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Dual Estimation of Fractional Variable Order Based on the Unscented Fractional Order Kalman Filter for Direct and Networked Measurements

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Abstract The paper is devoted to variable order estimation process when measurements are obtained in two different ways: directly and by lossy network. Since the problem of fractional order estimation is highly nonlinear, dual estimation algorithm based on Unscented Fractional Order Kalman filter has been used. In dual estimation process, state variable and order estimation have been divided into two sub-processes. For estimation state variables and variable fractional order, the Fractional Kalman filter and the Unscented Fractional Kalman filter have been used, respectively. The order estimation algorithms were applied to numerical examples and to real fractional variable order inertial system realized as an analog circuit.

Keywords Fractional calculus · Variable order derivative · Analog model · Estimation · Kalman filter

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1 Introduction

Recently, estimation problem in dynamical systems and control is widely considered. What is crucial, the order of estimated system is usually unknown and can be even fractional.

Fractional calculus is a generalization of traditional integer order integration and differentiation actions onto non-integer order. The idea of such a generalization has been mentioned in 1695 by Leibniz and L'Hospital. At the end of nineteenth century, Liouville and Riemann introduced first definition of fractional derivative. However, only just in late 60' of the twentieth century, this idea drew attention of the engineers. Fractional calculus was found a very useful tool for modeling behavior of many materials and systems, especially those based on the diffusion processes. The description and experimental results of modeling heat transfer processes were presented in [25,30]. Theoretical background of fractional calculus can be found in [9,11–13,15,18].

When the fractional order of derivative is not constant but depends on time, the various types of fractional variable order derivatives can be distinguished. In [14], nine different variable order derivative definitions have been given, and in [8,36], three general types of variable order definitions have been able to find, but without clear interpretation. In papers [27,28], the explanation of two main types and two recursive types of derivatives in the form of switching schemes are given. The equivalence between particular types of definitions and appropriate switching strategies are proven by authors. Moreover, based on these strategies, analog models of proper types derivatives were build and validated according to their numerical implementations. Based on these papers, it is possible to categorize fractional order derivatives into three switching strategies. The experimental results shown high accuracy for modeling the appropriate types of variable order definitions. In [29], analog realization of variable order derivative for multiple-switching order has been introduced; however, presented model gives non-stationary (variable parameter) system. Numerical routines for simulation of variable order derivatives based on different type definitions are given in [19].

When the state vector is not available directly from measurements, the Kalman filter algorithm can be used for estimate unknown states based on measurements and system dynamics [5,6,17]. More practical problem occurs when the physical data of a system are measured and analyzed through a network. Therefore, one of the practical areas are communication networks, where effort in analyzing the effect of packet losses has been highly considerable. To this kind of systems, generalization of Kalman filter algorithm can be applied [10,16,39]. For estimation of nonlinear systems, a set of generalized algorithms like Extended Kalman filter and Unscented Kalman filter are given in the literature [4,5,35]; especially, interesting algorithm is the Unscented Kalman filter that, in opposition to the Extended Kalman filter, not required differentiation of nonlinear function. In [5,17], UKF algorithm was used to teaching process of neural networks. In [1], the estimation results for fractional nonlinear systems based on Extended and Unscented Fractional Kalman filter (UFKF) were presented.

When mathematical model of dynamical systems is described by fractional order difference or derivative equations, the modified Fractional Kalman filter (FKF) algorithm should be used [23]. This algorithm has been used, for example, for estimation of state variables in the dynamical system with ultracapacitor [3], as well in a chaotic secure communication scheme [7]. In the case of systems with fractional order dynamics with data sending over lossy networks, where network-induced packet losses can become a source of degradation in estimation performance, the improved FKF has been investigated [31]. Fractional order estimation schemes for fractional and integer order systems with constant and variable fractional order colored noise are presented in [33]. Improved FKF for variable order systems is investigated in [40].

In practical application of fractional order systems, an identification of the system order plays a very important role, especially in the case of variable order systems. Usually, the parameters of the system were obtained during off-line numerical minimization routines [25,30]. In this paper, online dual estimation algorithms for state variable and order estimation, when measurements are obtained directly and by lossy network, are presented. For estimation state variables and variable fractional order, a FKF and UFKF have been used, respectively. Moreover, the verification of the developed estimation algorithm has been performed by testing it on a real electrical circuit analog model.

The paper is organized as follows. In Sect. 2, particular types of fractional variable order derivatives are introduced. In Sect. 3, basic properties of discrete fractional variable order state-space model are recalled. In Sect. 4, analog model of fractional variable order system is presented. In Sect. 5, dual estimation schemes based on UFKF for direct and networked measurements cases are presented. In Sect. 6, numerical results of modeling are presented. Finally, in Sect. 7, order estimation for analog model is presented.

2 Fractional Variable Order Grünwald–Letnikov Type Derivatives

As a base of generalization of the constant fractional order $\alpha \in \mathbb{R}$ difference onto variable order case, the following definition is taken into consideration:

$$ {}_0\Delta_k^{\alpha} f_k = \frac{1}{h^{\alpha}} \sum_{r=0}^{k} (-1)^r \binom{\alpha}{r} f_{k-r}, \tag{1}$$

where $h > 0$ is a step time.

For the case of order changing with time (variable order case), variety of definitions can be found in the literature [8,36]. Among them all, we present only two. The first one is obtained by replacing in (1) a constant order α by variable order $\alpha(t)$. In this approach, all coefficients for past samples are obtained for present value of the order and are given as follows:

Definition 1 The $\mathcal{A}$-type of fractional variable order difference is defined as follows:

$$ {}_0^{\mathcal{A}}\Delta_k^{\alpha_k} f_k = \frac{1}{h^{\alpha_k}} \sum_{r=0}^{k} (-1)^r \binom{\alpha_k}{r} f_{k-r}. \tag{2}$$

The definition of dual type of variable order derivative, that is consider in this paper, is given as follows:

Definition 2 [28] The $\mathcal{D}$-type of fractional variable order difference is defined as follows:

$$ {}^{\mathcal{D}}_{0}\Delta^{\alpha_k}_{k} f_k = \left(\frac{f_k}{h^{\alpha_k}} - \sum_{j=1}^{k} (-1)^j \binom{-\alpha_k}{j} {}^{\mathcal{D}}_{0}\Delta^{\alpha_{k-j}}_{k-j} f_{k-j} \right). \tag{3} $$

Remark 1 For a fractional constant order $\alpha = \text{const}$, the fractional differences given by Definitions 1 and 2 are numerically identical with constant order fractional difference given by (1).

3 Discrete Variable Fractional Order State-Space System

Let us consider a linear discrete fractional variable order state-space (DFVOSS) $\mathcal{A}$-type system

$$ {}^{\mathcal{A}}_{0}\Delta^{\Upsilon_{k+1}}_{k+1} x_{k+1} = A x_k + B u_k, \tag{4} $$

$$ x_{k+1} = h^{\Upsilon_{k+1}} {}^{\mathcal{A}}_{0}\Delta^{\alpha_{k+1}}_{k+1} x_{k+1} - \sum_{j=1}^{k+1} (-1)^j \Upsilon_{j,k+1} x_{k-j+1}, \tag{5} $$

$$ y_k = C x_k, \tag{6} $$

where

$$ \Upsilon_{j,k} = \text{diag}\left[\binom{\alpha_{1,k}}{j} \cdots \binom{\alpha_{N,k}}{j} \right], $$

$$ {}^{\mathcal{A}}\Delta^{\Upsilon_{k+1}} x_{k+1} = \begin{bmatrix} {}^{\mathcal{A}}\Delta^{\alpha_{1,k+1}} x_{1,k+1} \\ \vdots \\ {}^{\mathcal{A}}\Delta^{\alpha_{N,k+1}} x_{N,k+1} \end{bmatrix}, $$

$$ h^{\Upsilon_{k+1}} = \text{diag}\left[h^{\alpha_{1,k+1}} \cdots h^{\alpha_{N,k+1}} \right] $$

and $\alpha_{i,k} \in \mathbb{R}$ is the ith fractional variable order of the system, $u_k \in \mathbb{R}^d$ is a system input, $y_k \in \mathbb{R}^p$ is a system output, $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times d}$ and $C \in \mathbb{R}^{p\times N}$ are the state system, input, and output matrices, respectively, $x_k \in \mathbb{R}^N$ is a state vector, N is a number of state equations, and h is a time sampling. Basic properties of constant order discrete fractional variable state-space system (DFOSS) can be found in [2,21].

