

A LONG NECK PRINCIPLE FOR RIEMANNIAN SPIN MANIFOLDS WITH POSITIVE SCALAR CURVATURE

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Abstract. We develop index theory on compact Riemannian spin manifolds with boundary in the case when the topological information is encoded by bundles which are supported away from the boundary. As a first application, we establish a “long neck principle” for a compact Riemannian spin n -manifold with boundary X , stating that if $\text{scal}(X) \geq n(n-1)$ and there is a nonzero degree map into the sphere $f: X \rightarrow S^n$ which is strictly area decreasing, then the distance between the support of df and the boundary of X is at most π/n . This answers, in the spin setting and for strictly area decreasing maps, a question recently asked by Gromov. As a second application, we consider a Riemannian manifold X obtained by removing k pairwise disjoint embedded n -balls from a closed spin n -manifold Y . We show that if $\text{scal}(X) > \sigma > 0$ and Y satisfies a certain condition expressed in terms of higher index theory, then the radius of a geodesic collar neighborhood of ∂X is at most $\pi\sqrt{(n-1)/(n\sigma)}$. Finally, we consider the case of a Riemannian n -manifold V diffeomorphic to $N \times [-1, 1]$, with N a closed spin manifold with nonvanishing Rosenebrg index. In this case, we show that if $\text{scal}(V) \geq \sigma > 0$, then the distance between the boundary components of V is at most $2\pi\sqrt{(n-1)/(n\sigma)}$. This last constant is sharp by an argument due to Gromov.

1 Introduction and main results

The study of manifolds with positive scalar curvature has been a central topic in differential geometry in recent decades. On closed spin manifolds, the most powerful known obstruction to the existence of such metrics is based on the index theory for the spin Dirac operator. Indeed, the Lichnerowicz formula [Lic63] implies that, on a closed spin manifold Y with positive scalar curvature, the spin Dirac operator is invertible and hence its index must vanish.

When X is a compact Riemannian manifold with boundary of dimension at least three, it is well known by classical results of Kazdan and Warner [KW75a, KW75b, KW75c] that X always carries a metric of positive scalar curvature. In order to use topological information to study metrics of positive scalar curvature on X , we need extra geometric conditions. When X is equipped with a Riemannian metric with a product structure near the boundary, it is well known [APS75a, APS75b, APS76] that the Dirac operator with global boundary conditions is elliptic. This fact has been

extensively used in the past decades to study metrics of positive scalar curvature in the spin setting.

The purpose of this paper is to systematically extend the spin Dirac operator technique to the case when the metric does not necessarily have a product structure near the boundary and the topological information is encoded by bundles supported away from the boundary. As an application, we prove some metric inequalities with scalar curvature on spin manifolds with boundary, following the point of view recently proposed by Gromov.

1.1 Some questions by Gromov on manifolds with boundary. Recall that a map of Riemannian manifolds $f: M \rightarrow N$ is called ϵ -area contracting if $\|f^*\omega\| \leq \epsilon \|\omega\|$, for all two-forms $\omega \in \Lambda^2(N)$. When $\epsilon \leq 1$, we say that f is *area decreasing*. When $\epsilon < 1$, we say that f is *strictly area decreasing*.

Let (X, g) be a compact oriented n -dimensional Riemannian manifold with boundary and let $f: (X, g) \rightarrow (S^n, g_0)$ be a smooth area decreasing map, where g_0 denotes the standard round metric on the sphere. The “length of the neck” of (X, f) is defined as the distance between the support of the differential of f and the boundary of X . The *long neck problem* [Gro19, page 87] consists in the following question.

Question 1.1 (Long Neck Problem). What kind of a lower bound on scal_g and a lower bound on the “length of the neck” of (X, f) would make $\deg(f) = 0$?

REMARK 1.2. In this case, the topological obstruction is the existence of an area decreasing map $f: (X, g) \rightarrow (S^n, g_0)$ of nonzero degree. The extra geometric information is given by the “length of the neck” of (X, f) and the lower bound of scal_g .

REMARK 1.3. More precisely, Gromov [Gro19, page 87] conjectured the existence of a constant $c_n > 0$, depending only on the dimension n of the manifold X , such that

$$[\text{scal}_g \geq n(n-1)] \& [\text{dist}(\text{supp}(df), \partial X) \geq c_n] \Rightarrow \deg(f) = 0. \quad (1.1)$$

The main motivation of this paper is to prove this inequality in the case when X is spin.

We will now review two conjectures recently proposed by Gromov, which are related to the long neck problem. Let Y be a closed n -dimensional manifold. Let X be the n -dimensional manifold with boundary obtained by removing a small n -dimensional ball from Y . Observe that X is a manifold with boundary $\partial X \cong S^{n-1}$. Let g be a Riemannian metric on X . For $R > 0$ small enough, denote by $B_R(\partial X)$ the geodesic collar neighborhood of ∂X of width R . Gromov proposed the following conjecture [Gro18, Conjecture D’, 11.12].

CONJECTURE 1.4. *Let Y be a closed n -dimensional manifold such that Y minus a point admits no complete metric of positive scalar curvature. Let X be the manifold with boundary obtained by removing a small n -dimensional ball from Y . Let g be*

a Riemannian metric on X whose scalar curvature is bounded from below by a constant $\sigma > 0$. Then there exists a constant $c > 0$ such that if there exists a geodesic collar neighborhood $B_R(\partial X)$ of width R , then

$$R \leq \frac{c}{\sqrt{\sigma}}. \quad (1.2)$$

Let us now consider a second situation related to the long neck principle. Let N be a closed manifold. A band over N is a manifold V diffeomorphic to $N \times [-1, 1]$. If g is a Riemannian metric on V , we say that (V, g) is a Riemannian band over N and define the width of V by setting

$$\text{width}(V) := \text{dist}(\partial_- V, \partial_+ V), \quad (1.3)$$

where $\partial_- V$ and $\partial_+ V$ are the boundary components of V corresponding respectively to $N \times \{-1\}$ and $N \times \{1\}$. Recently, Gromov proposed the following conjecture [Gro18, Conjecture C, 11.12].

CONJECTURE 1.5. *Let N be a closed manifold of dimension $n - 1 \geq 5$ which does not admit a metric of positive scalar curvature. Suppose V is a Riemannian band over N whose scalar curvature is bounded from below by a constant $\sigma > 0$. Then*

$$\text{width}(V) \leq 2\pi \sqrt{\frac{n-1}{\sigma n}}. \quad (1.4)$$

REMARK 1.6. In general, one can ask whether, under the same hypotheses of Conjecture 1.5, there exists a constant c_n , depending only on the dimension n of the manifold N , such that the inequality

$$\text{width}(V) \leq \frac{c_n}{\sqrt{\sigma}} \quad (1.5)$$

holds. Gromov proved [Gro18, Optimality of $2\pi/n$, page 653] that the constant

$$c_n = 2\pi \sqrt{\frac{n-1}{n}}$$

is optimal.

1.2 Codimension zero obstructions. Let (X, g) be a compact n -dimensional Riemannian spin manifold with boundary whose scalar curvature is bounded from below by a constant $\sigma > 0$. The first main result of this paper consists in a “long neck principle” in this setting. Our method is based on the analysis of the incomplete Riemannian manifold $X^\circ = X \setminus \partial X$. The topological information is encoded by a pair of bundles with metric connections E and F over X° which have isomorphic typical fibers and are trivializable outside a compact submanifold with boundary $L \subset X^\circ$. Our topological invariant is given by the index of a twisted spin Dirac operator $D_{L_D}^{E,F}$ on the double L_D of L , constructed using the pair (E, F) .

In order to relate this invariant to the geometry of the manifold X , we make use of extra data. We use the distance function from the deleted boundary ∂X of X° to construct a rescaling function ρ in such a way that the Dirac operator of X° , rescaled by the function ρ , is essentially self-adjoint. We also make use of a potential, i.e. a smooth function $\phi: X^\circ \rightarrow [0, \infty)$ which vanishes on L and is locally constant in a neighborhood of the deleted boundary ∂X . Using these extra data, we construct a Fredholm operator $P_{\rho, \phi}^{E, F}$ on X° whose index coincides with the index of $D_{L_D}^{E, F}$. A vanishing theorem for the operator $P_{\rho, \phi}^{E, F}$ allows us to give conditions on scal_g and $\text{dist}(K, \partial V)$ in such a way that the index of $D_{L_D}^{E, F}$ must vanish. Our method can be regarded as an extension to a certain class of incomplete manifolds of the technique of Gromov and Lawson [GL80, GL83].

Theorem A. *Let (X, g) be a compact n -dimensional Riemannian spin manifold with boundary. Let $f: X \rightarrow S^n$ be a smooth strictly area decreasing map. If n is odd, we make the further assumption that f is constant in a neighborhood of ∂X . Suppose that the scalar curvature of g is bounded from below by a constant $\sigma > 0$. Moreover, suppose that*

$$\text{scal}_g \geq n(n - 1) \quad \text{on } \text{supp}(df) \tag{1.6}$$

and

$$\text{dist}(\text{supp}(df), \partial X) > \pi \sqrt{\frac{n - 1}{n\sigma}}. \tag{1.7}$$

Then $\text{deg}(f) = 0$.

REMARK 1.7. Theorem A answers Question 1.1 when X is spin and even dimensional and f is strictly area decreasing. The case when f is area decreasing can be treated with a slight modification of the techniques presented in this paper and will be discussed in a separated paper.

REMARK 1.8. Condition (1.7) implies that f is constant in a neighborhood of each connected component of ∂X so that the degree of f is well defined. The extra assumption when n is odd is needed, at least with the argument used in this paper, to reduce the odd-dimensional case to the even-dimensional case. We believe it is possible to drop this extra assumption.

REMARK 1.9. It is an interesting question whether, in dimension at most eight, it is possible to drop the spin assumption from Theorem A by using the minimal hypersurface technique of Schoen and Yau [SY79]. In fact, it is not clear whether this method can be used to approach the long neck problem, due to the difficulties, pointed out in [CS19], arising when the minimal hypersurface technique is used to treat maps that are area contracting.

We now consider a higher version of the long neck principle. Let Y be a closed n -dimensional spin manifold with fundamental group Γ . There is a canonical flat

bundle \mathcal{L}_Y over Y , called the Mishchenko bundle of Y , whose typical fiber is $C^*\Gamma$, the maximal real group C^* -algebra of Γ . The Rosenberg index [Ros83, Ros86a, Ros86b] of Y is the class $\alpha(Y) \in KO_n(C^*\Gamma)$, obtained as the index of the spin Dirac operator twisted with the bundle \mathcal{L}_Y . Here, $KO_n(C^*\Gamma)$ is the real K -theory of $C^*\Gamma$. The class $\alpha(Y)$ is the most general known obstruction to the existence of metrics of positive scalar curvature on Y . Denote by $\mathfrak{D}_{Y, \underline{C^*\Gamma}}$ the spin Dirac operator twisted with the bundle $\underline{C^*\Gamma}$, the trivial bundle on Y with typical fiber $C^*\Gamma$. We assume that

$$\text{the Rosenberg index } \alpha(Y) \text{ does not coincide with the index of } \mathfrak{D}_{Y, \underline{C^*\Gamma}}. \quad (1.8)$$

REMARK 1.10. From results of Hanke and Schick [HS06, HS07], closed enlargeable spin manifolds satisfy Condition (1.8). For the notion of enlargeable manifold, see [LM89, §IV.6]. Examples of closed enlargeable manifolds are the n -torus T^n and any closed spin manifold admitting a metric of nonpositive sectional curvature. Moreover, if M_1 and M_2 are closed spin manifolds and M_1 is enlargeable, then the connected sum $M_1 \# M_2$ is enlargeable as well. This provides us with a large class of examples satisfying Condition (1.8). For more details and examples of enlargeable manifolds, we refer the reader to [GL83, Section 5] and [LM89, § IV.6].

REMARK 1.11. An example of a manifold which is not enlargeable and satisfies Condition (1.8) is given by $T^4 \times N$, with N a $K3$ surface.

REMARK 1.12. Another interesting class of manifolds satisfying Condition (1.8) consists in aspherical spin manifolds whose fundamental group satisfies the strong Novikov conjecture.

We use Condition (1.8) to establish a “higher neck principle”. Let D_1, \dots, D_N be pairwise disjoint disks embedded in Y . Consider the compact manifold with boundary

$$X := Y \setminus (D_1^\circ \sqcup \dots \sqcup D_N^\circ),$$

where D_j° is the interior of D_j . Observe that the boundary of X is the disjoint union $\partial X = S_1^{n-1} \sqcup \dots \sqcup S_N^{n-1}$, where $S_j^{n-1} := \partial D_j$. If g is a Riemannian metric on X , the *normal focal radius* of ∂X , denoted by $\text{rad}_g^\circ(\partial X)$, is defined as follows. For $R > 0$ small enough, denote by $B_R(S_j^{n-1})$ the geodesic collar neighborhood of S_j^{n-1} of width R . Define $\text{rad}_g^\circ(\partial X)$ as the supremum of the numbers $R > 0$ such that there exist pairwise disjoint geodesic collar neighborhoods $B_R(S_1^{n-1}), \dots, B_R(S_N^{n-1})$.

Theorem B. *Let Y, Γ and X be as above. Suppose the Rosenberg index $\alpha(Y)$ does not coincide with the index of $\mathfrak{D}_{Y, \underline{C^*\Gamma}}$. Moreover, suppose g is a Riemannian metric on X whose scalar curvature is bounded from below by a constant $\sigma > 0$. Then*

$$\text{rad}_g^\circ(\partial X) \leq \pi \sqrt{\frac{n-1}{n\sigma}}. \quad (1.9)$$

In view of Conjecture 1.4, it is natural to consider, under Condition (1.8), the manifold Y with N points removed and ask whether it admits complete metrics of positive scalar curvature.

Theorem C. *Let Y be a closed spin manifold with fundamental group Γ and let P_1, \dots, P_N be distinct points in X . Suppose the Rosenberg index $\alpha(Y)$ does not coincide with the index of $\mathfrak{D}_{Y, C^*\Gamma}$. Then the open manifold $M := Y \setminus \{P_1, \dots, P_N\}$ cannot carry any complete metric of positive scalar curvature.*

REMARK 1.13. This theorem can be thought of as a “codimension zero” version of [Cec18, corollary B] and is proved with similar methods. Theorem C can also be regarded as a “higher version” of [Zha19, Theorem 1.1].

REMARK 1.14. When $N = 1$, Theorems B and C imply that Conjecture 1.4 holds with constant $c = \pi \sqrt{(n - 1)/n}$ for all closed n -dimensional spin manifolds satisfying Condition (1.8).

REMARK 1.15. When Y is simply connected, Condition (1.8) is vacuous and Theorems B and C are vacuous as well. The geometric interpretation of this fact could be related to the observation of Gromov [Gro19, page 723] that Conjecture 1.4 is probably vacuous for simply connected manifolds.

1.3 Codimension one obstructions. Let us now consider an n -dimensional Riemannian band (V, g) over a closed spin manifold N . Let $\partial_{\pm}V$ and $\text{width}(V)$ denote the same objects as in Subsection 1.1. In this case, our obstruction is the Rosenberg index of the $(n - 1)$ -dimensional spin manifold N . In analogy with the case of codimension zero obstructions, we consider the incomplete manifold $V^\circ = V \setminus \partial V$ and fix a rescaling function ρ and a potential ψ . We also assume that ψ is compatible with the band V . This means that there exist constants $\lambda_- < 0 < \lambda_+$ such that $\psi = \lambda_-$ in a neighborhood of the deleted negative boundary component ∂_-V and $\psi = \lambda_+$ in a neighborhood of the deleted positive boundary component ∂_+V . We use these extra data to construct a Fredholm operator $B_{\rho, \psi}$ on V° whose index coincide with $\alpha(N)$. From a vanishing theorem for the index of the operator $B_{\rho, \psi}$, we deduce the following result.

Theorem D. *Let N be a closed $(n - 1)$ -dimensional spin manifold with fundamental group Γ . Suppose the Rosenberg index $\alpha(N) \in KO_n(C^*\Gamma)$ does not vanish. Let V be a Riemannian band over N whose scalar curvature is bounded from below by a constant $\sigma > 0$. Then*

$$\text{width}(V) \leq 2\pi \sqrt{\frac{n - 1}{\sigma n}}.$$

REMARK 1.16. This theorem implies that conjecture 1.5 holds for all closed spin manifolds with nonvanishing Rosenberg index.

REMARK 1.17. In view of Remark 1.6, the inequality found in Theorem D is sharp.

