Continuous LTI Input–Output Stable Systems on $L^p(\mathbb{R})$ and $\mathcal{D}'_{L^p}(\mathbb{R})$ Associated with Differential Equations: Existence, Invertibility Conditions and Inversion

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Received: 13 August 2020 / Revised: 22 February 2021 / Accepted: 24 February 2021 /
Published online: 18 March 2021
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Abstract

A usual problem in analog signal processing is to ascertain the existence of a continuous single-input single-output linear time-invariant input–output stable system associated with a linear differential equation, i.e., of a continuous system such that, for every input signal in a given space of signals, yields an output, in the same space, which verifies the equation with known term the input, and to ascertain the existence of its inverse system. In this paper, we consider, as space of signals, the usual Banach space of L^p functions, or the space of distributions spanned by L^p functions and by their distributional derivatives, of any order (input spaces which include signals with not necessarily left-bounded support), we give a systematic theoretical analysis of the existence, uniqueness and invertibility of continuous linear time-invariant input–output stable systems (both causal and non-causal ones) associated with the differential equation and, in case of invertibility, we characterize the continuous inverse system. We also give necessary and sufficient conditions for causality. As an application, we consider the problem of finding a suitable *almost inverse* of a causal continuous linear time-invariant input–output stable *non-invertible* system, defined on the space of finite-energy functions, associated with a simple differential equation.

Keywords Analog signal processing · Mathematical foundations · Continuous linear time-invariant systems · Input–output stability · Inverse system · Distributional signals

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1 Introduction

An analog signal defined on $\mathbb{R}$ is, in this paper, a complex-valued distribution on $\mathbb{R}$, i.e., a member of the space $\mathcal{D}'(\mathbb{R})$.¹ This class includes usual signals like continuous functions and L^p functions, as well as less usual signals like distributional derivatives of any order of continuous functions and of L^p functions ($1 \leq p \leq \infty$).

A *space of analog signals* is a linear subspace $\mathcal{X}$ of $\mathcal{D}'(\mathbb{R})$ closed under translation and endowed with a notion of convergence and limit (denoted $\mathcal{X} - \lim$) for sequences, such that for every $f(t) \in \mathcal{X}$ and every sequence $f_k(t) \in \mathcal{X}$ we have

$$\mathcal{X} - \lim_{k \rightarrow \infty} f_k = f \quad \Rightarrow \quad \mathcal{D}' - \lim_{k \rightarrow \infty} f_k = f$$

Given two spaces of analog signals, $\mathcal{I}$ and $\mathcal{O}$, a (single-input single-output) system with input space $\mathcal{I}$ and output space $\mathcal{O}$ is, in this paper, a (single-valued) map:

$$\mathcal{L} : \mathcal{I} \rightarrow \mathcal{O}$$

As usual, a system $\mathcal{L} : \mathcal{I} \rightarrow \mathcal{O}$ is *linear time-invariant* (LTI) if the map $\mathcal{L}$ is linear and commutes with translations, and it is *continuous* if the map $\mathcal{L}$ is continuous, i.e., for every $f(t) \in \mathcal{I}$ and every sequence $f_k(t) \in \mathcal{I}$ it is:

$$\mathcal{I} - \lim_{k \rightarrow \infty} f_k = f \quad \Rightarrow \quad \mathcal{O} - \lim_{k \rightarrow \infty} \mathcal{L}(f_k) = \mathcal{L}(f)$$

Also as usual, a system $\mathcal{L} : \mathcal{I} \rightarrow \mathcal{O}$ is *invertible* if there exists an (obviously unique) *inverse system* $\mathcal{L}^{-1} : \mathcal{O} \rightarrow \mathcal{I}$ such that for every signal $f(t) \in \mathcal{I}$ it is:

$$\mathcal{L}^{-1}(\mathcal{L}(f)) = f$$

and for every signal $g(t) \in \mathcal{O}$ it is:

$$\mathcal{L}(\mathcal{L}^{-1}(g)) = g$$

and it is *causal* if for every $t_0 \in \mathbb{R}$ and every $f(t), g(t) \in \mathcal{I}$ it is:

$$\text{supp}(f - g) \subset [t_0, +\infty) \quad \Rightarrow \quad \text{supp}(\mathcal{L}(f) - \mathcal{L}(g)) \subset [t_0, +\infty)$$

We will say that a system $\mathcal{L} : \mathcal{I} \rightarrow \mathcal{O}$ is *input-output stable* (IOS) if $\mathcal{I} = \mathcal{O}$. In such case, the system $\mathcal{L} : \mathcal{I} \rightarrow \mathcal{I}$ will be called an *IOS system on $\mathcal{I}$* . Hence, a continuous IOS system $\mathcal{L}$ on $\mathcal{I}$ is such that for every $f(t) \in \mathcal{I}$ and every sequence $f_k(t) \in \mathcal{I}$ it is:

$$\mathcal{I} - \lim_{k \rightarrow \infty} f_k = f \quad \Rightarrow \quad \mathcal{I} - \lim_{k \rightarrow \infty} \mathcal{L}(f_k) = \mathcal{L}(f)$$

¹ For the reader's convenience, we collected in the Appendix, at the end of the paper, some notions on distributions used in the paper.

Moreover, we will say that a system $\mathcal{L} : \mathcal{I} \rightarrow \mathcal{O}$ is *bicontinuous invertible* if $\mathcal{L}$ is continuous, invertible, and the inverse system $\mathcal{L}^{-1} : \mathcal{O} \rightarrow \mathcal{I}$ is continuous.

A system $\mathcal{L} : \mathcal{I} \rightarrow \mathcal{O}$ with $\mathcal{O} = \mathcal{D}'(\mathbb{R})$, i.e., a system whose output space is the largest space of signals, will be referred to as *a system defined on $\mathcal{I}$* , letting that the expression implies that the output space is $\mathcal{D}'(\mathbb{R})$. Hence, a continuous system $\mathcal{L}$ defined on $\mathcal{I}$ is such that for every $f(t) \in \mathcal{I}$ and every sequence $f_k(t) \in \mathcal{I}$ it is:

$$\mathcal{I} - \lim_{k \rightarrow \infty} f_k = f \quad \Rightarrow \quad \mathcal{D}' - \lim_{k \rightarrow \infty} \mathcal{L}(f_k) = \mathcal{L}(f)$$

Finally, let

$$P(z) = a_n z^n + \cdots + a_0 z^0 \quad \text{and} \quad Q(z) = b_m z^m + \cdots + b_0 z^0$$

be two polynomials with coefficients in $\mathbb{C}$ such that $n \geq 0, m \geq 0, a_n \neq 0$ and $b_m \neq 0$, and let:²

$$P(D)x(t) = Q(D)f(t) \quad (1)$$

i.e.,

$$\sum_{k=0}^n a_k D^k x(t) = \sum_{k=0}^m b_k D^k f(t)$$

be the linear differential equation with constant coefficients defined by P and Q , in which $f(t) \in \mathcal{D}'(\mathbb{R})$ is a given signal and $x(t) \in \mathcal{D}'(\mathbb{R})$ is the unknown signal.

We will say that a system $\mathcal{L} : \mathcal{I} \rightarrow \mathcal{O}$ is *associated with the differential equation (1)* if:

$$\text{for every } f \in \mathcal{I} \text{ it is : } P(D)\mathcal{L}(f) = Q(D)f$$

i.e., if for every $f(t) \in \mathcal{I}$ the output $\mathcal{L}(f)$ is a solution in $\mathcal{O}$ of the differential equation (1).³

Let $\mathcal{I}$ be a given space of input signals. In this paper, we shall be concerned with the problem to find the continuous LTI IOS (causal as well as non-causal) systems on $\mathcal{I}$ associated with the differential equation (1) and, among them, to find the bicontinuous invertible ones.

LTI systems associated with an ordinary differential equation with constant coefficients are ubiquitous in engineering. Observe that we do not pose any restriction on the *degrees* of the polynomials defining the differential equation. Sometimes the restriction $\deg Q(z) < \deg P(z)$ or $\deg Q(z) \leq \deg P(z)$ is introduced on the basis of

² For every $n \in \mathbb{N}$ and $f(t) \in \mathcal{D}'(\mathbb{R})$, we indicate by $D^n f(t) \in \mathcal{D}'(\mathbb{R})$ the n -th distributional derivative of $f(t)$, see Sect. A-5 of the Appendix.

³ Since the elements of both $\mathcal{I}$ and $\mathcal{O}$ are distributions, $\mathcal{L}(f)$ is a *distributional solution* of the differential equation (1). For the existence and uniqueness of distributional solutions of a linear differential equation with constant coefficients, see Sect. A-12 of the Appendix.

some state-space realizability criteria,⁴ but we will not be concerned with state-space realizations. We also will find some restrictions on the degrees on the basis of the kind of input–output stability considered.

We restrict our attention to *continuous* LTI systems because continuity is an *essential* theoretical property, always *implicitly* assumed (see [17, Comment on p. 82, Ch. 5, Sect. 5-1] and [13, Sect. 1.1]).⁵

Input–output stable systems are implicitly commonly used. For example, a usual BIBO-stable system, i.e., a system which maps every bounded function into a bounded function, canonically defines an IOS system on $L^\infty(\mathbb{R})$. Analogously, a usual system which maps finite-energy signals into finite-energy signals, canonically defines an IOS system on $L^2(\mathbb{R})$, and a usual system which maps signals with left-bounded support into signals with left-bounded support, canonically defines an IOS system on $\mathcal{D}'_+(\mathbb{R})$.

The problem to find inverses of LTI IOS systems is encountered in many applications (e.g., channel equalization [19, Section 8.4.2, p. 579], deconvolution [10, Section 8.5.2], chaotic systems synchronization [15], design of compensators for measuring systems [16, Problem 4.52] and analog realization of inverse wavelet transform [3]). In particular, *non-causal* systems are needed to invert causal systems defined on spaces of signals with not necessarily left-bounded support [25].

The problem to find inverses of continuous LTI IOS systems on $\mathcal{D}'_+(\mathbb{R})$, i.e., in the natural context in which only *causal systems* and *signals with left-bounded support* are of interest, is easily solved through the classical representation as convolution of the continuous LTI IOS systems on $\mathcal{D}'_+(\mathbb{R})$ (see [29, Sect. 9.3 and 5.6]).

Precisely: for every $P(z)$ and $Q(z)$ there exists a *unique* continuous LTI IOS system $\mathcal{L}$ associated with the differential equation (1). The system $\mathcal{L}$ is a bicontinuous-invertible LTI IOS system, and the inverse system $\mathcal{L}^{-1}$ is the unique continuous LTI IOS system associated with the differential equation $Q(D)x(t) = P(D)g(t)$. The system $\mathcal{L}$ is defined by $\mathcal{L}(f) = h * f$, where $h(t)$, the impulse response of the system, is the inverse (unilateral) Laplace transform of the *transfer function* $Q(s)/P(s)$, and the system $\mathcal{L}^{-1}$ is defined by $\mathcal{L}^{-1}(g) = k * g$ where $k(t)$ is the inverse (unilateral) Laplace transform of the inverse transfer function $P(s)/Q(s)$.

The problem to find inverses of LTI IOS systems on the usual Banach spaces $L^p(\mathbb{R})$, and on their distributional generalizations $\mathcal{D}'_{L^p}(\mathbb{R})$, is very heavy and constitutes the main topic of this paper.

Concerning these two classes of spaces, *no complete theory* is available, and only some partial results are known.

In [9, Lesson 24 and Section 34.3], some *sufficient* conditions for the existence of a unique continuous LTI system mapping $L^p(\mathbb{R})$ into itself associated with the differential equation (1) are given. Precisely, it is proved that: if the polynomial $P(z)$ has no zero on the complex imaginary axis, and $\deg Q(z) \leq \deg P(z)$, then there exists

⁴ For example, in classical LTI state-space systems theory, an input–output description of the form $P(D)x(t) = Q(D)f(t)$ is *realizable* if and only if $\deg Q(z) \leq \deg P(z)$ [1, Th. 3.3, p. 391] (see also [11], where *minimal realizations* are discussed). However, when also *generalized state-space realizations* are taken into considerations, *any* input–output description of the form $P(D)x(t) = Q(D)f(t)$ is *realizable* [27, Sect. 3.1] (see also [12], where *minimal realizations* are discussed).

⁵ In [20] and [22], I. Sandberg gave some unexpected results that proved how a careless use of the notion of continuity may lead to the spread of unfounded beliefs in linear system theory.

a unique continuous LTI system mapping $L^1(\mathbb{R})$ (resp.: $L^2(\mathbb{R})$, $L^\infty(\mathbb{R})$) into itself associated with the differential equation (1). The system is that which maps any signal $f(t)$, whose Fourier transform is $\hat{f}(\omega)$, into the signal $y(t)$ whose Fourier transform is

$$\hat{y}(\omega) = \frac{Q(i\omega)}{P(i\omega)} \hat{f}(\omega)$$

Concerning LTI systems defined on $\mathcal{D}'_{L^p}(\mathbb{R})$ associated with differential equations, to the author's knowledge no paper besides [6] reports any result. Observe, however, that the space $\mathcal{D}'_{L^p}(\mathbb{R})$, i.e., the subspace of $\mathcal{D}'(\mathbb{R})$ spanned by $L^p(\mathbb{R})$ itself and by the distributional derivatives, of any order, of its members, is the minimal subspace of $\mathcal{D}'(\mathbb{R})$ which includes $L^p(\mathbb{R})$ and is closed with respect to differentiation, and hence that it is *the natural signal space* to be taken into consideration when studying systems associated with the differential equation (1) and $L^p(\mathbb{R})$ input signals are allowed.

The problem to find linear IOS systems has been faced also for systems associated to different kinds of equations. In [7], the author considers LTI systems associated with singular delay-differential equations with constant coefficients and discrete delays and discusses the existence of related LTI IOS systems on a certain class of functions with left-bounded support. In [8], the author considers linear time-varying systems associated with switched delay-differential equations with constant coefficients and discrete delays and discusses the existence of related linear time-varying IOS systems on $L^2(\mathbb{R})$.

In this paper, the input space $\mathcal{I}$ taken into consideration is always:

- either the usual Banach space $L^p(\mathbb{R})$, endowed with the notion of convergence and limit (denoted $L^p\text{-lim}$) for sequences induced by the usual norm $\|\cdot\|_p$
- or the space $\mathcal{D}'_{L^p}(\mathbb{R})$ endowed either with the notion of *weak*, or with the notion of *strong* convergence, and the corresponding notions of limit (denoted $w\text{-}\mathcal{D}'_{L^p}\text{-lim}$ and $s\text{-}\mathcal{D}'_{L^p}\text{-lim}$, respectively) for sequences,⁶

(all these input spaces include also signals whose support is not left-bounded) and we give a systematic theoretical analysis of the existence, uniqueness and bicontinuous invertibility of continuous (causal and non-causal) LTI IOS systems

$$\mathcal{L} : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R}), \quad \mathcal{L} : \mathcal{D}'_{L^p}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^p}(\mathbb{R}) \quad (1 \leq p \leq \infty)$$

associated with the differential equation (1) and, in case of bicontinuous invertibility, we characterize the inverse system.

Since two different notions of convergence (weak and strong) can be considered in $\mathcal{D}'_{L^p}(\mathbb{R})$, we will say that a system $\mathcal{L} : \mathcal{D}'_{L^p}(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R})$ defined on $\mathcal{D}'_{L^p}(\mathbb{R})$ is *w-continuous* (resp. *s-continuous*) when it is continuous with respect to the weak convergence (resp. with respect to the strong convergence) in $\mathcal{D}'_{L^p}(\mathbb{R})$, and to the convergence in $\mathcal{D}'(\mathbb{R})$, i.e., when for every $f(t) \in \mathcal{D}'_{L^p}(\mathbb{R})$ and every sequence

⁶ See Sect. A-9 of the Appendix.

$f_k(t) \in \mathcal{D}'_{L^p}(\mathbb{R})$ we have:

$$\begin{aligned} w\text{-}\mathcal{D}'_{L^p}\text{-}\lim_{k \rightarrow \infty} f_k = f &\Rightarrow \mathcal{D}'\text{-}\lim_{k \rightarrow \infty} \mathcal{L}(f_k) = \mathcal{L}(f) \\ (\text{resp. } s\text{-}\mathcal{D}'_{L^p}\text{-}\lim_{k \rightarrow \infty} f_k = f &\Rightarrow \mathcal{D}'\text{-}\lim_{k \rightarrow \infty} \mathcal{L}(f_k) = \mathcal{L}(f)) \end{aligned}$$

Analogously, we will say that an IOS system $\mathcal{L} : \mathcal{D}'_{L^p}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^p}(\mathbb{R})$ is *w-continuous* (resp. *s-continuous*) when it is continuous with respect to the weak convergence (resp. with respect to the strong convergence) in $\mathcal{D}'_{L^p}(\mathbb{R})$, i.e., when for every $f(t) \in \mathcal{D}'_{L^p}(\mathbb{R})$ and every sequence $f_k(t) \in \mathcal{D}'_{L^p}(\mathbb{R})$ we have:

$$\begin{aligned} w\text{-}\mathcal{D}'_{L^p}\text{-}\lim_{k \rightarrow \infty} f_k = f &\Rightarrow w\text{-}\mathcal{D}'_{L^p}\text{-}\lim_{k \rightarrow \infty} \mathcal{L}(f_k) = \mathcal{L}(f) \\ (\text{resp. } s\text{-}\mathcal{D}'_{L^p}\text{-}\lim_{k \rightarrow \infty} f_k = f &\Rightarrow s\text{-}\mathcal{D}'_{L^p}\text{-}\lim_{k \rightarrow \infty} \mathcal{L}(f_k) = \mathcal{L}(f)) \end{aligned}$$

and that it is *w-bicontinuous* (resp. *s-bicontinuous*) *invertible* if it is invertible and both $\mathcal{L}$ and $\mathcal{L}^{-1}$ are *w-continuous* (resp. *s-continuous*).

Finally, observe that continuity of a LTI IOS system $\mathcal{L} : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ is equivalent (see [24, Theorem A of Section 47]) to the existence of a real number $K \geq 0$ such that for every $f(t) \in L^p(\mathbb{R})$ it is:

$$\|\mathcal{L}(f)\|_p \leq K \|f\|_p$$

Here follows a brief summary of the content of the paper.