Let us consider the following DFVOSS system for commensurate case of order α_k

$$ {}^{\mathcal{A}}_{0}\Delta^{\alpha_{k+1}}_{k+1} x_{k+1} = u_k. \tag{7} $$

This can be expanded into (assuming $h = 1$)

$$\sum_{j=0}^{k+1} (-1)^j \binom{\alpha_{k+1}}{j} x_{k-j+1} = u_k \tag{8}$$

and rewritten as

$$x_{k+1} = u_k - \sum_{j=1}^{k+1} (-1)^j \binom{\alpha_{k+1}}{j} x_{k-j+1}. \tag{9}$$

The solution of the system given by the $\mathcal{A}$-type definition has the structure of $\mathcal{D}$-type definition [32], namely

$$^{\mathcal{D}}_{0}\Delta^{\beta_k}_{k+1} w_k = w_k - \sum_{j=1}^{k} (-1)^j \binom{-\beta_k}{j} {}^{\mathcal{D}}_{0}\Delta^{\beta_{k-j}}_{k+1} u_{k-j}. \tag{10}$$

Comparison of these two relations, along with substitutions

$$w_{k+1} = u_k, \quad -\alpha_{k+1} = \beta_{k+1},$$

and

$$x_{k+1} = {}^{\mathcal{D}}_{0}\Delta^{\alpha_{k+1}}_{k+1} w_{k+1}$$

yields

$$x_{k+1} = {}^{\mathcal{D}}_{0}\Delta^{-\alpha_{k+1}}_{k+1} u_k.$$

Remark 2 (Duality of variable order difference operators) In general case, order composition for variable order difference operators does not hold, e.g., ${}^{\mathcal{A}}_{0}\Delta^{\alpha_{k+1}}_{k+1} {}^{\mathcal{A}}_{0}\Delta^{-\alpha_{k+1}}_{k+1} u_k \neq u_k$. However, for dual operators [32], we have

$$\begin{aligned} {}^{\mathcal{A}}_{0}\Delta^{\alpha_{k+1}}_{k+1} {}^{\mathcal{D}}_{0}\Delta^{-\alpha_{k+1}}_{k+1} u_k &= u_k, \\ {}^{\mathcal{D}}_{0}\Delta^{\alpha_{k+1}}_{k+1} {}^{\mathcal{A}}_{0}\Delta^{-\alpha_{k+1}}_{k+1} u_k &= u_k. \end{aligned}$$

This leads to the conclusion (as it was presented in [32]), that in order to model system, that is built using $\mathcal{D}$-type integrals (discrete equivalence), a DFVOSS based on $\mathcal{A}$-type definition is needed, i.e.,

$$\begin{aligned} y_{k+1} &= {}^{\mathcal{D}}_{0}\Delta^{-\alpha_{k+1}}_{k+1} (u_k - a y_k), \\ {}^{\mathcal{A}}_{0}\Delta^{\alpha_{k+1}}_{k+1} y_{k+1} &= u_k - a y_k. \end{aligned}$$

4 Analog Model of Fractional Variable Order Integral System

An analog model of $\mathcal{D}$-type fractional variable order integral system can be realized in two ways: directly based on switching order scheme which is equivalent to such definition or through analogy between switching scheme and its parallel model introduced in [29]. The second method gives the possibility to build the n-switching model fully equivalent to the first one but in less complex way and that was the reason to take such model for further analysis. To realized $\mathcal{D}$-type fractional variable order definition, an multi-switching analog model presented in Fig. 1 was used. This model was widely described in [26].

The circuit branches with resistors R_1, R_2 and capacitors C_1, C_2 represent an approximation of half-order ($\alpha = 0.5$) impedance when electronic switches S_1 and S_2 are connected to terminals denotes as 2. The half-order impedance can be built according to algorithm described in [24] ($R_1 = 2.4\,\text{k}\Omega$, $R_2 = 8.2\,\text{k}\Omega$, $C_1 = 330\,\text{nF}$ and $C_2 = 220\,\text{nF}$). The quantity of resistors and capacitors determines the accuracy of whole impedance. This model approximation contains 200 passive elements. The frequency response of real half-order impedance and its model are overlapping in wide range frequency. Otherwise, when switch S_1 is connected to the terminal 1 and S_2 is grounded, then the voltage follower A_3 is charging the domino-ladder branches to the value of output signal. It is a necessary condition to keep the behavior of $\mathcal{D}$-type variable order definition. Finally, the branch with R_1 and C_1 elements connected to the negative input of amplifier A_1 represents a first-order impedance.

In fact, the order of system can be changed between -0.5 (half-order integral) and -1 (first-order integral) in any time and depends on position of switches (S_1, S_2

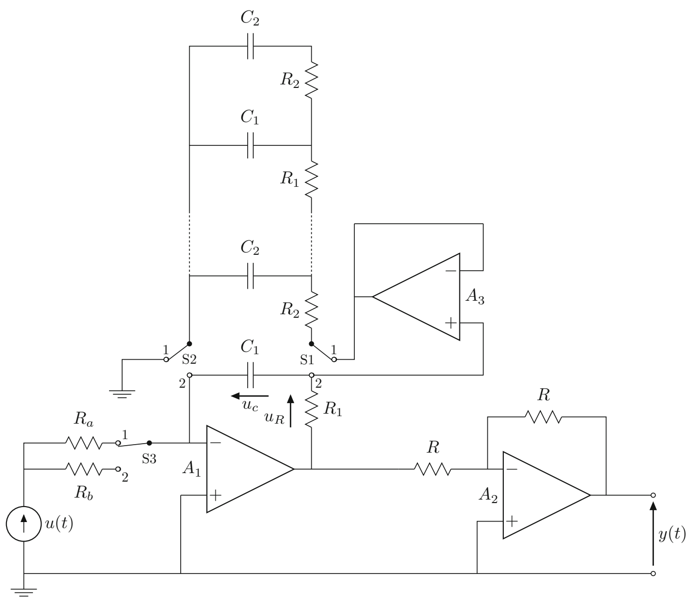


Fig. 1 Multi-switching analog realization of the $\mathcal{D}$-type fractional variable order integral

and S_3). Resistors R_a and R_b allow to sustain the constant value of integrator gain. Operational amplifier A_2 in configuration with resistors R gives voltage amplifier of a gain equal to -1 providing re-inversion of output signal (already inverted by integrator circuit).

5 Dual Estimation Based on UFKF Filter

Generally, dual estimation refers to the issue of simultaneously estimating the state of a dynamic system and its parameters. In our case, we will deal with estimation of a parameter changing in time, i.e., with estimation of the variable order. Dual estimation algorithms were already considered, e.g., in [37,38].

5.1 Variable Order Estimation Problem

Let us assume the simple autonomous scalar discrete variable order system

$$x_{k+1} = h^{\Upsilon_{k+1}} a x_k - \sum_{j=1}^{k+1} (-1)^j \Upsilon_{j,k+1} x_{k-j+1},$$

that can be rewritten in the matrix form as follows:

$$x_{k+1} = \left[h^{\Upsilon_{k+1}} a x_k + \Upsilon_{1,k+1} \; -\Upsilon_{2,k+1} \; \ldots \; \Upsilon_{k+1,k+1} \right] \left[x_k \; x_{k-1} \; \ldots \; x_0 \right].$$

Next, by expanding binomials as a polynomials of orders we obtain:

$$x_{k+1} = \left[\alpha_{k+1} \; \alpha_{k+1}^2 \; \alpha_{k+1}^3 \; \ldots \; \alpha_{k+1}^{k+1} \right] W \left[x_k \; x_{k-1} \; x_{k-2} \ldots x_0 \right],$$

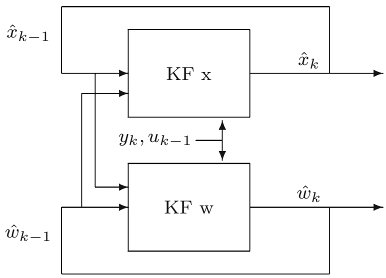
where W is a matrix with appropriate coefficients originated from binomial coefficients expansion and system parameters. As it can be noticed, the problem of order extraction from the measurements data is highly nonlinear. That is the reason, the estimation algorithm based on UFKF has been taken into consideration in this paper.

Moreover, in constant order (integer or fractional) systems the influence of the step time h can be easily incorporated into system matrices A and B. In variable order case, such incorporation leads to non-stationary system with variable in time system matrices. That is why incorporation of the step time has to be performed into the model itself, which provides the necessity for generalization of appropriate Kalman filter algorithm for this modification.

