

[Fau21]   D. Fauser. Integral foliated simplicial volume and S^1-actions. *Forum Math.*, **33**(3) (2021), 773–788.

[FFL19]   D. Fauser, S. Friedl, and C. Löh. Integral approximation of simplicial volume of graph manifolds. *Bull. Lond. Math. Soc.*, **51**(4) (2019), 715–731.

[Fed69]   H. Federer. *Geometric Measure Theory. Die Grundlehren der mathematischen Wissenschaften, Band 153*. Springer-Verlag New York Inc., New York (1969).

[FLPS16] R. Frigerio, C. Löh, C. Pagliantini, and R. Sauer. Integral foliated simplicial volume of aspherical manifolds. *Israel J. Math.*, *216*(2) (2016), 707–751,

[Gab02]   D. Gaboriau. Invariants l^2 de relations d'équivalence et de groupes. *Publ. Math. Inst. Hautes Études Sci.*, *95* (2002), 93–150 (French).

[GU09]   E. Glasner and V.V. Uspenskij. Effective minimal subflows of Bernoulli flows. *Proc. Am. Math. Soc.*, **137**(9) (2009), 3147–3154.

[Gro91]   M. Gromov. Foliated Plateau problem. I. Minimal varieties. *Geom. Funct. Anal.*, 1(1) (1991), 14–79.

[Gro82]   M. Gromov. Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.*, **56** (1982), 5–99 (1983).

[Gro86]   M. Gromov. Large Riemannian manifolds. Curvature and topology of Riemannian manifolds, (Katata, 1985), Lecture Notes in Mathematics, Vol. 1201. Springer-Verlag, Berlin (1986), pp. 108–121. https://doi.org/10.1007/BFb0075649.

[Gro93]   M. Gromov. *Asymptotic Invariants of Infinite Groups. Geometric Group Theory*, Vol. 2, (Sussex, 1991), *London Mathematical Society Lecture Note Series*, Vol. 182. Cambridge University Press, Cambridge (1993), pp. 1–295.

[Gro09]   M. Gromov. Singularities, expanders and topology of maps. I. Homology versus volume in the spaces of cycles. *Geom. Funct. Anal.*, **19**(3) (2009), 743–841.

[Gut10]   L. Guth. Metaphors in systolic geometry. In: *Proceedings of the International Congress of Mathematicians*, Vol. II. Hindustan Book Agency, New Delhi (2010), pp. 745–768.

[Gut11]   L. Guth. Volumes of balls in large Riemannian manifolds. *Ann. Math.*, **173**(1) (2011), 51–76.

[Gut17]   L. Guth. Volumes of balls in Riemannian manifolds and Uryson width. *J. Topol. Anal.*, **9**(2) (2017), 195–219.

[HM06]   G. Hjorth and M. Molberg. Free continuous actions on zero-dimensional spaces. *Topol. Appl.*, **153**(7) (2006), 1116–1131.

[LLNR]   Y. Liokumovich, B. Lishak, A. Nabutovsky, and R. Rotman. Filling metric spaces. 1704.08538.

[Loh06]   C. Löh. Measure homology and singular homology are isometrically isomorphic. *Math. Z.*, **253**(1) (2006), 197–218.

[Luc98]   W. Lück. Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers. I. Foundations. *J. Reine Angew. Math.*, **495** (1998), 135–162.

[Luc02]   W. Lück. L^2-invariants: theory and applications to geometry and K-theory. In: *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, Vol. 44. Springer-Verlag, Berlin (2002).

[Nab]   A. Nabutovsky. Linear bounds for constants in Gromov's systolic inequality and related results. 1909.12225.

[Pap20]   P. Papasoglu. Uryson width and volume. *Geom. Funct. Anal.*, **30**(2) (2020), 574–587.

[Sau05]   R. Sauer. L^2-Betti numbers of discrete measured groupoids. *Int. J. Algebra Comput.*, **15**(5-6) (2005), 1169–1188.

[Sau09]   R. Sauer. Amenable covers, volume and L^2-Betti numbers of aspherical manifolds. *J. Reine Angew. Math.*, **636** (2009), 47–92.

[Sau16]   R. Sauer. Volume and homology growth of aspherical manifolds. *Geom. Topol.*, **20**(2) (2016), 1035–1059.

[Sch05]   M. Schmidt. L^2-Betti numbers of $\mathcal{R}$-spaces and the integral foliated simplicial volume, WWU Münster, Ph.D. thesis (2005). https://nbn-resolving.org/urn:nbn:de:hbz:6-05699458563.

[Sch89]   R.M. Schoen. Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. In: *Topics in Calculus of Variations (Montecatini Terme, 1987). Lecture Notes in Mathematics*, Vol. 1365. Springer-Verlag, Berlin (1989), pp. 120–154.

[Ste85]   J. Steprāns. A characterization of free abelian groups. *Proc. Am. Math. Soc.*, **93**(2) (1985), 347–349.

[Tom08]   T. tom Dieck. *Algebraic Topology. EMS Textbooks in Mathematics.* European Mathematical Society (EMS), Zürich (2008).

[Tom87]   T. tom Dieck. *Transformation Groups. De Gruyter Studies in Mathematics*, Vol. 8. Walter de Gruyter & Co., Berlin (1987).

S. Braun and R. Sauer
Karlsruhe Institute of Technology, Karlsruhe, Germany.

sabine.r.braun@gmail.com
roman.sauer@kit.edu
http://topology.math.kit.edu