In Sect. 2, we give some results on IOS systems associated with differential equations, independent of linearity, time invariance and continuity, interesting *per se* and used to prove our main results of Sect. 4. In Sect. 3, we describe through a constructive procedure, the distribution $\Gamma(t)$, a key tool used in Sect. 4. In Sect. 4, we give a complete description, apart two pathologies arising when $p = \infty$, of the landscape of continuous LTI IOS systems on $\mathcal{D}'_{L^p}(\mathbb{R})$ and $L^p(\mathbb{R})$ associated with the differential equation (1), and of the landscape of bicontinuous invertible LTI IOS systems on $\mathcal{D}'_{L^p}(\mathbb{R})$ and $L^p(\mathbb{R})$ associated with the same differential equation, respectively. For each of the characterized systems, we also examine causality. In Sect. 5, we give the proofs of the Lemmas, Theorems and Corollaries stated in Sect. 4. These proofs rests heavily on the content of our previous papers [4], [5] and [6]. In Sect. 6, we apply our results to give, in a simple example, a theoretical foundation to the technique used in [3] to find a suitable continuous “almost-inverse” system of a noninvertible continuous LTI system associated with a given differential equation. Section 7 contains the conclusions. Finally, in the Appendix we summarize the definitions of the distributional spaces used in the paper, and some related notions concerning distributional solutions of ordinary differential equations with constant coefficients.

2 IOS Systems Associated with Differential Equations: Results Independent of Linearity, Time Invariance and Continuity

Let $P(z)$ and $Q(z)$ be as in Sect. 1, and let $\mathcal{X}$ be an arbitrary subspace of $\mathcal{D}'(\mathbb{R})$. An IOS system on $\mathcal{X}$ associated with the differential equation (1) is an arbitrary map

$$\mathcal{L} : \mathcal{X} \rightarrow \mathcal{X}$$

such that for every $f(t) \in \mathcal{X}$ it is: $P(D)\mathcal{L}(f) = Q(D)f$.

In this section, we state and prove three Propositions concerning properties of arbitrary IOS systems associated with the differential equation (1), interesting *per se* and used in the remaining part of the paper.

Two of them are related to the uniqueness of solution in $\mathcal{X}$ of the homogeneous differential equations $P(D)x(t) = 0$ and $Q(D)x(t) = 0$. Since we will use these two Propositions when $\mathcal{X} = L^p(\mathbb{R})$ or $\mathcal{X} = \mathcal{D}'_{L^p}(\mathbb{R})$, two additional Propositions study in particular the uniqueness of solution in $L^p(\mathbb{R})$ and $\mathcal{D}'_{L^p}(\mathbb{R})$ of a homogeneous differential equation with constant coefficients.

Proposition 1 Let $\mathcal{L} : \mathcal{X} \rightarrow \mathcal{X}$ be an IOS system on $\mathcal{X}$ associated with the differential equation (1).

Then: if $\mathcal{L}$ is invertible, its inverse map $\mathcal{L}^{-1} : \mathcal{X} \rightarrow \mathcal{X}$ is an IOS system on $\mathcal{X}$ associated with the differential equation

$$Q(D)x(t) = P(D)g(t) \quad (2)$$

Proof Let $g(t) \in \mathcal{X}$, and let $f = \mathcal{L}^{-1}(g)$. By assumption, it is $f(t) \in \mathcal{X}$. Since it is $P(D)\mathcal{L}(f) = Q(D)f$, then $Q(D)\mathcal{L}^{-1}(g) = Q(D)f = P(D)\mathcal{L}(f) = P(D)g$. $\square$

Proposition 2 Let $\mathcal{L} : \mathcal{X} \rightarrow \mathcal{X}$ be an IOS system on $\mathcal{X}$ associated with the differential equation (1), and let $\mathcal{M} : \mathcal{X} \rightarrow \mathcal{X}$ be an IOS system on $\mathcal{X}$ associated with the differential equation (2).

If $x(t) = 0$ is the unique solution in $\mathcal{X}$ of the homogeneous differential equation $P(D)x(t) = 0$ and also the unique solution in $\mathcal{X}$ of the homogeneous differential equation $Q(D)x(t) = 0$, then: $\mathcal{L}$ and $\mathcal{M}$ are each the inverse system of the other, i.e., for every $f(t) \in \mathcal{X}$ it is:

$$\mathcal{M}(\mathcal{L}(f)) = \mathcal{L}(\mathcal{M}(f)) = f$$

Proof Let $f(t) \in \mathcal{X}$. It is:

$$Q(D)\mathcal{M}(\mathcal{L}(f)) \stackrel{(i)}{=} P(D)\mathcal{L}(f) \stackrel{(ii)}{=} Q(D)f$$

where equality (i) holds since $\mathcal{M}$ is a map associated with the differential equation (2), and equality (ii) holds since $\mathcal{L}$ is a map associated with the differential equation (1). Hence: $Q(D)[\mathcal{M}(\mathcal{L}(f)) - f] = 0$.

Since $x(t) = 0$ is the unique solution in $\mathcal{X}$ of the homogeneous differential equation $Q(D)x(t) = 0$ and $\mathcal{M}(\mathcal{L}(f)) - f \in \mathcal{X}$, then: $\mathcal{M}(\mathcal{L}(f)) = f$.

Analogously, it is:

$$P(D)\mathcal{L}(\mathcal{M}(f(t))) = Q(D)\mathcal{M}(f(t)) = P(D)f(t)$$

Hence: $P(D)[\mathcal{L}(\mathcal{M}(f)) - f] = 0$.

Since $x(t) = 0$ is the unique solution in $\mathcal{X}$ of the homogeneous differential equation $P(D)x(t) = 0$ and $\mathcal{L}(\mathcal{M}(f)) - f \in \mathcal{X}$, then: $\mathcal{L}(\mathcal{M}(f)) = f$. $\square$

Proposition 3 Let $\mathcal{L}, \mathcal{M} : \mathcal{X} \rightarrow \mathcal{X}$ be two IOS systems on $\mathcal{X}$ associated with the differential equation (1).

If $x(t) = 0$ is the unique solution in $\mathcal{X}$ of the homogeneous differential equation $P(D)x(t) = 0$, then: $\mathcal{L} = \mathcal{M}$.

Proof Let $f(t) \in \mathcal{X}$. The hypothesis means that it is both $P(D)\mathcal{L}(f) = Q(D)f$ and $P(D)\mathcal{M}(f) = Q(D)f$. Hence, we have $P(D)\mathcal{L}(f) = P(D)\mathcal{M}(f)$, i.e., $P(D)(\mathcal{L}(f) - \mathcal{M}(f)) = 0$. Since $\mathcal{L}(f) - \mathcal{M}(f) \in \mathcal{X}$ and $x(t) = 0$ is the unique solution in $\mathcal{X}$ of the homogeneous differential equation $P(D)x(t) = 0$, then: $\mathcal{L}(f) = \mathcal{M}(f)$. $\square$

Let $q(z)$ be a nonzero polynomial with coefficients in $\mathbb{C}$. In the following Propositions, we study the uniqueness of solution in $L^p(\mathbb{R})$ and $\mathcal{D}'_{L^p}(\mathbb{R})$ of the homogeneous differential equation with constant coefficients $q(D)x(t) = 0$.

Proposition 4 Let $1 \leq p < \infty$. The distribution $x(t) = 0$ is the unique solution in $\mathcal{D}'_{L^p}(\mathbb{R})$ (resp. in $L^p(\mathbb{R})$) of the homogeneous differential equation $q(D)x(t) = 0$.

Proof Since $L^p(\mathbb{R}) \subset \mathcal{D}'_{L^p}(\mathbb{R})$, then it is obviously sufficient to prove the statement for $\mathcal{D}'_{L^p}(\mathbb{R})$.

If $\deg q(z) = 0$, the statement is obvious, hence, let $\deg q(z) \geq 1$.

First of all, we prove that:

(i) For every complex number α , the distribution $x(t) = 0$ is the unique solution in $\mathcal{D}'_{L^p}(\mathbb{R})$ of the differential equation $(D - \alpha)x(t) = 0$.

Indeed, if $f(t) \in \mathcal{D}'_{L^p}(\mathbb{R})$ is such that $(D - \alpha)f(t) = 0$, then for every $\varphi(t) \in \mathcal{D}(\mathbb{R})$ it is (for the convolution with a test function, see eq. (6) in Sect. 5):

$$((D - \alpha)f) * \varphi = (D - \alpha)(f * \varphi) = 0$$

Hence, there exists $c \in \mathbb{C}$ such that:

$$(f * \varphi)(t) = c e^{\alpha t}$$

Since, by Theorem 6 of the Appendix, it is $f * \varphi \in \mathcal{D}'_{L^p}(\mathbb{R})$, and, see Sect. A-6 of the Appendix, it is $\mathcal{D}'_{L^p}(\mathbb{R}) \subset \mathcal{D}'_{L^\infty}(\mathbb{R})$, then: $c e^{\alpha t} \in \mathcal{D}'_{L^\infty}(\mathbb{R})$, hence:

$$\lim_{|t| \rightarrow \infty} c e^{\alpha t} = 0$$

and this implies that $c = 0$. Hence, for every $\varphi(t) \in \mathcal{D}(\mathbb{R})$ it is $(f * \varphi)(t) = 0$. As a consequence, for every $\varphi(t) \in \mathcal{D}(\mathbb{R})$, defined $\tilde{\varphi}(t) = \varphi(-t) \in \mathcal{D}(\mathbb{R})$, by the last paragraph of Sect. 5 it is

$$\int f(t) \bullet \varphi(t) = \int f(t) \bullet \tilde{\varphi}(0 - t) = (f * \tilde{\varphi})(0) = 0$$

Hence, it is $f(t) = 0$.

The statement i) is equivalent to:

- (ii) For every complex number α , and every nonzero $f(t) \in \mathcal{D}'_{L^p}(\mathbb{R})$, the distribution $(D - \alpha)f(t)$ is a nonzero element of $\mathcal{D}'_{L^p}(\mathbb{R})$.

Now, write $q(z)$ in the form:

$$q(z) = c(z - s_1) \cdots (z - s_r)$$

where c is a nonzero complex number, and $s_1, \dots, s_r \in \mathbb{C}$, and let $f(t)$ be a nonzero element of $\mathcal{D}'_{L^p}(\mathbb{R})$. We have: $y_r(t) = (D - s_r)f(t)$ is a nonzero element of $\mathcal{D}'_{L^p}(\mathbb{R})$ and, for $j = r - 1, \dots, 1$: $y_j(t) = (D - s_j)y_{j+1}(t)$ is a nonzero element of $\mathcal{D}'_{L^p}(\mathbb{R})$. Thus:

$$q(D)f(t) = c(D - s_1) \cdots (D - s_r)f(t) = c y_1(t) \neq 0$$

This proves the statement. $\square$

Proposition 5 *The distribution $x(t) = 0$ is the unique solution in $\mathcal{D}'_{L^\infty}(\mathbb{R})$ (resp. in $L^\infty(\mathbb{R})$) of the homogeneous differential equation $q(D)x(t) = 0$ if and only if $q(z)$ has no zero on the complex imaginary axis.*

Proof If $\deg q(z) = 0$, the statement is obvious, hence, let $\deg q(z) \geq 1$.

If $\omega \in \mathbb{R}$ is such that $q(i\omega) = 0$, then $q(z) = R(z)(z - i\omega)$ for a suitable polynomial $R(z)$. As a consequence, for every $c \in \mathbb{C}$, the function $x(t) = c e^{i\omega t}$ is such that:

$$q(D)x(t) = R(D)(D - i\omega)x(t) = 0, \quad x(t) \in L^\infty(\mathbb{R}) \subset \mathcal{D}'_{L^\infty}(\mathbb{R})$$

Hence, if $q(z)$ has a zero on the complex imaginary axis, then the equation $q(D)x(t) = 0$ has an infinite number of solutions in $\mathcal{D}'_{L^\infty}(\mathbb{R})$ (resp. in $L^\infty(\mathbb{R})$).

Assume now that $q(z)$ has no zero on the complex imaginary axis.

First of all, we prove that:

- (i) For every complex number α with nonzero real part, the distribution $x(t) = 0$ is the unique solution in $\mathcal{D}'_{L^\infty}(\mathbb{R})$ of the differential equation $(D - \alpha)x(t) = 0$.

Indeed, if $f(t) \in \mathcal{D}'_{L^\infty}(\mathbb{R})$ is such that $(D - \alpha)f(t) = 0$, then, as in the proof of Proposition 4, for every $\varphi(t) \in \mathcal{D}(\mathbb{R})$ there exists $c \in \mathbb{C}$ such that:

$$(f * \varphi)(t) = c e^{\alpha t}$$

Since, by Theorem 6 of the Appendix, it is $f * \varphi \in \mathcal{D}_{L^\infty}(\mathbb{R})$, then we have: $c e^{\alpha t} \in \mathcal{D}_{L^\infty}(\mathbb{R})$, i.e., $c e^{\alpha t}$ is bounded. Since the real part of α is not zero, this implies that $c = 0$. Hence, for every $\varphi(t) \in \mathcal{D}(\mathbb{R})$ it is $(f * \varphi)(t) = 0$ and, as in the proof of Proposition 4, this implies $f(t) = 0$.

The statement i) is equivalent to:

- (ii) For every complex number α with nonzero real part, and every nonzero $f(t) \in \mathcal{D}'_{L^\infty}(\mathbb{R})$, the distribution $(D - \alpha)f(t)$ is a nonzero element of $\mathcal{D}'_{L^\infty}(\mathbb{R})$.

Proceeding as in the proof of Proposition 4, we conclude that if $f(t) \in \mathcal{D}'_{L^\infty}(\mathbb{R})$ is such that $q(D)f(t) = 0$ then $f(t) = 0$. $\square$

3 A Key Tool: The Distribution $\Gamma(t)$

Let $\mathcal{S}$ be as in Sect. 1. The analysis of the continuous, as well as of the bicontinuous invertible, LTI IOS systems on $\mathcal{S}$ associated with the differential equation (1), rests on a key tool: a suitable distribution $\Gamma(t)$.

Let $P(z)$ and $Q(z)$ be two nonzero polynomial as in Sect. 1. The following procedure **P** operates on the ordered pair $P(z)$, $Q(z)$ and returns the distribution $\Gamma(t)$.

Procedure P($P(z)$, $Q(z)$):

If $\deg P(z) = 0$, i.e., $P(z) = a_0 \neq 0$, set:

$$\Gamma(t) = \frac{1}{a_0} Q(\delta^{(1)}) = \frac{1}{a_0} (b_m \delta^{(m)}(t) + \dots + b_0 \delta^{(0)}(t))$$

If $n = \deg P(z) \geq 1$, execute the following steps:

- *Step 1.* Write $P(z)$ in the form:

$$P(z) = a_n (z - s_1)^{p_1} \dots (z - s_r)^{p_r}$$

where $s_1 = \sigma_1 + i\omega_1, \dots, s_r = \sigma_r + i\omega_r$ are pairwise distinct complex numbers, and $p_1, \dots, p_r$ are nonzero natural numbers, and write $Q(z)$ in the form:

$$Q(z) = (z - s_1)^{q_1} \dots (z - s_r)^{q_r} Q_1(z)$$

where $q_1, \dots, q_r \in \mathbb{N}$, and $Q_1(z)$ is relatively prime with $P(z)$.

- *Step 2.* Partial-fraction expand $Q(z)/P(z)$ in the uniquely determined form:

$$\frac{Q(z)}{P(z)} = \sum_{l: p_l > q_l} \left(\sum_{h=1}^{p_l - q_l} K_{lh} \frac{1}{(z - s_l)^h} \right) + R(z)$$

where—remember that $m = \deg Q(z)$ —it is:

$$R(z) = \begin{cases} 0 & \text{if } m < n \\ \sum_{h=0}^{m-n} K_h z^h & \text{if } m \geq n \end{cases}$$

and the complex numbers K_{lh} , K_h (when defined, i.e., when $p_l > q_l$, the first, and $m \geq n$, the second) are such that:

$$K_{lh} \neq 0 \text{ for } h = p_l - q_l \quad \text{and} \quad K_{m-n} \neq 0$$

(Concerning the first inequality, observe that $Q(z)/P(z)$ has a pole of order $p_l - q_l$ for every l such that $p_l > q_l$; the second is obvious.)

- *Step 3.* For every $l \in \{1, \dots, r\}$ such that $p_l > q_l$, define the polynomial function $\Lambda_l(t)$ of degree $p_l - q_l - 1$ by:

$$\Lambda_l(t) = \sum_{h=1}^{p_l - q_l} K_{lh} \frac{t^{h-1}}{(h-1)!}$$

and define:

$$\Lambda(t) = R(\delta^{(1)}) = \begin{cases} 0 & \text{if } m < n \\ \sum_{h=0}^{m-n} K_h \delta^{(h)}(t) & \text{if } m \geq n \end{cases}$$

- *Step 4.* Define the sets L_- , L_+ and L_0 of the index of the poles of the rational function $Q(z)/P(z)$ with negative, positive and zero real part, respectively, by:

$$L_- = \{l : p_l > q_l, \sigma_l < 0\}, \quad L_+ = \{l : p_l > q_l, \sigma_l > 0\}$$

and

$$L_0 = \{l : p_l > q_l, \sigma_l = 0\}$$

- *Step 5.* Set:

$$\Gamma(t) = \sum_{l \in L_-} \Lambda_l(t) e^{s_l t} H(t) - \sum_{l \in L_+} \Lambda_l(t) e^{s_l t} H(-t) + \Lambda(t)$$

Remark 1 The following statements subsist:

- (a) $\Gamma(t) \in \mathcal{D}'_{L^1}(\mathbb{R})$;
- (b) Let $P^*(z)$ and $Q^*(z)$ be two polynomials such that:

$$\frac{Q^*(z)}{P^*(z)} = \frac{Q(z)}{P(z)},$$

let $\Gamma(t)$ be the distribution obtained by $\mathbf{P}(P(z), Q(z))$, and $\Gamma^*(t)$ be the distribution obtained by $\mathbf{P}(P^*(z), Q^*(z))$. Then: $\Gamma(t) = \Gamma^*(t)$.