5.2 Dual Estimation Scheme

In dual estimation process of state variables and parameters, estimation is divided into two filters: The first filter estimates state variables vector $\hat{x}_k$, and the second parameters vector $\hat{w}_k$ of the system. The scheme of this type of estimation is given in Fig. 2, where KFx and KFw is a filter for state vector estimation and for parameters estimation, respectively. The Filter KFx is based on the past estimated value of parameter vector

Fig. 2 Dual estimation scheme



estimates $\hat{w}_{k-1}$ and data u_{k-1}, y_k in order to evaluate state estimate $\hat{x}_k$. On the other hand, Filter FKw uses past estimates obtained by KFx filter and data u_{k-1}, y_k to obtain its own state vector and output prediction $\tilde{\chi}_k^w$, $\mathcal{Y}_k^w$ to extract next parameters vector estimate $\hat{w}_k$.

Because the state vector estimation problem (KFx filter) is linear, the fractional variable order Kalman filter, given below, has been used.

Proposition 1 *For the discrete fractional variable order system state vector estimation in dual estimation algorithm, the Kalman filter (KFx) is given by the following set of equations*

$$ {}_0^{\mathcal{A}}\Delta_{k+1}^{\Upsilon_{k+1}} \tilde{x}_{k+1} = A\hat{x}_k + Bu_k, \tag{11}$$

$$\tilde{x}_{k+1} = h^{\Upsilon_{k+1}} {}_0^{\mathcal{A}}\Delta_{k+1}^{\Upsilon_{k+1}} \tilde{x}_{k+1} - \sum_{j=1}^{k+1} (-1)^j \Upsilon_{j,k+1} \hat{x}_{k+1-j}, \tag{12}$$

$$\tilde{P}_k = \left(h^{\Upsilon_k} A + \Upsilon_{1,k}\right) P_{k-1} \left(h^{\Upsilon_k} A + \Upsilon_{1,k}\right)^T \tag{13}$$

$$+ h^{\Upsilon_k} Q_{k-1} h^{\Upsilon_k} + \sum_{j=2}^{k} \Upsilon_{j,k} P_{k-j} \Upsilon_{j,k}^T, \tag{14}$$

$$K_k = \tilde{P}_k C^T (C \tilde{P}_k C^T + R_k)^{-1}, \tag{15}$$

$$\hat{x}_k = \tilde{x}_k + K_k (y_k - C\tilde{x}_k), \tag{16}$$

$$P_k = (I - K_k C)\tilde{P}_k, \tag{17}$$

where initial conditions are

$$x_0 \in \mathbb{R}^N, \quad P_0 = E[(\tilde{x}_0 - x_0)(\tilde{x}_0 - x_0)^T],$$

and v_k and ω_k are assumed to be independent with zero expected value. □

Proof The proof is similar to this presented in [22] with including step time h. The main differences are in equations for state and covariance matrix prediction defined by the following relation

$$\tilde{x}_k = E[x_k | z_{k-1}^*], \tag{18}$$

which is a random variable x_k conditioned on the measurements stream z_{k-1}^* that contains values of the measurements output $y_0, y_1, \ldots, y_k$ and input signal $u_0, u_1, \ldots, u_k$.

The prediction of the state vector can be obtained as follows:

$$\tilde{x}_{k+1} = E\left[h^{\Upsilon_{k+1}}(Ax_k + Bu_k + \omega_k) - \sum_{j=1}^{k+1}(-1)^j \Upsilon_{j,k+1} x_{k+1-j} | z_k^*\right]$$
$$= h^{\Upsilon_{k+1}}(AE\left[x_k|z_k^*\right] + Bu_k) - \sum_{j=1}^{k+1}(-1)^j \Upsilon_{j,k+1} E\left[x_{k+1-j}|z_k^*\right].$$

Under assumption $E[x_{k+1-j}|z_k^*] \approx E[x_{k+1-j}|z_{k+1-j}^*]$, which means the past estimates not be updated using newer measurements, the relation for state prediction is obtained.

The prediction of an estimation error covariance matrix is defined as follows

$$\tilde{P}_k = E\left[(\tilde{x}_k - x_k)(\tilde{x}_k - x_k)^T\right]. \tag{19}$$

The term $(\tilde{x}_k - x_k)$, used in prediction of the covariance error matrix, is evaluated as follows:

$$(\tilde{x}_k - x_k) = h^{\Upsilon_k}(A\hat{x}_{k-1} + Bu_{k-1}) - \sum_{j=1}^{k}\left((-1)^j \Upsilon_{j,k}\hat{x}_{k-j}\right)$$
$$- h^{\Upsilon_k}(Ax_{k-1} - Bu_{k-1} - \omega_{k-1}) + \sum_{j=1}^{k}\left((-1)^j \Upsilon_{j,k} x_{k-j}\right)$$
$$= (h^{\Upsilon_k}A - \Upsilon_{1,k})(\hat{x}_{k-1} - x_{k-1}) - h^{\Upsilon_k}\omega_{k-1}$$
$$- \sum_{j=2}^{k}\left[(-1)^j \Upsilon_{j,k}(\hat{x}_{k-j} - x_{k-j})\right].$$

In order to obtain this relation, similar assumption like in FKF derivation has been used. It is assumed that the expected values of terms $(\hat{x}_l - x_l)(\hat{x}_m - x_m)^T$ are equal to zero when $l \neq m$, which finally gives the following equation

$$\tilde{P}_k = E\left[(\tilde{x}_k - x_k)(\tilde{x}_k - x_k)^T\right]$$
$$= (h^{\Upsilon_k}A - \Upsilon_{1,k})E[(\hat{x}_{k-1} - x_{k-1})(\hat{x}_{k-1} - x_{k-1})^T](h^{\Upsilon_k}A - \Upsilon_{1,k})^T$$
$$+ h^{\Upsilon_k}E[\omega_{k-1}\omega_{k-1}^T](h^{\Upsilon_k})^T + \sum_{j=2}^{k}\Upsilon_{j,k}E[(\hat{x}_{k-j} - x_{k-j})(\hat{x}_{k-j} - x_{k-j})^T]\Upsilon_{j,k}^T$$
$$= \left(h^{\Upsilon_k}A + \Upsilon_{1,k}\right)P_{k-1}\left(h^{\Upsilon_k}A + \Upsilon_{1,k}\right)^T + h^{\Upsilon_k}Q_{k-1}(h^{\Upsilon_k})^T + \sum_{j=2}^{k}\Upsilon_{j,k}P_{k-j}\Upsilon_{j,k}^T.$$

The rest of the proof is analogical to the proof of variable order fractional Kalman filter (VOFKF) presented in [22]. □

Due to high nonlinearity of the order estimation problem, the Unscented Fractional Order Kalman filter is used (as the KFw filter in dual estimation scheme presented in Fig. 2).

Proposition 2 *For the discrete fractional variable order system order estimation in dual estimation algorithm, the Unscented Kalman filter (called KFw) is given by the following set of equations*

$$\begin{aligned}
\tilde{\alpha}_k &= \hat{\alpha}_{k-1}, \\
\tilde{P}^w_k &= \hat{P}^w_{k-1} + hQ^w_{k-1}h, \\
\tilde{\mathcal{W}}_k &= \left[\, \tilde{\alpha}_k \quad \tilde{\alpha}_k \pm \left(\sqrt{(L+\lambda)\tilde{P}^w_k}\right)_i \,\right], \\
\Delta^{\tilde{\mathcal{W}}_{k,i}} \tilde{\chi}^w_{k,i} &= A\hat{x}_{k-1} + Bu_{k-1}, \\
\tilde{\chi}^w_{k,i} &= h^{\tilde{\mathcal{W}}_{k,i}} \Delta^{\tilde{\mathcal{W}}_{k,i}} \tilde{\chi}^w_{k,i} - \sum_{j=1}^{k} (-1)^j \binom{\tilde{\mathcal{W}}_{k,i}}{j} \hat{x}_{k-j}, \\
\tilde{Y}^w_{k,i} &= C(\tilde{\mathcal{W}}_k)\tilde{\chi}^w_{k,i}, \\
\tilde{y}^w_k &= \sum_{i=0}^{2L} W^{(m)} \tilde{Y}_{k,i}, \\
P^w_{y_k y_k} &= \sum_{i=1}^{2L} W^{(c)}_i [\tilde{\mathcal{Y}}_{i,k} - \tilde{y}_k][\tilde{\mathcal{Y}}_{i,k} - \tilde{y}_k]^T + R^w, \\
P^w_{w_k y_k} &= \sum_{i=1}^{2L} W^{(c)}_i [\tilde{\mathcal{W}}_{i,k} - \tilde{w}_k][\tilde{\mathcal{Y}}_{i,k} - \tilde{y}_k]^T, \\
\mathcal{K}^w_k &= P^w_{w_k y_k} (P^w_{y_k y_k})^{-1}, \\
\hat{\alpha}_k &= \tilde{\alpha}_k + \mathcal{K}^w_k (y_k - \tilde{y}^w_k), \\
P^w_k &= \hat{P}^w_k - \mathcal{K}^w_k P^w_{y_k y_k} \mathcal{K}^w_k, \\
Q^w_k &= (1-\delta)Q^w_{k-1} + h\delta(\mathcal{K}^w_k)(y_k - \tilde{y}^w_k)(y_k - \tilde{y}^w_k)^T (\mathcal{K}^w_k)^T h,
\end{aligned}$$