REMARK 1.18. Zeidler [Zei19, Theorem 1.4] recently proved that, under the same hypotheses of Theorem D, there exists a constant c , independent of n , such that Inequality (1.5) holds. This constant is numerically close to 20.51: see Remark [Zei19, Remark 1.9]. Therefore, it is not optimal or asymptotically optimal (the asymptotically optimal constant would be 2π). Theorem D strengthens [Zei19, Theorem 1.4] with the optimal constant. This answers a question asked by Zeidler: see [Zei19, Remark 1.9].

Theorem D implies the following relevant case of Gromov's Conjecture 1.5.

COROLLARY E. *Conjecture 1.5 holds when N is a closed simply connected manifold of dimension at least 5.*

REMARK 1.19. This corollary strengthens [Zei19, corollary 1.5] with the optimal constant. It follows from Theorem D by the same argument used in [Zei19] so we do not repeat it here.

The paper is organized as follows. In Section 2, we prove a K -theoretic additivity formula for the index in the setting of manifolds complete for a differential operator. In Section 3, we study rescaled Dirac operators and prove a Lichnerowicz-type inequality in this situation. In Section 4, we construct the operator $P_{\rho,\phi}^{E,F}$ and prove a formula to compute its index. In Section 5, we prove a vanishing theorem for the operator $P_{\rho,\phi}^{E,F}$ and use it to prove Theorems A, B, and C. Finally, in Section 6 we construct the operator $B_{\rho,\psi}$ and use it to prove Theorem D.

2 A K -theoretic additivity formula for the index

This section is devoted to the analytical background of this paper. In Subsection 2.1, we recall some preliminary notions on differential operators acting on bundles of modules over C^* -algebras and fix notation. In Subsection 2.2, we consider a differential operator P on a not necessarily complete Riemannian manifold M . In order to ensure that P has self-adjoint and regular closure, we make use of the notion of completeness of M for P , developed by Higson and Roe [HR00] and extended to the C^* -algebra setting by Ebert [Ebe16]. When P^2 is uniformly positive at infinity, by results of Ebert [Ebe16] the closure of P is Fredholm and its index is well defined. In Subsection 2.3, we extend to this slightly more general class of operators a K -theoretic additivity formula due to Bunke [Bun95].

2.1 Differential operators linear over C^* -algebras. Throughout this paper, A denotes a complex unital C^* -algebra. We will also consider the case when A is endowed with a Real structure. We are mostly interested in the following two types of Real C^* -algebras. The first one is the Real Clifford algebra $Cl_{n,m}$: see [Sch93, Section 1.2] and [Ebe16, page 4] for details. The second one is the maximal group C^* -algebra $C^*\Gamma$ associated to a countable discrete group Γ . This is the completion of the group algebra $\mathbb{C}[\Gamma]$ with respect to the maximal norm and is endowed with a

canonical Real structure induced by complex conjugation: see [Ebe16, Section 1.1] and [HR00, Definition 3.7.4].

For Hilbert A -modules H and H' , we denote by $\mathcal{L}_A(H, H')$ the space of adjointable operators from H to H' and by $\mathcal{K}_A(H, H')$ the subspace of the compact ones. We also use the notation $\mathcal{L}_A(H) := \mathcal{L}_A(H, H)$ and $\mathcal{K}_A(H) := \mathcal{K}_A(H, H)$. For the properties of Hilbert A -modules and adjointable operators, we refer to [Lan95] and [WO93, Section 15].

Let (M, g) be a Riemannian manifold. Let W be a bundle of finitely generated projective Hilbert A -modules with inner product on M and let $P: \Gamma(M; W) \rightarrow \Gamma(M; W)$ be a formally self-adjoint differential operator of order one. If W is \mathbb{Z}_2 -graded, we require that the operator P is odd with respect to the grading. If A has a Real structure, we require that W is a bundle of finitely generated projective Real Hilbert A -modules and the operator P is real, i.e. $P\kappa(w) = \kappa(Pw)$ for all $w \in \Gamma(M; W)$, where κ is the involution defining the Real structure. For more details, we refer to [Ebe16, Sections 1.1 and 1.2]. We are mostly interested in the two types of operators described in the following examples.

EXAMPLE 2.1. Let (M, g) be a Riemannian spin manifold and let E be a Hermitian vector bundle over M endowed with a metric connection. Let \mathcal{S}_M and \mathcal{D}_M be the associated complex spinor bundle and complex spin Dirac operator. Denote by $\mathcal{D}_{M,E}: \Gamma(M; \mathcal{S}_M \otimes E) \rightarrow \Gamma(M; \mathcal{S}_M \otimes E)$ the operator \mathcal{D}_M twisted with the bundle E . If M is even dimensional, \mathcal{S}_M is \mathbb{Z}_2 -graded and the operator $\mathcal{D}_{M,E}$ is odd with respect to the induced \mathbb{Z}_2 -grading on $\mathcal{S}_M \otimes E$. If in addition M is closed, the operator $\mathcal{D}_{M,E}$ defines a class index $(\mathcal{D}_{M,E})$ in $K_0(\mathbb{C}) = \mathbb{Z}$. For more details on this construction, we refer to [LM89, §II.5].

EXAMPLE 2.2. Let (M, g) be an n -dimensional Riemannian spin manifold. Let E be a bundle of finitely generated projective Real Hilbert A -modules with inner product and metric connection on M . Let \mathcal{S}_M be the $\text{Cl}_{n,0}$ -spinor bundle on (M, g) with associated $\text{Cl}_{n,0}$ -linear spin Dirac operator \mathcal{D}_M . The bundle \mathcal{S}_M is endowed with a $\text{Cl}_{n,0}$ -valued inner product and is equipped with canonical Real structure and \mathbb{Z}_2 -grading. Let $\mathcal{D}_{M,E}: \Gamma(\mathcal{S}_M \otimes E) \rightarrow \Gamma(\mathcal{S}_M \otimes E)$ be the operator \mathcal{D}_M twisted with the bundle E . The \mathbb{Z}_2 -grading on \mathcal{S}_M induces a \mathbb{Z}_2 -grading on $\mathcal{S}_M \otimes E$ and the operator $\mathcal{D}_{M,E}$ is odd with respect to this grading. When M is closed, the operator $\mathcal{D}_{M,E}$ defines a class index $(\mathcal{D}_{M,E}) \in \text{KO}_n(A)$. For more details, see [LM89, §II.7] and [Ebe16, Section 1]. For the background material on Dirac operators twisted with bundles of Hilbert A -modules, we refer to [Sch05, Section 6.3]. We finally recall a particular instance of this construction, which is relevant for the geometric applications of this paper. Let M be a closed n -dimensional spin manifold with fundamental group Γ . Let \mathcal{L}_Γ be the Mishchenko bundle over M . The bundle \mathcal{L}_Γ has typical fiber $C^*\Gamma$ and is equipped with a canonical flat connection. The class index $(\mathcal{D}_{M,\mathcal{L}_\Gamma}) \in \text{KO}_n(C^*\Gamma)$ is called the *Rosenberg index* of M and is denoted by $\alpha(M)$. For more details, see [Ros07, Sto02].

REMARK 2.3. To be precise, index $\left(\mathcal{D}_{M,E}\right)$ is a class in $KO_n(A_{\mathbb{R}})$, where $A_{\mathbb{R}}$ is the real C^* -algebra consisting of the fixed points of the involution of A . With a slight abuse of notation, we denote a Real C^* -algebra and its fixed point algebra by the same symbol.

REMARK 2.4. The fixed point algebra of $C^*\Gamma$ with respect to the canonical involution is the maximal real C^* -algebra of Γ , which in this paper will be denoted by the same symbol.

2.2 Manifolds which are complete for a differential operator. Let (M, g) be a Riemannian manifold. Let $W \rightarrow M$ be a bundle of finitely generated projective Hilbert A -modules with inner product and let $P: \Gamma(M; W) \rightarrow \Gamma(M; W)$ be a formally self-adjoint differential operator of order one. We regard P as a symmetric unbounded operator on $L^2(M; W)$ with initial domain $\Gamma_c(M; W)$. We will now give a condition so that its closure $\bar{P}: \text{dom}(\bar{P}) \rightarrow L^2(M; W)$ is self-adjoint and regular. For the background material on unbounded operators on Hilbert A -modules and the notion of regularity, see [Lan95].

DEFINITION 2.5. A coercive function is a proper smooth function $h: M \rightarrow \mathbb{R}$ which is bounded from below.

DEFINITION 2.6. We say that the pair (M, P) is complete, or that M is complete for P , if there exists a coercive function $h: M \rightarrow \mathbb{R}$ such that the commutator $[P, h]$ is bounded.

REMARK 2.7. The notion of completeness of a manifold for an operator depends only on the principal symbol of the operator. This means that if (M, P) is complete and $\Phi: W \rightarrow W$ is a fiberwise self-adjoint bundle map, then $(M, P + \Phi)$ is also complete.

REMARK 2.8. Suppose h is a coercive function on M and $\hat{h}: M \rightarrow \mathbb{R}$ is a smooth function coinciding with h outside of a compact set. Then \hat{h} is a coercive function as well. Moreover, $[P, h]$ is bounded if and only if $[P, \hat{h}]$ is bounded.

The next theorem, due to Ebert, gives the wanted sufficient condition. It is a generalization to operators linear over C^* -algebras of a result of Higson and Roe [HR00, Proposition 10.2.10].

Theorem 2.9 (Ebert, [Ebe16, Theorem 1.14]). *If (M, P) is complete, then the closure of P is self-adjoint and regular.*

Assume (M, P) is complete and denote the self-adjoint and regular closure of P by the same symbol. Assume also there is a \mathbb{Z}_2 -grading $W = W^+ \oplus W^-$ and the operator P is odd with respect to this grading, i.e. it is of the form

$$P = \begin{pmatrix} 0 & P^- \\ P^+ & 0 \end{pmatrix}, \quad (2.1)$$

where $P^\pm: \Gamma(M; W^\pm) \rightarrow \Gamma(M; W^\mp)$ are formally adjoint to one another. Finally, assume P is elliptic.

To simplify the notation, in the remaining part of this section we set $H := L^2(M; W)$. We say that the operator P^2 is *uniformly positive at infinity* if there exist a compact subset $K \subset M$ and a constant $c > 0$ such that

$$\langle P^2 w, w \rangle \geq c \langle w, w \rangle, \quad w \in \Gamma_c(M \setminus K; W|_{M \setminus K}). \tag{2.2}$$

In this case, by [Ebe16, Theorem 2.41] the operator $P(P^2 + 1)^{-1/2} \in \mathcal{L}_A(H)^{\text{odd}}$ is Fredholm. We denote its index by $\text{index}(P)$.

In the next lemma, we collect some properties of the operator P that will be needed in the proof of the additivity formula.

LEMMA 2.10. *The operator $P^2 + 1 + t^2$ is invertible for every $t \geq 0$. Moreover, $(P^2 + 1 + t^2)^{-1}$ is a positive element of $\mathcal{L}_A(H)$ and there is the absolutely convergent integral representation*

$$(P^2 + 1)^{-1/2} = \frac{2}{\pi} \int_0^\infty (P^2 + 1 + t^2)^{-1} dt. \tag{2.3}$$

Finally, we have the estimates

$$\|(P^2 + 1 + t^2)^{-1}\|_{\mathcal{L}_A(H)} \leq (1 + t^2)^{-1}, \tag{2.4}$$

$$\|P(P^2 + 1 + t^2)^{-1}\|_{\mathcal{L}_A(H)} \leq \frac{1}{2\sqrt{1 + t^2}}, \tag{2.5}$$

$$\|P^2(P^2 + 1 + t^2)^{-1}\|_{\mathcal{L}_A(H)} \leq 1, \tag{2.6}$$

for all $t \geq 0$.

Proof. The first part of the lemma and Inequality (2.4) follow from [Ebe16, Proposition 1.21]. Inequalities (2.5) and (2.6) follow from Part (2) of [Ebe16, Theorem 1.19]. \square

2.3 Cut-and-paste invariance. For $i = 1, 2$, let M_i be a Riemannian manifold, let $W_i = W_i^+ \oplus W_i^-$ be a \mathbb{Z}_2 -graded bundle of finitely generated projective Hilbert A -modules with inner product and let P_i be an odd formally self-adjoint elliptic differential operator of order one. We assume that (M_i, P_i) is complete and that P_i^2 is uniformly positive at infinity so that its index is well defined. Let $U_i \cup_{N_i} V_i$ be a partition of M_i , where N_i is a closed separating hypersurface. This means that $M_i = U_i \cup V_i$ and $U_i \cap V_i = N_i$. We make the following assumption.

Assumption 2.11. *The operators coincide near the separating hypersurfaces.* This means that there exist tubular neighborhoods $\mathcal{U}(N_1)$ and $\mathcal{U}(N_2)$ respectively of N_1 and N_2 and an isometry $\Gamma: \mathcal{U}(N_1) \rightarrow \mathcal{U}(N_2)$ such that $\Gamma|_{N_1}: N_1 \rightarrow N_2$ is a diffeomorphism and Γ is covered by a bundle isometry

$$\tilde{\Gamma}: W_1|_{\mathcal{U}(N_1)} \rightarrow W_2|_{\mathcal{U}(N_2)} \quad \text{so that} \quad P_2 = \tilde{\Gamma} \circ P_1 \circ \tilde{\Gamma}^{-1}$$

in $\mathcal{U}(N_2)$.

This assumption allows us to do the following cut-and-paste construction. Cut the manifolds M_i and the bundles W_i along N_i . Use the map Γ to interchange the boundary components and construct the Riemannian manifolds

$$M_3 := U_1 \cup_N V_2 \quad \text{and} \quad M_4 := U_2 \cup_N V_1,$$

where $N \cong N_1 \cong N_2$. Moreover, using the map $\tilde{\Gamma}$ to glue the bundles, we obtain \mathbb{Z}_2 -graded bundles

$$W_3 := W_1|_{U_1} \cup_N W_2|_{V_2} \quad \text{and} \quad W_4 := W_2|_{U_2} \cup_N W_1|_{V_1},$$

and odd formally self-adjoint elliptic differential operators of order one P_3 and P_4 . Observe that, using Remark 2.8, the pairs (M_3, P_3) and (M_4, P_4) are complete and that the operators P_3^2 and P_4^2 are uniformly positive at infinity. Therefore, the indices of P_3 and P_4 are well defined. The next theorem is a slight generalization of [Bun95, Theorem 1.2].

Theorem 2.12. $\text{index}(P_1) + \text{index}(P_2) = \text{index}(P_3) + \text{index}(P_4)$.

Proof. Use the notation

$$\mathcal{H} = H_1 \oplus H_2 \oplus H_3^{\text{op}} \oplus H_4^{\text{op}} \quad \text{and} \quad \mathcal{F} = F_1 \oplus F_2 \oplus F_3 \oplus F_4,$$

where $H_i = L^2(M_i; W_i)$, $F_i = P_i(P_i^2 + 1)^{-1/2}$ and $R_i(t) = (P_i^2 + 1 + t^2)^{-1}$. In order to prove the thesis, we need to show that $\text{index}(\mathcal{F}) = 0$.

Pick cutoff functions χ_{U_i} and χ_{V_i} such that

$$\text{supp}(\chi_{U_i}) \subset U_i \cup \mathcal{U}(N_i) \quad \text{supp}(\chi_{V_i}) \subset V_i \cup \mathcal{U}(N_i) \quad \chi_{U_i}^2 + \chi_{V_i}^2 = 1.$$

Moreover, we assume that $\chi_{U_1} = \chi_{U_2}$ and $\chi_{V_1} = \chi_{V_2}$ when restricted to $\mathcal{U}(N) \cong \mathcal{U}(N_1) \cong \mathcal{U}(N_2)$. Multiplication by χ_{U_1} defines an operator $a \in \mathcal{L}_A(H_3, H_1)$. Similarly, use the cutoff functions to define operators $b \in \mathcal{L}_A(H_1, H_4)$, $c \in \mathcal{L}_A(H_2, H_3)$, and $d \in \mathcal{L}_A(H_2, H_4)$. Consider the operator

$$\mathcal{X} := z \begin{pmatrix} 0 & 0 & -a^* & -b^* \\ 0 & 0 & -c^* & d^* \\ a & c & 0 & 0 \\ b & -d & 0 & 0 \end{pmatrix} \in \mathcal{L}_A(\mathcal{H}),$$

where $z \in \mathcal{L}_A(H)$ is the \mathbb{Z}_2 -grading. As explained in [CB18, Subsection 3.1] and in the proof of [Bun95, Theorem 1.14], in order to show that $\text{index}(\mathcal{F}) = 0$, it suffices to show that $\mathcal{X}\mathcal{F} + \mathcal{F}\mathcal{X} \in \mathcal{K}_A(\mathcal{H})$. To this end, it is enough to verify the compactness of operators of the form $a^*F_3 - F_1a^* \in \mathcal{L}_A(H_3, H_1)$.