- (c) If $Q(z)/P(z)$ has no pole on the complex imaginary axis, i.e., if L_0 is empty, then:

- (i) $\Gamma(t)$ is the unique fundamental solution in $\mathcal{D}'_{L^1}(\mathbb{R})$ of the differential equation (1), i.e., it is the unique solution in $\mathcal{D}'_{L^1}(\mathbb{R})$ of the differential equation $P(D)x(t) = Q(D)\delta(t)$;⁷
- (ii) $\Gamma(t)$ is the inverse bilateral Laplace transform of the rational function

$$\frac{Q(s)}{P(s)}, \quad \sigma_- < \operatorname{Re}(s) < \sigma_+$$

where:

$$\sigma_- = \begin{cases} \max\{\sigma_l : l \in L_-\} & \text{if } \deg P(z) > 0 \text{ and } L_- \text{ is not empty} \\ -\infty & \text{if } \deg P(z) = 0 \text{ or } L_- \text{ is empty} \end{cases}$$

and

$$\sigma_+ = \begin{cases} \min\{\sigma_l : l \in L_+\} & \text{if } \deg P(z) > 0 \text{ and } L_+ \text{ is not empty} \\ +\infty & \text{if } \deg P(z) = 0 \text{ or } L_+ \text{ is empty} \end{cases}$$

- (iii) $\Gamma(t)$ is the inverse Fourier transform of the rational function $Q(i\omega)/P(i\omega)$.

4 Characterization of the Continuous, and of the Bicontinuous Invertible, LTI IOS Systems on $\mathcal{D}'_{L^p}(\mathbb{R})$ and $L^p(\mathbb{R})$ Associated with Differential Equations, and of their Inverses

Let $P(z)$, $Q(z)$ be as in Sect. 1. In this section, we give a complete description, apart two pathologies arising when $p = \infty$, of the landscape of continuous LTI IOS systems, and of bicontinuous invertible LTI IOS systems, and of their inverses, on $\mathcal{D}'_{L^p}(\mathbb{R})$ and $L^p(\mathbb{R})$ associated with the differential equation (1).

The pathology of the space $L^\infty(\mathbb{R})$ and, respectively, of the space $\mathcal{D}'_{L^\infty}(\mathbb{R})$ endowed with the strong convergence, which prevents us from giving a complete description of the systems when $p = \infty$ consists in the existence of different continuous LTI systems defined on $L^\infty(\mathbb{R})$ and, respectively, on $\mathcal{D}'_{L^\infty}(\mathbb{R})$, with a same impulse response (see Section 6 of [6]).⁸

The section is structured in four triads “Lemma–Theorem–Corollary.” In each triad, the Lemma indirectly gives a condition necessary to the existence of continuous LTI IOS systems, the Theorem describes the continuous LTI IOS systems, and the Corollary discusses invertibility and describes the inverse systems. In the first triad, we analyze systems defined on $\mathcal{D}'_{L^p}(\mathbb{R})$, with $1 \leq p < \infty$, in the second one we analyze systems defined on $L^p(\mathbb{R})$, with $1 \leq p < \infty$. In the last two triads, we analyze systems defined on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ and on $L^\infty(\mathbb{R})$, respectively.

To make the section easier to read, the proofs are given separately in the subsequent dedicated Sect. 5.

⁷ The statement follows by Lemma 4 of [6].

⁸ In the engineering literature, the pathology of $L^\infty(\mathbb{R})$ has been made known by I. Sandberg in his paper [21].

Let $\Gamma(t)$ and $\Theta(t)$ be the distributions obtained, respectively, by $\mathbf{P}(P(z), Q(z))$ and $\mathbf{P}(Q(z), P(z))$ (see Sect. 3).

4.1 Systems Defined on $\mathcal{D}'_{L^p}(\mathbb{R})$, $1 \leq p < \infty$

Lemma 1 *Let $1 \leq p < \infty$. If*

- *the rational function $Q(z)/P(z)$ has at least one pole on the complex imaginary axis,*

then: there are neither w –continuous nor s –continuous LTI IOS systems on $\mathcal{D}'_{L^p}(\mathbb{R})$ associated with the differential equation (1).

Theorem 1 *Let $1 \leq p < \infty$. If*

- *the rational function $Q(z)/P(z)$ has no pole on the complex imaginary axis,*

then:

- (a) *the system $\mathcal{L}_\Gamma : \mathcal{D}'_{L^p}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^p}(\mathbb{R})$ defined by:*

$$\mathcal{L}_\Gamma(f) = \Gamma * f$$

is both the unique w –continuous, and the unique s –continuous, LTI IOS system on $\mathcal{D}'_{L^p}(\mathbb{R})$ associated with the differential equation (1);

- (b) *for every $f(t) \in \mathcal{D}'_{L^p}(\mathbb{R})$, the distribution $\mathcal{L}_\Gamma(f)$ is the unique solution in $\mathcal{D}'_{L^p}(\mathbb{R})$ of the differential equation (1);*
 (c) *the system $\mathcal{L}_\Gamma$ is causal if and only if the rational function $Q(z)/P(z)$ has no pole with positive real part.*

Corollary 1 *Let $1 \leq p < \infty$, let*

- *the rational function $Q(z)/P(z)$ have no pole on the complex imaginary axis,*

and take into account the unique w –continuous, as well as the unique s –continuous, LTI IOS system

$$\mathcal{L}_\Gamma : \mathcal{D}'_{L^p}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^p}(\mathbb{R})$$

on $\mathcal{D}'_{L^p}(\mathbb{R})$ associated with the differential equation (1).

The system $\mathcal{L}_\Gamma$ is w –bicontinuous (resp.: s –bicontinuous) invertible if and only if

- *also the rational function $P(z)/Q(z)$ has no pole on the complex imaginary axis.*

Let the previous condition hold. Then,

- (a) *the system $\mathcal{L}_\Gamma^{-1}$ is the system $\mathcal{L}_\Theta : \mathcal{D}'_{L^p}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^p}(\mathbb{R})$ defined by:*

$$\mathcal{L}_\Theta(f) = \Theta * f$$

and it is the unique w –continuous, as well as the unique s –continuous, LTI IOS system on $\mathcal{D}'_{L^p}(\mathbb{R})$ associated with the differential equation (2);

- (b) for every $g(t) \in \mathcal{D}'_{L^p}(\mathbb{R})$, the distribution $\mathcal{L}_\Gamma^{-1}(g)$ is the unique solution in $\mathcal{D}'_{L^p}(\mathbb{R})$ of the differential equation (2);
- (c) the system $\mathcal{L}_\Gamma^{-1}$ is causal if and only if the rational function $P(z)/Q(z)$ has no pole with positive real part.

4.2 Systems Defined on $L^p(\mathbb{R})$, $1 \leq p < \infty$.

Lemma 2 Let $1 \leq p < \infty$. If

- the rational function $Q(z)/P(z)$ has at least one pole on the complex imaginary axis, or $\deg Q(z) > \deg P(z)$,

then there are no continuous LTI IOS systems on $L^p(\mathbb{R})$ associated with the differential equation (1).

Theorem 2 Let $1 \leq p < \infty$. If

- the rational function $Q(z)/P(z)$ has no pole on the complex imaginary axis, and $\deg Q(z) \leq \deg P(z)$,

then:

- (a) the system $\mathcal{L}_\Gamma : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ defined by

$$\mathcal{L}_\Gamma(f) = \Gamma * f$$

is the unique continuous LTI IOS system on $L^p(\mathbb{R})$ associated with the differential equation (1);

- (b) for every $f(t) \in L^p(\mathbb{R})$, the distribution $\mathcal{L}_\Gamma(f)$ is the unique solution in $L^p(\mathbb{R})$ of the differential equation (1);
- (c) the system $\mathcal{L}_\Gamma$ is causal if and only if the rational function $Q(z)/P(z)$ has no pole with positive real part.

Corollary 2 Let $1 \leq p < \infty$, let

- the rational function $Q(z)/P(z)$ have no pole on the complex imaginary axis, and $\deg Q(z) \leq \deg P(z)$,

and take into account the unique continuous LTI IOS system

$$\mathcal{L}_\Gamma : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$$

on $L^p(\mathbb{R})$ associated with the differential equation (1).

The system $\mathcal{L}_\Gamma$ is bicontinuous invertible if and only if

- also the rational function $P(z)/Q(z)$ has no pole on the complex imaginary axis, and $\deg Q(z) = \deg P(z)$.

Let the previous condition hold. Then,

(a) the system $\mathcal{L}_\Gamma^{-1}$ is the system $\mathcal{L}_\Theta : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ defined by:

$$\mathcal{L}_\Theta(f) = \Theta * f$$

and it is the unique continuous LTI IOS system on $L^p(\mathbb{R})$ associated with the differential equation (2);

- (b) for every $g(t) \in L^p(\mathbb{R})$, the distribution $\mathcal{L}_\Gamma^{-1}(g)$ is the unique solution in $L^p(\mathbb{R})$ of the differential equation (2);
- (c) the system $\mathcal{L}_\Gamma^{-1}$ is causal if and only if the rational function $P(z)/Q(z)$ has no pole with positive real part.

4.3 Systems Defined on $\mathcal{D}'_{L^\infty}(\mathbb{R})$.

Lemma 3 *If*

- the rational function $Q(z)/P(z)$ has at least one pole on the complex imaginary axis,

then: there are neither w –continuous nor s –continuous LTI IOS systems on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1).

Theorem 3 *If*

- the rational function $Q(z)/P(z)$ has no pole on the complex imaginary axis,

then the following statements hold:

- concerning w –convergence:
 - (a) the system $\mathcal{L}_\Gamma : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^\infty}(\mathbb{R})$ defined by

$$\mathcal{L}_\Gamma(f) = \Gamma * f$$

is the unique w –continuous LTI IOS on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1);

- (b) *for every $f(t) \in \mathcal{D}'_{L^\infty}(\mathbb{R})$, the distribution $\mathcal{L}_\Gamma(f)$ is a solution in $\mathcal{D}'_{L^\infty}(\mathbb{R})$ of the differential equation (1), and such solution is the unique in $\mathcal{D}'_{L^\infty}(\mathbb{R})$ if and only if $P(z)$ has no zero on the complex imaginary axis;*
- concerning s –convergence:
 - (c) the system $\mathcal{L}_\Gamma : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^\infty}(\mathbb{R})$ defined by

$$\mathcal{L}_\Gamma(f) = \Gamma * f$$

is a s –continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1);

- (d) *if $\mathcal{L} : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^\infty}(\mathbb{R})$ is a s –continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1), then the impulse response of $\mathcal{L}$ is $\Gamma(t)$ —the pathology of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ doesn't allow us to say that it is $\mathcal{L} = \mathcal{L}_\Gamma$;*

- (e) if $P(z)$ has no zero on the complex imaginary axis, then: the system $\mathcal{L}_\Gamma$ is the unique s –continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1);
- concerning causality:
 - (f) the system $\mathcal{L}_\Gamma$ is causal if and only if the rational function $Q(z)/P(z)$ has no pole with positive real part.

Corollary 3 *Let*

- the rational function $Q(z)/P(z)$ have no pole on the complex imaginary axis.

There exist w –bicontinuous (resp.: s –bicontinuous) invertible LTI IOS systems on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1) if and only if

- also the rational function $P(z)/Q(z)$ has no pole on the complex imaginary axis.

Let the previous condition hold. Then:

- concerning w –convergence:
 - (a) the system $\mathcal{L}_\Gamma : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^\infty}(\mathbb{R})$ defined by

$$\mathcal{L}_\Gamma(f) = \Gamma * f$$

is the unique w –bicontinuous invertible LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1);

- (b) the system $\mathcal{L}_\Gamma^{-1}$ is the system $\mathcal{L}_\Theta : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^\infty}(\mathbb{R})$ defined by:

$$\mathcal{L}_\Theta(f) = \Theta * f$$

and it is the unique w –continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (2);

- (c) for every $g(t) \in \mathcal{D}'_{L^\infty}(\mathbb{R})$, the distribution $\mathcal{L}_\Gamma^{-1}(g)$ is a solution in $\mathcal{D}'_{L^\infty}(\mathbb{R})$ of the differential equation (2), and such solution is the unique in $\mathcal{D}'_{L^\infty}(\mathbb{R})$ if and only if $Q(z)$ has no zero on the complex imaginary axis;

- concerning s –convergence:

- (d) the system $\mathcal{L}_\Gamma : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^\infty}(\mathbb{R})$ defined by

$$\mathcal{L}_\Gamma(f) = \Gamma * f$$

is a s –bicontinuous invertible LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1)—the pathology of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ doesn't allow us to say that it is the unique (see d) of Theorem 3;

- (e) the system $\mathcal{L}_\Gamma^{-1}$ is the system $\mathcal{L}_\Theta : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^\infty}(\mathbb{R})$ defined by:

$$\mathcal{L}_\Theta(f) = \Theta * f$$

and it is a s –continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (2)—the pathology of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ doesn't allow us to say that it is the unique;

- (f) if $P(z)$ and $Q(z)$ have no zero on the complex imaginary axis, then: $\mathcal{L}_\Gamma$ is the unique s –bicontinuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1), and $\mathcal{L}_\Gamma^{-1}$ is the unique s –continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (2);

– concerning causality:

- (g) the system $\mathcal{L}_\Gamma^{-1}$ is causal if and only if the rational function $P(z)/Q(z)$ has no pole with positive real part.

4.4 Systems Defined on $L^\infty(\mathbb{R})$.

Lemma 4 *If*

- the rational function $Q(z)/P(z)$ has at least one pole on the complex imaginary axis, or $\deg Q(z) > \deg P(z)$,

then: there are no continuous LTI IOS systems on $L^\infty(\mathbb{R})$ associated with the differential equation (1).

Theorem 4 *If*

- the rational function $Q(z)/P(z)$ has no pole on the complex imaginary axis, and $\deg Q(z) \leq \deg P(z)$,

then the following statements hold:

- (a) the system $\mathcal{L}_\Gamma : L^\infty(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ defined by

$$\mathcal{L}_\Gamma(f) = \Gamma * f$$

is a continuous LTI IOS system on $L^\infty(\mathbb{R})$ associated with the differential equation (1);

- (b) *if $\mathcal{L} : L^\infty(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ is a continuous LTI IOS system on $L^\infty(\mathbb{R})$ associated with the differential equation (1), then the impulse response of $\mathcal{L}$ is $\Gamma(t)$ —the pathology of $L^\infty(\mathbb{R})$ doesn't allow us to say that $\mathcal{L} = \mathcal{L}_\Gamma$;*
- (c) *if $P(z)$ has no zero on the complex imaginary axis, then $\mathcal{L}_\Gamma$ is the unique continuous LTI IOS system on $L^\infty(\mathbb{R})$ associated with the differential equation (1);*
- (d) *for every $f(t) \in L^\infty(\mathbb{R})$, the function $\mathcal{L}_\Gamma(f)$ is a solution in $L^\infty(\mathbb{R})$ of the differential equation (1), and such solution is the unique in $L^\infty(\mathbb{R})$ if and only if $P(z)$ has no zero on the complex imaginary axis;*
- (e) *the system $\mathcal{L}_\Gamma$ is causal if and only if the rational function $Q(z)/P(z)$ has no pole with positive real part.*

Corollary 4 *Let*

- the rational function $Q(z)/P(z)$ have no pole on the complex imaginary axis, and $\deg Q(z) \leq \deg P(z)$.

The following statement holds: there exist bicontinuous invertible LTI IOS systems on $L^\infty(\mathbb{R})$ associated with the differential equation (1) if and only if

- also the rational function $P(z)/Q(z)$ has no pole on the complex imaginary axis, and $\deg Q(z) = \deg P(z)$.

Let the previous condition hold. Then:

- (a) the system $\mathcal{L}_\Gamma : L^\infty(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ defined by

$$\mathcal{L}_\Gamma(f) = \Gamma * f$$

is a bicontinuous invertible LTI IOS system on $L^\infty(\mathbb{R})$ associated with the differential equation (1) — the pathology of $L^\infty(\mathbb{R})$ doesn't allow us to say that it is the unique (see b) of Theorem 4);

- (b) the system $\mathcal{L}_\Gamma^{-1}$ is the system $\mathcal{L}_\Theta : L^\infty(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ defined by:

$$\mathcal{L}_\Theta(f) = \Theta * f$$

and it is a continuous LTI IOS system on $L^\infty(\mathbb{R})$ associated with the differential equation (2)—the pathology of $L^\infty(\mathbb{R})$ doesn't allow us to say that it is the unique;

- (c) if $P(z)$ and $Q(z)$ have no zero on the complex imaginary axis, then: $\mathcal{L}_\Gamma$ is the unique bicontinuous LTI IOS system on $L^\infty(\mathbb{R})$ associated with the differential equation (1), and $\mathcal{L}_\Gamma^{-1}$ is the unique continuous LTI IOS system on $L^\infty(\mathbb{R})$ associated with the differential equation (2);
- (d) the system $\mathcal{L}_\Gamma^{-1}$ is causal if and only if the rational function $P(z)/Q(z)$ has no pole with positive real part.

5 Proofs of the Statements of Previous Sect. 4

This section contains the proofs of the Lemmas, Theorems and Corollaries stated in Sect. 4.

As a premise to the proofs, we observe that the notions of *continuous LTI system defined on $\mathcal{I}$* associated with the differential equation (1) and of *continuous LTI IOS system on $\mathcal{I}$* associated with the differential equation (1) are related by the following obvious, but formally essential, remark.

Remark 2 The following statements subsist:

- (a) Let $\mathcal{L} : \mathcal{I} \rightarrow \mathcal{I}$ be a continuous LTI IOS system on $\mathcal{I}$ associated with the differential equation (1).