where

$$\hat{\chi}^w_k = \left[\, \hat{x}_k \quad \hat{x}_k \pm (\sqrt{(L+\lambda)P_k}) \,\right],$$

what means that

$$\hat{\chi}^w_{i,k} = \begin{cases} \hat{x}_k, & i = 0 \\ [\hat{x}_k + (\sqrt{(L+\lambda)P_k})_i], & i = 1 \ldots L \\ [\hat{x}_k - (\sqrt{(L+\lambda)P_k})_{2L-i}], & i = L+1 \ldots 2L \end{cases}$$

and where $(\sqrt{(L+\lambda)P_k})_i$ *is* i*th column of matrix square root (e.g., Cholesky factorization), and coefficients of Unscented transformation* W *are equal to*

$$\begin{aligned} W_0^{(m)} &= \lambda/(L+\lambda), \\ W_0^{(c)} &= \lambda/(L+\lambda) + (1-\mathfrak{A}^2+\mathfrak{B}), \\ W_i^{(m)} &= W_i^{(c)} = 1/(2(L+\lambda)), \end{aligned}$$

where $\lambda = \mathfrak{A}^2(L+\kappa) - L$, $\mathfrak{A}$ *is a coefficient describing width of point expansion during the transformation (in literature is chosen from the range* $1 \leq \mathfrak{A} \leq 1e-4$, *usually denoted as* α*, but in this article, because of using order* α *this notation has been changed),* κ *is an additional scaling coefficient usually chosen as 3-L,* $\mathfrak{B}$ *is a coefficient that corresponds with our knowledge about type of noise, for Gaussian noise is chosen as* $\mathfrak{B} = 2$ *(in the literature usually denoted as* β*). The* δ *coefficient is a "forgetting factor" according to RobbinsMonro stochastic approximation scheme for estimating the innovations (see* [5]*, p. 240). For more intuitive choosing of parameters* Q^w_{k-1}*, let us define* ${}^*Q^w_{k-1} = hQ^w_{k-1}h$*, which represents covariance of order noise in each sample time.*

Proof The algorithm is a generalization of the Fractional Unscented Kalman filter given in [20], while the step time h and $\mathcal{A}$-type variable order difference definition is taken into consideration.

Because order dynamics of estimated system is unknown (for estimation of arbitrary order function), the following equation of the estimated order dynamics is used

$$\alpha_k = \alpha_{k-1} + h\omega^w_{k-1},$$

where ω^w_{k-1} is a zero mean noise that represents possible order changes. The order prediction is given by the following relation

$$\hat{\alpha}_k = E[\alpha_k | z^*_{k-1}] = E[\alpha_{k-1} + h\omega^w_{k-1} | z^*_{k-1}] = \hat{\alpha}_{k-1}.$$

Due to linearity of the dynamics of the order, the Unscented transformation is not needed to obtain prediction order covariance matrix $\tilde{P}^w_k$, and this covariance matrix will be evaluated as follows:

$$\begin{aligned} \tilde{P}^w_k &= E\left[(\tilde{\alpha}_k - \alpha_k)(\tilde{\alpha}_k - \alpha_k)^T\right] \\ &= E\left[(\tilde{\alpha}_{k-1} - \alpha_{k-1} - h\omega^w_{k-1})(\tilde{\alpha}_{k-1} - \alpha_{k-1} - h\omega^w_{k-1})^T\right] \\ &= \tilde{P}^w_{k-1} + hQ^w_{k-1}h. \end{aligned}$$

The rest of equations are the same as for Fractional Unscented Kalman filter given in [20]. □

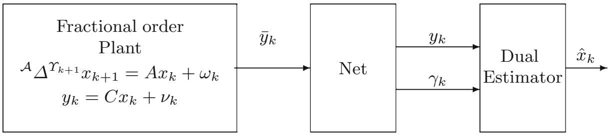


Fig. 3 Estimation process over a network

5.3 Dual Estimation for Networked Measurements

For the case of measurements over lossy network (see Fig. 3), some parts of packets are lost during transmission, which has negative influence on the efficiency of the estimation process. In order to improve estimation algorithms, not only measurement values but also information about packets losing γ_k are needed. The $\gamma_k \in \{1, 0\}$ has value 1 when packet y_k is obtained, and 0 when y_k is lost.

Analogously as in direct measurement case, for state vector estimation (KFx filter), the fractional variable order Kalman filter for the networked systems case has been used.

Proposition 3 *For the discrete fractional variable order networked system state vector estimation in dual estimation algorithm the Kalman filter (called KFx) given by the following set of equations has been used*

$$ {}_0^{\mathcal{A}}\Delta_{k+1}^{\Upsilon_{k+1}} \tilde{x}_{k+1} = A\hat{x}_k + Bu_k, \tag{20}$$

$$\tilde{x}_{k+1} = h^{\Upsilon_{k+1}} {}_0^{\mathcal{A}}\Delta_{k+1}^{\Upsilon_{k+1}} \tilde{x}_{k+1} - \sum_{j=1}^{k+1} (-1)^j \Upsilon_{j,k+1} \hat{x}_{k+1-j}, \tag{21}$$

$$\tilde{P}_k = \left(h^{\Upsilon_k} A + \Upsilon_{1,k}\right) P_{k-1} \left(h^{\Upsilon_k} A + \Upsilon_{1,k}\right)^T \tag{22}$$

$$+ h^{\Upsilon_k} Q_{k-1} h^{\Upsilon_k} + \sum_{j=2}^{k} \Upsilon_{j,k} P_{k-j} \Upsilon_{j,k}^T, \tag{23}$$

$$K_k = \tilde{P}_k C^T (C\tilde{P}_k C^T + R_k)^{-1}, \tag{24}$$

$$\hat{x}_k = \tilde{x}_k + \gamma_k K_k (y_k - C\tilde{x}_k), \tag{25}$$

$$P_k = (I - \gamma_k K_k C)\tilde{P}_k, \tag{26}$$

where γ_k represents the knowledge of packet losses and initial conditions are

$$x_0 \in \mathbb{R}^N, \quad P_0 = E[(\tilde{x}_0 - x_0)(\tilde{x}_0 - x_0)^T]$$

and ν_k and ω_k are assumed to be independent with zero expected value. □

Proof The algorithm is a modification of the algorithm given in Proposition 1, including information about packages losing γ_k in last two equations and is similar to this presented in [34] with additionally including step time h. □

For order estimation in network systems as a KFw filter the Unscented Variable Fractional Order Kalman filter has been used. For simplicity is presented the algorithm for estimation simple one order; however, it can be easily extended for multiple orders estimation. The algorithm is given in the form of following theorem:

Proposition 4 *For the order estimation in discrete fractional variable order networked system in dual estimation algorithm, the Unscented Kalman filter (called KFw) is given by the following set of equations*

$$
\begin{aligned}
\tilde{\alpha}_k &= \hat{\alpha}_{k-1},\\
\tilde{P}^w_k &= \hat{P}^w_{k-1} + hQ^w_{k-1}h,\\
\tilde{\mathcal{W}}_k &= \left[\tilde{\alpha}_k \quad \tilde{\alpha}_k \pm \left(\sqrt{(L+\lambda)\tilde{P}^w_k}\right)_i\right],\\
\Delta^{\tilde{\mathcal{W}}_{k,i}}\tilde{\chi}^w_{k,i} &= A\hat{x}_{k-1} + Bu_{k-1},\\
\tilde{\chi}^w_{k,i} &= h^{\tilde{\mathcal{W}}_{k,i}}\Delta^{\tilde{\mathcal{W}}_{k,i}}\tilde{\chi}^w_{k,i} - \sum_{j=1}^{k}(-1)^j\binom{\tilde{\mathcal{W}}_{k,i}}{j}\hat{x}_{k-j},\\
\tilde{Y}^w_{k,i} &= C(\tilde{\mathcal{W}}_k)\tilde{\chi}^w_{k,i},\\
\tilde{y}^w_k &= \sum_{i=0}^{2L} W^{(m)}\tilde{Y}_{k,i},\\
P^w_{y_k y_k} &= \sum_{i=1}^{2L} W_i^{(c)}[\tilde{\mathcal{Y}}_{i,k} - \tilde{y}_k][\tilde{\mathcal{Y}}_{i,k} - \tilde{y}_k]^T + R^w,\\
P^w_{w_k y_k} &= \sum_{i=1}^{2L} W_i^{(c)}[\tilde{\mathcal{W}}_{i,k} - \tilde{w}_k][\tilde{\mathcal{Y}}_{i,k} - \tilde{y}_k]^T,\\
\mathcal{K}^w_k &= P^w_{w_k y_k}(P^w_{y_k y_k})^{-1},\\
\hat{\alpha}_k &= \tilde{\alpha}_k + \gamma_k\mathcal{K}^w_k(y_k - \tilde{y}^w_k),\\
P^w_k &= \hat{P}^w_k - \gamma_k\mathcal{K}^w_k P^w_{y_k y_k}\mathcal{K}^w_k,\\
Q^w_k &= (1-\delta)Q^w_{k-1} + \gamma_k h\delta(\mathcal{K}^w_k)(y_k - \tilde{y}^w_k)(y_k - \tilde{y}^w_k)^T(\mathcal{K}^w_k)^T h,
\end{aligned}
$$

where the meaning of occurring above terms is the same as in Proposition 2. □

Proof The algorithm is a modification of the algorithm given in Proposition 2, with including information about packages losing γ_k in last three equations. □

6 Numerical Results

Numerical results, presented in following section, have been obtained in MATLAB/Simulink environment.

6.1 Order Estimation for Direct Measurements

Example 1 (Order estimation for direct measurements and single system order switch) Let us consider the following discrete variable order state-space system:

$$A = -3.04, \quad B = 3.03, \quad C = 1, \tag{27}$$

the variable order is a single switch between two values and is defined as

$$\alpha_k = \begin{cases} 0.8 & \text{for} \quad t \leq 1, \\ 0.4 & \text{for} \quad t > 1, \end{cases}$$

and noise has the following parameters:

$$E[\omega\omega^T] = 8.82 \cdot 10^{-8},$$
$$E[\nu\nu^T] = 1.47 \cdot 10^{-7}.$$

Parameters of KFx filter are:

$$P_0 = \left[10\right], \quad Q_0 = \left[8.82 \cdot 10^{-8}\right],$$
$$x_0 = [0], \quad R = [1.47 \cdot 10^{-7}].$$

Parameters of KFw filter are:

$$P_0^w = \left[5.79 \cdot 10^{-7}\right], \quad {}^*Q_0^w = \left[3.71 \cdot 10^{-11}\right],$$
$$\hat{\alpha}_0 = [0.5], \quad R^w = [0.1], \quad \mathfrak{A} = 0.0001, \quad \delta = 0.5.$$

Figure 4 presents input and output of the analog system—the data for estimation process, and results of applying dual estimation algorithms to these data. As it can be seen, the order is estimated with very high accuracy, and algorithm needed very short time to adjust for order changing.

Example 2 (Order estimation for direct measurements and higher noise) In this case, the noise has the following parameters:

$$E[\omega\omega^T] = 9.01 \cdot 10^{-8},$$
$$E[\nu\nu^T] = 1.58 \cdot 10^{-5}$$

and initial values for the filter are

$$P_0 = \left[10\right], \quad Q_0 = \left[9.01 \cdot 10^{-8}\right],$$
$$x_0 = [0], \quad R = [1.58 \cdot 10^{-5}].$$

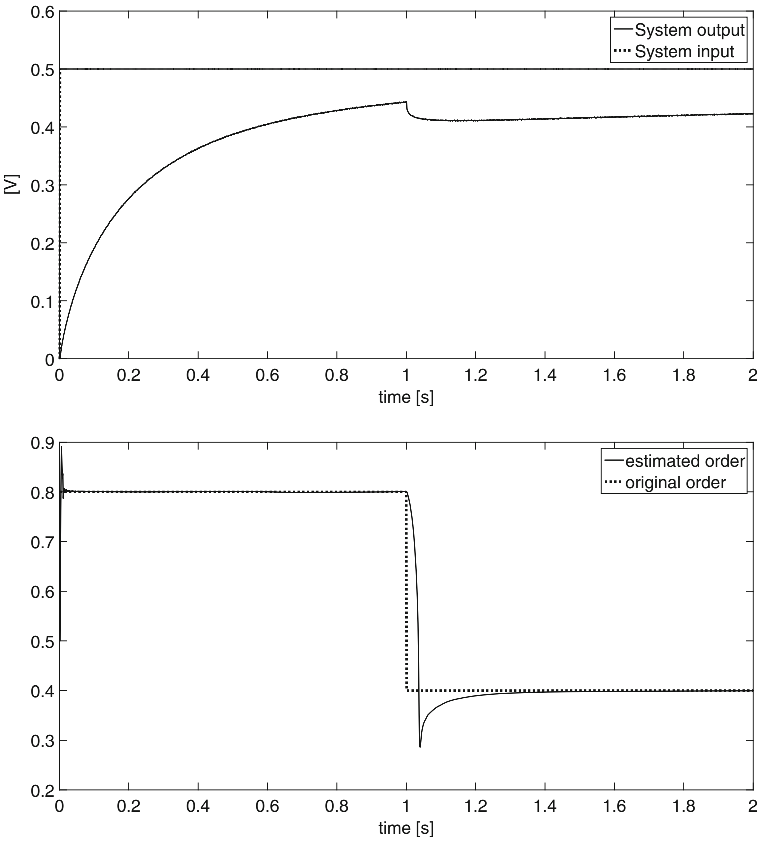


Fig. 4 Results for dual estimation of variable order α_k, input and output of the system—*upper figure*, estimated order—*bottom figure*

Figure 5 presents input and output of the analog system with higher output noise than in example before. It also presents results of applying dual estimation algorithm to these data. As it can be seen, the order is estimated with quite high accuracy; however, accuracy is lower than for a case of lower noise.

Example 3 (Order estimation for direct measurements and different output noise variances) Let us consider the system from Example 1 with the same parameters, except for the variable order, which takes in the form of sinusoidal function. Below the results of series of experiments for estimation of variable order, performed for different output noise variances R, are presented. What is expected, with increasing variance noise, the accuracy of estimation error decreases (see Table 1; Fig. 6).

Example 4 (Order estimation for different values of ${}^*Q_k^w$) To investigate influence of parameter ${}^*Q_k^w$, which is expected variability of estimated order, into estimation accuracy, let us consider the same system as in Example 1 with different values of matrix ${}^*Q_k^w$.

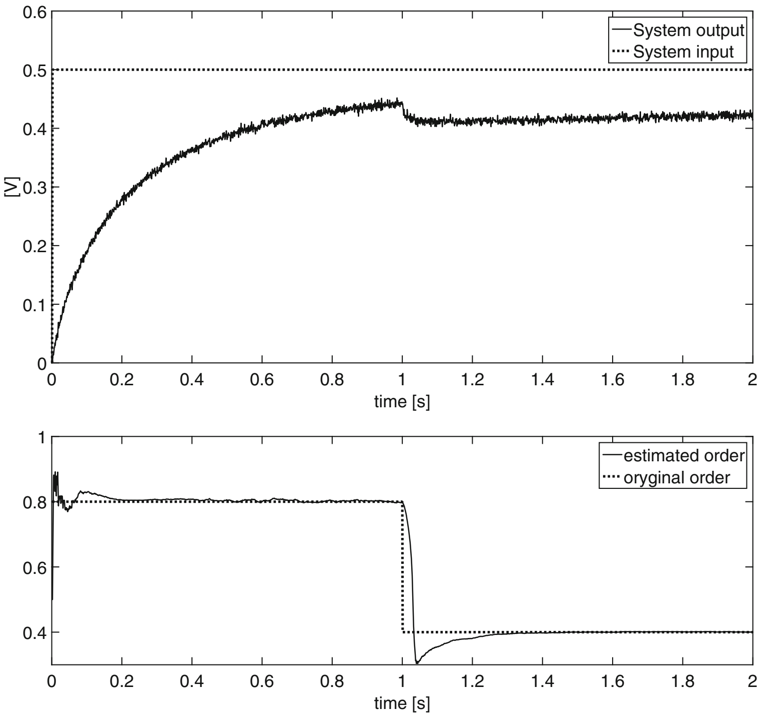


Fig. 5 Results for dual estimation of variable order α_k, input and output of the system—*upper figure*, estimated order—*bottom figure*

Table 1 Comparison of order estimation error norms (where $e_k = \alpha_k - \tilde{\alpha}_k$, $k = 200, \ldots, T/h$), during time T for different output noise variances R

Output noise variance R	Estimation error $h \sum_{i=200}^{T/h} e_i^2$
0.00000016	0.0001
0.00001504	0.0003
0.00152402	0.0039

The comparison of order estimation results is presented in Fig. 7. As it can be noticed, the smaller value of ${}^*Q_k^w$ the faster estimated order approach to the original one, however differences between results are not so significant. In order to thoroughly explain the differences, let us analyze the value of $\sqrt{(L+\lambda)\tilde{P}_k^w}$, which define a spread of sigma points obtained in UFKF algorithm (Fig. 8).