Let $\chi = \chi_{U_1}$ and let $\rho \in C_c^\infty(U_i \cup \mathcal{U}(N_i))$ be such that $\rho\chi = \chi$. Using Assumption 2.11, the operators $\chi P_3 - P_1\chi$ and $(\chi P_1 - P_1\chi)\rho$ define the same element in

$\mathcal{L}_A(H_3, H_1)$, that we denote by $[P, \chi]$. Using the integral representation (2.3) and the computations in [Bun95, page 13], we obtain

$$\chi F_3 - F_1 \chi = \frac{2}{\pi} \int_0^\infty (\chi P_3 R_3(t) - P_1 R_1(t) \chi) dt = \frac{2}{\pi} \int_0^\infty Q_{3,1}(t) dt, \tag{2.7}$$

where

$$Q_{3,1}(t) := -[P, \chi] R_3(t) + P_1^2 R_1(t) [P, \chi] R_3(t) + P_1 R_1(t) [P, \chi] P_3 R_3(t).$$

Using Inequalities (2.4), (2.6) and (2.5) and [Ebe16, Theorem 2.33 and Remark 2.35], we deduce that the operator $Q_{3,1}(t)$ is compact and absolutely integrable. By (2.7), $a^* F_3 - F_1 a^* \in \mathcal{K}_A(H_3, H_1)$, which concludes the proof. \square

3 A rescaled dirac operator

In this section, we present a general method to construct a complete pair on a Riemannian spin manifold. Our method is based on rescaling the possibly twisted spin Dirac operator. Moreover, we prove an estimate from below for the square of the rescaled twisted Dirac operator. Finally, in order to obtain a slight improvement of this estimate, we extend to operators linear over C^* -algebras an inequality due to Friedrich [Fri80, Thm.A] on closed manifolds and generalized by Bär [Bär09, Theorem 3.1] to open manifolds. This improvement will be used in Sections 5 and 6 to obtain the factor $\sqrt{(n-1)/n}$ in Theorems A, B, and D. Even if we mostly focus on the spin case, all the results of this section hold with the obvious modifications for any operator of Dirac type.

3.1 Admissible rescaling functions. Let (M, g) be a Riemannian manifold. Let $V \rightarrow M$ be a bundle of finitely generated projective Hilbert A -modules with inner product and let $Z: \Gamma(M; V) \rightarrow \Gamma(M; V)$ be a formally self-adjoint elliptic differential operator of order one such that

$$\|[Z, \xi]_x\| \leq |d\xi_x|, \quad \xi \in C^\infty(M), \quad x \in M. \tag{3.1}$$

Here, $\|[Z, \xi]_x\|$ is the norm of the adjointable map $[Z, \xi]_x: V_x \rightarrow V_x$. For a function $\rho: M \rightarrow (0, \infty)$, define the rescaled operator $Z_\rho: \Gamma(M; V) \rightarrow \Gamma(M; V)$ as

$$Z_\rho := \rho Z \rho. \tag{3.2}$$

Observe that Z_ρ is a formally self-adjoint differential operator of order one and

$$[Z_\rho, \xi] = \rho^2 [Z, \xi], \quad \xi \in C^\infty(M). \tag{3.3}$$

Therefore, Z_ρ is elliptic.

DEFINITION 3.1. A smooth function $\rho: M \rightarrow (0, 1]$ is called an admissible rescaling function for M if there exists a coercive (see Definition 2.5) function h such that $\rho^2 |dh|$ is in $L^\infty(M)$.

REMARK 3.2. The property for a smooth function ρ of being an admissible rescaling function depends only on its behaviour at infinity. Moreover, suppose $\rho_1, \rho_2: M \rightarrow (0, 1]$ are smooth functions such that ρ_1 is admissible and $\rho_2 = b\rho_1$ outside of a compact set for some constant $b > 0$. Then ρ_2 is admissible as well.

PROPOSITION 3.3. *Let ρ be an admissible rescaling function. Then the pair (M, Z_ρ) is complete.*

Proof. Since ρ is admissible, choose a coercive function h such that $\rho^2 |dh|$ is in $L^\infty(M)$. By (3.1) and (3.3), we deduce

$$\| [Z_\rho, h] v \| \leq \| \rho^2 |dh| \|_\infty \| v \|, \quad v \in \Gamma_c(M; V). \quad \square$$

REMARK 3.4. When (M, g) is a complete Riemannian manifold, the function $\rho = 1$ is admissible and Proposition 3.3 implies the classical fact that a Dirac operator on (M, g) is essentially self-adjoint.

We now describe a method for constructing admissible rescaling functions on open Riemannian manifolds. In Sections 5 and 6, we will use this method together with the geometry at infinity of the manifolds to construct complete pairs.

PROPOSITION 3.5. *Let $\tau: M \rightarrow (0, \infty)$ be a smooth function such that*

$$\lim_{x \rightarrow \infty} \tau(x) = 0 \quad (3.4)$$

and there exists a constant $c > 0$ satisfying

$$|d\tau_x| \leq c, \quad x \in M. \quad (3.5)$$

Suppose $\gamma_\alpha: (0, \infty) \rightarrow (0, 1]$ is a smooth function such that $\gamma_\alpha(t) = t^\alpha$ for t near 0. Then $\rho_\alpha := \gamma_\alpha \circ \tau$ is an admissible rescaling function for all $\alpha \geq 1/2$.

Proof. Observe, using (3.4), that $h(x) = \log(1/\tau(x))$ is a coercive function. By (3.5) and since $\gamma_\alpha(t) = t^\alpha$ for t near 0, there exists a compact subset $K \subset M$ such that

$$\rho_\alpha^2(x) |dh_x| = \tau^{2\alpha-1}(x) |d\tau_x| \leq c \cdot \tau^{2\alpha-1}(x), \quad x \in M \setminus K.$$

Since $\tau^{2\alpha-1} \in L^\infty(M)$ for $2\alpha \geq 1$, the previous inequality and Remark 3.2 imply the thesis. \square

3.2 A Friedrich inequality for operators linear over C^* -algebras. Let (M, g) be an n -dimensional Riemannian spin manifold with associated spinor bundle S_M and Dirac operator D_M . Let (E, ∇^E) be a bundle of finitely generated projective Hilbert A -modules with inner product and metric connection. Denote by $Z: \Gamma(M; S_M \otimes E) \rightarrow \Gamma(M; S_M \otimes E)$ the Dirac operator D_M twisted with the bundle E . We consider the following two situations:

- (1) S_M is the complex spinor bundle \mathcal{S}_M , (E, ∇^E) is a Hermitian vector bundle with metric connection and Z is the twisted complex spin Dirac operator $\mathcal{D}_{M,E}$ described in Example 2.1;
- (2) A is a Real C^* -algebra, S_M is the $Cl_{n,0}$ -linear spinor bundle \mathcal{S}_M , (E, ∇^E) is a bundle of finitely generated projective Real Hilbert A -modules with inner product and metric connection and Z is the twisted $Cl_{n,0}$ -linear Dirac operator $\mathcal{D}_{M,E}$ described in Example 2.2.

When there is no danger of confusion, we will denote the bundle S_M simply by S . The operator Z is related to the scalar curvature of g through the classical Lichnerowicz formula

$$Z^2 = \nabla^* \nabla + \frac{1}{4} \text{scal}_g + \mathcal{R}^E, \tag{3.6}$$

where $\nabla^* \nabla$ is the connection Laplacian of $S \otimes E$ and $\mathcal{R}^E: S \otimes E \rightarrow S \otimes E$ is a bundle map depending linearly on the components of the curvature tensor $F(\nabla^E)$ of ∇^E . In particular, if $F(\nabla^E) = 0$ in a region $\Omega \subset M$, then $\mathcal{R}^E = 0$ on Ω . See [LM89, §II.8] for more details. The next theorem provides a slight improvement of the estimate from below of Z^2 directly following from (3.6).

Theorem 3.6. *Let (M, g) , (E, ∇^E) and Z be as above. Set*

$$\bar{n} := \frac{n}{n-1} \quad \text{and} \quad \overline{\text{scal}}_g(x) := \frac{1}{4} \text{scal}_g(x). \tag{3.7}$$

Then the inequality

$$\langle Z^2 u, u \rangle \geq \bar{n} \langle \overline{\text{scal}}_g u, u \rangle + \bar{n} \langle \mathcal{R}^E u, u \rangle \tag{3.8}$$

holds for all $u \in \Gamma_c(M; S \otimes E)$.

In order to prove Theorem 3.6, we first establish the following abstract inequality for Hilbert C^* -modules.

LEMMA 3.7. *Let H be a Hilbert module over a C^* -algebra A . For $x_1, \dots, x_N \in H$, we have*

$$\left(\sum_{i=1}^N x_i \left| \sum_{i=1}^N x_i \right. \right) \leq N \sum_{i=1}^N (x_i | x_i)$$

where $(\cdot | \cdot)$ is the A -valued inner product of H .

Proof. For $x, y \in H$, we have

$$(x | y) + (y | x) \leq (x - y | x - y) + (x | y) + (y | x) = (x | x) + (y | y). \tag{3.9}$$

Therefore,

$$\left(\sum_{i=1}^N x_i \left| \sum_{i=1}^N x_i \right. \right) = \sum_{i=1}^N (x_i | x_i) + \sum_{i < j} \{ (x_i | x_j) + (x_j | x_i) \} \leq N \sum_{i=1}^N (x_i | x_i)$$

where the last inequality is obtained by applying Inequality (3.9) to the terms $(x_i | x_j) + (x_j | x_i)$. □

Proof of Theorem 3.6. Let $u \in \Gamma_c(M; S \otimes E)$. Recall that the operator Z has the local expression

$$Zu = \sum_{i=1}^n c(e_i) \nabla_{e_i} u$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis of $T_x M$, $c(e_i)$ is Clifford multiplication by e_i , and ∇ is the connection on $S \otimes E$ induced by the connections on S and E . At a point $x \in M$, using Lemma 3.7 we obtain

$$\begin{aligned} \langle Zu, Zu \rangle_x &= \left\langle \sum_{i=1}^n c(e_i) \nabla_{e_i} u, \sum_{i=1}^n c(e_i) \nabla_{e_i} u \right\rangle_x \\ &\leq n \sum_{i=1}^n \langle c(e_i) \nabla_{e_i} u, c(e_i) \nabla_{e_i} u \rangle_x = n \sum_{i=1}^n \langle \nabla_{e_i} u, \nabla_{e_i} u \rangle_x = n \langle \nabla u, \nabla u \rangle_x. \end{aligned}$$

By integrating the previous inequality, we get

$$\langle Z^2 u, u \rangle = \langle Zu, Zu \rangle \leq n \langle \nabla u, \nabla u \rangle = n \langle \nabla^* \nabla u, u \rangle.$$

Using the last inequality together with the Lichnerowicz formula (3.6), we deduce

$$\langle Z^2 u, u \rangle \geq \frac{1}{n} \langle Z^2 u, u \rangle + \langle \overline{\text{scal}}_g u, u \rangle + \langle \mathcal{R}^E u, u \rangle,$$

from which Inequality (3.8) follows. \square

3.3 A Lichnerowicz-type inequality for the rescaled operator. Let (M, g) , S , E and Z be as in Subsection 3.2. For a smooth function ρ , let Z_ρ be the rescaled operator defined by (3.2). In the next proposition, we state a Lichnerowicz-type inequality for the rescaled operator.

PROPOSITION 3.8. *Suppose the scalar curvature of g is bounded from below by a constant $\sigma > 0$. Let $\rho: M \rightarrow (0, \infty)$ be a smooth function and let \bar{n} , $\bar{\sigma}$ be the constants defined in (3.7). Then the inequality*

$$\langle Z_\rho^2 u, u \rangle \geq \frac{\bar{n}\omega}{1+\omega} \langle \overline{\text{scal}}_g \rho^4 u, u \rangle + \frac{\bar{n}\omega}{1+\omega} \langle \mathcal{R}^E \rho^4 u, u \rangle - \omega \langle \rho^2 |d\rho|^2 u, u \rangle \quad (3.10)$$

holds for every $\omega > 0$ and every $u \in \Gamma_c(M; S \otimes E)$.

The proof of this proposition is based on the following lemma.

LEMMA 3.9. *Let $\xi: M \rightarrow \mathbb{R}$ be a smooth function. Then the inequality*

$$\langle \xi Z(v), \xi Z(v) \rangle \geq \frac{\omega}{1+\omega} \langle Z(\xi v), Z(\xi v) \rangle - \omega \langle |d\xi|^2 v, v \rangle \quad (3.11)$$

holds for every $\omega > 0$ and every $v \in \Gamma_c(M; S \otimes E)$.

Proof. By direct computation,

$$Z \xi^2 Z = \xi Z^2 \xi - |\mathrm{d}\xi|^2 - Z \xi c(\mathrm{d}\xi) + c(\mathrm{d}\xi) \xi Z.$$

Hence,

$$\begin{aligned} \langle \xi Z(v), \xi Z(v) \rangle &= \langle Z(\xi v), Z(\xi v) \rangle - \langle |\mathrm{d}\xi|^2 v, v \rangle \\ &\quad - \langle c(\mathrm{d}\xi)v, \xi Z v \rangle - \langle \xi Z v, c(\mathrm{d}\xi)v \rangle. \end{aligned} \tag{3.12}$$

Fix $\omega > 0$ and observe that

$$\begin{aligned} 0 &\leq \left\langle \frac{\xi}{\sqrt{\omega}} Z(v) - \sqrt{\omega} c(\mathrm{d}\xi)v, \frac{\xi}{\sqrt{\omega}} Z(v) - \sqrt{\omega} c(\mathrm{d}\xi)v \right\rangle \\ &= \frac{1}{\omega} \langle \xi Z(v), \xi Z(v) \rangle + \omega \langle |\mathrm{d}\xi|^2 v, v \rangle - \langle c(\mathrm{d}\xi)v, \xi Z v \rangle - \langle \xi Z v, c(\mathrm{d}\xi)v \rangle. \end{aligned}$$

This inequality together with (3.12) yields

$$\langle \xi Z(v), \xi Z(v) \rangle \geq \langle Z(\xi v), Z(\xi v) \rangle - \frac{1}{\omega} \langle \xi Z(v), \xi Z(v) \rangle - (1 + \omega) \langle |\mathrm{d}\xi|^2 v, v \rangle,$$

which implies (3.11). □

Proof of Proposition 3.8. It follows from Theorem 3.6 and Lemma 3.9, with $\xi = \rho$ and $v = \rho u$. □

4 Generalized Gromov–Lawson operators

In this section, we study the geometric situation when M is a Riemannian spin manifold and (E, F) is a pair of bundles with isomorphic typical fibers and whose supports are contained in the interior of a compact submanifold with boundary $L \subset M$. In Subsection 4.1, we define the class $\text{rel-ind}(M; E, F)$ as the index of a suitable elliptic differential operator $D_{L_D}^{E,F}$ over the double L_D of L . In Subsection 4.2, we use an admissible rescaling function ρ and a potential ϕ to define a Fredholm operator $P_{\rho,\phi}^{E,F}$ on M . Finally, in Subsection 4.3 we show that the index of $P_{\rho,\phi}^{E,F}$ coincides with $\text{rel-ind}(M; E, F)$.

4.1 Localized Dirac obstructions. Let (M, g) be an n -dimensional Riemannian spin manifold with associated \mathbb{Z}_2 -graded spinor bundle $S_M = S_M^+ \oplus S_M^-$. Let (E, ∇^E) be a bundle of finitely generated projective Hilbert A -modules with inner product and metric connection on M . Denote by $D_{M,E}: \Gamma(M; S_M \otimes E) \rightarrow \Gamma(M; S_M \otimes E)$ the spin Dirac operator twisted with the bundle E . Observe that $D_{M,E}$ is odd with respect to the grading

$$S_M \otimes E = (S_M^+ \otimes E) \oplus (S_M^- \otimes E). \tag{4.1}$$

We consider the following two situations.

- (I) M is even dimensional and (E, ∇^E) is a Hermitian vector bundle with metric connection. In this case, S_M is the complex spinor bundle \mathcal{S}_M and $D_{M,E}$ is the twisted complex spin Dirac operator $\mathcal{D}_{M,E}$ described in Example 2.1.
- (II) A is a Real C^* -algebra, and E is a bundle of finitely generated projective Real Hilbert A -modules with inner product and metric connection. In this case, S_M is the $\text{Cl}_{n,0}$ -linear spinor bundle \mathcal{S}_M and $D_{M,E}$ is the twisted $\text{Cl}_{n,0}$ -linear spin Dirac operator $\mathcal{D}_{M,E}$ described in Example 2.2.

When there is no danger of confusion, we will use the notation S and S^\pm instead of S_M and S_M^\pm .

Let (F, ∇^F) be a second bundle of finitely generated projective Hilbert A -modules with inner product and metric connection over M . We make the following assumption.