Since $\mathcal{I} \subset \mathcal{D}'(\mathbb{R})$ and convergence in $\mathcal{I}$ implies convergence in $\mathcal{D}'(\mathbb{R})$, then the system

$$\mathcal{L}^* : \mathcal{I} \rightarrow \mathcal{D}'(\mathbb{R})$$

defined by $\mathcal{L}^*(f) = \mathcal{L}(f)$ is a continuous LTI system defined on $\mathcal{I}$ associated with the differential equation (1). Moreover,

- (i) $\mathcal{L}^*(\mathcal{I}) \subset \mathcal{I}$;
- (ii) for every $f(t) \in \mathcal{I}$ and every sequence $f_k(t) \in \mathcal{I}$ we have:

$$\mathcal{I} - \lim_{k \rightarrow \infty} f_k = f \Rightarrow \mathcal{I} - \lim_{k \rightarrow \infty} \mathcal{L}^*(f_k) = \mathcal{L}^*(f)$$

- (b) Let $\mathcal{L}^* : \mathcal{I} \rightarrow \mathcal{D}'(\mathbb{R})$ be a continuous LTI system defined on $\mathcal{I}$ associated with the differential equation (1), and let:

- (i) $\mathcal{L}^*(\mathcal{I}) \subset \mathcal{I}$;
- (ii) for every $f(t) \in \mathcal{I}$ and every sequence $f_k(t) \in \mathcal{I}$ we have:

$$\mathcal{I} - \lim_{k \rightarrow \infty} f_k = f \Rightarrow \mathcal{I} - \lim_{k \rightarrow \infty} \mathcal{L}^*(f_k) = \mathcal{L}^*(f)$$

Then, the system $\mathcal{L} : \mathcal{I} \rightarrow \mathcal{I}$ defined by $\mathcal{L}(f) = \mathcal{L}^*(f)$ is a continuous LTI IOS system on $\mathcal{I}$ associated with the differential equation (1).

Proof of Lemma 1 We prove the statements by contradiction, i.e., we prove that: if there exists either a w -continuous or a s -continuous LTI IOS system on $\mathcal{D}'_{LP}(\mathbb{R})$ associated with the differential equation (1), then the rational function $Q(z)/P(z)$ has no pole on the complex imaginary axis.

If $\deg P(z) = 0$, then there is nothing to prove. Hence, assume that $\deg P(z) \geq 1$.

Let $\mathcal{L} : \mathcal{D}'_{LP}(\mathbb{R}) \rightarrow \mathcal{D}'_{LP}(\mathbb{R})$ be a w -continuous or a s -continuous LTI IOS system on $\mathcal{D}'_{LP}(\mathbb{R})$ associated with the differential equation (1), and let $f(t) = \delta(t) \in \mathcal{D}'_{LP}(\mathbb{R})$. By hypothesis, $\mathcal{L}(\delta)$ is a solution in $\mathcal{D}'_{LP}(\mathbb{R})$ of the differential equation (1). Hence, by Lemma 4 of [6], the rational function $Q(z)/P(z)$ has no pole on the complex imaginary axis. $\square$

Proof of Theorem 1 The proof has two steps.

Step 1: Proof in the case $n = \deg P(z) = 0$.

If $n = \deg P(z) = 0$, then $P(z) = a_0 \neq 0$, the rational function $Q(z)/a_0$ has no pole on the complex imaginary axis, and the differential equation (1) reduces to: $a_0 x(t) = Q(D)f(t)$. Then, obviously:

- (1) the system $\mathcal{L}_\Gamma : \mathcal{D}'_{LP}(\mathbb{R}) \rightarrow \mathcal{D}'_{LP}(\mathbb{R})$ defined by

$$\mathcal{L}_\Gamma(f) = \frac{1}{a_0} Q(D)f$$

is the unique w -continuous, as well as the unique s -continuous, LTI IOS system on $\mathcal{D}'_{LP}(\mathbb{R})$ associated with the differential equation $a_0 x(t) = Q(D)f(t)$,

- (2) for every $f(t) \in \mathcal{D}'_{LP}(\mathbb{R})$, the distribution $\mathcal{L}_\Gamma(f)$ is the unique solution in $\mathcal{D}'_{LP}(\mathbb{R})$ of the differential equation $a_0 x(t) = Q(D)f(t)$, and
- (3) the rational function $Q(z)/a_0$ has no pole with positive real part, and the system $\mathcal{L}_\Gamma$ is causal.

Hence, all the statements hold.

Step 2: Proof in the case $n = \deg P(z) \geq 1$.

Since, by item a) of Remark 1, $\Gamma(t) \in \mathcal{D}'_{L^1}(\mathbb{R})$ then, by Theorem 10 in the Appendix, $\mathcal{L}_\Gamma$ is a s -continuous (resp.: w -continuous) LTI IOS system on $\mathcal{D}'_{LP}(\mathbb{R})$.

Since $Q(z)/P(z)$ has no pole on the complex imaginary axis, then, by item c) of Remark 1, $\Gamma(t)$ is a fundamental solution of the differential equation (1) in $\mathcal{D}'_{L^1}(\mathbb{R})$. By the *Third idea* of the Introduction of [6], $\mathcal{L}_\Gamma$ is then a s –continuous (resp.: w –continuous) LTI system associated with the differential equation (1).

Finally, by Proposition 4, $x(t) = 0$ is the unique solution in $\mathcal{D}'_{L^p}(\mathbb{R})$ of the homogeneous differential equation $P(D)x(t) = 0$. Hence, by Proposition 3, $\mathcal{L}_\Gamma$ is the unique s –continuous (resp.: w –continuous) LTI IOS system on $\mathcal{D}'_{L^p}(\mathbb{R})$ associated with the differential equation (1). Thus, a) holds.

To prove that b) holds, observe that, since a) holds, then $\mathcal{L}_\Gamma$ is a IOS system on $\mathcal{D}'_{L^p}(\mathbb{R})$ associated with the differential equation $P(D)x(t) = Q(D)f(t)$. Hence, for every $f(t) \in \mathcal{D}'_{L^p}(\mathbb{R})$, the distribution $\mathcal{L}_\Gamma(f)$ is a solution in $\mathcal{D}'_{L^p}(\mathbb{R})$ of $P(D)x(t) = Q(D)f(t)$. By Proposition 4, it is the unique.

The statement c) is a straightforward consequence of e) of Theorem 5 of [6]. $\square$

Proof of Corollary 1 The proof has two steps.

Step 1: $\mathcal{L}_\Gamma$ is w –bicontinuous as well as s –bicontinuous invertible $\Leftrightarrow P(z)/Q(z)$ has no pole on the complex imaginary axis.

($\Leftarrow$) By Theorem 1 applied to the differential equation (2), the system $\mathcal{L}_\Theta : \mathcal{D}'_{L^p}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^p}(\mathbb{R})$ defined by $\mathcal{L}_\Theta(g) = \Theta * g$ is the unique w –continuous, as well as the unique s –continuous, LTI IOS system on $\mathcal{D}'_{L^p}(\mathbb{R})$ associated with the differential equation (2).

Since, by Proposition 4, $x(t) = 0$ is the unique solution in $\mathcal{D}'_{L^p}(\mathbb{R})$ both of the homogeneous differential equation $P(D)x(t) = 0$ and of the homogeneous differential equation $Q(D)x(t) = 0$, then, by Proposition 2, it is

$$\mathcal{L}_\Gamma \circ \mathcal{L}_\Theta = \mathcal{L}_\Theta \circ \mathcal{L}_\Gamma = \iota : \mathcal{D}'_{L^p}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^p}(\mathbb{R})$$

i.e., $\mathcal{L}_\Gamma$ is invertible and its inverse is $\mathcal{L}_\Gamma^{-1} = \mathcal{L}_\Theta$.

Since $\mathcal{L}_\Theta$ is w –continuous, as well as s –continuous, then $\mathcal{L}_\Gamma$ is w –bicontinuous as well as s –bicontinuous invertible.

($\Rightarrow$) Since, by assumption and Proposition 1, $\mathcal{L}_\Gamma^{-1}$ is a w –continuous (resp. s –continuous) LTI IOS system on $\mathcal{D}'_{L^p}(\mathbb{R})$ associated with the differential equation (2), then, by Lemma 1, the rational function $P(z)/Q(z)$ has no pole on the complex imaginary axis.

Step 2: $P(z)/Q(z)$ has no pole on the complex imaginary axis $\Rightarrow$ a), b), c).

The proof that a) holds has been given in ($\Leftarrow$) of Step 1 above.

To prove that b) and c) hold, observe that, as proved in ($\Leftarrow$) of Step 1 above, it is: $\mathcal{L}_\Gamma^{-1} = \mathcal{L}_\Theta$. Hence, statements b) and c) follow by b) and c), respectively, of Theorem 1 applied to the differential equation (2). $\square$

Proof of Lemma 2 The proof is by contradiction, i.e., we prove that: if there exists an LTI IOS continuous system on $L^p(\mathbb{R})$ associated with the differential equation (1), then $Q(z)/P(z)$ has no pole on the complex imaginary axis and $\deg Q(z) \leq \deg P(z)$.

The proof has three steps.

Step 1: Proof in the case $n = \deg P(z) = 0$.

Assume that $\mathcal{L}$ is an LTI IOS continuous system on $L^p(\mathbb{R})$ associated with the differential equation (1). Since $P(z) = a_0 \neq 0$, then $Q(z)/P(z)$ has no pole on

the complex imaginary axis. Moreover, the differential equation (1) now reduces to: $a_0x(t) = Q(D)f(t)$. It is straightforward to prove that: if $\deg Q(z) > \deg P(z) = 0$, i.e., $\deg Q(z) \geq 1$, then there exists $f(t) \in L^p(\mathbb{R})$ such that the differential equation has no solution in $L^p(\mathbb{R})$. But, by assumption, $\mathcal{L}(f)$ is such a solution. Thus, $\deg Q(z) \leq \deg P(z)$.

Step 2: Proof in the case $n = \deg P(z) \geq 1$, $p > 1$.

Assume that $\mathcal{L}$ is an LTI IOS continuous system on $L^p(\mathbb{R})$ associated with the differential equation (1). By a) of Remark 2, the system $\mathcal{L}^* : L^p(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R})$ defined by $\mathcal{L}^*(f) = \mathcal{L}(f)$ is a continuous LTI differential system defined on $L^p(\mathbb{R})$ associated with the differential equation (1). By Theorem 1 of [6], the rational function $Q(z)/P(z)$ has no pole on the complex imaginary axis.

Now, since $Q(z)/P(z)$ has no pole on the complex imaginary axis, then, by b) of Theorem 1 of [6], we have that the unique continuous LTI differential system defined on $L^p(\mathbb{R})$ associated with the differential equation (1) is the restriction to $L^p(\mathbb{R})$ of the system $\mathcal{L}_\Gamma^* : \mathcal{D}'_{L^p}(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R})$ defined by $\mathcal{L}_\Gamma^*(f) = \Gamma * f$. Hence, $\mathcal{L}^*$ is the restriction to $L^p(\mathbb{R})$ of $\mathcal{L}_\Gamma^*$. Since, by assumption, it is $\mathcal{L}(L^p(\mathbb{R})) \subset L^p(\mathbb{R})$, then $\mathcal{L}_\Gamma^*(L^p(\mathbb{R})) = \mathcal{L}^*(L^p(\mathbb{R})) = \mathcal{L}(L^p(\mathbb{R})) \subset L^p(\mathbb{R})$. Thus, by d) of Theorem 5 of [6] applied to $\mathcal{L}_\Gamma^*$, it is $\deg Q(z) \leq \deg P(z)$.

Step 3: Proof in the case $n = \deg P(z) \geq 1$ and $p = 1$.

Assume that $\mathcal{L}$ is an LTI IOS continuous system on $L^1(\mathbb{R})$ associated with the differential equation (1). By a) of Remark 2, the system $\mathcal{L}^* : L^1(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R})$ defined by $\mathcal{L}^*(f) = \mathcal{L}(f)$ is a continuous LTI system defined on $L^1(\mathbb{R})$ associated with the differential equation (1). By Section II and Theorem 5.1 of [5], there exists $h(t) \in \mathcal{D}'_{L^\infty}(\mathbb{R})$ such that, for every $f(t) \in L^1(\mathbb{R})$ it is $\mathcal{L}^*(f) = h * f$. Since, by assumption, for every $\varphi(t) \in \mathcal{D}(\mathbb{R}) \subset L^1(\mathbb{R})$ it is $h * \varphi = \mathcal{L}^*(\varphi) = \mathcal{L}(\varphi) \in L^1(\mathbb{R})$, then $h(t) \in \mathcal{D}'_{L^1}(\mathbb{R})$. Moreover, by the *Second idea*, Section 1 of [6], $h(t)$ is a fundamental solution, in $\mathcal{D}'_{L^1}(\mathbb{R})$, of the differential equation (1). Hence, by Lemma 4 of [6], $Q(z)/P(z)$ has no pole on the complex imaginary axis.

Now, since $Q(z)/P(z)$ has no pole on the imaginary axis, then, by d) of Theorem 4 of [6], there exists a family $\mu = (\mu_l \in \mathbb{C}, l : \sigma_l = 0)$ of complex numbers such that:

$$h(t) = \Gamma(t) + \sum_{l:\sigma_l=0} \mu_l e^{i\omega_l t}$$

Since $\Gamma(t) \in \mathcal{D}'_{L^1}(\mathbb{R})$ and, as shown above, $h(t) \in \mathcal{D}'_{L^1}(\mathbb{R})$, then

$$\sum_{l:\sigma_l=0} \mu_l e^{i\omega_l t} \in \mathcal{D}'_{L^1}(\mathbb{R})$$

Since, by Corollary 2 of [6], for every $\mu \neq 0$ it is

$$\sum_{l:\sigma_l=0} \mu_l e^{i\omega_l t} \notin \dot{\mathcal{D}}'_{L^\infty}(\mathbb{R}) \supset \mathcal{D}'_{L^1}(\mathbb{R})$$

then we have: $h(t) = \Gamma(t)$. This proves that $\mathcal{L}^*$ is the system defined by $\mathcal{L}^*(f) = \Gamma * f$. Since, by assumption, $\mathcal{L}(L^1(\mathbb{R})) \subset L^1(\mathbb{R})$, then $\mathcal{L}^*(L^1(\mathbb{R})) = \mathcal{L}(L^1(\mathbb{R})) \subset$

$L^1(\mathbb{R})$. By *d*) of Theorem 5 of [6] applied to $\mathcal{L}^*$, this implies that $\deg Q(z) \leq \deg P(z)$. $\square$

Proof of Theorem 2 The proof has three steps.

Step 1: Proof in the case $n = \deg P(z) = 0$.

Since $\deg Q(z) \leq \deg P(z)$ and $n = \deg P(z) = 0$, then $m = \deg Q(z) = 0$. Hence, $P(z) = a_0 \neq 0$, $Q(z) = b_0 \neq 0$, and the differential equation (1) reduces to: $a_0 x(t) = b_0 f(t)$. Then, obviously:

(1) the system $\mathcal{L}_\Gamma : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ defined by

$$\mathcal{L}_\Gamma(f) = \frac{b_0}{a_0} f$$

is the unique continuous LTI IOS system on $L^p(\mathbb{R})$ associated with the differential equation $a_0 x(t) = b_0 f(t)$, hence *a*) holds;

(2) for every $f(t) \in L^p(\mathbb{R})$, the function $\mathcal{L}_\Gamma(f)$ is the unique solution in $L^p(\mathbb{R})$ of the differential equation $a_0 x(t) = b_0 f(t)$, hence *b*) holds; and

(3) the rational function b_0/a_0 has no pole with positive real part, and the system $\mathcal{L}_\Gamma$ is causal, hence *c*) holds.

Step 2: Proof in the case $n = \deg P(z) \geq 1$.

Since $Q(z)/P(z)$ has no pole on the complex imaginary axis, and $\deg Q(z) \leq \deg P(z)$, then, see Sect. 3, it is: $\Gamma(t) = c\delta(t) + \gamma(t)$ where $c \in \mathbb{C}$ and $\gamma(t) \in L^1(\mathbb{R})$. Hence, for every $f(t) \in L^p(\mathbb{R})$ we have: $\mathcal{L}_\Gamma(f) = cf + \gamma * f$. Since, by Theorem 1.2, Chapter VIII, § 1 of [14], the map $f \mapsto \gamma * f$ is a continuous map from $L^p(\mathbb{R})$ into itself, then $\mathcal{L}_\Gamma$ is a continuous LTI IOS system on $L^p(\mathbb{R})$.

Since $Q(z)/P(z)$ has no pole on the complex imaginary axis, then (see Remark 1) $\Gamma(t)$ is a fundamental solution of the differential equation (1) in $\mathcal{D}'_{L^1}(\mathbb{R})$. By the *Third idea* of the Introduction of [6], $\mathcal{L}_\Gamma$ is then a continuous LTI IOS system on $L^p(\mathbb{R})$ associated with the differential equation (1).

Finally, by Proposition 4, $x(t) = 0$ is the unique solution in $L^p(\mathbb{R})$ of the homogeneous differential equation $P(D)x(t) = 0$. Hence, by Proposition 3, $\mathcal{L}_\Gamma$ is the unique continuous LTI IOS system on $L^p(\mathbb{R})$ associated with the differential equation (1). Thus, *a*) holds.

To prove that *b*) holds, observe that, since *a*) holds, then $\mathcal{L}_\Gamma$ is a IOS system on $L^p(\mathbb{R})$ associated with the differential equation $P(D)x(t) = Q(D)f(t)$. Hence, for every $f(t) \in L^p(\mathbb{R})$, the function $\mathcal{L}_\Gamma(f)$ is a solution in $L^p(\mathbb{R})$ of $P(D)x(t) = Q(D)f(t)$. By Proposition 4, it is the unique.

The statement *c*) is a straightforward consequence of *e*) of Theorem 5 of [6]. $\square$

Proof of Corollary 2 Use the same arguments as in the proof of Corollary 1, substituting $\mathcal{D}'_{L^p}(\mathbb{R})$ with $L^p(\mathbb{R})$, Theorem 1 with Theorem 2 and Lemma 1 with Lemma 2. $\square$

Proof of Lemma 3 We prove the statements *by contradiction*, i.e., we prove that: *if there exists either a w -continuous or a s -continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1), then the rational function $Q(z)/P(z)$ has no pole on the complex imaginary axis.*

If $\deg P(z) = 0$, then there is nothing to prove. Hence, assume that $\deg P(z) \geq 1$.

Let $\mathcal{L} : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^\infty}(\mathbb{R})$ be a w -continuous (resp.: a s -continuous) LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1). By a) of Remark 2, the system $\mathcal{L}^* : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R})$ defined by $\mathcal{L}^*(f) = \mathcal{L}(f)$ is a w -continuous (resp. a s -continuous) LTI system defined on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1). By Theorem 2 of [6], the rational function $Q(z)/P(z)$ has no pole on the complex imaginary axis. $\square$

Proof of Theorem 3 The proof has two steps.