Example 5 (Order estimation for direct measurements and rapidly changed system order) Let us consider the DFVOSS system (4)–(6) with the following matrices:

$$A = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}. \tag{28}$$

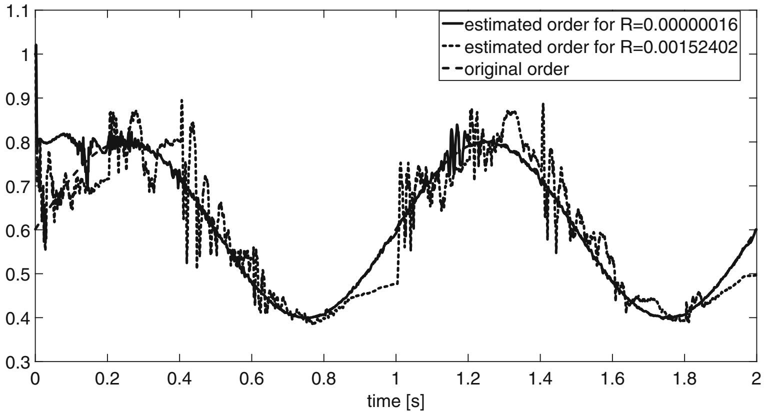
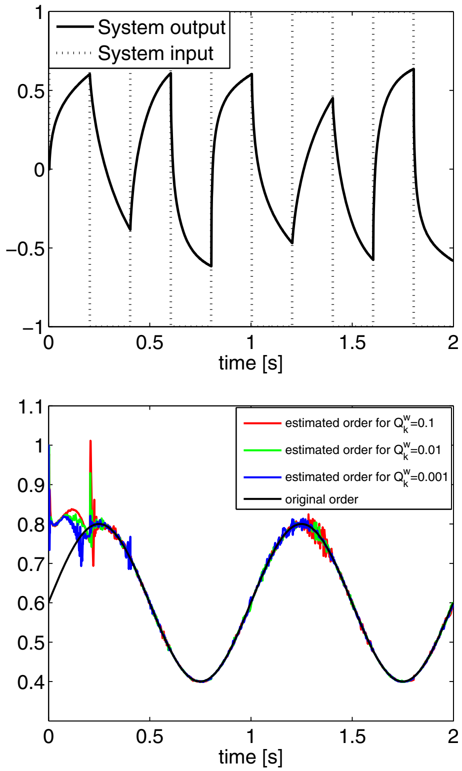


Fig. 6 Estimated order for different variance noises

Fig. 7 Results for dual estimation of variable order α_k for different parameters Q_k^w, input and output of the system—*upper figure*, estimated order—*bottom figure*



To verify the effectiveness of the estimation algorithm in terms of the rate of change of the order, the variable orders for both state variables are assumed to be the following chirp functions

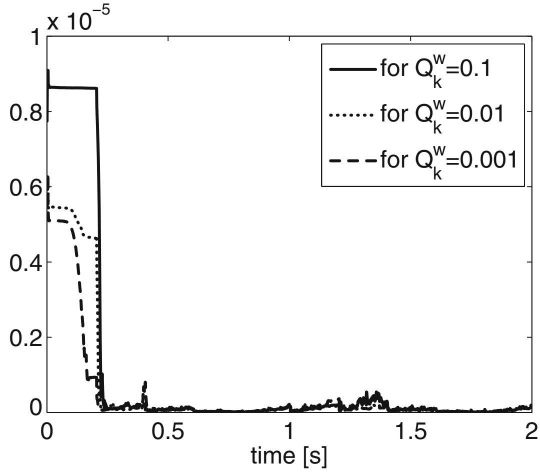


Fig. 8 Spread of sigma points during estimation of variable order α_k process for different parameters $^*Q_k^w$

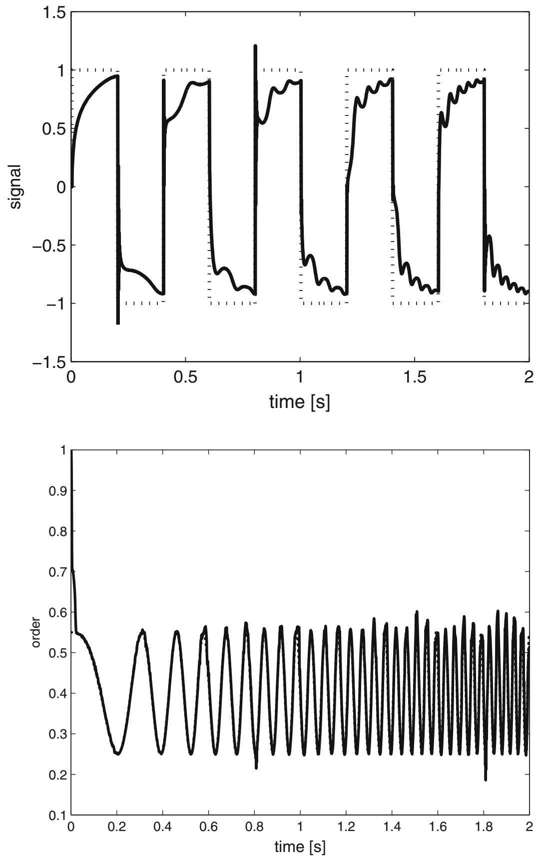


Fig. 9 Results for dual estimation of variable order α_k: input (*dotted line*) and output (*solid line*) of the system—*upper figure*, original order (*dotted line*) and estimated order (*solid line*) of the system—*bottom figure*

$$\alpha_{i,k} = 0.4 + 0.15\cos\left(2\pi\left(f_0 t_k + \frac{\beta t_k^2}{2}\right)\right), \quad \beta = \frac{f_1 - f_0}{t_N}, \quad i = 1, 2,$$

for $f_0 = 1$, $f_1 = 30$, $t_N = 2$. The noise has the following parameters:

$$E[\omega\omega^T] = \begin{bmatrix} 0.90 \cdot 10^{-7} & 0 \\ 0 & 0.90 \cdot 10^{-7} \end{bmatrix},$$
$$E[\nu\nu^T] = 1.5810^{-7}.$$

Parameters of KFx filter are:

$$P_0 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \quad Q_0 = \begin{bmatrix} 0.90 \cdot 10^{-7} & 0 \\ 0 & 0.90 \cdot 10^{-7} \end{bmatrix},$$
$$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R = [1.58 \cdot 10^{-7}].$$

Parameters of KFw filter are:

$$P_0^w = \left[5.79 \cdot 10^{-7}\right], \quad {}^*Q_0^w = \left[3.71 \cdot 10^{-11}\right],$$
$$\hat{\alpha}_0 = [0.5], \quad R^w = [1.58 \cdot 10^{-7}], \quad \mathfrak{A} = 0.001, \quad \delta = 0.5.$$

Figure 9 presents input and output of the DFVOSS system (28)—the data for estimation process, and results of applying dual estimation algorithms to these data. As it can be seen, despite the high rate of changing the order of the system, its estimation is performed with very high accuracy, and algorithm needed very short time to adjust for order changing.

6.2 Order Estimation for Networked Measurements

Example 6 (Order estimation for direct measurements and low noise) Parameters of the system and filters are the same as in Example 1, and the transmission rate for measurements is 30 %.

As it can be seen in Fig. 10, accuracy of order estimation is lower than it was obtained in direct measurements case, but still it shows high accuracy of the dual estimation algorithm. The losing of accuracy is caused by losing information during transmission by the communication network.