Assumption 4.1. *The bundles have isomorphic typical fibers and are trivializable at infinity.* This means that there exist a finitely generated projective Hilbert A -module \mathcal{V} and a compact subset $K \subset M$ such that

$$(E, \nabla^E)|_{M \setminus K} \cong (F, \nabla^F)|_{M \setminus K} \cong (\underline{\mathcal{V}}, d_{\underline{\mathcal{V}}})|_{M \setminus K}$$

where $\underline{\mathcal{V}} \rightarrow M$ denotes the trivial bundle with fiber \mathcal{V} and $d_{\underline{\mathcal{V}}}$ denotes the trivial connection on $\underline{\mathcal{V}}$. In this case, we say that K is an *essential support* for (E, F) and that $M \setminus K$ is a *neighborhood of infinity*.

In this setting, we define a relative index following Gromov and Lawson [GL83]. Let $L \subset M$ be a smooth compact submanifold with boundary, whose interior contains an essential support of (E, F) . Deform the metric and the spinor bundle in such a way that they have a product structure in a tubular neighborhood of ∂L . Form the double $L_D := L \cup_{\partial L} L^-$ of L , where L^- denotes the manifold L with opposite orientation. The Riemannian metric g induces a Riemannian metric g_1 on L_D which is symmetric with respect to ∂L and has a product structure in a tubular neighborhood of ∂L . The double L_D is a closed manifold carrying a natural spin structure induced by the spin structure of L . The associated spinor bundle S_{L_D} has a reflection symmetry with respect to ∂L . Using Assumption (4.1), define $(V(E, F), \nabla^{V(E,F)})$ as the bundle with connection on L_D coinciding with (E, ∇^E) over L and with (F, ∇^F) over L^- . Denote by $D_{L_D}^{E,F}$ the Dirac operator D_{L_D} twisted with the bundle $V(E, F)$. In the next lemma, we collect some properties of the index of the operator $D_{L_D}^{E,F}$.

LEMMA 4.2. *Let (E, F) and (G, H) be two pairs of bundles of finitely generated projective Hilbert A -modules with inner product and metric connection over M satisfying Assumption 4.1. Let $L \subset M$ be a compact submanifold with boundary whose interior contains an essential support of both (E, F) and (G, H) . Then*

$$\text{index} \left(D_{L_D}^{E,E} \right) = 0; \tag{4.2}$$

$$\text{index} \left(D_{L_D}^{E,F} \right) + \text{index} \left(D_{L_D}^{H,G} \right) = \text{index} \left(D_{L_D}^{E,G} \right) + \text{index} \left(D_{L_D}^{H,F} \right); \quad (4.3)$$

and

$$\text{index} \left(D_{L_D}^{E,F} \right) + \text{index} \left(D_{L_D}^{F,E} \right) = 0. \quad (4.4)$$

Proof. Identity (4.2) follows from the fact that the operator $D_{L_D}^{E,E}$ is symmetric with respect to the separating hypersurface ∂L . For Identity (4.3), consider the partition $L \cup_{\partial L} L^-$ and apply Theorem 2.12 to the operators $D_{L_D}^{E,F}$ and $D_{L_D}^{G,H}$. Finally, Identity (4.4) follows from (4.2) and (4.3). \square

Observe that the index of $D_{L_D}^{E,F}$ does not depend on the metric. The next proposition states that it does not depend on the choice of the submanifold L .

PROPOSITION 4.3. *Let (E, ∇^E) and (F, ∇^F) be a pair of bundles of finitely generated projective Hilbert A -modules with inner product and metric connection over M satisfying Assumption 4.1. Suppose L and L' are smooth compact submanifolds with boundary of M whose interiors contain an essential support of (E, F) . Then the indices of $D_{L_D}^{E,F}$ and $D_{L'_D}^{E,F}$ coincide.*

Proof. Observe first that it suffices to prove the thesis when one of the submanifolds is contained in the interior of the other. To see this, consider a compact submanifold with boundary $L'' \subset M$ whose interior contains both L and L' .

Using this observation, we will prove the theorem under the assumption that L is contained in the interior of L' . Consider the Riemannian spin manifolds (L_D, g_1) and (L'_D, g_2) , where g_1 and g_2 are induced by g as explained above. Consider the operators $D_{L_D}^{F,F}$ on L_D and $D_{L'_D}^{E,F}$ on L'_D . Observe we have partitions $L_D = L \cup_{\partial L} L^-$ and $L'_D = L \cup_{\partial L} W$, where $W = \overline{L' \setminus L} \cup_{\partial L'} (L')^-$. Deform all structures to be a product in a tubular neighborhood of ∂L in such a way that Assumption 2.11 is satisfied. Using the cut-and-paste construction described in Subsection 2.3, we obtain the operator $D_{L'_D}^{F,F}$ on L'_D and the operator $D_{L_D}^{E,F}$ on L_D . By (4.2), the indices of $D_{L_D}^{F,F}$ and $D_{L'_D}^{F,F}$ vanish. Using Theorem 2.12, we obtain

$$\begin{aligned} \text{index} \left(D_{L'_D}^{E,F} \right) &= \text{index} \left(D_{L'_D}^{F,F} \right) + \text{index} \left(D_{L'_D}^{E,F} \right) \\ &= \text{index} \left(D_{L_D}^{F,F} \right) + \text{index} \left(D_{L_D}^{E,F} \right) = \text{index} \left(D_{L_D}^{E,F} \right), \end{aligned}$$

which concludes the proof. \square

Proposition 4.3 allows us to define the *relative index* of the pair (E, F) as the class

$$\text{rel-ind}(M; E, F) := \text{index} \left(D_{L_D}^{E,F} \right), \quad (4.5)$$

where $L \subset M$ is a submanifold with boundary whose interior contains an essential support of (E, F) .

REMARK 4.4. In the case (I) from the beginning of this subsection, $\text{rel-ind}(M; E, F) \in \mathbb{Z}$. In the case (II), $\text{rel-ind}(M; E, F) \in \text{KO}_n(A)$.

This class will be used as a localized obstruction for the metric g to have positive scalar curvature under some extra geometric conditions. To this end, we will need information on the endomorphisms \mathcal{R}^E and \mathcal{R}^F that appear in the Lichnerowicz formula (3.6). We conclude this subsection presenting two examples where we can determine whether the class $\text{rel-ind}(M; E, F)$ vanishes and we have control on the lower bound of the endomorphisms \mathcal{R}^E and \mathcal{R}^F . These two examples will be used in the geometric applications of Sections 5 and 6.

EXAMPLE 4.5. Let (M, g) be an even-dimensional Riemannian spin manifold and let $f: (M, g) \rightarrow (S^n, g_0)$ be a smooth map which is strictly area decreasing and locally constant at infinity. This last condition means that there exists a compact subset $K \subset M$ such that f is constant on the connected components of $M \setminus K$. Then, using a construction of Gromov and Lawson [GL80, GL83] and estimates by Llarull [Lla98], there exist Hermitian vector bundles with metric connections (E, ∇^E) and (F, ∇^F) satisfying Assumption (4.1) and such that

- (i) (F, ∇^F) is the trivial bundle endowed with the trivial connection;
- (ii) (E, ∇^E) is pulled back from S^n and satisfies

$$\mathcal{R}_x^E > -\frac{n(n-1)}{4}, \quad x \in \text{supp}(df);$$

- (iii) the support of df is an essential support of (E, F) ;
- (iv) if $\text{rel-ind}(M; E, F)$ vanishes, then $\deg(f) = 0$.

EXAMPLE 4.6. Our second example makes use of higher index theory. Let Y be a closed n -dimensional spin manifold with fundamental group Γ . Let $(\mathcal{L}_Y, \nabla^{\mathcal{L}_Y})$ be the Mishchenko bundle over Y endowed with the canonical flat metric connection. Recall that \mathcal{L}_Y has typical fiber $C^*\Gamma$. Suppose Condition (1.8) is satisfied. Pick distinct points $P_1, \dots, P_N \in Y$ and consider the open manifold $M := Y \setminus \{P_1, \dots, P_N\}$. Let D_1, \dots, D_N be pairwise disjoint n -dimensional disks embedded in Y such that P_j is in the interior of D_j . Choose embedded n -dimensional disks D'_1, \dots, D'_N such that D'_j lies in the interior of D_j and P_j is in the interior of D'_j . Let $f: M \rightarrow Y$ be a smooth map collapsing each end $D'_j \setminus P_j$ to the point P_j and being the identity map outside of $(D_1 \setminus P_1) \sqcup \dots \sqcup (D_N \setminus P_N)$. Let (E, ∇^E) be the flat bundle over M obtained as the pullback of $(\mathcal{L}_Y, \nabla^{\mathcal{L}_Y})$ through f . Let (F, ∇^F) be the trivial bundle over M with fiber $C^*\Gamma$, equipped with the trivial connection. Then

- (i) (E, ∇^E) and (F, ∇^F) satisfy Assumption (4.1);
- (ii) $\text{rel-ind}(M; E, F) \neq 0$.

Property (ii) follows from Condition (1.8) using Theorem 2.12. Notice that, since the connections ∇^E and ∇^F are flat, $\mathcal{R}^E = \mathcal{R}^F = 0$. A similar construction works if we pick pairwise disjoint embedded n -dimensional disks $D_1, \dots, D_N \subset Y$ and consider the open manifold $Y \setminus \bigsqcup_{j=1}^N D_j$.

4.2 Compatible potentials. Let (M, g) , (E, ∇^E) and (F, ∇^F) denote the same objects as in Subsection 4.1. Suppose Assumption 4.1 is satisfied. Consider the twisted Dirac operators $Q: \Gamma_c(M; S \otimes E) \rightarrow \Gamma_c(M; S \otimes E)$ and $R: \Gamma_c(M; S \otimes F) \rightarrow \Gamma_c(M; S \otimes F)$. Recall from Subsection 4.1 that we have \mathbb{Z}_2 -gradings

$$S \otimes E = (S^+ \otimes E) \oplus (S^- \otimes E) \quad \text{and} \quad S \otimes F = (S^+ \otimes F) \oplus (S^- \otimes F) \quad (4.6)$$

and that the operators Q and R are odd with respect to these gradings. Fix an admissible rescaling function ρ for M and consider the rescaled operators Q_ρ and R_ρ defined by (3.2). Recall from Subsection 3.1 that Q_ρ and R_ρ are first order formally self-adjoint elliptic differential operators. Finally, observe that the operators Q_ρ and R_ρ are odd with respect to the grading (4.6), i.e. they are of the form

$$Q_\rho = \begin{pmatrix} 0 & Q_\rho^- \\ Q_\rho^+ & 0 \end{pmatrix} \quad \text{and} \quad R_\rho = \begin{pmatrix} 0 & R_\rho^- \\ R_\rho^+ & 0 \end{pmatrix}$$

where $Q_\rho^+: \Gamma_c(M; S^+ \otimes E) \rightarrow \Gamma_c(M; S^- \otimes E)$, $R_\rho^+: \Gamma_c(M; S^+ \otimes F) \rightarrow \Gamma_c(M; S^- \otimes F)$ and Q_ρ^-, R_ρ^- are formally adjoint respectively to Q_ρ^+, R_ρ^+ .

In order to construct a Fredholm operator out of the operators Q_ρ and R_ρ , we make use of a potential.

DEFINITION 4.7. *We say that a smooth function $\phi: M \rightarrow [0, \infty)$ is a compatible potential if $\phi = 0$ in a neighborhood of an essential support of (E, F) and ϕ is constant and nonzero in a neighborhood of infinity.*

Fix a compatible potential ϕ . By Assumption 4.1, the bundles E and F are isomorphic in a neighborhood of the support of ϕ . Therefore, ϕ defines bundle maps

$$\phi: S^\pm \otimes E \longrightarrow S^\pm \otimes F \quad \text{and} \quad \phi: S^\pm \otimes F \longrightarrow S^\pm \otimes E.$$

Set

$$W^+ := (S^+ \otimes E) \oplus (S^- \otimes F) \quad \text{and} \quad W^- := (S^+ \otimes F) \oplus (S^- \otimes E).$$

Define the operator $P_{\rho, \phi}^+: \Gamma(W^+) \rightarrow \Gamma(W^-)$ through the formula

$$P_{\rho, \phi}^+ := \begin{pmatrix} \phi & R_\rho^- \\ Q_\rho^+ & -\phi \end{pmatrix}.$$

Denote by $P_{\rho, \phi}^-$ its formal adjoint and consider the graded bundle $W := W^+ \oplus W^-$. The *generalized Gromov-Lawson operator* associated to our data is the operator $P_{\rho, \phi}^{E, F}: \Gamma(W) \rightarrow \Gamma(W)$ defined as

$$P_{\rho, \phi}^{E, F} := \begin{pmatrix} 0 & P_{\rho, \phi}^- \\ P_{\rho, \phi}^+ & 0 \end{pmatrix}.$$

By construction, $P_{\rho, \phi}^{E, F}$ is an odd formally self-adjoint elliptic differential operator of order one. When there is no danger of confusion, we will denote $P_{\rho, \phi}^{E, F}$ simply by $P_{\rho, \phi}$.

Theorem 4.8. *For every admissible rescaling function ρ and every compatible potential ϕ , the pair $(M, P_{\rho, \phi})$ is complete and the operator $P_{\rho, \phi}^2$ is uniformly positive at infinity.*

The proof of this theorem is based on the following two lemmas.

LEMMA 4.9. *Let U, V be Hilbert A -modules and let $T: U \rightarrow V$ be an adjointable operator such that $T^*T = b^2 \text{id}_U$, for some constant $b > 0$. Then for every $\eta \in U$ and $\theta \in V$ we have*

$$(T\eta | \theta)_V + (\theta | T\eta)_V \geq -(b\eta | \eta)_U - (b\theta | \theta)_V, \quad (4.7)$$

where $(\cdot | \cdot)_U$ and $(\cdot | \cdot)_V$ are the A -valued inner products respectively of U and V .

Proof. Pick $\eta \in U$ and $\theta \in V$. We have

$$\begin{aligned} 0 &\leq \left(b^{-1/2}T\eta + b^{1/2}\theta \mid b^{-1/2}T\eta + b^{1/2}\theta \right)_V \\ &= (b^{-1}T^*T\eta | \eta)_U + (b\theta | \theta)_V + (T\eta | \theta)_V + (\theta | T\eta)_V \\ &= (b\eta | \eta)_U + (b\theta | \theta)_V + (T\eta | \theta)_V + (\theta | T\eta)_V, \end{aligned}$$

from which (4.7) follows. \square

LEMMA 4.10. *Let $w \in \Gamma_c(M; W^+)$, $u \in \Gamma_c(M; S^+ \otimes E)$, and $v \in \Gamma_c(M; S^- \otimes F)$ be such that $w = u \oplus v$. Then*

$$\begin{aligned} \left\langle P_{\rho, \phi}^+ w, P_{\rho, \phi}^+ w \right\rangle &\geq \left\langle Q_\rho^+ u, Q_\rho^+ u \right\rangle + \langle \phi u, \phi u \rangle - \langle \rho^2 |d\phi| u, u \rangle \\ &\quad + \left\langle R_\rho^- v, R_\rho^- v \right\rangle + \langle \phi v, \phi v \rangle - \langle \rho^2 |d\phi| v, v \rangle. \end{aligned} \quad (4.8)$$

Analogously,

$$\begin{aligned} \left\langle P_{\rho, \phi}^- \bar{w}, P_{\rho, \phi}^- \bar{w} \right\rangle &\geq \left\langle R_\rho^+ \bar{v}, R_\rho^+ \bar{v} \right\rangle + \langle \phi \bar{v}, \phi \bar{v} \rangle - \langle \rho^2 |d\phi| \bar{v}, \bar{v} \rangle \\ &\quad + \left\langle Q_\rho^- \bar{u}, Q_\rho^- \bar{u} \right\rangle + \langle \phi \bar{u}, \phi \bar{u} \rangle - \langle \rho^2 |d\phi| \bar{u}, \bar{u} \rangle \end{aligned} \quad (4.9)$$

for every $\bar{w} \in \Gamma_c(M; W^-)$, $\bar{u} \in \Gamma_c(M; S^- \otimes E)$, and $\bar{v} \in \Gamma_c(M; S^+ \otimes F)$ such that $\bar{w} = \bar{v} \oplus \bar{u}$.