Step 1: Proof in the case $n = \deg P(z) = 0$.

To prove the statement, set $p = \infty$ in Step 1 of the proof of Theorem 1.

Step 2: Proof in the case $n = \deg P(z) \geq 1$.

Since, by a) of Remark 1, $\Gamma(t) \in \mathcal{D}'_{L^1}(\mathbb{R})$ then, by Theorem 10 in the Appendix, $\mathcal{L}_\Gamma$ is a s -continuous (resp.: w -continuous) LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$. Moreover, by Theorem 4.1 of [4], $\mathcal{L}_\Gamma$ is the unique w -continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$.

Since $Q(z)/P(z)$ has no pole on the complex imaginary axis, then (see Remark 1) $\Gamma(t)$ is the unique fundamental solution of the differential equation (1) in $\mathcal{D}'_{L^1}(\mathbb{R})$. By the Third idea of the Introduction of [6], $\mathcal{L}_\Gamma$ is then a s -continuous (resp.: the unique w -continuous) LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1). This proves a) and c).

To prove that b) holds, observe that, since a) holds, then $\mathcal{L}_\Gamma$ is a IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation $P(D)x(t) = Q(D)f(t)$. Hence, for every $f(t) \in \mathcal{D}'_{L^\infty}(\mathbb{R})$, the distribution $\mathcal{L}_\Gamma(f)$ is a solution in $\mathcal{D}'_{L^p}(\mathbb{R})$ of $P(D)x(t) = Q(D)f(t)$. By Proposition 4, it is the unique if and only if $P(z)$ has no zero on the complex imaginary axis.

To prove statement d), let us consider the system $\mathcal{L}^* : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R})$ defined by $\mathcal{L}^*(f) = \mathcal{L}(f)$. By Section 2 of [4], $\mathcal{L}$ and $\mathcal{L}^*$ have the same impulse response. Moreover, by First idea and Second idea in Section 1 of [6], the impulse response $\Delta(t)$ of $\mathcal{L}^*$ is a solution in $\mathcal{D}'_{L^1}(\mathbb{R})$ of the differential equation $P(D)x(t) = Q(D)\delta(t)$. Since the rational function $Q(z)/P(z)$ has no pole on the complex imaginary axis, by i) of item c) of Remark 1, it is $\Delta(t) = \Gamma(t)$. Hence, the impulse response of $\mathcal{L}$ is $\Gamma(t)$.

To prove statement e), observe that, by Proposition 5, since $P(z)$ has no zero on the complex imaginary axis then $x(t) = 0$ is the unique solution in $\mathcal{D}'_{L^\infty}(\mathbb{R})$ of the homogeneous differential equation $P(D)x(t) = 0$. Hence, by Proposition 3, since $P(z)$ has no zero on the complex imaginary axis, then $\mathcal{L}_\Gamma$ is the unique s -continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1).

The statement f) is a straightforward consequence of e) of Theorem 5 of [6]. $\square$

Proof of Corollary 3 Remember that, by hypothesis, $Q(z)/P(z)$ has no pole on the complex imaginary axis, and that, by Theorem 3, $\mathcal{L}_\Gamma$ is the unique w -continuous (resp.: a s -continuous) LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1).

The proof has three steps.

Step 1: $\mathcal{L}_\Gamma$ is w -bicontinuous invertible $\Leftrightarrow P(z)/Q(z)$ has no pole on the complex imaginary axis.

($\Leftarrow$) Since the rational function $P(z)/Q(z)$ has no pole on the complex imaginary axis, then, by Theorem 3:

I_w) the system $\mathcal{L}_\Theta : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^\infty}(\mathbb{R})$ defined by $\mathcal{L}_\Theta(g) = \Theta * g$ is the unique w -continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (2).

We prove now that:

Π_w) the system $\mathcal{L}_\Gamma$ is invertible, and $\mathcal{L}_\Gamma^{-1} = \mathcal{L}_\Theta$.

Indeed, since $\mathcal{L}_\Gamma$ is associated with the differential equation (1), then, for every $g(t) \in \mathcal{D}'_{L^\infty}(\mathbb{R})$ we have:

$$P(D)\mathcal{L}_\Gamma(\mathcal{L}_\Theta(g)) = Q(D)\mathcal{L}_\Theta(g)$$

Moreover, since $\mathcal{L}_\Theta$ is associated with the differential equation (2), then:

$$Q(D)\mathcal{L}_\Theta(g) = P(D)g$$

Hence,

$$P(D)\mathcal{L}_\Gamma(\mathcal{L}_\Theta(g)) = P(D)g$$

Thus, the system $\mathcal{L}_\Gamma \circ \mathcal{L}_\Theta : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^\infty}(\mathbb{R})$ is a w -continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation $P(D)x(t) = P(D)g(t)$. Since $P(z)/P(z)$ has no pole on the complex imaginary axis, then, by Theorem 3, it is $\mathcal{L}_\Gamma \circ \mathcal{L}_\Theta = \iota : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^\infty}(\mathbb{R})$.

Analogously, for every $f(t) \in \mathcal{D}'_{L^\infty}(\mathbb{R})$ we have:

$$Q(D)\mathcal{L}_\Theta(\mathcal{L}_\Gamma(f)) = P(D)\mathcal{L}_\Gamma(f) = Q(D)f$$

Thus, the system $\mathcal{L}_\Theta \circ \mathcal{L}_\Gamma : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^\infty}(\mathbb{R})$ is a w -continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation $Q(D)x(t) = Q(D)f(t)$. Since $Q(z)/Q(z)$ has no pole on the complex imaginary axis, then, by Theorem 3, it is $\mathcal{L}_\Theta \circ \mathcal{L}_\Gamma = \iota : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^\infty}(\mathbb{R})$. Statement Π_w) is proved.

Now, since $\mathcal{L}_\Gamma$ has the following properties:

- by hypothesis, it is the unique w -continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1),
- by Π_w), it is invertible,
- by I_w), its inverse $\mathcal{L}_\Gamma^{-1} = \mathcal{L}_\Theta$ is the unique w -continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$,

then $\mathcal{L}_\Gamma$ is the unique w -bicontinuous invertible LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1).

($\Rightarrow$) Since, by assumption and Proposition 1, $\mathcal{L}_\Gamma^{-1}$ is a w -continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (2), then, by Lemma 3, the rational function $P(z)/Q(z)$ has no pole on the complex imaginary axis.

Step 2: $\mathcal{L}_\Gamma$ is s -bicontinuous invertible $\Leftrightarrow P(z)/Q(z)$ has no pole on the complex imaginary axis.

($\Leftarrow$) Since the rational function $P(z)/Q(z)$ has no pole on the complex imaginary axis, then, by Theorem 3:

I_s) the system $\mathcal{L}_\Theta : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^\infty}(\mathbb{R})$ defined by $\mathcal{L}_\Theta(g) = \Theta * g$ is a s -continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (2).

We prove now that:

II_s) the system $\mathcal{L}_\Gamma$ is invertible, and $\mathcal{L}_\Gamma^{-1} = \mathcal{L}_\Theta$.

Indeed, as shown in *Step 1* above, $\mathcal{L}_\Gamma$ is a w -continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1), and $\mathcal{L}_\Theta$ is a w -continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (2). Hence, the arguments used to prove II_w) above, prove II_s).

Since $\mathcal{L}_\Gamma$ has the following properties:

- by hypothesis, it is a s -continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1),
- by II_s), it is invertible,
- by I_s), its inverse $\mathcal{L}_\Gamma^{-1} = \mathcal{L}_\Theta$ is a s -continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$,

then $\mathcal{L}_\Gamma$ is a s -bicontinuous invertible LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1).

($\Rightarrow$) Since, by assumption and Proposition 1, $\mathcal{L}_\Gamma^{-1}$ is a s -continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (2), then, by Lemma 3, the rational function $P(z)/Q(z)$ has no pole on the complex imaginary axis.

Step 3: $P(z)/Q(z)$ has no pole on the complex imaginary axis $\Rightarrow a$)– g).

Statements a) and b) have been proved in ($\Leftarrow$) of *Step 1* above.

To prove c), observe that, by I_w), the system $\mathcal{L}_\Theta$ is associated with the differential equation (2). Hence, for every $g(t) \in \mathcal{D}'_{L^\infty}(\mathbb{R})$, the distribution $\mathcal{L}_\Theta(g)$ is a solution in $\mathcal{D}'_{L^\infty}(\mathbb{R})$ of the differential equation (2). By Proposition 5, such solution is the unique in $\mathcal{D}'_{L^\infty}(\mathbb{R})$ if and only if $Q(z)$ has no zero on the complex imaginary axis. Since $\mathcal{L}_\Theta(g) = \mathcal{L}_\Gamma^{-1}(g)$, then c) holds.

Statements d) and e) have been proved in ($\Leftarrow$) of *Step 2* above.

To prove that f) holds, observe that, since the rational function $Q(z)/P(z)$ (resp. $P(z)/Q(z)$) has no pole on the complex imaginary axis, then: if $P(z)$ (resp. $Q(z)$) has no zero on the complex imaginary axis, then, by e) of Theorem 3, the system $\mathcal{L}_\Gamma$ is the unique s -continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (1) (resp. $\mathcal{L}_\Gamma^{-1} = \mathcal{L}_\Theta$ is the unique s -continuous LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$ associated with the differential equation (2)).

Finally, to prove that statement g) holds observe that, as proved in ($\Leftarrow$) of *Step 1* above, it is: $\mathcal{L}_\Gamma^{-1} = \mathcal{L}_\Theta$. Hence, statement g) follows by f) of Theorem 3 applied to the differential equation (2). $\square$

Proof of Lemma 4 The proof is by contradiction, i.e., we prove that: if there exists a continuous LTI IOS system on $L^\infty(\mathbb{R})$ associated with the differential equation (1), then $Q(z)/P(z)$ has no pole on the complex imaginary axis and $\deg Q(z) \leq \deg P(z)$.

The proof has two steps.

Step 1: Proof in the case $n = \deg P(z) = 0$.

To prove the statement, set $p = \infty$ in *Step 1* of the proof of Lemma 2.

Step 2: Proof in the case $n = \deg P(z) \geq 1$.

Let $\mathcal{L}$ be a continuous LTI IOS system on $L^\infty(\mathbb{R})$ associated with the differential equation (1). By a) of Remark 2, the system $\mathcal{L}^* : L^\infty(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R})$ defined by $\mathcal{L}^*(f) = \mathcal{L}(f)$ is a continuous LTI system defined on $L^\infty(\mathbb{R})$ associated with the differential equation (1). Hence, by Theorem 2 of [6], the rational function $Q(s)/P(s)$ has no pole on the complex imaginary axis.

We prove by contradiction that $\deg Q(z) \leq \deg P(z)$.

Assume the contrary, i.e., assume that $\deg Q(z) = \deg P(z) + \nu$, $\nu \geq 1$. Then, see Sect. 3, it is:

$$\Gamma(t) = \Gamma_0(t) + K_0\delta(t) + \sum_{h=1}^{\nu} K_h\delta^{(h)}(t)$$

where:

$$\Gamma_0(t) = \sum_{l \in L_-} \Lambda_l(t) e^{slt} H(t) - \sum_{l \in L_+} \Lambda_l(t) e^{slt} H(-t),$$

and $K_\nu \neq 0$. Let $r(t) \in C^{\nu-1}(\mathbb{R})$ be the function defined by:

$$r(t) = \begin{cases} 0 & \text{for } t < 0 \\ t^{\nu-1/2} & \text{for } t \geq 0 \end{cases}$$

and $\phi(t)$ be a $C^\infty(\mathbb{R})$ function such that:

$$\phi(t) = \begin{cases} 0 & \text{for } t > 3 \\ 1 & \text{for } t < 2 \end{cases}$$

Defined $g(t) = r(t)\phi(t)$, it is:

- (1) $g(t) \in \mathcal{C}_0(\mathbb{R})$ (the subspace of $L^\infty(\mathbb{R})$ of the continuous functions null at infinity in the usual sense),
- (2) for every $j = 0, \dots, \nu - 1$, it is $g^{(j)}(t) \in L^\infty(\mathbb{R})$ (indeed: for $j = 0, \dots, \nu - 1$, the function $g^{(j)}(t)$ is continuous and it has compact support), and (3) $g^{(\nu)}(t) \notin L^\infty(\mathbb{R})$.

Since:

$$(\Gamma * g)(t) = (\Gamma_0 * g)(t) + K_0g(t) + \sum_{h=1}^{\nu} K_hg^{(h)}(t)$$

and $\Gamma_0(t) \in L^1(\mathbb{R})$, then $\Gamma_0 * g + K_0g \in L^\infty(\mathbb{R})$. Moreover, since $K_\nu \neq 0$, for the last summand it is: $\sum_{h=1}^{\nu} K_hg^{(h)} \notin L^\infty(\mathbb{R})$. Thus, $\Gamma * g \notin L^\infty(\mathbb{R})$.

Since, by statement f) of Theorem 2 of [6], for every $f(t) \in \mathcal{C}_0(\mathbb{R})$ it is $\mathcal{L}(f) = \Gamma * f$, then we have $\mathcal{L}(g) \notin L^\infty(\mathbb{R})$. This contradicts the hypothesis that $\mathcal{L}(L^\infty(\mathbb{R})) \subset L^\infty(\mathbb{R})$. $\square$

Proof of Theorem 4 The proof has two steps.

Step 1: Proof in the case $n = \deg P(z) = 0$.

Set $p = \infty$ in *Step 1* of the proof of Theorem 2.

Step 2: Proof in the case $n = \deg P(z) \geq 1$.

Since $Q(z)/P(z)$ has no pole on the complex imaginary axis, and $\deg Q(z) \leq \deg P(z)$, then, see Sect. 3, $\Gamma(t) = c\delta(t) + \gamma(t)$ where $c \in \mathbb{C}$ and $\gamma(t) \in L^1(\mathbb{R})$. Hence, for every $f(t) \in L^\infty(\mathbb{R})$ we have: $\mathcal{L}_\Gamma(f) = cf + \gamma * f$. Since, by Theorem 1.2, Chapter VIII, § 1 of [14], the map $f \mapsto \gamma * f$ is a continuous map from $L^\infty(\mathbb{R})$ into itself, then $\mathcal{L}_\Gamma$ is a continuous LTI IOS system on $L^\infty(\mathbb{R})$.

Since $Q(z)/P(z)$ has no pole on the complex imaginary axis, then (see Remark 1) $\Gamma(t)$ is a fundamental solution of the differential equation (1) in $\mathcal{D}'_{L^1}(\mathbb{R})$. By the *Third idea* of the Introduction of [6], $\mathcal{L}_\Gamma$ is then a continuous LTI system associated with the differential equation (1). This proves a).

The proof of b), c) and d) is obtained by replacing $\mathcal{D}'_{L^\infty}(\mathbb{R})$ with $L^\infty(\mathbb{R})$ in the proof of statements d), e) and b), respectively, of Theorem 3.

The statement e) is a straightforward consequence of e) of Theorem 5 of [6]. $\square$

Proof of Corollary 4 Remember that, by hypothesis, $Q(z)/P(z)$ has no pole on the complex imaginary axis and $\deg Q(z) \leq \deg P(z)$, and that, by Theorem 4, $\mathcal{L}_\Gamma$ is a continuous LTI IOS system on $L^\infty(\mathbb{R})$ associated with the differential equation (1).

The proof has two steps.

Step 1: $\mathcal{L}_\Gamma$ is bicontinuous invertible $\Leftrightarrow P(z)/Q(z)$ has no pole on the complex imaginary axis and $\deg Q(z) = \deg P(z)$.

($\Leftarrow$) Since $P(z)/Q(z)$ has no pole on the complex imaginary axis, and $\deg Q(z) = \deg P(z)$ then, by a) of Theorem 4:

(I) the system $\mathcal{L}_\Theta : L^\infty(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ defined by $\mathcal{L}_\Theta(g) = \Theta * g$ is a continuous LTI IOS system on $L^\infty(\mathbb{R})$ associated with the differential equation (2)—the pathology of $L^\infty(\mathbb{R})$ doesn't allow us to say that it is the unique.

We prove now that:

(II) the system $\mathcal{L}_\Gamma$ is invertible, and $\mathcal{L}_\Gamma^{-1} = \mathcal{L}_\Theta$.

Indeed, let $\mathcal{M}_\Gamma : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^\infty}(\mathbb{R})$ be the system defined by $\mathcal{M}_\Gamma(f) = \Gamma * f$, and $\mathcal{M}_\Theta : \mathcal{D}'_{L^\infty}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^\infty}(\mathbb{R})$ be the system defined by $\mathcal{M}_\Theta(f) = \Theta * f$ (remember that $\Gamma(t)$ and $\Theta(t)$ are in $\mathcal{D}'_{L^1}(\mathbb{R})$ and see Sect. A-10 of the Appendix).

Since $Q(z)/P(z)$ and $P(z)/Q(z)$ have no pole on the complex imaginary axis, by Corollary 3, the system $\mathcal{M}_\Gamma$ is a w -bicontinuous invertible LTI IOS system on $\mathcal{D}'_{L^\infty}(\mathbb{R})$, and its inverse system is $\mathcal{M}_\Gamma^{-1} = \mathcal{M}_\Theta$. Hence, for every $f(t) \in \mathcal{D}'_{L^\infty}(\mathbb{R})$ it is:

$$\mathcal{M}_\Gamma(\mathcal{M}_\Theta(f)) = \mathcal{M}_\Theta(\mathcal{M}_\Gamma(f)) = f$$

Since for every $f(t) \in L^\infty(\mathbb{R})$ it is: $\mathcal{L}_\Gamma(f) = \mathcal{M}_\Gamma(f)$ and $\mathcal{L}_\Theta(f) = \mathcal{M}_\Theta(f)$, then for every $f(t) \in L^\infty(\mathbb{R})$ it is:

$$\mathcal{L}_\Gamma(\mathcal{L}_\Theta(f)) = \mathcal{M}_\Gamma(\mathcal{M}_\Theta(f)) = f \quad \text{and} \quad \mathcal{L}_\Theta(\mathcal{L}_\Gamma(f)) = \mathcal{M}_\Theta(\mathcal{M}_\Gamma(f)) = f$$

i.e., $\mathcal{L}_\Gamma$ is invertible, and $\mathcal{L}_\Gamma^{-1} = \mathcal{L}_\Theta$. Statement II) is proved.