Example 7 (Order estimation for direct measurements and high noise) Parameters of the system and filters are the same as in Example 2, and the transmission rate for measurements is 30 %.

Figure 11 presents estimation results for higher noise than in the previous example. As it can be notice, higher noise caused lower accuracy of order estimation, but it is still on the reasonable level.

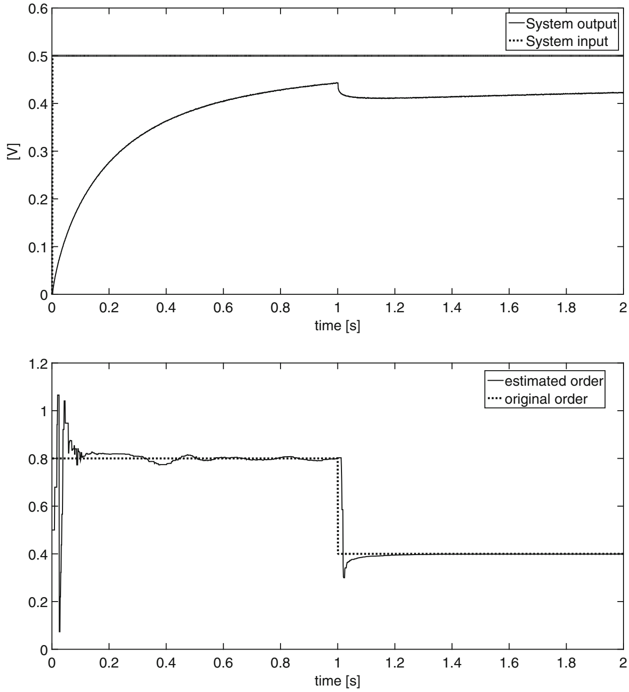


Fig. 10 Results for dual estimation of variable order α_k for networked measurements, input and output of the system—*upper figure*, estimated order—*bottom figure*

7 Order Estimation for Analog Model

In order to validate proposed algorithm in real application, experimental data obtained from variable order inertial system will be used. Such a system is realized by putting fractional variable order integrator in unity feedback system, as shown in Fig. 12.

All measurement data have been gathered with time sample equals to 0.001 sec and input signal equal to $0.5 \cdot H(t)$, where $H(t)$ is a Heaviside step function.

7.1 Analog Model of Variable Order System

The analog model of the fractional variable order inertial system has been realized based on fractional variable order integral, given in Fig. 1, in unity feedback system presented in Fig. 12, and consists of the following parts:

- data acquisition card dSPACE 1104;
- operational amplifiers TL071;
- electronic switches DG303;
- passive elements such as: resistors $R_1 = 2.4\,\text{k}\Omega$, $R_2 = 8.2\,\text{k}\Omega$, $R = 100\,\text{k}\Omega$, $R_a = 43\,\text{k}\Omega$ and $R_b = 33\,\text{k}\Omega$, capacitors $C_1 = 330\,\text{nF}$ and $C_2 = 220\,\text{nF}$.

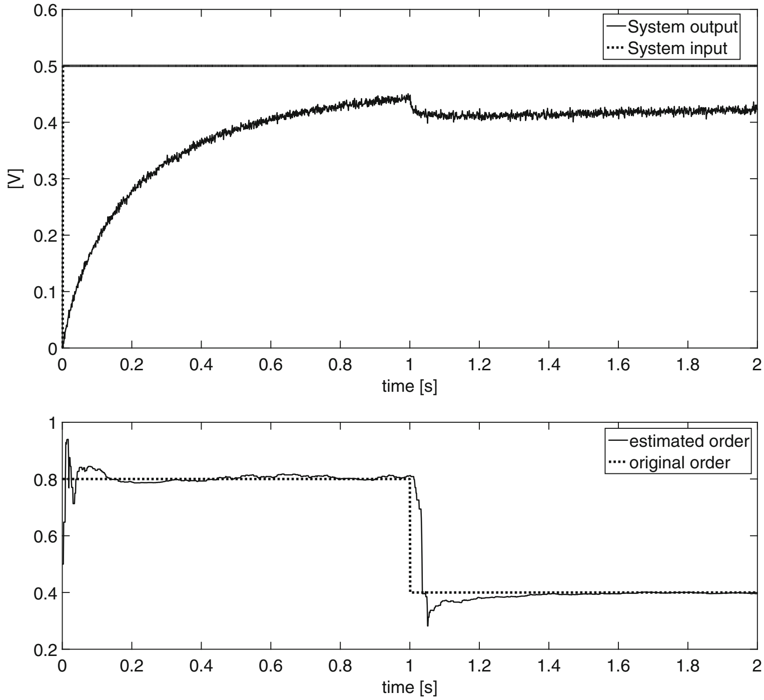
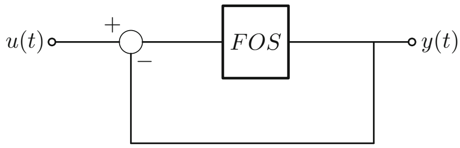


Fig. 11 Results for dual estimation of variable order α_k for networked measurements, input and output of the system—*upper figure*, estimated order—*bottom figure*

Fig. 12 Realization of the fractional variable order inertial system based on fractional variable order system (FOS) presented in Fig. 1



The variable order integral system has been denoted as FOS block in Fig. 12. To build the electronic circuit board corresponding to fractional variable order inertial system has been used a universal electronic board specially prepared for testing a variable order systems. The overview of the real circuit board with fractional variable order inertial system has been shown in Fig. 13.

The order of such variable inertial system depends only on switches positions and can be changed many times during experimental process. When all switches (S_1, S_2 and S_3) in Fig. 1 are connected to terminals 1, the first-order inertial system is considering. Otherwise, when switches are connected to terminals 2, the half-order inertial system has been achieved.

7.2 Order Estimation for Analog Model and Direct Measurements

The identification process of analog model parameters was conducted according to the algorithm presented in [32]. The parameters to be identified were obtained for

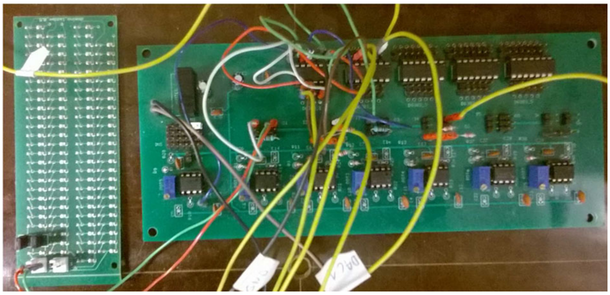


Fig. 13 A circuit board of fractional variable order inertial system

constant value of order. It was done due to the fact that the system was designed to keep the constant value of parameters for each order. Finally, identified parameters for discrete variable order state-space system are as following:

$$A = -3.1937, \quad B = 3.2129, \quad C = 1. \tag{29}$$

Parameters of KFx filter are:

$$P_0 = \left[10\right], \quad Q_0 = \left[8.82 \cdot 10^{-8}\right],$$
$$x_0 = [0], \quad R = [1.6 \cdot 10^{-5}].$$

Parameters of KFw filter are:

$$P_0^w = \left[0.5\right], \quad {}^*Q_0^w = \left[0.01\right],$$
$$\hat{\alpha}_0 = [0.5], \quad R^w = [1.6 \cdot 10^{-5}], \quad \mathfrak{A} = 0.0001, \quad \delta = 0.5.$$

Figure 14 presents input and output of the analog system—the data for estimation process. Figure 15 shows results of applying dual estimation algorithm to these data. As it can be seen, the order is estimated with a high accuracy.

Example 8 In Table 2 and Fig. 16, the results of series of experiments for estimation of variable order, performed for different step times h, have been presented. What was expected, the larger is the step time, the accuracy of estimation error decreases.

7.3 Order Estimation for Analog Model and Networked Measurements

Parameters of the system and filters are the same as in Sect. 7.2, and the transmission rate for measurements is 30 %.

Figure 17 shows results of applying dual estimation algorithms for data obtained by lossy network. As it can be seen, the order is estimated with quite high accuracy, however with lower accuracy than for direct measurements case.