Proof. We have

$$P_{\rho, \phi}^+ w = (\phi u + R_\rho^- v) \oplus (Q_\rho^+ u - \phi v),$$

from which

$$\begin{aligned} \left\langle P_{\rho, \phi}^+ w, P_{\rho, \phi}^+ w \right\rangle &= \left\langle Q_\rho^+ u, Q_\rho^+ u \right\rangle + \langle \phi u, \phi u \rangle + \left\langle (R_\rho^+ \phi - \phi Q_\rho^+) u, v \right\rangle \\ &\quad + \left\langle R_\rho^- v, R_\rho^- v \right\rangle + \langle \phi v, \phi v \rangle + \left\langle v, (R_\rho^+ \phi - \phi Q_\rho^+) u \right\rangle. \end{aligned} \quad (4.10)$$

Let us now analyze the operator $(R_\rho^+ \phi - \phi Q_\rho^+): \Gamma(M; S^+ \otimes E) \rightarrow \Gamma(M; S^- \otimes F)$. By Assumption 4.1, we have isomorphisms

$$S^+ \otimes E \cong S^+ \otimes F \quad S^- \otimes E \cong S^- \otimes F \quad \text{on } M \setminus K. \quad (4.11)$$

Since ϕ vanishes in a neighborhood of K , using the isomorphisms (4.11) Clifford multiplication by $d\phi$ defines a bundle map $\tilde{c}(d\phi): S^+ \otimes E \rightarrow S^- \otimes F$. When $d\phi_x = 0$, $\tilde{c}(d\phi_x)$ is the zero map. When $d\phi_x \neq 0$, $\tilde{c}(d\phi_x)^*\tilde{c}(d\phi_x) = -|d\phi_x|^2$. Observe that, under the isomorphisms (4.11), the operators Q_ρ^+ and Q_ρ^- correspond respectively to the operators R_ρ^+ and R_ρ^- . Therefore,

$$R_\rho^+ \phi - \phi Q_\rho^+ = \rho^2 \tilde{c}(d\phi).$$

Moreover, we have

$$\begin{aligned} & \langle \rho^2(x) \tilde{c}(d\phi_x) u(x) \mid v(x) \rangle_x + \langle v(x) \mid \rho^2(x) \tilde{c}(d\phi_x) u(x) \rangle_x \\ & \geq -\langle \rho^2(x) |d\phi_x| u(x) \mid u(x) \rangle_x - \langle \rho^2(x) |d\phi_x| v(x) \mid v(x) \rangle_x \end{aligned} \tag{4.12}$$

for all $x \in M$. When $d\phi_x = 0$, this inequality is trivial. When $d\phi_x \neq 0$, it follows from Lemma 4.9 by setting $U = S_x^+ \otimes E_x$, $V = S_x^- \otimes F_x$, $\eta = u(x)$, $\theta = v(x)$, $T = \rho^2(x) c(d\phi_x)$ and $b = \rho^2(x) |d\phi_x|$. Using (4.12), we obtain

$$\begin{aligned} & \langle (R_\rho^+ \phi - \phi Q_\rho^+) u, v \rangle + \langle v, (R_\rho^+ \phi - \phi Q_\rho^+) u \rangle \\ & = \int_M \left\{ \langle \rho^2(x) \tilde{c}(d\phi_x) u(x) \mid v(x) \rangle_x + \langle v(x) \mid \rho^2(x) \tilde{c}(d\phi_x) u(x) \rangle_x \right\} d\mu_g(x) \\ & \geq - \int_M \left\{ \langle \rho^2(x) |d\phi_x| u(x) \mid u(x) \rangle_x + \langle \rho^2(x) |d\phi_x| v(x) \mid v(x) \rangle_x \right\} d\mu_g(x) \\ & = - \langle \rho^2 |d\phi| u, u \rangle - \langle \rho^2 |d\phi| v, v \rangle. \end{aligned}$$

Finally, Inequality (4.8) follows from this last inequality and (4.10). Inequality (4.9) is proved in a similar way. \square

Proof of Theorem 4.8. The completeness of the pair $(M, P_{\rho,\phi})$ follows from Proposition 3.3 and Remark 2.7. Moreover, since $\rho \leq 1$, from Lemma 4.10 we deduce

$$\langle P_{\rho,\phi}^2 w, w \rangle \geq \langle (\phi^2 - |d\phi|) w, w \rangle, \quad w \in \Gamma_c(M; W).$$

Since ϕ is a compatible potential, the previous inequality implies that $P_{\rho,\phi}^2$ is uniformly positive at infinity. \square

From Theorem 4.8 and the results of Subsection 2.2, the class index $(P_{\rho,\phi})$ is well defined, for every admissible rescaling function ρ and every compatible potential ϕ . In the case (I) from Subsection 4.1, $\text{index}(P_{\rho,\phi}) \in \mathbb{Z}$. In the case (II), $\text{index}(P_{\rho,\phi}) \in \text{KO}_n(A)$.

4.3 The index theorem. Let (M, g) , (E, ∇^E) and (F, ∇^F) denote the same objects as in Subsection 4.1. Suppose Assumption 4.1 is satisfied.

Theorem 4.11. *For every admissible rescaling function ρ and every compatible potential ϕ , the classes $\text{index}(P_{\rho,\phi})$ and $\text{rel-ind}(M; E, F)$ coincide.*

In order to prove this theorem, we first establish some stability properties of the index of $P_{\rho,\phi}$.

LEMMA 4.12. *Let ρ be an admissible rescaling function and let ϕ be a compatible potential. Then,*

- (a) *if ρ' is a second admissible rescaling function coinciding with ρ in a neighborhood of infinity, then $\text{index}(P_{\rho',\phi}) = \text{index}(P_{\rho,\phi})$;*
- (b) *if ϕ' is a second compatible potential, then $\text{index}(P_{\rho,\phi'}) = \text{index}(P_{\rho,\phi})$;*
- (c) *if $E = F$, then $\text{index}(P_{\rho,\phi}) = 0$.*

Proof. By Remark 3.2, the function $\rho_t := t\rho' + (1 - t)\rho$ is an admissible rescaling function, for $t \in [0, 1]$. Part (a) follows by observing that $\left\{P_{\rho_t,\phi} (P_{\rho_t,\phi}^2 + 1)^{-1/2}\right\}$, with $0 \leq t \leq 1$, is a continuous path of Fredholm operators.

Observe now that the function $\phi_t := t\phi' + (1 - t)\phi$ is a compatible potential, for $t \in [0, 1]$. Part (b) follows by observing that $\left\{P_{\rho,\phi_t} (P_{\rho,\phi_t}^2 + 1)^{-1/2}\right\}$, with $0 \leq t \leq 1$, is a continuous path of Fredholm operators.

Finally, suppose that $E = F$. In this case, the operator $P_{\rho,1}$ is well defined and Fredholm. Moreover, by the computations of Lemma 4.10, the operator $P_{\rho,1}^2$ is uniformly positive and $\text{index}(P_{\rho,1}) = 0$. By considering the functions $\phi_t = t + (1 - t)\phi$ and arguing as in Part (b), we deduce Part (c). □

We will now establish an additivity formula for generalized Gromov-Lawson operators, from which we will deduce Theorem 4.11. Let us consider the following situation. Let (E, F) and (G, H) be two pairs over (M, g) satisfying Assumption 4.1. Let $K \subset M$ be an essential support of both (E, F) and (G, H) and let $L \subset M$ be a compact submanifold with boundary whose interior contains K . Let ϕ be a compatible potential vanishing in a neighborhood of L and let ρ be an admissible rescaling function such that $\rho = 1$ in a neighborhood of L . Denote respectively by $P_{\rho,\phi}^{E,F}$ and $P_{\rho,\phi}^{G,H}$ the generalized Gromov-Lawson operators associated to these data.

LEMMA 4.13. *In the above situation, we have*

$$\text{index}\left(P_{\rho,\phi}^{E,F}\right) + \text{index}\left(D_{L_D}^{G,H}\right) = \text{index}\left(P_{\rho,\phi}^{G,H}\right) + \text{index}\left(D_{L_D}^{E,F}\right). \tag{4.13}$$

Proof. Let S_{L_D} and D_{L_D} be the \mathbb{Z}_2 -graded spinor bundle and odd Dirac operator associated to the spin manifold L_D . Consider the \mathbb{Z}_2 -graded bundle $V = V^+ \oplus V^-$ over L_D , where

$$\begin{aligned} V^+ &:= (S_{L_D}^+ \otimes V(G, E)) \oplus (S_{L_D}^- \otimes V(H, F)), \\ V^- &:= (S_{L_D}^+ \otimes V(H, F)) \oplus (S_{L_D}^- \otimes V(G, E)), \end{aligned}$$

and where $V(G, E)$ and $V(H, F)$ are the bundles defined in Subsection 4.1. Define the operator $T^+ : \Gamma(L_D; V^+) \rightarrow \Gamma(L_D; V^-)$ through the formula

$$T^+ := \begin{pmatrix} 0 & \left(D_{L_D}^{H,F}\right)^- \\ \left(D_{L_D}^{G,E}\right)^+ & 0 \end{pmatrix}.$$

Let $P_{H,F}^{G,E}: \Gamma(V) \rightarrow \Gamma(V)$ be the operator defined as

$$P_{H,F}^{G,E} := \begin{pmatrix} 0 & T^- \\ T^+ & 0 \end{pmatrix},$$

where T^- is the formal adjoint of T^+ . Observe that $P_{H,F}^{G,E}$ is an odd formally self-adjoint elliptic differential operator of order one. By construction,

$$\text{index} \left(P_{H,F}^{G,E} \right) = \text{index} \left(D_{L_D}^{G,E} \right) - \text{index} \left(D_{L_D}^{H,F} \right).$$

Using (4.3) and (4.4), we obtain

$$\text{index} \left(P_{H,F}^{G,E} \right) = \text{index} \left(D_{L_D}^{G,H} \right) - \text{index} \left(D_{L_D}^{E,F} \right). \tag{4.14}$$

From (4.3) and (4.14), we have

$$\text{index} \left(P_{F,F}^{E,E} \right) = 0. \tag{4.15}$$

We now relate the operator $P_{H,F}^{G,E}$ to the operator $P_{\rho,\phi}^{E,F}$ on M . Consider the partitions $M = L \cup_{\partial L} (M \setminus L)$ and $L_D = L \cup_{\partial L} L^-$. Modify the Riemannian metrics and the Clifford structures on M and L_D in a tubular neighborhood of ∂L in such a way that Assumption 2.11 is satisfied. Using the cut-and-paste construction described in Subsection 2.3, we obtain the operator $P_{\rho,\phi}^{E,F}$ on M and the operator $P_{H,F}^{G,E}$ on L_D . From Identities (4.14), (4.15) and Theorem 2.12, we deduce

$$\begin{aligned} & \text{index} \left(P_{\rho,\phi}^{E,F} \right) + \text{index} \left(D_{L_D}^{G,H} \right) - \text{index} \left(D_{L_D}^{E,F} \right) \\ &= \text{index} \left(P_{\rho,\phi}^{E,F} \right) + \text{index} \left(P_{H,F}^{G,E} \right) \\ &= \text{index} \left(P_{\rho,\phi}^{G,H} \right) + \text{index} \left(P_{F,F}^{E,E} \right) = \text{index} \left(P_{\rho,\phi}^{G,H} \right), \end{aligned}$$

which implies Identity (4.13). □

Proof of Theorem 4.11. Let ρ be an admissible rescaling function, let ϕ be a compatible potential, and let $L \subset M$ be a compact submanifold with boundary whose interior contains an essential support of (E, F) . Using Part (a) and Part (b) of Lemma 4.12, we assume that, in a neighborhood of L , we have $\phi = 0$ and $\rho = 1$. Let $(\underline{\mathcal{V}}, d_{\underline{\mathcal{V}}})$ be as in Assumption 4.1. By Part (c) of Lemma 4.12 and Identity (4.2), the indices of $P_{\rho,\phi}^{\underline{\mathcal{V}},\underline{\mathcal{V}}}$ and $D_{L_D}^{\underline{\mathcal{V}},\underline{\mathcal{V}}}$ vanish. Using Lemma 4.13, we deduce

$$\begin{aligned} \text{index} \left(P_{\rho,\phi}^{E,F} \right) &= \text{index} \left(P_{\rho,\phi}^{E,F} \right) + \text{index} \left(D_{L_D}^{\underline{\mathcal{V}},\underline{\mathcal{V}}} \right) \\ &= \text{index} \left(P_{\rho,\phi}^{\underline{\mathcal{V}},\underline{\mathcal{V}}} \right) + \text{index} \left(D_{L_D}^{E,F} \right) = \text{index} \left(D_{L_D}^{E,F} \right), \end{aligned}$$

which concludes the proof. □

5 The long neck problem

This section is devoted to proving a long neck principle in the spin setting. Suppose (X, g) is a compact Riemannian spin manifold with boundary and let (E, F) be a pair of bundles with isomorphic typical fibers and whose supports are contained in the interior of X . We will give conditions on the lower bound of scal_g and the distance between the supports of E, F and ∂X so that $\text{rel-ind}(X^\circ; E, F)$ must vanish, where X° is the interior of X . To this end, we will use the distance function from ∂X to construct a generalized Gromov-Lawson operator $P_{\rho, \phi}^{E, F}$ on X° . In Subsection 5.1, we prove a vanishing theorem for the operator $P_{\rho, \phi}^{E, F}$, from which we deduce an abstract long neck principle. As applications, we prove Theorems A and B respectively in Subsections 5.2 and 5.3. Finally, in Subsection 5.4 we use a generalized Gromov-Lawson operator on a complete manifold to prove Theorem C.

5.1 A vanishing theorem on compact manifolds with boundary. We consider the following setup. Let (X, g) be a compact n -dimensional Riemannian spin manifold with boundary ∂X . By removing the boundary, we obtain the open manifold $X^\circ := X \setminus \partial X$. The metric g induces an incomplete metric on X° , that we denote by the same symbol. Let (E, ∇^E) and (F, ∇^F) be bundles of finitely generated projective Hilbert A -modules with inner product and metric connection over (X°, g) . Suppose Assumption 4.1 is satisfied.

DEFINITION 5.1. *For an essential support K of (E, F) , a K -bounding function is a smooth function $\nu: X^\circ \rightarrow [0, \infty)$ such that $\nu = 0$ on $X^\circ \setminus K$ and*

$$\mathcal{R}_x^E \geq -\nu(x) \quad \text{and} \quad \mathcal{R}_x^F \geq -\nu(x), \quad x \in K. \quad (5.1)$$

We say that ν is a bounding function if it is a K -bounding function for some essential support K of (E, F) .

The next theorem states an abstract “long neck principle” for compact Riemannian spin manifolds with boundary.

Theorem 5.2. *Let K be an essential support of (E, F) , let ν be a K -bounding function, and let σ be a positive constant. Suppose that*

$$\frac{\text{scal}_g(x)}{4} > \nu(x), \quad x \in K; \quad (5.2)$$

$$\text{scal}_g(x) \geq \sigma, \quad x \in X^\circ \setminus K; \quad (5.3)$$

and

$$\text{dist}(K, \partial X) > \pi \sqrt{\frac{n-1}{n\sigma}}. \quad (5.4)$$

Then $\text{rel-ind}(X^\circ; E, F) = 0$.

In order to prove this theorem, we will make use of the index theory developed in Section 4. For an admissible rescaling function ρ and a compatible potential ϕ on X° , let $P_{\rho,\phi}: \Gamma_c(X^\circ; W) \rightarrow \Gamma_c(X^\circ; W)$ be the associated generalized Gromov-Lawson operator. For the definition of the bundle W and the operator $P_{\rho,\phi}$ and their properties, see Subsection 4.2. We start with proving an estimate for the operator $P_{\rho,\phi}^2$ in this setting. As in Section 3, we use the notation $\bar{n} = n/(n - 1)$ and $\overline{\text{scal}}_g(x) = \text{scal}_g(x)/4$.

LEMMA 5.3. *Let ρ be an admissible rescaling function, let ϕ be a compatible potential and let ν a bounding function. Then, for every $w \in \Gamma_c(M; W)$ and every $\omega > 0$, we have*

$$\langle P_{\rho,\phi}^2 w, w \rangle \geq \langle \Phi_{\rho,\phi}^{\omega,\nu} w, w \rangle,$$

where $\Phi_{\rho,\phi}^{\omega,\nu}: M \rightarrow \mathbb{R}$ is the smooth function defined by the formula

$$\begin{aligned} \Phi_{\rho,\phi}^{\omega,\nu}(x) := & \frac{\bar{n}\omega}{1 + \omega} \rho^4(x) \overline{\text{scal}}_g(x) - \frac{\bar{n}\omega}{1 + \omega} \rho^4(x) \nu(x) \\ & - \omega \rho^2(x) |d\rho_x|^2 + \phi^2(x) - \rho^2(x) |d\phi_x| \end{aligned} \tag{5.5}$$

for $x \in X^\circ$.

Proof. It follows from Lemma 4.10 and Proposition 3.8. □

COROLLARY 5.4. *Suppose there exist an admissible rescaling function ρ , a compatible potential ϕ , a bounding function ν , and positive constants ω and c such that*

$$\Phi_{\rho,\phi}^{\omega,\nu}(x) \geq c, \quad x \in X^\circ. \tag{5.6}$$

Then the class $\text{rel-ind}(X^\circ; E, F)$ vanishes.