Now, since $\mathcal{L}_\Gamma$ has the following properties:

- by hypothesis, it is a continuous LTI IOS system on $L^\infty(\mathbb{R})$ associated with the differential equation (1),
- by II), it is invertible,
- by I), its inverse $\mathcal{L}_\Gamma^{-1} = \mathcal{L}_\Theta$ is a continuous LTI IOS system on $L^\infty(\mathbb{R})$,

then $\mathcal{L}_\Gamma$ is a bicontinuous invertible LTI IOS system on $L^\infty(\mathbb{R})$ associated with the differential equation (1).

($\Rightarrow$) By assumption and Proposition 1, $\mathcal{L}_\Gamma^{-1}$ is a continuous LTI IOS system on $L^\infty(\mathbb{R})$ associated with the differential equation (2). By Lemma 4, the rational function $P(z)/Q(z)$ has no pole on the complex imaginary axis, and $\deg P(z) \leq \deg Q(z)$, hence $\deg Q(z) = \deg P(z)$.

Step 2: $P(z)/Q(z)$ has no pole on the complex imaginary axis and $\deg Q(z) = \deg P(z) \Rightarrow a), b), c)$.

Statements *a)* and *b)* have been proved in ($\Leftarrow$) of *Step 1* above.

To prove that *c)* holds, observe that, since the rational function $Q(z)/P(z)$ (resp. $P(z)/Q(z)$) has no pole on the complex imaginary axis and $\deg P(z) = \deg Q(z)$, then: if $P(z)$ (resp. $Q(z)$) has no zero on the complex imaginary axis, then, by *c)* of Theorem 4, the system $\mathcal{L}_\Gamma$ is the unique continuous LTI IOS system on $L^\infty(\mathbb{R})$ associated with the differential equation (1) (resp. $\mathcal{L}_\Gamma^{-1} = \mathcal{L}_\Theta$ is the unique continuous LTI IOS system on $L^\infty(\mathbb{R})$ associated with the differential equation (2)).

To prove that *d)* holds, observe that, as proved in ($\Leftarrow$) of *Step 1* above, it is: $\mathcal{L}_\Gamma^{-1} = \mathcal{L}_\Theta$. Hence, statement *d)* follows by *e)* of Theorem 4 applied to the differential equation (2). $\square$

6 An Application: Almost-Inverse of a Noninvertible System

In [3], the authors consider a space A of bandlimited functions (used to model some EEG signals), and face the problem to reconstruct a signal $f(t) \in A$, given the continuous wavelet transform of $f(t)$, defined by a suitable wavelet, at a given scale. The wavelet transform they consider is realized, at the given scale, by a continuous LTI IOS system $\mathcal{H}$ on $L^2(\mathbb{R})$ associated with a differential equation. Since the system $\mathcal{H}$ has not a continuous LTI IOS inverse system, they find a continuous LTI IOS system $\mathcal{G}$ on $L^2(\mathbb{R})$, associated with a suitably modified differential equation, such that, for every signal $f(t) \in A$, the signal $\mathcal{G}(\mathcal{H}(f))$ is an *accurate approximation* of $f(t)$.

As an application of the theoretical results given in the previous sections, we give, on a simple example, a theoretical foundation to the technique used in [3]. Precisely, we show how to find a suitable continuous “almost-inverse” system of a noninvertible continuous LTI system associated with a given differential equation.

Let $P(z) = (z + 1)(z + 2)$, $Q(z) = z + 3$ and consider the differential equation:

$$(D + 1)(D + 2)x(t) = (D + 3)f(t) \quad (3)$$

By *a*) of Theorem 2, there exists a unique continuous LTI IOS system on $L^2(\mathbb{R})$ associated with the differential equation (3): the system $\mathcal{L}_\Gamma : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ defined by $\mathcal{L}_\Gamma(f) = \Gamma * f$, where

$$\Gamma(t) = 2e^{-t}H(t) - e^{-2t}H(t)$$

is the distribution obtained by $\mathbf{P}(P(z), Q(z))$, see Sect. 3. Moreover, by *c*) of Theorem 2, $\mathcal{L}_\Gamma$ is a causal system.

Since $\deg P(z) \neq \deg Q(z)$, by Corollary 2 the system $\mathcal{L}_\Gamma$ is not a bicontinuous invertible LTI IOS system. Hence, either it is not invertible, or its inverse is not continuous.

Actually, the system $\mathcal{L}_\Gamma : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is (a) one-to-one, but (b) *not* onto—indeed:

- (a) if $f(t), g(t) \in L^2(\mathbb{R})$ are such that $\mathcal{L}_\Gamma(f) = \mathcal{L}_\Gamma(g)$, then it is: $P(D)\mathcal{L}_\Gamma(f) = Q(D)f$ and $P(D)\mathcal{L}_\Gamma(g) = Q(D)g$, hence: $Q(D)f = Q(D)g$, i.e., $Q(D)(f - g) = 0$; by Proposition 4, this implies $f(t) = g(t)$;
- (b) let $g(t) = e^{-t}H(t) \in L^2(\mathbb{R})$; if there would exist a function $f(t) \in L^2(\mathbb{R})$ such that $\mathcal{L}_\Gamma(f) = g$, then $f(t)$ would verify $(D+1)(D+2)e^{-t}H(t) = (D+3)f(t)$; as a consequence, by applying the Fourier transform⁹ we would obtain $\hat{f}(\omega) = (i\omega+2)/(i\omega+3)$, and hence, by Plancherel's Theorem, it would be $(i\omega+2)/(i\omega+3) \in L^2(\mathbb{R})$, which is false.

Hence $\mathcal{L}_\Gamma : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ has not an inverse system, i.e., there does not exist an LTI IOS system $\mathcal{L} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ such that for every $f(t) \in L^2(\mathbb{R})$ it is:

$$\mathcal{L}_\Gamma(\mathcal{L}(f)) = f \quad \text{and} \quad \mathcal{L}(\mathcal{L}_\Gamma(f)) = f$$

Now, for every positive real number B , let:

$$\Omega_B = \{f(t) \in L^2(\mathbb{R}) \text{ such that } \text{supp } \hat{f}(\omega) \subset [-B, B]\}$$

be the vector space of the *finite energy B-bandlimited* signals.

The idea to find a continuous “*almost inverse*” of $\mathcal{L}_\Gamma$ rests on the introduction of a suitable polynomial $r(z)$ such that, defined:

$$Q_*(z) = Q(z)r(z)$$

it is:

- (i) there exists a bicontinuous invertible LTI IOS system $\mathcal{L} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ associated with the differential equation $P(D)x(t) = Q_*(D)f(t)$, and
- (ii) $\mathcal{L}^{-1} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is an “*almost inverse*” of $\mathcal{L}_\Gamma$ in the sense that: given positive real numbers ϵ and B , for every $f(t) \in \Omega_B$, it is:

$$\frac{\|\mathcal{L}_\Gamma(\mathcal{L}^{-1}(f)) - f\|_2}{\|f\|_2} \leq \epsilon \quad \text{and} \quad \frac{\|\mathcal{L}^{-1}(\mathcal{L}_\Gamma(f)) - f\|_2}{\|f\|_2} \leq \epsilon$$

⁹ For every $f(t) \in L^2(\mathbb{R})$, we denote by $\hat{f}(\omega) = L^2\text{-}\lim_{r \rightarrow \infty} \int_{-r}^r e^{-i\omega t} f(t) dt$ the Fourier transform of $f(t)$.

i.e., the L^2 -relative error made by approximating f by both $\mathcal{L}_\Gamma(\mathcal{L}^{-1}(f))$ and $\mathcal{L}^{-1}(\mathcal{L}_\Gamma(f))$ is upper-bounded by ϵ .

By Corollary 2, a polynomial $Q_*(z)$ satisfying condition (i) must be such that the rational functions $Q_*(z)/P(z)$ and $P(z)/Q_*(z)$ have no pole on the complex imaginary axis, and $\deg P(z) = \deg Q_*(z)$. Hence, an almost canonical choice is:

$$Q_*(z) = K(z + 3)(z + \omega_*)$$

where K and ω_* are two nonzero real numbers.

The following considerations show that: if $K = 1/\omega_*$ and ω_* is such that

$$\sqrt{\frac{B^2}{B^2 + \omega_*^2}} \leq \epsilon \quad (4)$$

then also condition (ii) is satisfied.

From a merely heuristic point of view, observe that: the previous choices imply that, when ω_* approaches infinity, then the factor

$$r(z) = K(z + \omega_*) = \frac{z}{\omega_*} + 1$$

approaches the polynomial 1, and hence suggest that the *modified differential equation* approaches the *original differential equation*.

By Corollary 2, there exists a unique bicontinuous invertible LTI IOS system $\mathcal{L} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ associated with the differential equation $P(D)x(t) = Q_*(D)f(t)$, its inverse $\mathcal{L}^{-1} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is defined by $\mathcal{L}^{-1}(f) = h * f$, where $h(t)$ is the distribution obtained by $\mathbf{P}(Q_*(z), P(z))$, and, for every $f(t) \in L^2(\mathbb{R})$, the function $\mathcal{L}^{-1}(f)$ is the unique solution in $L^2(\mathbb{R})$ of the differential equation $Q_*(D)x(t) = P(D)f(t)$.

Let $f(t) \in \Omega_B$ be given, let $y = \mathcal{L}^{-1}(f)$ and let $s = \mathcal{L}_\Gamma(y) = \mathcal{L}_\Gamma(\mathcal{L}^{-1}(f))$. It is:

$$Q_*(D)y(t) = P(D)f(t) \quad \text{and} \quad P(D)s(t) = Q(D)y(t)$$

and hence:

$$\hat{y}(\omega) = \frac{P(i\omega)}{Q_*(i\omega)} \hat{f}(\omega) \quad \text{and} \quad \hat{s}(\omega) = \frac{Q(i\omega)}{P(i\omega)} \hat{y}(\omega) = \frac{\omega_*}{i\omega + \omega_*} \hat{f}(\omega)$$

Since for every $g(t) \in L^2(\mathbb{R})$ it is $\|g(t)\|_2 = \sqrt{1/2\pi} \|\hat{g}(\omega)\|_2$ (Plancherel's Theorem), then:

$$\|\mathcal{L}_\Gamma(\mathcal{L}^{-1}(f)) - f\|_2 = \|s - f\|_2 = \frac{1}{\sqrt{2\pi}} \|\hat{s} - \hat{f}\|_2 = \frac{1}{\sqrt{2\pi}} \left\| \left(\frac{\omega_*}{i\omega + \omega_*} - 1 \right) \hat{f}(\omega) \right\|_2$$

Since $\text{supp } \hat{f}(\omega) \subset [-B, B]$, then:

$$\left\| \left(\frac{\omega_*}{i\omega + \omega_*} - 1 \right) \hat{f}(\omega) \right\|_2 \leq \max_{|\omega| \leq B} \left| \frac{\omega_*}{i\omega + \omega_*} - 1 \right| \|\hat{f}(\omega)\|_2$$

Moreover,

$$\max_{|\omega| \leq B} \left| \frac{\omega_*}{i\omega + \omega_*} - 1 \right| = \sqrt{\frac{B^2}{B^2 + \omega_*^2}}$$

Finally, using (4), we get:

$$\|\mathcal{L}_\Gamma(\mathcal{L}^{-1}(f)) - f\|_2 \leq \sqrt{\frac{B^2}{B^2 + \omega_*^2}} \frac{\|\hat{f}(\omega)\|_2}{\sqrt{2\pi}} \leq \epsilon \|f(t)\|_2$$

Hence, $\mathcal{L}$ verifies the first inequality of condition (ii).

Similar arguments prove that $\mathcal{L}$ verifies also the second inequality of condition (ii).

Observe that, since (see [18, Section 6.3, p. 197]) for every $g(t) \in \Omega_B$ it is:

$$\|g(t)\|_\infty \leq \sqrt{\frac{B}{\pi}} \|g(t)\|_2$$

then, given positive real numbers ϵ_* and E , if ω_* satisfies:

$$\sqrt{\frac{B^2}{B^2 + \omega_*^2}} \leq \sqrt{\frac{\pi}{BE}} \epsilon_* \quad (5)$$

then, for every $f(t) \in \Omega_B$ with energy $\|f(t)\|_2^2 \leq E$, it is also:

$$\|\mathcal{L}_\Gamma(\mathcal{L}^{-1}(f)) - f\|_\infty \leq \epsilon_* \quad \text{and} \quad \|\mathcal{L}^{-1}(\mathcal{L}_\Gamma(f)) - f\|_\infty \leq \epsilon_*$$

i.e., the L^∞ -absolute error made by approximating f by both $\mathcal{L}_\Gamma(\mathcal{L}^{-1}(f))$ and $\mathcal{L}^{-1}(\mathcal{L}_\Gamma(f))$ is upper-bounded by ϵ_* .

Finally, observe that if $\omega_* > 0$, then, by c) of Corollary 2, also the system $\mathcal{L}^{-1}$ is causal.

7 Conclusions

In the context of analog signal processing, given a space $\mathcal{S}$ of input signals and a differential equation with constant coefficients $P(D)x(t) = Q(D)f(t)$, we considered the usual problem of the existence of continuous single-input single-output LTI IOS systems on $\mathcal{S}$ associated with the differential equation, of their invertibility, and of the determination of their inverse.

When the input space $\mathcal{S}$ is the space $\mathcal{D}'_+(\mathbb{R})$ of the distributional signals with left-bounded support, the answer is well known. Precisely, there always exists a *unique* continuous LTI IOS system on $\mathcal{S}$ associated with the differential equation: the system $\mathcal{L}$ defined by $\mathcal{L}(f) = h * f$ where $h(t)$ is the inverse (unilateral) Laplace transform of the transfer function $Q(s)/P(s)$. This unique system $\mathcal{L}$ has a continuous inverse, and this inverse is *the unique* continuous LTI IOS systems on $\mathcal{S}$ associated with the differential equation $Q(D)x(t) = P(D)g(t)$.

The space $\mathcal{D}'_+(\mathbb{R})$ of all the signals with left-bounded support is not, obviously, the unique space of analog signals of interest in analog signal processing. Other usually considered spaces are the spaces $L^p(\mathbb{R})$. However, concerning the previous problem, only sporadic and partial results are found in the literature.

Besides there are other spaces that should have been considered, and instead are totally ignored in the literature on signal theory (apart [4]–[6]): the $\mathcal{D}'_{L^p}(\mathbb{R})$, i.e., the spaces spanned by each $L^p(\mathbb{R})$ and the derivatives of any order of its members. The reason why they should have been used and studied is obvious: working with $L^p(\mathbb{R})$ and with differential equations, $\mathcal{D}'_{L^p}(\mathbb{R})$ is the least space in which computations can be performed.

Then, we considered as input space $\mathcal{S}$ the space of L^p functions and the space of $\mathcal{D}'_{L^p}$ distributions, input spaces which include signals with not necessarily left-bounded support. In each case, we gave a *systematic* theoretical analysis of the existence, uniqueness and invertibility of continuous LTI IOS systems (both causal and non-causal) on $\mathcal{S}$ associated with the differential equation and, in every case of invertibility, we characterized the inverse system.

This paper offers to researchers engaged in problems of differential equations and L^p signals, the landscape of the related continuous, and of bicontinuous invertible, LTI IOS systems, on $\mathcal{D}'_{L^p}(\mathbb{R})$ and $L^p(\mathbb{R})$.

This landscape, for the equation $P(D)x(t) = Q(D)f(t)$, can be summarized as follows.

When the input space $\mathcal{S}$ is $\mathcal{D}'_{L^\infty}(\mathbb{R})$ endowed with the *weak* convergence, or the input space $\mathcal{S}$ is $\mathcal{D}'_{L^p}(\mathbb{R})$ or $L^p(\mathbb{R})$ with $1 \leq p < \infty$, the landscape is simple. Precisely, if the rational function $Q(z)/P(z)$ satisfies a simple condition *necessary* to the existence of continuous LTI IOS systems on $\mathcal{S}$ associated with the differential equation $P(D)x(t) = Q(D)f(t)$, then there exists a *unique* such system: the system $\mathcal{L}_\Gamma$ defined by $\mathcal{L}_\Gamma(f) = \Gamma * f$, where $\Gamma(t)$ is the distribution defined in Sect. 3 by the procedure $\mathbf{P}(P(z), Q(z))$. Moreover, if (and only if) also the rational function $P(z)/Q(z)$ satisfies the same simple condition satisfied by $Q(z)/P(z)$, $\mathcal{L}_\Gamma$ has a continuous inverse, and this inverse is *the unique* continuous LTI IOS systems on $\mathcal{S}$ associated with the differential equation $Q(D)x(t) = P(D)g(t)$.

On the other hand, when the input space $\mathcal{S}$ is $\mathcal{D}'_{L^\infty}(\mathbb{R})$ endowed with the *strong* convergence, or the input space $\mathcal{S}$ is $L^\infty(\mathbb{R})$, by the pathology of the space $\mathcal{S}$, the landscape is not completely definite: when the rational function $Q(z)/P(z)$ satisfies the necessary condition, $\mathcal{L}_\Gamma$ is now *simply one* of the continuous LTI IOS systems on $\mathcal{S}$ associated with the differential equation $P(D)x(t) = Q(D)f(t)$, *not necessarily* the unique. Moreover, if (and only if) also the rational function $P(z)/Q(z)$ satisfies the same simple condition satisfied by $Q(z)/P(z)$, $\mathcal{L}_\Gamma$ has still a continuous inverse,

but this inverse is now *simply one* of the continuous LTI IOS systems on $\mathcal{S}$ associated with the differential equation $Q(D)x(t) = P(D)g(t)$, *not necessarily* the unique.

Acknowledgements The author would like to thank the anonymous reviewers for their comments and suggestions which prompted him to greatly improve the presentation of the results, and Prof. M. Poletti for his valuable discussions.

Funding Open access funding provided by Università di Pisa within the CRUI-CARE Agreement.

Compliance with Ethical Standards

Conflicts of interest The authors declare that they have no conflict of interest.

Data Availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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A Appendix: Distributions

In this section, for the reader's convenience, we describe the spaces of distributions used in the paper and recall some notions concerning distributional solutions of ordinary differential equations with constant coefficients.

A-1 Identification of Function Equal Almost Everywhere (a.e.)