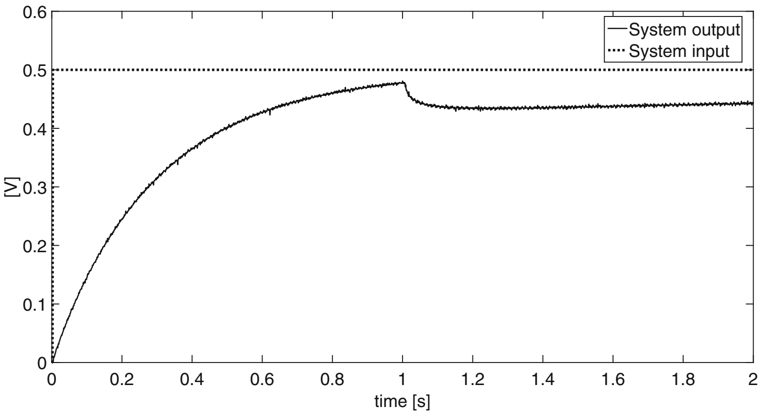


Fig. 14 Input and output of the analog system

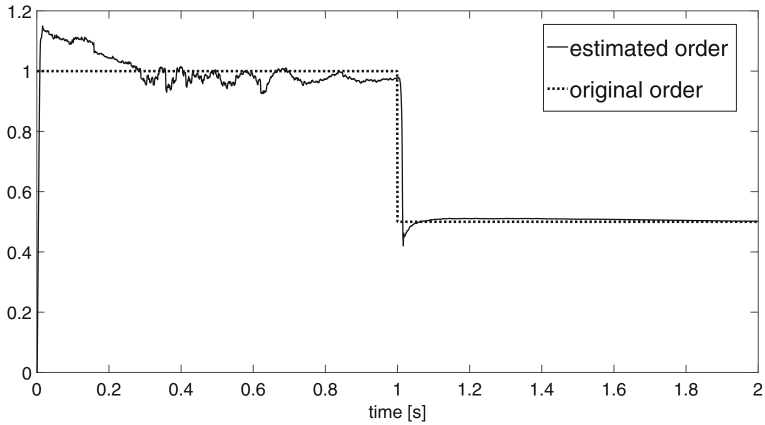


Fig. 15 Results for dual estimation of variable order α_k for analog model—estimated order

Table 2 Comparison of order estimation error norms (where $e_k = \alpha_k - \tilde{\alpha}_k, k = 1, \ldots, T/h$), during time T for different step times h

Step time h (s)	Estimation error $h \sum_{i=1}^{T/h} e_i^2$
0.0010	0.0113
0.0020	0.0277
0.0050	0.0572
0.0080	0.0665
0.0100	0.0505
0.0200	0.0904

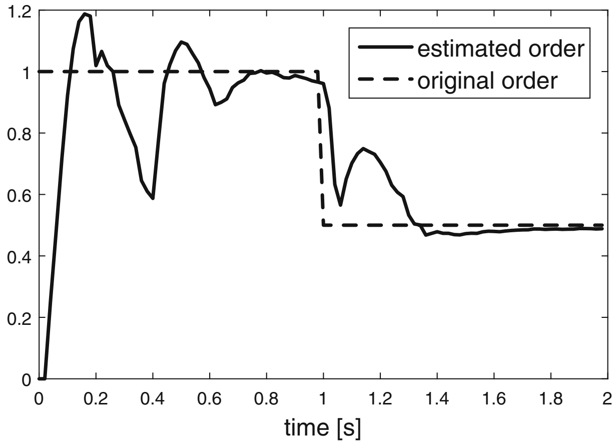


Fig. 16 Estimated order for step time $h = 0.02$ s

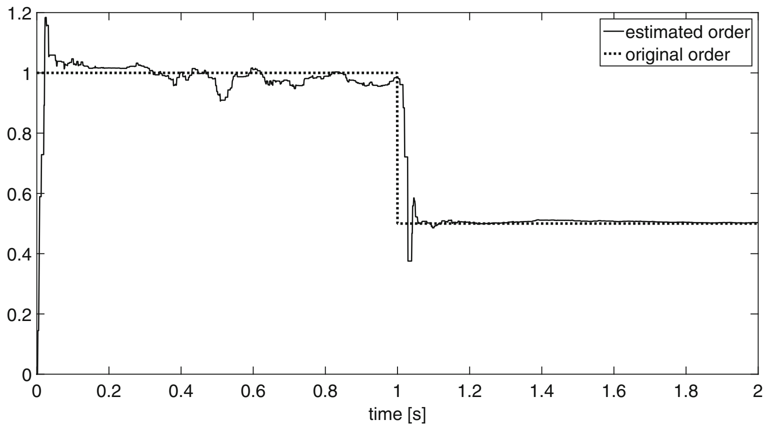


Fig. 17 Results for dual estimation of variable order α_k for analog model and networked measurements—estimated order

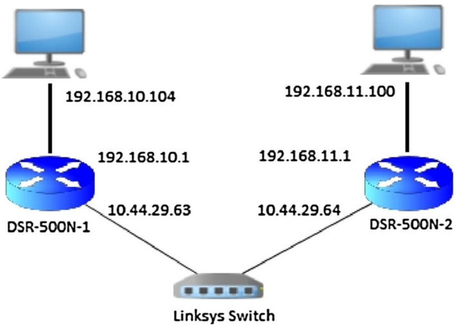
7.4 Order Estimation for Analog Model and Networked Measurements Transmitted by Real Network

In this section, the measurements obtained by specially build separate computer network contained two computers and two D-Link DGS-1100-16 routers Fig. 18. In Fig. 19, the schema of lossy network has been presented.

In experiment, the package of 2000 samples was sent through the network with the sampling time 0.001 s. The transmission was realized in use of the UDP protocol. This protocol uses a simply connectionless transmission model without transmission tracking which means that it does not have any mechanism for flow or transmission control. In effect of overloaded network, the package can be lost or received out of order. This kind of issues has been simulated. For this purpose, the bandwidth for transmission channel was limited. The signal from the source was transmission

Fig. 18 D-Link routers

Fig. 19 Schema of lossy network



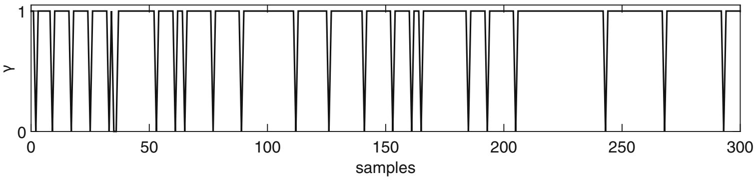


Fig. 20 Samples lost during transmission

with exceeded available bandwidth. As a result, the transmission rate about 90 % was achieved. In Fig. 20, information about packages losing γ_k for the first 300 samples is shown. The value equal to 0 means that this sample was missing or it had come in out of order. In this experiment, the parameters used for estimation are the same as in Sect. 7.2. Results of estimation are presented in Fig. 21.

8 Conclusions

In this paper, the variable order estimation algorithms for a case when measurements are obtained directly and by lossy network have been presented. The order estimation algorithms were applied to numerical examples and to real fractional variable order inertial system. Since the problem of fractional order estimation is highly nonlinear, the dual estimation algorithm has been used. For state variables and variable fractional order estimation, the Fractional Kalman filter and the Unscented Fractional Kalman filter have been used, respectively. Numerical results shown efficiency of proposed

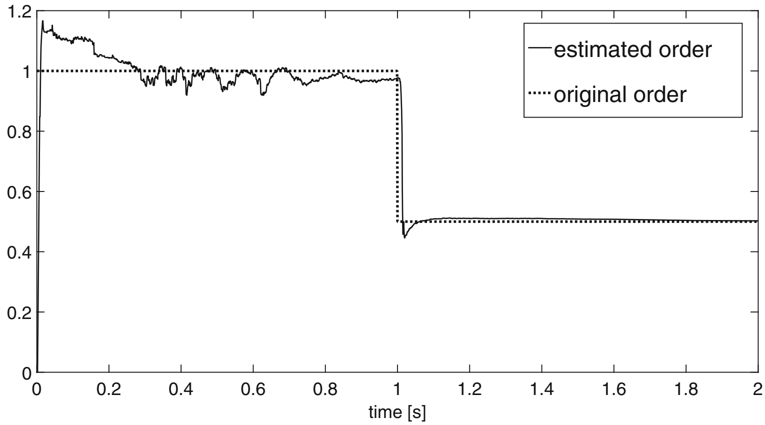


Fig. 21 Results for dual estimation of variable order α_k for analog model and networked measurements obtained from real network—estimated order

algorithms for direct and networked measurements as well. The estimation algorithm has been also tested on a real object being electrical circuit analog model. The proposed algorithms have confirmed the possibility of further use in case of order estimation of real objects of unknown order both constant or variable.

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