Proof. From Lemma 5.3, Condition (5.6) implies that the operator $P_{\rho,\phi}^2$ is invertible. Now the thesis follows from Theorem 4.11. □

For the remaininig part of this section, we use the notation

$$\bar{\sigma} := \frac{\sigma}{4}$$

for a positive constant σ .

LEMMA 5.5. *Suppose $\Lambda > \pi/\sqrt{\bar{n}\bar{\sigma}}$ for some constant $\sigma > 0$. Then there exist constants $\omega > 0$, $L \in (0, \Lambda)$ and a smooth function $Y: (0, \infty) \rightarrow [0, \infty)$ satisfying*

- (1) $Y = 0$ in a neighborhood of $[\Lambda, \infty)$;
- (2) $Y = Y_0$ in a neighborhood of $(0, L]$, where Y_0 is a constant such that $Y_0^2 > \omega/4$;
- (3) there exists a constant $c > 0$ such that

$$\frac{\bar{n}\bar{\sigma}\omega}{1 + \omega} L^2 + Y^2(t) - L |Y'(t)| \geq c,$$

for t varying in a neighborhood of $[L, \Lambda]$.

Moreover, L can be chosen arbitrary small.

Proof. Write $\sigma = \sigma_1 + \sigma_2$, with σ_1 and σ_2 positive numbers. We use the notation $\bar{\sigma}_1 := \sigma_1/4$ and $\bar{\sigma}_2 := \sigma_2/4$. Observe that $\frac{1+\omega}{\omega} \rightarrow 1$, as $\omega \rightarrow \infty$. Since $\Lambda > \pi/\sqrt{\bar{n}\sigma}$, we choose $\omega > 0$ and $\sigma_1 \in (0, \sigma)$ such that

$$\Lambda > \pi\sqrt{\frac{1+\omega}{\bar{n}\sigma_1\omega}}. \tag{5.7}$$

This is achieved by taking ω large enough and σ_1 close enough to σ .

For positive constants A and B , consider the function

$$Y_{A,B}(t) = AB \tan(A(\Lambda - t)), \quad \Lambda - \frac{\pi}{2A} < t < \Lambda + \frac{\pi}{2A}.$$

Observe that $Y_{A,B}$ satisfies

$$A^2B^2 + Y_{A,B}^2(t) - B|Y'_{A,B}(t)| = 0 \quad \text{and} \quad Y_{A,B}(\Lambda) = 0. \tag{5.8}$$

Moreover,

$$Y_{A,B}(t) \rightarrow \infty, \quad \text{as } t \rightarrow \left(\Lambda - \frac{\pi}{2A}\right)^+. \tag{5.9}$$

Observe that the point at which the function $Y_{A,B}$ goes to infinity is uniquely determined by the constant A .

We now make the following choice for A and B . Set

$$A := \sqrt{\frac{\bar{n}\bar{\sigma}_1\omega}{1+\omega}}.$$

Notice that

$$\frac{\pi}{2A} = \frac{\pi}{2}\sqrt{\frac{1+\omega}{\bar{n}\bar{\sigma}_1\omega}} = \pi\sqrt{\frac{1+\omega}{\bar{n}\sigma_1\omega}}.$$

Using (5.7), we deduce

$$\Lambda - \frac{\pi}{2A} = \Lambda - \pi\sqrt{\frac{1+\omega}{\bar{n}\sigma_1\omega}} > 0.$$

Choose

$$L \in \left(0, \Lambda - \pi\sqrt{\frac{1+\omega}{\bar{n}\sigma_1\omega}}\right)$$

and set $B := L$. With this choice for A and B , we obtain the function

$$\tilde{Y}(t) := \sqrt{\frac{\bar{n}\bar{\sigma}_1\omega}{1+\omega}}L \tan\left(\sqrt{\frac{\bar{n}\bar{\sigma}_1\omega}{1+\omega}}(\Lambda - t)\right), \quad \Lambda - \pi\sqrt{\frac{1+\omega}{\bar{n}\sigma_1\omega}} < t < \Lambda + \pi\sqrt{\frac{1+\omega}{\bar{n}\sigma_1\omega}}.$$

From (5.8), we deduce that

$$\tilde{Y}(\Lambda) = 0 \tag{5.10}$$

and

$$\frac{\bar{n}\bar{\sigma}\omega}{1+\omega}L^2 + \tilde{Y}^2(t) - L \left| \tilde{Y}'(t) \right| = \frac{\bar{n}\bar{\sigma}_2\omega}{1+\omega}\eta^2\Lambda^2 > 0. \tag{5.11}$$

Moreover, by (5.9) we deduce that we can choose $T \in \left(\Lambda - \pi\sqrt{\frac{1+\omega}{\bar{n}\sigma_1\omega}}, \Lambda \right)$ such that

$$\tilde{Y}^2(T) > \frac{\omega}{4}.$$

For $\epsilon > 0$ small enough, let $\tilde{Y}_\epsilon: (0, \infty) \rightarrow [0, \infty)$ be a smooth function such that

- \tilde{Y}_ϵ is constant in a neighborhood of $(0, T]$;
- $\tilde{Y}_\epsilon = 0$ in a neighborhood of $[\Lambda, \infty)$;
- \tilde{Y}_ϵ is ϵ -close to \tilde{Y} on $[T, \Lambda]$;
- $\left| \tilde{Y}'_\epsilon(t) \right| \leq \left| \tilde{Y}'(t) \right|$, for $t \in (T, \Lambda)$.

Since $L < T$, for ϵ small enough, we obtain a function satisfying Properties (1), (2) and (3). The final statement follows from the fact that L can be arbitrarily chosen in the set $\left(0, \Lambda - \pi\sqrt{\frac{1+\omega}{\bar{n}\sigma_1\omega}} \right)$, where σ_1 and ω satisfy (5.8). □

LEMMA 5.6. *Suppose $\Lambda > \pi/\sqrt{\bar{n}\sigma}$. Then there exist a constant $\omega > 0$ and smooth functions $Z: (0, \infty) \rightarrow (0, 1]$ and $Y: (0, \infty) \rightarrow [0, \infty)$ satisfying*

- (a) $Z(t) = d\sqrt{t}$ near 0, for some constant $d > 0$;
- (b) Z is constant in a neighborhood of $[\Lambda, \infty)$;
- (c) Y is constant near 0;
- (d) $Y = 0$ in a neighborhood of $[\Lambda, \infty)$;
- (e) there exists a constant $c > 0$ such that the functions Z and Y satisfy the differential inequality

$$\frac{\bar{n}\bar{\sigma}\omega}{1+\omega}Z^4(t) - \omega Z^2(t) \left| Z'(t) \right|^2 + Y^2(t) - Z^2(t) \left| Y'(t) \right| \geq c,$$

for all $t > 0$.

Proof. Pick constants $\omega > 0$, $L \in (0, \Lambda]$, with $L \leq 1$, and a smooth function $Y: (0, \infty) \rightarrow [0, \infty)$ satisfying Properties (1)–(3) of Lemma 5.5. Observe that Properties (1) and (2) imply that Y satisfies Properties (c) and (d). Using Property (2) of Lemma 5.5, pick a smooth function $Z: (0, \infty) \rightarrow (0, 1]$ and a constant $\delta > 0$ such that

- (i) $Z = \sqrt{L}$ in a neighborhood of $[L, \infty)$;
- (ii) $Z(t) = (1 + \delta)\sqrt{t}$ for t near 0 and $Z(t) \leq (1 + \delta)\sqrt{t}$ in a neighborhood of $(0, L]$;
- (iii) $0 \leq Z'(t) \leq (1 + \delta)/(2\sqrt{t})$ for all $t > 0$;
- (iv) $Y_0^2 > (1 + \delta)^4\omega/4$.

By Properties (i) and (ii), it follows that Z satisfies (a) and (b). In order to conclude the proof, it remains to show that Y and Z satisfy (e).

Let U_0 be an open neighborhood of $(0, L]$ such that $Y = Y_0$ on U_0 . Let U_1 be an open neighborhood of $[L, \infty)$ such that $Z = \sqrt{L}$ on U_1 . We will prove Property (e) by analyzing separately these two open sets. Let us begin with U_0 . On this set, $Y = Y_0$. Moreover, by Properties (ii) and (iii) we have

$$Z^2(t) |Z'(t)|^2 \leq \frac{(1 + \delta)^4}{4}, \quad t \in U_0.$$

Therefore, using Property (iv) we obtain

$$\frac{\bar{n}\bar{\sigma}\omega}{1 + \omega} Z^4(t) - \omega Z^2(t) |Z'(t)|^2 + Y^2(t) - Z^2(t) |Y'(t)| \geq Y_0^2 - (1 + \delta)^4 \frac{\omega}{4} > 0, \quad (5.12)$$

for every $t \in U_0$. Let us now analyze U_1 . On this set, $Z = \sqrt{L}$. Therefore, using Property (3) of Lemma 5.5, we deduce that there exists a constant $c > 0$ such that

$$\begin{aligned} & \frac{\bar{n}\bar{\sigma}\omega}{1 + \omega} Z^4(t) - \omega Z^2(t) |Z'(t)|^2 + Y^2(t) - Z^2(t) |Y'(t)| \\ &= \frac{\bar{n}\bar{\sigma}\omega}{1 + \omega} L^2 + Y^2(t) - L |Y'(t)| \geq c, \end{aligned} \quad (5.13)$$

for every $t \in U_1$. Since $U_0 \cup U_1 = (0, \infty)$, (5.12) and (5.13) imply that Y and Z satisfy Property (e). This concludes the proof. \square

Proof of Theorem 5.2. By Corollary 5.4, it suffices to show that, when Conditions (5.2) and (5.4) are satisfied, there exist an admissible rescaling function ρ , a compatible potential ϕ and positive constants ω and c such that

$$\Phi_{\rho, \phi}^{\omega, \nu}(x) \geq c, \quad x \in X^\circ, \quad (5.14)$$

where $\Phi_{\rho, \phi}^{\omega, \nu}$ is the function defined by (5.5).

Using Condition (5.4), pick a constant Λ satisfying

$$\frac{\pi}{\sqrt{\bar{n}\bar{\sigma}}} < \Lambda < \text{dist}(K, \partial X). \quad (5.15)$$

Choose a constant $\omega > 0$ and smooth functions $Z: (0, \infty) \rightarrow (0, 1]$ and $Y: (0, \infty) \rightarrow [0, \infty)$ satisfying Conditions (a)–(e) of Lemma 5.6.

Let τ be the distance function from the boundary of X . We regard τ as a function on X° . Fix a constant $\delta > 0$ and consider the function

$$\mu(x) := \min(\tau(x)/2, \delta), \quad x \in X^\circ.$$

By [GW79, Proposition 2.1], there exists a smooth function τ_δ on X° such that

$$|\tau(x) - \tau_\delta(x)| < \mu(x) \quad \forall x \in X^\circ \quad \text{and} \quad \|\text{d}\tau_\delta\|_\infty < 1 + \delta. \quad (5.16)$$

Observe that these conditions imply that τ_δ is positive and goes to 0 at infinity. Choose a constant Λ_1 such that $\Lambda < \Lambda_1 < \text{dist}(K, \partial X)$. Consider the open sets

$$\Omega_0 := \{x \in X^\circ \mid \tau_\delta(x) > \Lambda\} \quad \text{and} \quad \Omega_1 := \{x \in X^\circ \mid \tau_\delta(x) < \Lambda_1\}. \quad (5.17)$$

Observe that $X^\circ = \Omega_0 \cup \Omega_1$ and that, by taking δ small enough, $K \subset \Omega_0$ and $\Omega_1 \subset X^\circ \setminus K$.

Define functions $\rho: X^\circ \rightarrow (0, 1]$ and $\phi: X^\circ \rightarrow [0, \infty)$ by setting

$$\rho := Z \circ \tau_\delta \quad \text{and} \quad \phi := Y \circ \tau_\delta. \quad (5.18)$$

By Proposition 3.5, Remark 3.2 and Property (a) of Lemma 5.6, ρ is an admissible rescaling function. Moreover, by Property (b) of Lemma 5.6, we have

(i) $\rho = \rho_0$ on Ω_0 , for a constant $\rho_0 > 0$.

By Properties (c) and (d) of Lemma 5.6, ϕ is a compatible potential and satisfies

(ii) $\phi = 0$ on Ω_0 .

Using (i) and (ii), we deduce

$$\Phi_{\rho, \phi}^{\omega, \nu}(x) = \frac{\bar{n}\omega}{1 + \omega} \rho_0^4 \overline{\text{scal}}_g(x) - \frac{\bar{n}\omega}{1 + \omega} \rho_0^4 \nu(x), \quad x \in \Omega_0. \quad (5.19)$$

Using Condition (5.2) and Condition (5.3), from (5.19) we obtain

$$\Phi_{\rho, \phi}^{\omega, \nu}(x) \geq \frac{\rho_0^4 \bar{n}\omega}{1 + \omega} \inf_{y \in K} (\overline{\text{scal}}_g(y) - \nu(y)) > 0, \quad x \in K$$

and

$$\Phi_{\rho, \phi}^{\omega, \nu}(x) \geq \frac{\bar{n}\bar{\sigma}\omega}{1 + \omega} \rho_0^4 > 0, \quad x \in \Omega_0 \setminus K.$$

Therefore, Inequality (5.14) holds on Ω_0 .

In order to complete the proof, we will show that, for δ small enough, Inequality (5.14) holds on Ω_1 as well. Using (5.16), we have

$$|\text{d}\rho_x| \leq (1 + \delta) |Z'(\tau_\delta(x))| \quad \text{and} \quad |\text{d}\phi_x| \leq (1 + \delta) |Y'(\tau_\delta(x))|.$$

Since $\Omega_1 \subset X^\circ \setminus K$, $\text{scal}_g \geq \sigma$ on Ω_1 . Since ν is a K -bounding function, $\nu = 0$ on Ω_1 . By taking δ small enough and using Property (e) of Lemma 5.6, we deduce that there exists a constant $c_1 > 0$ such that

$$\begin{aligned} \Phi_{\rho, \phi}^{\omega, \nu}(x) &\geq \frac{\bar{n}\bar{\sigma}\omega}{1 + \omega} \rho^4(x) - \omega \rho^2(x) |\text{d}\rho_x|^2 + \phi^2(x) - \rho^2(x) |\text{d}\phi_x| \\ &\geq \frac{\bar{n}\bar{\sigma}\omega}{1 + \omega} Z^4(\tau_\delta(x)) - \omega Z^2(\tau_\delta(x)) (1 + \delta)^2 |Z'(\tau_\delta(x))|^2 \\ &\quad + Y^2(\tau_\delta(x)) - Z^2(\tau_\delta(x)) (1 + \delta) |Y'(\tau_\delta(x))| \geq c_1 \end{aligned}$$

for every $x \in \Omega_1$. This concludes the proof. □

5.2 Proof of Theorem A. Assume first that n is even. Let (E, ∇^E) and (F, ∇^F) be the Hermitian bundles with metric connections constructed in Example 4.5 using the map f . Recall that (E, ∇^E) and (F, ∇^F) satisfy Assumption (4.1), with $\text{supp}(df)$ an essential support of (E, F) , and

$$\text{rel-ind}(X^\circ; E, F) = 0 \text{ implies } \deg(f) = 0. \quad (5.20)$$

Moreover, $\mathcal{R}^F = 0$ everywhere, $\mathcal{R}^E = 0$ on $M \setminus \text{supp}(df)$ and, since f is strictly area decreasing, $\mathcal{R}^E \geq -\theta n(n-1)/4$ on $\text{supp}(df)$, for some $\theta \in (0, 1)$. By Condition (1.6), there exists a compact subset $K \subset X^\circ$ such that $\text{supp}(df) \subset K^\circ$ and

$$\text{scal}_g(x) \geq \theta_1 n(n-1), \quad x \in K \quad (5.21)$$

for some constant $\theta_1 \in (\theta, 1)$. Observe that K is an essential support of (E, F) . Let $\nu: X^\circ \rightarrow [0, \theta n(n-1)/4]$ be a smooth function such that $\nu = \theta n(n-1)/4$ on $\text{supp}(df)$ and $\nu = 0$ on $X^\circ \setminus K$. Then ν is a K -bounding function and

$$\frac{\text{scal}_g}{4} \geq \frac{\theta_1 n(n-1)}{4} > \frac{\theta n(n-1)}{4} \geq \nu \quad \text{on } K.$$

The last inequality, (1.7) and Theorem 5.2 imply that $\text{rel-ind}(X^\circ; E, F) = 0$. The thesis now follows from (5.20).