Let $f(t)$ and $g(t)$ be two functions defined on $\mathbb{R}$. The functions are *almost everywhere equal* if for *almost every* $t \in \mathbb{R}$ it is $f(t) = g(t)$, i.e., if there exists a set $\Omega \subset \mathbb{R}$ with *Lebesgue measure equal to zero* such that for every $t \in \mathbb{R} \setminus \Omega$ it is $f(t) = g(t)$. As usual, $f(t)$ and $g(t)$ will be tacitly considered as a *unique function*, and the equality $f(t) = g(t)$ will be said to hold.

For instance, for the Heaviside step function $H(t)$ defined as

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

the following equalities hold:

$$H(t) = \chi_{[0, +\infty)}(t) = \chi_{(0, +\infty)}(t) = \chi_{(0, +\infty) \cup \mathbb{Z}}(t) = \chi_{[0, +\infty) \cup \mathbb{Q}}(t)$$

(here $\chi_A(t)$ denotes *the characteristic function of the set A*).¹⁰

This usual tacit identification avoids to substitute the familiar notion of *function* with a new unfamiliar notion of—let's say—*essential function*, where the essential functions are the equivalence classes of functions, modulo the equivalence relation defined by: $f(t) \sim g(t)$ if and only if $f(t)$ and $g(t)$ are almost everywhere equal.

Once established this identification, the only functions $f(t)$ for which the value $f(t_0)$ at each single point $t_0 \in \mathbb{R}$ is *still an actually meaningful information* are the $f(t) \in C^0(\mathbb{R})$, i.e., *the continuous functions*.

A-2 The Space $L^1_{\text{loc}}(\mathbb{R})$

$L^1_{\text{loc}}(\mathbb{R})$ is the space of the functions $f(t)$ locally integrable on $\mathbb{R}$, i.e., of the Lebesgue measurable functions $f: \mathbb{R} \rightarrow \mathbb{C}$ which are Lebesgue integrable on every compact set $K \subset \mathbb{R}$.

A-3 The spaces $L^p(\mathbb{R})$, $1 \leq p \leq \infty$

For every p such that $1 \leq p < \infty$, $L^p(\mathbb{R})$ is the space of Lebesgue measurable functions $f(t)$ on $\mathbb{R}$, such that $|f(t)|^p$ is a Lebesgue integrable function; for every $f(t) \in L^p(\mathbb{R})$ let:

$$\|f(t)\|_p = \left(\int_{\mathbb{R}} |f(t)|^p dt \right)^{1/p}$$

$L^\infty(\mathbb{R})$ is the space of Lebesgue measurable bounded functions on $\mathbb{R}$, i.e., the space of Lebesgue measurable functions $f(t)$ such that: there exists $\lambda \in \mathbb{R}$ such that for *almost every* $t \in \mathbb{R}$ it is $|f(t)| \leq \lambda$ (such a λ is called an *essential upper bound* of $|f(t)|$); for every $f(t) \in L^\infty(\mathbb{R})$ let¹¹

$$\|f(t)\|_\infty = \text{essential sup}_{t \in \mathbb{R}} |f(t)|$$

It is well known that, for every p such that $1 \leq p \leq \infty$, $\|\cdot\|_p$ is a *norm* in $L^p(\mathbb{R})$, and that, endowed with such norm, $L^p(\mathbb{R})$ is a *Banach space* (pay attention: this is the first time we are using the *identification* of Sect. A-1).

A-4 The Space $\mathcal{D}(\mathbb{R})$

$\mathcal{D}(\mathbb{R})$ is the space of the functions $\varphi(t) \in C^\infty(\mathbb{R})$ such that $\text{supp } \varphi(t)$ is a compact subset of $\mathbb{R}$.

¹⁰ If $A \subset \mathbb{R}$, the characteristic function of A is the function $\chi_A: \mathbb{R} \rightarrow \mathbb{C}$ defined by: $\chi_A(t) = 1$ if $t \in A$ and $\chi_A(t) = 0$ if $t \notin A$

¹¹ Let $f(t)$ be a function on $\mathbb{R}$. The *essential sup* of $f(t)$ is the infimum of the set of the essential upper bounds of $f(t)$.

Let $\varphi_k(t)$, $\varphi(t)$ be a sequence and an element, respectively, in $\mathcal{D}(\mathbb{R})$. The sequence $\varphi_k(t)$ converges to $\varphi(t)$ in $\mathcal{D}(\mathbb{R})$, or

$$\mathcal{D}\text{-}\lim_{k \rightarrow \infty} \varphi_k(t) = \varphi(t),$$

mean that:

- there exists a compact set $K \subset \mathbb{R}$ such that: for every $k \in \mathbb{N}$ it is $\text{supp } \varphi_k(t) \subset K$,
- for every $q \in \mathbb{N}$, the sequence $D^q \varphi_k(t)$ converges to $D^q \varphi(t)$ uniformly on $\mathbb{R}$.

A subset B of $\mathcal{D}(\mathbb{R})$ is called a *bounded subset of $\mathcal{D}(\mathbb{R})$* if there exist a compact $K \subset \mathbb{R}$, and positive real numbers M_q , $q \in \mathbb{N}$, such that:

- for every $\varphi(t) \in B$ it is $\text{supp } \varphi(t) \subset K$,
- for every $q \in \mathbb{N}$ it is

$$\sup\{\|D^q \varphi(t)\|_\infty : \varphi(t) \in B\} \leq M_q$$

A-5 The Space $\mathcal{D}'(\mathbb{R})$ of the Distributions on $\mathbb{R}$ and the Convolution by $\mathcal{D}$ Functions

By L. Schwartz's definition, $\mathcal{D}'(\mathbb{R})$ is the space of the continuous linear functionals $T : \mathcal{D}(\mathbb{R}) \rightarrow \mathbb{C}$. These functionals are called *distributions (on $\mathbb{R}$)*. If $T \in \mathcal{D}'(\mathbb{R})$ and $\varphi(t) \in \mathcal{D}(\mathbb{R})$, the number $T(\varphi(t)) \in \mathbb{C}$ is usually denoted by $\langle T, \varphi(t) \rangle$.

For every $f(t) \in L^1_{\text{loc}}(\mathbb{R})$, the map $T_f : \mathcal{D}(\mathbb{R}) \rightarrow \mathbb{C}$ defined by

$$T_f(\varphi(t)) = \int_{\mathbb{R}} f(t)\varphi(t)dt$$

is a distribution.

By [28, Lemma in Section 1.6 of Chapter 1], the following theorem holds (*pay attention*: this is the second time we are using the *identification* of Sect. A-1):

Theorem 5 (Du Bois-Reymond) *Let $f(t), g(t) \in L^1_{\text{loc}}(\mathbb{R})$. The following statement holds:*

$$T_f = T_g \Leftrightarrow f(t) = g(t)$$

As a consequence of Theorem 5, the linear map

$$\begin{aligned} L^1_{\text{loc}}(\mathbb{R}) &\rightarrow \mathcal{D}'(\mathbb{R}) \\ f(t) &\rightarrow T_f \end{aligned}$$

is injective, hence the image of $L^1_{\text{loc}}(\mathbb{R})$ through this map is an *isomorphic copy in $\mathcal{D}'(\mathbb{R})$* of $L^1_{\text{loc}}(\mathbb{R})$. As a consequence, for every $f(t) \in L^1_{\text{loc}}(\mathbb{R})$, the distribution T_f (the copy of $f(t)$ in $\mathcal{D}'(\mathbb{R})$) will be identified with $f(t)$ and denoted by the same symbol $f(t)$. This allows to handle $L^1_{\text{loc}}(\mathbb{R})$ as a subspace of $\mathcal{D}'(\mathbb{R})$.

If $f(t) \in L^1_{\text{loc}}(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$ and $\varphi(t) \in \mathcal{D}(\mathbb{R})$, the complex number $\langle f(t), \varphi(t) \rangle$ verifies:

$$\langle f(t), \varphi(t) \rangle = \langle T_f, \varphi(t) \rangle = \int_{\mathbb{R}} f(t) \varphi(t) dt$$

hence it is the φ -weighted average of the signal $f(t)$ and can be consequently interpreted as the result of the measure of $f(t)$ by the measuring instrument $\varphi(t)$ (see [10, Section 11.3.5, p. 279]).

From here on, all the distributions will be denoted by function-like symbols $f(t)$, the elements $\varphi(t) \in \mathcal{D}(\mathbb{R})$ will be interpreted as the allowed measuring instruments—distributed on $\mathbb{R}$ —for the signals in $\mathcal{D}'(\mathbb{R})$ (see [17, Appendix I, Section I-3, p. 281], [26, Section 3.3.1, p. 35]), and the measure of a signal $f(t) \in \mathcal{D}'(\mathbb{R})$ by the measuring instrument $\varphi(t) \in \mathcal{D}(\mathbb{R})$, i.e., $\langle f(t), \varphi(t) \rangle \in \mathbb{C}$, will be denoted by the integral-like symbol

$$\int f(t) \bullet \varphi(t)$$

Hence, the “ideal experiment” to know a distribution $f(t)$ is to take into account all the measuring instruments $\varphi(t)$ distributed on $\mathbb{R}$ and, for each of them, to record the result of the measure of $f(t)$.

The following items summarize some usual “calculus on distributions.”

- If $f(t) \in \mathcal{D}'(\mathbb{R})$, the continuous linear functional $T : \mathcal{D}(\mathbb{R}) \rightarrow \mathbb{C}$ defined by:

$$T(\varphi(t)) = - \int f(t) \bullet (D\varphi)(t)$$

is a distribution on $\mathbb{R}$. Well-known non-trivial arguments allow to discover that T plays the role of the derivative of $f(t)$. Consequently, T is denoted by $Df(t)$.

- Let $f(t) \in \mathcal{D}'(\mathbb{R})$; for every $\tau \in \mathbb{R}$, $f(t - \tau) \in \mathcal{D}'(\mathbb{R})$ is the distribution defined by:

$$\int f(t - \tau) \bullet \varphi(t) = \int f(t) \bullet \varphi(t + \tau)$$

- Let $f(t) \in \mathcal{D}'(\mathbb{R})$; for every $q \in \mathbb{N}$, $D^q f(t)$ is the distribution defined by:

$$\int D^q f(t) \bullet \varphi(t) = (-1)^q \int f(t) \bullet (D^q \varphi)(t)$$

- The Dirac impulse $\delta(t)$ is the distribution defined by:

$$\int \delta(t) \bullet \varphi(t) = \varphi(0)$$

- Let $\tau \in \mathbb{R}$; $\delta(t - \tau)$ —i.e., *the Dirac impulse at τ* —is the distribution such that:

$$\int \delta(t - \tau) \bullet \varphi(t) = \int \delta(t) \bullet \varphi(t + \tau) = \varphi(\tau)$$

- Let $q \in \mathbb{N}$; $D^q \delta(t)$ is the distribution such that:

$$\int D^q \delta(t) \bullet \varphi(t) = (-1)^q \int \delta(t) \bullet (D^q \varphi)(t) = (-1)^q (D^q \varphi)(0)$$

Finally, let $f(t) \in \mathcal{D}'(\mathbb{R})$ and $\varphi(t) \in \mathcal{D}(\mathbb{R})$. By [23, Ch. VI, §4, Th. XI, p. 166], the *convolution* $(f * \varphi)(t)$ is the function defined, for every $t \in \mathbb{R}$, by:

$$(f * \varphi)(t) = \int f(\tau) \bullet \varphi(t - \tau) \quad (6)$$

and it is: $(f * \varphi)(t) \in C^\infty(\mathbb{R})$.

Observe that, for every $t_0 \in \mathbb{R}$, the complex number $(f * \varphi)(t_0)$ is *the measure of $f(t)$ by the measuring instrument*

$$\tilde{\varphi}(t) = \varphi(-t) \in \mathcal{D}(\mathbb{R}) \quad (\text{“thought as centered in } 0 \in \mathbb{R}\text{”})$$

“with its center translated in $t_0 \in \mathbb{R}$.”

A-6 The Spaces $\mathcal{D}_{L^p}(\mathbb{R})$

For $1 \leq p \leq \infty$, $\mathcal{D}_{L^p}(\mathbb{R})$ is (see [23, Ch. 6, §8, p.199]) the space of the $\varphi(t) \in C^\infty(\mathbb{R})$ such that:

$$\text{for every } q \in \mathbb{N} : \quad D^q \varphi(t) \in L^p(\mathbb{R})$$

Moreover, $\dot{\mathcal{D}}_{L^\infty}(\mathbb{R})$ is the subspace of $\mathcal{D}_{L^\infty}(\mathbb{R})$ of the $\varphi(t)$ such that:

$$\text{for every } q \in \mathbb{N} : \quad \lim_{|t| \rightarrow \infty} D^q \varphi(t) = 0$$

Let $\varphi_k(t)$, $\varphi(t)$ be a sequence and an element, respectively, in $\mathcal{D}_{L^p}(\mathbb{R})$. *The sequence $\varphi_k(t)$ converges to $\varphi(t)$ in $\mathcal{D}_{L^p}(\mathbb{R})$, or*

$$\mathcal{D}_{L^p} - \lim_{k \rightarrow \infty} \varphi_k(t) = \varphi(t)$$

mean that:

$$\text{for every } q \in \mathbb{N} : \quad L^p - \lim_{k \rightarrow \infty} D^q \varphi_k(t) = D^q \varphi(t)$$

The space $\mathcal{D}(\mathbb{R})$ is dense in $\mathcal{D}_{L^p}(\mathbb{R})$ for $1 \leq p < \infty$, and in $\dot{\mathcal{D}}_{L^\infty}(\mathbb{R})$, but is not dense in $\mathcal{D}_{L^\infty}(\mathbb{R})$.

Let $1 < p < q < \infty$: the following relations hold

$$\mathcal{D}(\mathbb{R}) \subset \mathcal{D}_{L^1}(\mathbb{R}) \subset \mathcal{D}_{L^p}(\mathbb{R}) \subset \mathcal{D}_{L^q}(\mathbb{R}) \subset \dot{\mathcal{D}}_{L^\infty}(\mathbb{R}) \subset \mathcal{D}_{L^\infty}(\mathbb{R})$$

A subset B of one of the $\mathcal{D}_{L^p}(\mathbb{R})$, $1 \leq p \leq \infty$ is called a *bounded subset* of $\mathcal{D}_{L^p}(\mathbb{R})$ if there exist positive real numbers $M_q, q \in \mathbb{N}$, such that:

$$\text{for every } q \in \mathbb{N} : \sup\{\|D^q \varphi(t)\|_p : \varphi(t) \in B\} \leq M_q$$

A subset B of $\dot{\mathcal{D}}_{L^\infty}(\mathbb{R})$ is called a *bounded subset* of $\dot{\mathcal{D}}_{L^\infty}(\mathbb{R})$ if there exist positive real numbers $M_q, q \in \mathbb{N}$, such that:

$$\text{for every } q \in \mathbb{N} : \sup\{\|D^q \varphi(t)\|_\infty : \varphi(t) \in B\} \leq M_q$$

A-7 The Spaces $\mathcal{D}'_{L^p}(\mathbb{R})$

By [23, Ch. VI, §8, p. 199], $\mathcal{D}'_{L^1}(\mathbb{R})$ is the space of the continuous linear functionals on $\dot{\mathcal{D}}_{L^\infty}(\mathbb{R})$. If $f(t) \in \mathcal{D}'_{L^1}(\mathbb{R})$, and $\varphi(t) \in \dot{\mathcal{D}}_{L^\infty}(\mathbb{R})$, the number $f(\varphi(t)) \in \mathbb{C}$, usually denoted by $\langle f(t), \varphi(t) \rangle$, will be denoted by:

$$\int f(t) \bullet \varphi(t)$$

The element $\varphi(t)$ will be interpreted as a *measuring instrument* on the members of $\mathcal{D}'_{L^1}(\mathbb{R})$, and the members of $\dot{\mathcal{D}}_{L^\infty}(\mathbb{R})$ will be called the *allowed measuring instruments* on $\mathcal{D}'_{L^1}(\mathbb{R})$.

For $1 < p \leq \infty$, $\mathcal{D}'_{L^p}(\mathbb{R})$ is the space of the continuous linear functionals on $\mathcal{D}_{L^{p'}}(\mathbb{R})$, $1/p + 1/p' = 1$, and similar notations and notions as above will be adopted.

Observe that every $f(t) \in \mathcal{D}'_{L^p}(\mathbb{R})$ induces a continuous linear functional on $\mathcal{D}(\mathbb{R})$, precisely:

$$\begin{aligned} \mathcal{D}(\mathbb{R}) &\rightarrow \mathbb{C} \\ \varphi(t) &\rightarrow \int f(t) \bullet \varphi(t) \end{aligned}$$

As a consequence, every $f(t) \in \mathcal{D}'_{L^p}(\mathbb{R})$ becomes a *particular* member of $\mathcal{D}'(\mathbb{R})$.

By the previous remark, $\mathcal{D}'_{L^p}(\mathbb{R})$ is a subspace of $\mathcal{D}'(\mathbb{R})$. The following theorem characterizes the $f(t) \in \mathcal{D}'(\mathbb{R})$ which are members of $\mathcal{D}'_{L^p}(\mathbb{R})$ (for the proof see [23, Ch. VI, §8, Th. XXV, p. 201]).

Theorem 6 *Let $1 \leq p \leq \infty$. For every $f(t) \in \mathcal{D}'(\mathbb{R})$, the following statements are equivalent:*

- $f(t) \in \mathcal{D}'_{L^p}(\mathbb{R})$,

– there exist $m \in \mathbb{N}$, and a family $f_q^\circ(t) \in L^p(\mathbb{R})$, $q \in \mathbb{N}$, $q \leq m$, such that

$$f(t) = \sum_{0 \leq q \leq m} D^q f_q^\circ(t)$$

– for every $\varphi(t) \in \mathcal{D}(\mathbb{R})$ it is $(f * \varphi)(t) \in L^p(\mathbb{R})$ (see Eq. (6) in Sect. A-5),
 – for every $\varphi(t) \in \mathcal{D}(\mathbb{R})$ it is $(f * \varphi)(t) \in \mathcal{D}_{L^p}(\mathbb{R})$.