Suppose now that n is odd and the map f is constant in a neighborhood of ∂X . As in the proof of [LM89, Proposition 6.10], we fix a 1-contracting map $\mu: S^n \times S^1 \rightarrow S^{n+1}$ of degree one, which is constant on $\{*\} \times S^n \cup S^1 \times \{*\}'$. Here, $*$ and $*'$ are distinguished points respectively in S^n and S^1 with $f(\partial X) \subset \{*\}$. For $R > 0$, let S_R^1 be the circle of radius R and consider the manifold $X_1 := X \times S_R^1$ equipped with the product metric, denoted by g_1 . Let $f_1: X_1 \rightarrow S^{n+1}$ be the map given by the composition $\mu \circ (f \times 1/R)$. Then f_1 is area decreasing for R large enough. Moreover, $\text{scal}_{g_1} = \text{scal}_g \geq \sigma$ and $\text{dist}(\text{supp}(df_1), \partial X_1) = \text{dist}(\text{supp}(df), \partial X)$. Now the thesis follows from the even dimensional case. \square

5.3 Proof of Theorem B. Let g be a Riemannian metric on X whose scalar curvature is bounded from below by a constant $\sigma > 0$. Consider the incomplete Riemannian manifold X° . Let (E, ∇^E) and (F, ∇^F) be the flat bundles on X° with typical fiber $C^*\Gamma$ constructed in Example 4.6. Recall that, since Y satisfies Condition (1.8), the class $\text{rel-ind}(X^\circ; E, F) \in \text{KO}_n(C^*\Gamma)$ does not vanish. Moreover, for $0 < R < \text{rad}_g^\circ(\partial X)$, the geodesic collar neighborhoods $B_R(S_1^{n-1}), \dots, B_R(S_N^{n-1})$ are pairwise disjoint and the closure of the set

$$X \setminus \bigsqcup_{i=1}^N B_R(S_i^{n-1})$$

is an essential support of (E, F) that we denote by K_R . Since $\text{dist}(K_R, \partial X) = R$ and $\text{rel-ind}(X^\circ; E, F) \neq 0$, by Theorem 5.2 we deduce

$$R \leq \pi \sqrt{\frac{n-1}{n\sigma}}, \quad 0 < R < \text{rad}_g^\circ(\partial X)$$

from which Inequality (1.9) follows.

5.4 Proof of Theorem C. Suppose there exists a complete Riemannian metric g on M such that $\text{scal}_g > 0$ everywhere. Let (E, ∇^E) and (F, ∇^F) be the flat bundles on M with typical fiber $C^*\Gamma$ constructed in Example 4.6. Since Y satisfies Condition (1.8), we have

$$\text{rel-ind}(M; E, F) \neq 0. \tag{5.22}$$

In order to obtain a contradiction, we will construct a generalized Gromov-Lawson operator on M .

Since the metric g is complete, the function $\rho = 1$ is admissible. Let K be an essential support of (E, F) and let $\phi: M \rightarrow [0, \infty)$ be a smooth function such that $\phi = 0$ in a neighborhood of K and $\phi = 1$ in a neighborhood of infinity. Observe that, for $\lambda > 0$, $\lambda\phi$ is a compatible potential. Denote by P_λ the generalized Gromov-Lawson operator $P_{1,\lambda\phi}$. Let $\Phi_\lambda: M \rightarrow \mathbb{R}$ be the smooth function defined by the formula

$$\Phi_\lambda(x) := \frac{1}{4} \text{scal}_g(x) + \lambda^2 \phi^2(x) - \lambda |\text{d}\phi_x|, \quad x \in M. \tag{5.23}$$

Since the connections ∇^E and ∇^F are flat, from Lemma 4.10 and the Lichnerowicz formula (3.6) we deduce

$$\langle P_\lambda^2 w, w \rangle \geq \langle \Phi_\lambda w, w \rangle, \quad w \in \Gamma_c(M; W). \tag{5.24}$$

In order to prove the thesis, we will show that the function Φ_λ is uniformly positive for λ small enough. In fact, in this case (5.24) and Theorem 4.11 imply $\text{rel-ind}(M; E, F) = 0$, contradicting (5.22).

Let Ω_0 and Ω_1 be open subsets of M such that $M = \Omega_0 \cup \Omega_1$, Ω_0 is relatively compact, and $\phi = 1$ on Ω_1 . Since Ω_0 is relatively compact, there exists a constant $\sigma > 0$ such that $\text{scal}_g \geq \sigma$ on Ω_0 . Choose λ satisfying

$$0 < \lambda < \frac{\sigma}{4 \|\text{d}\phi\|_\infty}. \tag{5.25}$$

With this choice, we have

$$\Phi_\lambda(x) \geq \frac{\sigma}{4} - \lambda \|\text{d}\phi\|_\infty > 0, \quad x \in \Omega_0. \tag{5.26}$$

Since $\phi = 1$ on Ω_1 , we also have

$$\Phi_\lambda(x) \geq \lambda^2, \quad x \in \Omega_1. \tag{5.27}$$

Therefore, when λ satisfies (5.25), the function Φ_λ is uniformly positive. □

6 Estimates on band widths

This last section is devoted to the proof of Theorem D. In Subsection 6.1, using rescaling functions and potentials in a similar fashion as in Subsection 4, we extend the theory of Callias-type operators to Riemannian manifolds which are not necessarily complete. More precisely, we develop a rescaled version of the real Callias-type operators used by Zeidler in [Zei19]. In Subsection 6.2, we focus on compact Riemannian spin bands and prove a Callias-type index theorem, stating that the index of a Callias-type operator on a compact Riemannian spin band coincides with the index of an elliptic differential operator on a separating hypersurface. Finally, in Subsection 6.3 we prove a vanishing theorem yielding Theorem D.

6.1 Generalized Callias-type operators. Let (M, g) be an n -dimensional Riemannian spin manifold. Let A be a Real unital C^* -algebra and let (E, ∇^E) be a bundle of finitely generated projective Hilbert A -modules endowed with a metric connection. Let \mathfrak{S}_M be the associated $\text{Cl}_{n,0}$ -linear spinor bundle with Dirac operator \mathfrak{D}_M . Denote by $Z: \Gamma(M; \mathfrak{S}_M \hat{\otimes} E) \rightarrow \Gamma(M; \mathfrak{S}_M \hat{\otimes} E)$ the operator \mathfrak{D}_M twisted with the bundle E .

DEFINITION 6.1. *We say that a smooth function $\psi: M \rightarrow \mathbb{R}$ is a Callias potential if there exist a compact subset $K \subset M$ and a constant $c > 0$ such that $\psi^2 - |d\psi| > c$ on $M \setminus K$.*

Fix an admissible rescaling function ρ and a Callias potential ψ . Let Z_ρ be the operator Z rescaled with the function ρ defined by Formula (3.2). The *generalized Callias-type operator* associated to these data is the first order elliptic differential operator $B_{\rho,\psi}: \Gamma(M; \mathfrak{S}_M \hat{\otimes} E \hat{\otimes} \text{Cl}_{0,1}) \rightarrow \Gamma(M; \mathfrak{S}_M \hat{\otimes} E \hat{\otimes} \text{Cl}_{0,1})$ defined as

$$B_{\rho,\psi} := Z_\rho \hat{\otimes} 1 + \psi \hat{\otimes} \epsilon, \quad (6.1)$$

where ϵ denotes left-multiplication by the Clifford generator of $\text{Cl}_{0,1}$.

REMARK 6.2. When the metric g is complete, $\rho = 1$ is admissible. If the Callias potential ψ is a proper map with bounded gradient, $B_{1,\psi}$ coincides with the operator used in [Zei19].

We now show that the operator $B_{\rho,\psi}$ has a well-defined index.

Theorem 6.3. *For every admissible rescaling function ρ and every Callias potential ψ , the pair $(M, B_{\rho,\psi})$ is complete and the operator $B_{\rho,\psi}^2$ is uniformly positive at infinity.*

Proof. The completeness of the pair $(M, B_{\rho,\psi})$ follows from Proposition 3.3 and Remark 2.7. Moreover, we have

$$B_{\rho,\psi}^2 = Z_\rho^2 \hat{\otimes} 1 + \rho^2 c(d\psi) \hat{\otimes} \epsilon + \psi^2. \quad (6.2)$$

Since $\rho \leq 1$, from the previous identity we deduce

$$\langle B_{\rho,\psi}^2 w, w \rangle \geq \langle (\psi^2 - |\mathrm{d}\psi|) w, w \rangle, \quad w \in \Gamma_c(M; \mathfrak{S}_M \hat{\otimes} E \hat{\otimes} \mathrm{Cl}_{0,1}). \tag{6.3}$$

Since ψ is a Callias potential, the last inequality implies that $B_{\rho,\psi}^2$ is uniformly positive at infinity. \square

From Theorem 6.3 and the results of Subsection 2.2, the class index $(B_{\rho,\psi})$ in $\mathrm{KO}_n(A)$ is well defined, for every admissible rescaling function ρ and every Callias potential ψ .

REMARK 6.4. When (M, g) is complete, $\rho = 1$ and the Callias potential ψ is a proper function with uniformly bounded gradient, the index of $B_{1,\psi}$ coincides with the class $\mathrm{index}_{\mathrm{XPM}}(\mathfrak{D}_{M,E}, \psi)$ used in [Zei19].

We conclude this subsection with some stability properties of the index of $B_{\rho,\psi}$.

PROPOSITION 6.5. *Suppose ρ is an admissible rescaling function and ψ is a Callias potential. Then*

- (a) *if ρ' is a second admissible rescaling function coinciding with ρ outside of a compact set, then the indices of $B_{\rho,\psi}$ and $B_{\rho',\psi}$ coincide;*
- (b) *if ψ' is a second Callias potential coinciding with ψ outside of a compact set, then the indices of $B_{\rho,\psi}$ and $B_{\rho,\psi'}$ coincide;*
- (c) *if ψ is constant and nonzero outside of a compact set, then the index of $B_{\rho,\psi}$ vanishes.*

Proof. For Parts (a) and (b), it suffices to consider the linear homotopies $\rho_t := t\rho' + (1-t)\rho$ and $\psi_t = t\psi' + (1-t)\psi$, with $0 \leq t \leq 1$, and argue as in the proof of Lemma 4.12. Let us prove Part (c). Let ψ_0 be a constant nonzero function such that $\psi = \psi_0$ outside of a compact set. Observe that ψ_0 is a Callias potential. By Identity (6.2), the operator B_{ρ,ψ_0}^2 is uniformly positive and the index of B_{ρ,ψ_0} vanishes. Using Part (b), we deduce that the index of $B_{\rho,\psi}$ vanishes as well. \square

6.2 Callias-type operators on Riemannian spin bands. We start with recalling the notion of band due to Gromov [Gro18, Section 2]. A *band* is a manifold V with two distinguished subsets $\partial_{\pm}V$ of the boundary ∂V . It is called *proper* if each $\partial_{\pm}V$ is a union of connected components of the boundary and $\partial V = \partial_-V \sqcup \partial_+V$. If V is a Riemannian manifold, we define the width of V as $\mathrm{width}(V) := \mathrm{dist}(\partial_-V, \partial_+V)$.

In order to define a generalized Callias-type operator in this setting, we proceed as in Subsection 5.1 and consider the open manifold $V^\circ := V \setminus \partial V$. The metric g induces an incomplete metric on V° , that we denote by the same symbol. Given a collar neighborhood $U_+ \subset V$ of ∂_+V , we say that $U_+^\circ := U_+ \setminus \partial_+V \subset V^\circ$ is a neighborhood of the positive boundary at infinity of V° . In a similar way, define the notion of a neighborhood of the negative boundary at infinity of V° .

DEFINITION 6.6. A Callias potential ψ on (V°, g) is called band compatible if there exist constants λ_- and λ_+ , with $\lambda_- < 0 < \lambda_+$, such that the image of ψ is contained in $[\lambda_-, \lambda_+]$, $\psi = \lambda_-$ in a neighborhood of the negative boundary at infinity of V° and $\psi = \lambda_+$ in a neighborhood of the positive boundary at infinity of V° .

Let (E, ∇^E) be a bundle of finitely generated projective Real Hilbert A -modules with inner product and metric connection. We now study the properties of the generalized Callias-type operator $B_{\rho, \psi}$, where ρ is an admissible rescaling function and ψ is a band compatible Callias potential.

LEMMA 6.7. Let ρ be an admissible rescaling function. Suppose ψ_1 and ψ_2 are two band compatible Callias potentials. Then the indices of B_{ρ, ψ_1} and B_{ρ, ψ_2} coincide.

Proof. Consider the linear homotopy $\psi_t = t\psi_1 + (1 - t)\psi_2$, with $0 \leq t \leq 1$. Since ψ_1 and ψ_2 are band compatible, ψ_t is a band compatible Callias potential for all $t \in [0, 1]$. The thesis follows by arguing as in the proof of Lemma 4.12. \square

Theorem 6.8. Let ρ be an admissible rescaling function and let $\psi: V^\circ \rightarrow [\lambda_-, \lambda_+]$ be a band compatible Callias potential. If $a \in (\lambda_-, \lambda_+)$ is a regular value of ψ , then

$$\text{index}(B_{\rho, \phi}) = \text{index}\left(\mathcal{D}_{\phi^{-1}(a), E|_{\phi^{-1}(a)}}\right) \in \text{KO}_{n-1}(A).$$

Proof. Let $U \cong \partial V \times [0, 1)$ be a collar neighborhood of ∂V such that ψ is constant on $U \setminus \partial V$. Using Part (a) of Proposition 6.5, assume there exists a constant $\rho_0 \in (0, 1]$ such that $\rho = \rho_0$ on the complement of $U' \cong \partial V \times [0, 1/4)$. Observe that the manifold $N \cong \partial V \times \{1/2\}$ is a closed separating hypersurface of V° . Moreover, $N = N_+ \sqcup N_-$, where $N_+ \simeq \partial_+ V$ and $N_- \cong \partial_- V$. Therefore, we have the partition

$$V^\circ = Y \cup_N W.$$

Here, Y is a compact manifold with $\partial Y = N_+ \sqcup N_-$ and $W = W_- \sqcup W_+$, where W_- is a neighborhood of the negative boundary at infinity with $\partial W_- \cong N_-$ and W_+ is a neighborhood of the positive boundary at infinity with $\partial W_+ \cong N_+$. Let us assume (after deformation near N) that our data respect the product structure of a tubular neighborhood of N where the function ψ is constant. Consider the half-cylinders

$$Z_- := (-\infty, 0] \times N_- \quad \text{and} \quad Z_+ := [0, \infty) \times N_+.$$

Observe that the bundles $E|_{N_\pm}$ extend to bundles E_{Z_\pm} with metric connections on Z_\pm . Consider the manifolds

$$(M_2)_- := Z_- \cup_{N_-} W_-^- \quad (M_2)_+ := W_+^- \cup_{N_+} Z_+ \quad M_2 := (M_2)_- \sqcup (M_2)_+$$

where W_-^- and W_+^- denote respectively the manifolds W_- and W_+ with opposite orientations. Let g_2 be the Riemannian metric coinciding with g on W_\pm^- and being a product on the half cylinders Z_\pm . Let E_2 be the bundle with metric connection coinciding with E on $W_-^- \sqcup W_+^-$ and with E_{Z_\pm} respectively on Z_\pm . Let $\rho_2^+ : M_2^+ \rightarrow (0, 1]$

be the smooth function coinciding with ρ on W_+^- and with ρ_0 on Z_+ . Observe that ρ_2^+ is admissible, since the metric g_2 is complete on the cylindrical end Z_+ . Finally, let μ_+ be a positive constant such that $\lambda_+^2 > \mu_+$. Then there exists a smooth function $\psi_2^+ : (M_2)_+ \rightarrow [\lambda_+, \infty)$ such that

- (i) $\psi_2^+ = \lambda_+$ on a neighborhood of W_+^- ;
- (ii) $\psi_2^+(t, x) \geq \mu_+ t$ for all $(t, x) \in [0, \infty) \times N_+ \subset Z_+$;
- (iii) $|\mathrm{d}\psi_2^+| \leq \mu_+$.

By Properties (i) and (ii), ψ_2^+ is a Callias potential. Let $B_{\rho_2^+, \psi_2^+}$ be the associated generalized Callias-type operator. Since $\rho_2^+ \leq 1$ and $(\psi_2^+)^2 \geq (\lambda_+)^2 > \mu_+$, Properties (i)–(iii) imply that the function $(\psi_2^+)^2 - |\mathrm{d}\psi_2^+|$ is uniformly positive on $(M_2)_+$. By Inequality (6.3), we deduce that the index class of $B_{\rho_2^+, \psi_2^+}$ vanishes. Finally, observe that ψ_2^+ is proper with uniformly bounded gradient on the cylindrical end Z_+ .

In a similar way, construct an admissible rescaling function ρ_2^- and a Callias potential $\psi_2^- : (M_2)_- \rightarrow (-\infty, \lambda_-]$ such that the index of the associated generalized Callias-type operator $B_{\rho_2^-, \psi_2^-}$ vanishes and the function ψ_2^- is proper with uniformly bounded gradient on the cylindrical end Z_- . Finally, observe that ρ_2^\pm induce an admissible rescaling function ρ_2 on M_2 and ψ_2^\pm induce a Callias potential ψ_2 on M_2 . Since $M_2 = M_2^- \sqcup M_2^+$, the index of the associated operator B_{ρ_2, ψ_2} vanishes.

Observe that the manifolds V° and M_2 satisfy Assumption 2.11. Using the cut-and-paste construction described in Subsection 2.3, we obtain Riemannian manifolds

$$M_3 := Z_- \cup_{N_-} Y \cup_{N_+} Z_+ \quad \text{and} \quad M_4 := W \cup_N W^-$$

and generalized Callias-type operators B_{ρ_3, ψ_3} and B_{ρ_4, ψ_4} respectively on M_3 and M_4 . Notice that the potential ψ_4 is constant and nonzero on both connected components of B_{ρ_4, ψ_4} . Therefore, the index of B_{ρ_4, ψ_4} vanishes by Part (c) of Proposition 6.5.

Let us now analyze the operator B_{ρ_3, ψ_3} . By construction, ψ_3 is a proper smooth function whose gradient is uniformly bounded. Moreover, since $a \in (\lambda_-, \lambda_+)$, $\psi_3^{-1}(a) = \psi^{-1}(a)$. Using Remark 6.4 and [Zei19, Theorem A.1], the indices of B_{ρ_3, ψ_3} and $\mathcal{D}_{\phi^{-1}(a), E|_{\phi^{-1}(a)}}$ coincide. Therefore, using Theorem 2.12, we obtain

$$\begin{aligned} \text{index}(B_{\rho, \psi}) &= \text{index}(B_{\rho, \psi}) + \text{index}(B_{\rho_2, \psi_2}) \\ &= \text{index}(B_{\rho_3, \psi_3}) + \text{index}(B_{\rho_4, \psi_4}) = \text{index}\left(\mathcal{D}_{\phi^{-1}(a), E|_{\phi^{-1}(a)}}\right), \end{aligned}$$

which concludes the proof. □

We finally specialize to the case when (V, g) is a Riemannian band over a closed spin manifold N , i.e. V is diffeomorphic to $N \times [-1, 1]$. Let $(\mathcal{L}_V, \nabla^{\mathcal{L}_V})$ be the Mishchenko bundle of V endowed with the canonical flat connection. For an admissible rescaling function ρ and a band compatible Callias potential ψ , denote by $B_{\rho, \psi}^{\mathcal{L}_V}$ the generalized Callias-type operator associated to these data.

COROLLARY 6.9. *Let N be a closed $(n - 1)$ -dimensional spin manifold with fundamental group Γ . Let (V, g) be a Riemannian spin band over N . For an admissible rescaling function ρ and a band compatible Callias potential ψ , we have*

$$\text{index} \left(B_{\rho, \psi}^{\mathcal{L}_V} \right) = \alpha(N) \in \text{KO}_{n-1}(C^*\Gamma). \tag{6.4}$$

Proof. For $\lambda > 0$ small enough, let $\psi_1: V \rightarrow [-\lambda, \lambda]$ be a smooth function such that 0 is a regular value, $\psi_1^{-1}(0) = N \times \{0\}$, the support of $d\psi_1$ is contained in the interior of a closed geodesic tubular neighborhood of $N \times \{0\}$, $\psi_1 = \lambda$ in a neighborhood of $\partial_+ V$, and $\psi_1 = -\lambda$ in a neighborhood of $\partial_- V$. Observe that ψ_1 is a band compatible Callias potential. Since the inclusion $N \hookrightarrow V$ induces an isomorphism on π_1 , Identity (6.4) follows from Lemma 6.7 and Theorem 6.8. \square

6.3 Proof of Theorem D. Suppose

$$\text{width}(V) > 2\pi \sqrt{\frac{n-1}{\sigma n}}. \tag{6.5}$$

In order to prove the thesis, we need to show that $\alpha(N) = 0$.

For an admissible rescaling function ρ and a band compatible Callias potential ψ , consider the generalized Callias-type operator $B_{\rho, \psi}^{\mathcal{L}_V}$ used in Corollary 6.9. Let $\Psi_{\rho, \phi}^\omega: V^\circ \rightarrow \mathbb{R}$ be the smooth function defined by the formula

$$\Psi_{\rho, \phi}^\omega(x) := \frac{\bar{n}\bar{\sigma}\omega}{1 + \omega} \rho^4(x) - \omega \rho^2(x) |d\rho_x|^2 + \psi^2(x) - \rho^2(x) |d\psi_x|, \quad x \in V^\circ. \tag{6.6}$$

Here, we use the notation $\bar{n} = n/(n - 1)$ and $\bar{\sigma} = \sigma/4$ introduced respectively in Sections 3 and 5. Since the bundle \mathcal{L}_V is flat and $\text{scal}_g \geq \sigma$, from Proposition 3.8 and Identity (6.2) we deduce

$$\left\langle (B_{\rho, \phi}^{\mathcal{L}_V})^2 w, w \right\rangle \geq \left\langle \Psi_{\rho, \phi}^\omega w, w \right\rangle, \tag{6.7}$$

for every $w \in \Gamma_c(V^\circ; \mathfrak{S}_M \hat{\otimes} \mathcal{L}_V \hat{\otimes} \text{Cl}_{0,1})$ and every $\omega > 0$. By Corollary 6.9 and Inequality (6.7), it suffices to show that there exist an admissible rescaling function ρ , a band compatible Callias potential ψ and positive constants ω, c such that

$$\Psi_{\rho, \psi}^\omega(x) \geq c, \quad x \in V^\circ. \tag{6.8}$$

To this end, we will use Lemma 5.6 in a similar way as in the proof of Theorem 5.2.

Using Condition (6.5), choose a constant Λ satisfying

$$\frac{2\pi}{\sqrt{\bar{n}\sigma}} < 2\Lambda < \text{width}(V). \tag{6.9}$$

Pick a constant $\omega > 0$ and smooth functions $Z: (0, \infty) \rightarrow (0, \infty)$ and $Y: (0, \infty) \rightarrow [0, \infty)$ satisfying Conditions (a)–(e) of Lemma 5.6.

Let τ^+ be the distance function from ∂_+V and let τ^- be the distance function from ∂_-V . We regard τ^\pm as functions on V° . Fix a constant $\delta > 0$ and consider the functions μ^+ and μ^- defined as

$$\mu^\pm(x) := \min(\tau^\pm(x)/2, \delta), \quad x \in V^\circ.$$

By [GW79, Proposition 2.1], there exist smooth functions τ_δ^+ and τ_δ^- on V° such that

$$|\tau^\pm(x) - \tau_\delta^\pm(x)| < \mu^\pm(x), \quad \forall x \in V^\circ \tag{6.10}$$

and

$$\|d\tau_\delta^\pm\|_\infty \leq 1 + \delta. \tag{6.11}$$

Choose constants Λ_1, Λ_2 such that $\Lambda < \Lambda_1 < \Lambda_2 < \text{width}(V)/2$ and consider the open sets

$$\Omega_\pm := \{x \in V^\circ \mid \tau_\delta^\pm(x) < \Lambda_2\}.$$

By taking δ small enough, $\bar{\Omega}_- \cap \bar{\Omega}_+ = \emptyset$. Define the set

$$\Omega := \{x \in V^\circ \mid \min(\tau_\delta^+(x), \tau_\delta^-(x)) > \Lambda_1\}.$$

Let τ_δ be a smooth function on V° such that $\tau_\delta = \tau_\delta^-$ on Ω_- , $\tau_\delta = \tau_\delta^+$ in Ω_+ and $\tau_\delta(x) \geq \Lambda_1$ when $x \in \Omega$. Observe that Properties (6.10) and (6.11) imply that τ_δ is positive and goes to 0 at infinity.

Define the smooth function $\rho: V^\circ \rightarrow (0, 1]$ by setting

$$\rho := Z \circ \tau_\delta. \tag{6.12}$$

By Proposition 3.5, Remark 3.2 and Property (a) of Lemma 5.6, ρ is an admissible rescaling function. Moreover, by Property (b) of Lemma 5.6, we have

$$\rho = \rho_0 \text{ on } \Omega, \text{ for some constant } \rho_0 \in (0, 1]. \tag{6.13}$$

Define a function $\psi: V^\circ \rightarrow [0, \infty)$ by setting

$$\psi = \pm Y \circ \tau_\delta \quad \text{on } \Omega_\pm \quad \text{and} \quad \psi = 0 \quad \text{on } \Omega. \tag{6.14}$$

By Properties (c) and (d) of Lemma 5.6, ψ is well defined and is a band compatible Callias potential. Using (6.13) and (6.14), we deduce

$$\Psi_{\rho, \psi}^\omega(x) = \frac{\bar{n}\bar{\sigma}\omega}{1 + \omega} \rho_0^4 > 0, \quad x \in \Omega. \tag{6.15}$$

Therefore, Inequality (6.8) holds on Ω .

In order to complete the proof, we will show that, for δ small enough, Inequality (6.8) holds on Ω_+ and Ω_- as well. Using (6.11), (6.12) and (6.14) we have

$$|d\rho_x| \leq (1 + \delta) |Z'(\tau_\delta(x))| \quad \text{and} \quad |d\psi_x| \leq (1 + \delta) |Y'(\tau_\delta(x))|$$

for every $x \in \Omega_- \cup \Omega_+$. By taking δ small enough and using Property (e) of Lemma 5.6, we deduce that there exists a constant $c_1 > 0$ such that

$$\begin{aligned} \Psi_{\rho,\phi}^\omega(x) &= \frac{\bar{n}\bar{\sigma}\omega}{1+\omega} \rho^4(x) - \omega \rho^2(x) |d\rho_x|^2 + \psi^2(x) - \rho^2(x) |d\psi_x| \\ &\geq \frac{\bar{n}\bar{\sigma}\omega}{1+\omega} Z^4(\tau_\delta(x)) - \omega Z^2(\tau_\delta(x))(1+\delta)^2 |Z'(\tau_\delta(x))|^2 \\ &\quad + Y^2(\tau_\delta(x)) - Z^2(\tau_\delta(x))(1+\delta) |Y'(\tau_\delta(x))| \geq c_1 \end{aligned}$$

for every $x \in \Omega_- \cup \Omega_+$. This concludes the proof. \square

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References

- [APS75a] M. F. ATIYAH, V. K. PATODI, and I. M. SINGER. Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Cambridge Philos. Soc.*, 77:43–69, 1975.
- [APS75b] M. F. ATIYAH, V. K. PATODI, and I. M. SINGER. Spectral asymmetry and Riemannian geometry. II. *Math. Proc. Cambridge Philos. Soc.*, 78(3):405–432, 1975.
- [APS76] M. F. ATIYAH, V. K. PATODI, and I. M. SINGER. Spectral asymmetry and Riemannian geometry. III. *Math. Proc. Cambridge Philos. Soc.*, 79(1):71–99, 1976.
- [CB18] M. BRAVERMAN and S. CECCHINI. Callias-type operators in von Neumann algebras. *J. Geom. Anal.*, 28(1):546–586, 2018.
- [Bun95] U. BUNKE. A K -theoretic relative index theorem and Callias-type Dirac operators. *Math. Ann.*, 303(2):241–279, 1995.

- [Bär09] C. BÄR. Spectral bounds for Dirac operators on open manifolds. *Ann. Global Anal. Geom.*, 36(1):67–79, 2009.
- [Cec18] S. CECCHINI. Callias-type operators in C^* -algebras and positive scalar curvature on noncompact manifolds. *J. Topol. Anal.*, Advance online publication (2018). <https://doi.org/10.1142/S1793525319500687>
- [CS19] S. CECCHINI and T. SCHICK. Enlargeable metrics on nonspin manifolds. *Proc. Amer. Math. Soc.*, Advance online publication (2019). <https://doi.org/10.1090/proc/14706>
- [Ebe16] J. EBERT. Elliptic regularity for Dirac operators on families of noncompact manifolds (2016). *arXiv e-prints* [arXiv:1608.01699](https://arxiv.org/abs/1608.01699)
- [Fri80] T. FRIEDRICH. Der erste Eigenwert des Dirac-Operators einer kompakten, Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung. *Math. Nachr.*, 97:117–146, 1980.
- [GL80] M. GROMOV and H. B. LAWSON. The classification of simply connected manifolds of positive scalar curvature. *Ann. of Math.*, 111(3):423–434, 1980.
- [GL83] M. GROMOV and H. B. LAWSON. Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Inst. Hautes Études Sci. Publ. Math.*, 58:83–196, 1983.
- [Gro18] M. GROMOV. Metric inequalities with scalar curvature. *Geom. Funct. Anal.*, 28(3):645–726, 2018.
- [Gro19] M. GROMOV. Four Lectures on Scalar Curvature (2019). *arXiv e-prints* [arXiv:1908.10612](https://arxiv.org/abs/1908.10612)
- [GW79] R. E. GREENE and H. WU. C^∞ approximations of convex, subharmonic, and plurisubharmonic functions. *Ann. Sci. École Norm. Sup. (4)*, 12(1):47–84 (1979).
- [HR00] N. HIGSON and J. ROE. *Analytic K-homology*. Oxford Mathematical Monographs. Oxford University Press, Oxford (2000).
- [HS06] B. HANKE and T. SCHICK. Enlargeability and index theory. *J. Differential Geom.*, 74(2):293–320, 2006.
- [HS07] B. HANKE and T. SCHICK. Enlargeability and index theory: infinite covers. *K-Theory*, 38(1):23–33, 2007.
- [KW75a] J. L. KAZDAN and F. W. WARNER. A direct approach to the determination of Gaussian and scalar curvature functions. *Invent. Math.*, 28:227–230, 1975.
- [KW75b] J. L. KAZDAN and F. W. WARNER. Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures. *Ann. of Math.*, 101:317–331, 1975.
- [KW75c] J. L. KAZDAN and F. W. WARNER. Scalar curvature and conformal deformation of Riemannian structure. *J. Differential Geom.*, 10:113–134, 1975.
- [Lan95] E. C. LANCE. *Hilbert C^* -modules*. volume 210 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge. A toolkit for operator algebras (1995).
- [Lic63] A. LICHTNEROWICZ. Spineurs harmoniques. *C. R. Acad. Sci. Paris*, 257:7–9, 1963.
- [Lla98] M. LLARULL. Sharp estimates and the Dirac operator. *Math. Ann.*, 310(1):55–71, 1998.
- [LM89] H. B. LAWSON JR. and M.-L. MICHELSON. *Spin geometry*. volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ (1995).
- [Ros83] J. ROSENBERG. C^* -algebras, positive scalar curvature, and the Novikov conjecture. *Inst. Hautes Études Sci. Publ. Math.*, 58:197–212, 1983.

- [Ros86a] J. ROSENBERG. C^* -algebras, positive scalar curvature and the Novikov conjecture. II. In *Geometric methods in operator algebras (Kyoto, 1983)*, volume 123 of *Pitman Res. Notes Math. Ser.*, Longman Sci. Tech., Harlow, pp. 341–374 (1986).
- [Ros86b] J. ROSENBERG. C^* -algebras, positive scalar curvature, and the Novikov conjecture. III. *Topology*, 25(3):319–336 (1986).
- [Ros07] J. ROSENBERG. Manifolds of positive scalar curvature: a progress report. In *Surveys in differential geometry. Vol. XI*, volume 11 of *Surv. Differ. Geom.*, Int. Press, Somerville, MA, pp. 259–294 (2007).
- [Sch93] H. SCHRÖDER. *K-theory for real C^* -algebras and applications*, volume 290 of *Pitman Res. Notes Math. Ser.*. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York (1993).
- [Sch05] T. SCHICK. L^2 -index theorems, KK -theory, and connections. *New York J. Math.*, 11:387–443, 2005.
- [Sto02] S. STOLZ. Manifolds of positive scalar curvature. In *Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001)*, volume 9 of *ICTP Lect. Notes*, Abdus Salam Int. Cent. Theoret. Phys., Trieste, pp. 661–709 (2002).
- [SY79] R. SCHOEN and S. T. YAU. On the structure of manifolds with positive scalar curvature. *Manuscripta Math.*, 28(1-3):159–183, 1979.
- [WO93] N. E. WEGGE-OLSEN. *K-theory and C^* -algebras*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York (1993).
- [Zei19] R. ZEIDLER. Band width estimates via the Dirac operator. *J. Differential Geom.* (to appear) (2019).
- [Zha19] W. ZHANG. Nonnegative scalar curvature and area decreasing maps (2019). *arXiv e-prints* [arXiv:1912.03649](https://arxiv.org/abs/1912.03649)

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