$\dot{\mathcal{D}}'_{L^\infty}(\mathbb{R})$ is the space of the *distributions null at infinity*, i.e., of the $f(t) \in \mathcal{D}'(\mathbb{R})$ such that: for every $\varphi(t) \in \mathcal{D}(\mathbb{R})$ the smooth function $(f * \varphi)(t)$ is null at infinity, i.e., verifies:

$$\lim_{|t| \rightarrow \infty} (f * \varphi)(t) = 0$$

(see [23, Sect. VI, §8, *Les espaces de distributions* ($\mathcal{D}'_{L^p}$), p. 200, and *Autre définition de distributions bornées*, p. 205]).

The following theorem gives the inclusion relations among the spaces $\mathcal{D}'_{L^p}(\mathbb{R})$ (see [23, Ch. VI, §8, p. 201]).

Theorem 7 For every $1 < r < p < \infty$ it is:

$$\mathcal{D}'_{L^1}(\mathbb{R}) \subset \mathcal{D}'_{L^r}(\mathbb{R}) \subset \mathcal{D}'_{L^p}(\mathbb{R}) \subset \dot{\mathcal{D}}'_{L^\infty}(\mathbb{R}) \subset \mathcal{D}'_{L^\infty}(\mathbb{R})$$

A-8 Convergence in $\mathcal{D}'(\mathbb{R})$

Remember that we are interpreting the members of $\mathcal{D}(\mathbb{R})$ as the *allowed measuring instruments* on the signals $f(t) \in \mathcal{D}'(\mathbb{R})$, and:

$$\int f(t) \bullet \varphi(t) \in \mathbb{C}$$

as the result of the measure of $f(t)$ by the measuring instrument $\varphi(t) \in \mathcal{D}(\mathbb{R})$.

Let $f_k(t)$, $f(t)$ be a sequence and a member, respectively, of $\mathcal{D}'(\mathbb{R})$. By [23, Ch. III, §2, remark on Th. VII, p. 70], the following two statements are equivalent:

– the sequence $f_k(t)$ is *weakly convergent* to $f(t)$ in $\mathcal{D}'(\mathbb{R})$, i.e., for every measuring instrument $\varphi(t) \in \mathcal{D}(\mathbb{R})$ it is:

$$\lim_{k \rightarrow \infty} \int f_k(t) \bullet \varphi(t) = \int f(t) \bullet \varphi(t)$$

– the sequence $f_k(t)$ is *strongly convergent* to $f(t)$ in $\mathcal{D}'(\mathbb{R})$, i.e., for every measuring instrument $\varphi(t) \in \mathcal{D}(\mathbb{R})$ it is:

$$\lim_{k \rightarrow \infty} \int f_k(t) \bullet \varphi(t) = \int f(t) \bullet \varphi(t)$$

and moreover this convergence is *uniform* on every bounded subset B of the allowed measuring instruments.

Due to this result, when one of the two conditions above is verified, the terms *weakly* and *strongly* are omitted; we simply write:

$$\mathcal{D}' - \lim_{k \rightarrow \infty} f_k(t) = f(t)$$

and say that *the sequence* $f_k(t)$ $\mathcal{D}'$ -converges to $f(t)$.

This equivalence does not subsist for the spaces $\mathcal{D}'_{L^p}(\mathbb{R})$, so in these spaces *two notions* of convergence for sequences must be considered and cannot be identified. The next subsection gives the details.

A-9 Weak and Strong Convergence in $\mathcal{D}'_{L^p}(\mathbb{R})$

Remember that we are interpreting the members of $\dot{\mathcal{D}}_{L^\infty}(\mathbb{R})$ as the *allowed measuring instruments* on the signals $f(t) \in \mathcal{D}'_{L^1}(\mathbb{R})$, and the members of $\mathcal{D}_{L^{p'}}(\mathbb{R})$ as the *allowed measuring instruments* on the signals $f(t) \in \mathcal{D}'_{L^p}(\mathbb{R})$, $1 < p \leq \infty$.

Definition 1 Let $f_k(t)$, $f(t)$ be a sequence and a member, respectively, of $\mathcal{D}'_{L^p}(\mathbb{R})$, with $1 \leq p \leq \infty$.

- The sequence $f_k(t)$ is said to be *weakly convergent* to $f(t)$ in $\mathcal{D}'_{L^p}(\mathbb{R})$, and we write:

$$w\text{-}\mathcal{D}'_{L^p} - \lim_{k \rightarrow \infty} f_k(t) = f(t)$$

if for every allowed measuring instrument $\varphi(t)$ on $\mathcal{D}'_{L^p}(\mathbb{R})$ (see above) it is:

$$\lim_{k \rightarrow \infty} \int f_k(t) \bullet \varphi(t) = \int f(t) \bullet \varphi(t)$$

- The sequence $f_k(t)$ is said to be *strongly convergent* to $f(t)$ in $\mathcal{D}'_{L^p}(\mathbb{R})$, and we write:

$$s\text{-}\mathcal{D}'_{L^p} - \lim_{k \rightarrow \infty} f_k(t) = f(t)$$

if for every allowed measuring instrument $\varphi(t)$ on $\mathcal{D}'_{L^p}(\mathbb{R})$ (see above) it is:

$$\lim_{k \rightarrow \infty} \int f_k(t) \bullet \varphi(t) = \int f(t) \bullet \varphi(t)$$

and moreover this convergence is *uniform* on every bounded subset B of the allowed measuring instruments.

It is easily seen that:

- if $s - \mathcal{D}'_{L^p} - \lim_{k \rightarrow \infty} f_k(t) = f(t)$, then: $w - \mathcal{D}'_{L^p} - \lim_{k \rightarrow \infty} f_k(t) = f(t)$;
- $w - \mathcal{D}'_{L^p} - \lim_{k \rightarrow \infty} \delta(t - k) = 0$; indeed, for every allowed measuring instrument $\varphi(t)$ it is:

$$\lim_{k \rightarrow \infty} \int \delta(t - k) \bullet \varphi(t) = \lim_{k \rightarrow \infty} \varphi(k) = 0$$

(remember that, see Sect. A-6, it is: $\varphi(t) \in \dot{\mathcal{D}}_{L^\infty}(\mathbb{R})$);

- the statement $s - \mathcal{D}'_{L^p} - \lim_{k \rightarrow \infty} \delta(t - k) = 0$ is *false*. Indeed: let $\varphi(t) \in \mathcal{D}(\mathbb{R})$ verify $\varphi(0) = 1$, then: $B = \{\varphi(t - \beta) : \beta \in \mathbb{N}\}$ is a bounded set of allowed measuring instruments on which the convergence in the previous item is *not uniform*.

In particular, weak and strong convergences in the spaces $\mathcal{D}'_{L^p}(\mathbb{R})$ are not equivalent, and cannot be identified (as instead it happened in $\mathcal{D}'(\mathbb{R})$).

By [23, Ch. VI, §8, Remarque 2° to Th. XXV, p. 202] the following extremely deep characterization of *strongly convergent* sequences in $\mathcal{D}'_{L^p}(\mathbb{R})$ is obtained. The same reference gives the far less significant information (and hence here omitted) that can be given on weakly convergent sequences.

Theorem 8 *Let $f_k(t)$, $f(t)$ be a sequence and a member, respectively, of $\mathcal{D}'_{L^p}(\mathbb{R})$. The following statements are equivalent:*

- $s - \mathcal{D}'_{L^p} - \lim_{k \rightarrow \infty} f_k(t) = f(t)$,
- there exist $m \in \mathbb{N}$, and:

$$g_{jk}(t), g_j(t) \in L^p(\mathbb{R}), \quad q_j \in \mathbb{N}, \quad j = 1, \dots, m; \quad k \in \mathbb{N}$$

such that:

- for every $k \in \mathbb{N}$ it is: $f_k(t) = \sum_{j=1}^m D^{q_j} g_{jk}(t)$,
- $f(t) = \sum_{j=1}^m D^{q_j} g_j(t)$,
- for every $j = 1, \dots, m$ it is: $L^p - \lim_{k \rightarrow \infty} g_{jk}(t) = g_j(t)$.

A-10 Convolution Between $\mathcal{D}'_{L^p}(\mathbb{R})$ Spaces

Let $1 \leq p, q \leq \infty$ verify:

$$\frac{1}{p} + \frac{1}{q} \geq 1 \quad (\text{equivalent to: } 1 \leq q \leq p')$$

and let $1 \leq r \leq \infty$ verify:

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \quad \left(\text{equivalent to: } r = \frac{p'q}{p' - q} \right)$$

In [23, Ch. VI, §8, Th. XXVI, p. 203] it is defined the *convolution product*:

$$* : \mathcal{D}'_{L^p}(\mathbb{R}) \times \mathcal{D}'_{L^q}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^r}(\mathbb{R})$$

The following two theorems supply the information sufficient for handling this operation. In particular, the first one makes clear that the new notion is the *canonical distributional extension* of the usual convolution $*$: $L^p(\mathbb{R}) \times L^q(\mathbb{R}) \rightarrow L^r(\mathbb{R})$.

Theorem 9 Let $f(t) \in \mathcal{D}'_{L^p}(\mathbb{R})$, $g(t) \in \mathcal{D}'_{L^q}(\mathbb{R})$, and let (see Theorem 6):

$$f(t) = \sum_{i \in I} D^{p_i} f_i(t), \quad g(t) = \sum_{j \in J} D^{q_j} g_j(t)$$

where: I, J are two finite sets of natural numbers, each $p_i, q_j \in \mathbb{N}$, each $f_i(t) \in L^p(\mathbb{R})$ and each $g_j(t) \in L^q(\mathbb{R})$.

Then:

$$(f * g)(t) = \sum_{(i,j) \in I \times J} D^{p_i + q_j} (f_i * g_j)(t)$$

(observe that: for every $(i, j) \in I \times J$ it is $(f_i * g_j)(t) \in L^r(\mathbb{R})$).

Theorem 10 Let $\gamma(t) \in \mathcal{D}'_{L^q}(\mathbb{R})$, and let $\mathcal{L}_\gamma : \mathcal{D}'_{L^p}(\mathbb{R}) \rightarrow \mathcal{D}'_{L^r}(\mathbb{R})$ be the convolution operator defined by

$$\mathcal{L}_\gamma(f) = \gamma * f$$

(observe that in Item (2°) of the previous reference, coherently with the proof, the term “continue” must be substituted by “hypocontinue”).

The following statements hold:

- (a) the map $\mathcal{L}_\gamma$ is strongly continuous, i.e., it transforms strongly convergent sequences in $\mathcal{D}'_{L^p}(\mathbb{R})$ into strongly convergent sequences in $\mathcal{D}'_{L^r}(\mathbb{R})$;
- (b) the map $\mathcal{L}_\gamma$ is weakly continuous under each of the following assumptions (see [5, Theorem 3.3]):
 - (i) if $p \neq 1$ (independently of the particular $\gamma(t) \in \mathcal{D}'_{L^q}(\mathbb{R})$),
 - (ii) if $p = 1$, $q \neq \infty$ (independently of the particular $\gamma(t) \in \mathcal{D}'_{L^q}(\mathbb{R})$),
 - (iii) if $p = 1$, $q = \infty$, and $\gamma(t) \in \mathcal{D}'_{L^\infty}(\mathbb{R})$.

A-11 The Space $\mathcal{D}'_+(\mathbb{R})$

$\mathcal{D}'_+(\mathbb{R})$ is the space of the $f(t) \in \mathcal{D}'(\mathbb{R})$ with left-bounded support. A sequence $f_k(t) \in \mathcal{D}'_+(\mathbb{R})$ is $\mathcal{D}'_+$ -convergent to $f(t) \in \mathcal{D}'_+(\mathbb{R})$, and we write

$$\mathcal{D}'_+ - \lim_{k \rightarrow \infty} f_k(t) = f(t)$$

if $\mathcal{D}' - \lim_{k \rightarrow \infty} f_k(t) = f(t)$ and there exists $a \in \mathbb{R}$ such that for every k it is $\text{supp } f_k \subset (a, +\infty)$.

In [23, Ch. VI, §5, p. 170], for every $f(t), g(t) \in \mathcal{D}'_+(\mathbb{R})$ it is defined the convolution $(f * g)(t) \in \mathcal{D}'_+(\mathbb{R})$. By [23, Ch. VI, §5, *Les opérations du calcul symbolique à une variable*, p. 171], if

$$\mathcal{D}'_+ - \lim_{k \rightarrow \infty} g_k(t) = g(t)$$

then, by [23, Ch. VI, §5, Th. XIII, p. 172], it is

$$\mathcal{D}'_+ - \lim_{k \rightarrow \infty} (f * g_k)(t) = (f * g)(t)$$

The previous result, apart from its intrinsic meaning and theoretical applications, allows the following easy to think and to handle alternative way to define the convolution in $\mathcal{D}'_+(\mathbb{R})$: let $f(t), g(t) \in \mathcal{D}'_+(\mathbb{R})$, write $g(t)$ in the form¹²

$$g(t) = \mathcal{D}'_+ - \lim_{k \rightarrow \infty} \varphi_k(t) \quad \text{with } \varphi_k(t) \in \mathcal{D}(\mathbb{R}) \text{ for every } k$$

and use the definition

$$(f * g)(t) = \mathcal{D}'_+ - \lim_{k \rightarrow \infty} (f * \varphi_k)(t)$$

(see Eq. (6) in Sect. A-5).

By [23, Ch. VI, §5, Th. XIV, p. 173], $(\mathcal{D}'_+(\mathbb{R}); +, *)$ is a commutative algebra without zero divisors, and $\delta(t)$ is its identity.

A-12 A Remark on Initial Conditions for Differential Equations

Let $P(z) = a_n z^n + \cdots + a_0 z^0$ be a polynomial with coefficients in $\mathbb{C}$, such that $n \geq 1$ and $a_n \neq 0$, and write it in the form:

$$P(z) = c(z - s_1)^{q_1} \cdots (z - s_r)^{q_r}$$

where $c \neq 0$ is a complex number, each $s_k \in \mathbb{C}$, r and each q_k are natural numbers greater or equal to one.

For $f(t) \in \mathcal{D}'(\mathbb{R})$, take into account the differential equation:

$$P(D)x(t) = f(t) \tag{7}$$

¹² It is always possible. Indeed: Write $g(t) = \mathcal{D}' - \lim_{k \rightarrow \infty} \theta_k(t)$ with $\theta_k(t) \in \mathcal{D}(\mathbb{R})$ for every k (remember that $\mathcal{D}(\mathbb{R})$ is dense in $\mathcal{D}'(\mathbb{R})$, see [2, Th. 1.20]), and let $\eta(t)$ be a $C^\infty(\mathbb{R})$ function with left-bounded support and such that $\eta(t) = 1$ on an open set containing $\text{supp } g(t)$. Then it is: for every k it is $\eta(t)\theta_k(t) \in \mathcal{D}(\mathbb{R})$, there exists $a \in \mathbb{R}$ such that for every k it is $\text{supp } \eta(t)\theta_k(t) \subset (a, +\infty)$, and:

$$g(t) = \eta(t)g(t) = \eta(t) \mathcal{D}' - \lim_{k \rightarrow \infty} \theta_k(t) \stackrel{(a)}{=} \mathcal{D}' - \lim_{k \rightarrow \infty} \eta(t)\theta_k(t) \stackrel{(b)}{=} \mathcal{D}'_+ - \lim_{k \rightarrow \infty} \eta(t)\theta_k(t)$$

where equality (a) follows by the continuity of the multiplication of a distribution by a smooth function, and (b) by the definition of $\mathcal{D}'_+ - \lim$.

in the unknown $x(t) \in \mathcal{D}'(\mathbb{R})$.

It is well known that the equation (7) has solutions, and that, if $\xi(t)$ is a solution, the set of all the solutions is:

$$S(f) = \xi(t) + \text{span}\{t^0 e^{s_1 t}, \dots, t^{q_1-1} e^{s_1 t}, \dots, t^0 e^{s_r t}, \dots, t^{q_r-1} e^{s_r t}\}$$

A condition $\mathcal{C}(x(t))$, meaningful for every solution $x(t) \in S(f)$, may be considered an *initial condition* for the equation (7) if the problem:

$$\begin{cases} P(D)x(t) = f(t) \\ x(t) \text{ such that } \mathcal{C}(x(t)) \text{ holds} \end{cases}$$

has a unique solution.

If $f(t) \in L^1_{\text{loc}}(\mathbb{R})$, it is well known that every solution $x(t)$ of the equation (7) is in $C^{n-1}(\mathbb{R})$. Moreover, for every $t_0 \in \mathbb{R}$, and every $c_0, \dots, c_{n-1} \in \mathbb{C}$, the *usual condition*:

$$\mathcal{C}(x(t)) = (D^0 x(t_0) = c_0, D^1 x(t_0) = c_1, \dots, D^{n-1} x(t_0) = c_{n-1}) \quad (8)$$

is meaningful for every $x(t) \in S(f)$, and it is an *existence and uniqueness condition* for the equation (7).

Instead, for a generic $f(t) \in \mathcal{D}'(\mathbb{R})$, the initial condition (8) is, in general, meaningless (there may be solutions $x(t) \notin C^{n-1}(\mathbb{R})$). Other types of conditions may however play the role of existence and uniqueness condition. For instance:

- if $f(t) \in \mathcal{D}'_+(\mathbb{R})$, then (as it is well known) the condition $x(t) \in \mathcal{D}'_+(\mathbb{R})$ is an existence and uniqueness condition (it corresponds to the usual “condition of initial rest”);
- if $f(t) \in \mathcal{D}'_{L^p}(\mathbb{R})$, $1 \leq p < \infty$, and $P(z)$ has no zero on the complex imaginary axis, then the condition $x(t) \in \mathcal{D}'_{L^\infty}(\mathbb{R})$ is an existence and uniqueness condition (see [6, Theorem 1 and 3]);
- if $f(t) \in \mathcal{D}'_{L^p}(\mathbb{R})$, $1 \leq p \leq \infty$, and $P(z)$ has no zero on the complex imaginary axis, then the condition $x(t) \in \mathcal{D}'_{L^p}(\mathbb{R})$ is an existence and uniqueness condition (see Theorem 1 and 3 of Sect. 4);
- if $f(t) \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, and $P(z)$ has no zero on the complex imaginary axis, then the condition $x(t) \in L^p(\mathbb{R})$ is an existence and uniqueness condition (see Theorem 2 and 4 of Sect. 4).

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