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Global attractivity for reaction–diffusion equations with periodic coefficients and time delays

Alfonso Ruiz-Herrera and Tarik Mohammed Touaoula

Abstract. In this paper, we provide sharp criteria of global attraction for a class of non-autonomous reaction–diffusion equations with delay and Neumann conditions. Our methodology is based on a subtle combination of some dynamical system tools and the maximum principle for parabolic equations. It is worth mentioning that our results are achieved under very weak and verifiable conditions. We apply our results to a wide variety of classical models, including the non-autonomous variants of Nicholson's equation or the Mackey–Glass model. In some cases, our technique gives the optimal conditions for the global attraction.

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1. Introduction

In the last decades, delay differential equations have been largely employed in the study of population dynamics, infectious diseases, or neural networks [4,16,19,21,22]. The use of time delays naturally arises to describe the maturation time of biological species, the incubation time of some diseases, or the maturation time of blood cells. One of the most celebrated equations is the Nicholson's blowfly equation

$$u'(t) = pu(t-\tau)e^{-qu(t-\tau)} - ru(t). \tag{1}$$

In (1), $u(t)$ is the size of the mature blowfly population at time t, p is the maximum per-capita daily egg production rate, $\frac{1}{q}$ is the size at which the fly reproduces at its maximum rate, τ is the maturation time and r is a per-capita death rate. With remarkable precision, Gurney et al. [7] used model (1) to explain the oscillations recorded in Nicholson's laboratory experiments [12]. A remarkable fact is that (1) can be derived from the classical McKendrick–von Forester equations

$$\begin{cases} \frac{\partial u}{\partial a}(a,t) + \frac{\partial u}{\partial t}(a,t) = -\mu(a)u(a,t), \\ u(0,t) = b(M(t)), \\ u(a,0) = u_0(a), \end{cases} \tag{2}$$

with

$$M(t) = \int_{\tau}^{+\infty} u(a,t)\,\mathrm{d}a,$$

see [16]. In (2), $b(M(t))$ is the birth rate function, and $M(t)$ denotes the total population size of mature individuals. This remark indicates that (1) is a solid modeling framework, not restricted to the evolution of blowflies. However, when (1) is employed to study a population in a non-laboratory ecosystem, the introduction of diffusion terms associated with the movement of individuals seems a natural modification

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of (1), see [18,24–27]. In this context, if the immature individuals do not diffuse but the mature ones do, we arrive at the model

$$\begin{cases} u_t(t,x) = \mathrm{d}\nabla^2 u(t,x) + pu(t-\tau,x)e^{-qu(t-\tau,x)} - ru(t,x) & x \in \Omega \\ n(x)\cdot\nabla u(t,x) = 0 & x \in \partial\Omega \end{cases} \tag{3}$$

for all $t \geq 0$ where $\Omega \subset \mathbb{R}^N$ is a connected and bounded domain with smooth boundary. We stress that the Neumann boundary condition indicates that the individuals of the population do not cross the boundary of the domain. It is worth noting that equations of the form

$$u'(t) = f(u(t-\tau)) - ru(t) \tag{4}$$

with a general $f : [0,+\infty) \longrightarrow [0,+\infty)$ are of paramount importance in many contexts, see, for instance, the celebrated Mackey–Glass equation [5,19]. For (4), we can introduce diffusion terms as we did with (1) and we arrive at model

$$\begin{cases} u_t(t,x) = \mathrm{d}\nabla^2 u(t,x) + f(u(t-\tau,x)) - ru(t,x) & x \in \Omega \\ n(x)\cdot\nabla u(t,x) = 0 & x \in \partial\Omega \end{cases} \tag{5}$$

A key issue associated with (5) is to provide sharp sufficient conditions for the existence of a globally attracting non-trivial equilibrium. When f is increasing, the powerful theory of monotone systems can be employed and (5) typically exhibits simple dynamics [19,20]. However, when f is non-monotone, the theory of monotone dynamical systems does not work, and the study of (5) becomes more subtle. In a series of papers, Yi and Zou [24–27] focused on this problem by combining dynamical systems arguments and some sharp inequalities. Their approach allows us to deduce nice results on the global dynamical picture of model (3), (see also [28] and the references therein). For example, it was shown that the best delay-independent condition of global attraction in (1) is also valid in (3). From this analysis, the model exhibits a transcritical bifurcation at the origin. If $\frac{p}{q} < 1$, there is global extinction, whereas if $\frac{p}{q} \in (1, e^2)$, there is a globally attracting non-trivial equilibrium.

It is well-known that interactive populations often live in fluctuating environments. For example, biological conditions such as temperature, humidity, or the availability of food resources play a critical role in the evolution of the species and they vary in time in a seasonal or daily manner [1,10,13,17]. On the other hand, seasonality has been observed in the incidence of many infectious diseases, such as measles, chickenpox, etc, [2]. Actually, it is becoming clear the importance of considering nonautonomous variants of (5) in any biological model, see [9,11] and the references therein. However, the introduction of seasonality adds many difficulties in the mathematical analysis of (5). In particular, the arguments given by Yi and Zou [24–27] are not valid for nonautonomous equations. At first sight, it is very hard to establish a connection with a discrete equation as they did in [25].

Given the importance of reaction–diffusion equations with delay in practical problems, the literature on this topic is vast, see [6,10,11,21,22,24–27,30] and the references therein. However, to the best of our knowledge, there is no general and systematic methodology in the literature concerning criteria of global attractivity in (5) with periodic coefficients. To fill this gap, the main objective of this paper is to extend to non-autonomous equations the research conducted by Yi and Zou [24–27]. Here, there are the four leading ideas:

- Providing sufficient conditions for the permanence in the non-autonomous counterpart of equation (5).
- Identifying a class of amenable nonlinearities. This class has been already mentioned by Yi and Zou [24,27].
- Employing suitable changes of variables.
- Using the maximum principle for parabolic equations subtly.

Our results are achieved under very weak and verifiable assumptions. Moreover, the results in [24,25] are recovered by our general methodology when applied to autonomous equations. From a practical point of view, this paper allows us to investigate the effects of seasonal forcing on classical reaction–diffusion models.

The structure of the paper is as follows. In Sect. 2, we provide some preliminary and basic results. In Sect. 3, we focus on the permanence and global extinction in the nonautonomous counterpart of equation (5). We give the proof of the main results in Sect. 4. Finally, we conclude the paper with the application of our results in classical models. We stress that our approach includes the results deduced from the theory of monotone equations.

2. Mathematical framework, global well-posedness and dissipativity

Consider

$$\begin{cases} u_t(t,x) = \mathrm{d}\nabla^2 u(t,x) - a(t)u(t,x) + b(t)h(u(t-\tau,x)) & x \in \Omega \\ n(x)\cdot\nabla u(t,x) = 0 & x \in \partial\Omega \end{cases} \tag{6}$$

for all $t \geq 0$ where $\Omega \subset \mathbb{R}^N$ is a connected and bounded domain with smooth boundary. The constants d and τ are strictly positive. The functions $a, b : \mathbb{R} \longrightarrow (0, +\infty)$ are continuous, T-periodic and strictly positive. Moreover, $h : [0, +\infty) \longrightarrow [0, +\infty)$ is a bounded function of class $\mathcal{C}^1$ of the form $h(x) = xf(x)$ with $f : [0, +\infty) \longrightarrow (0, +\infty)$ decreasing and $\lim_{x\to+\infty} f(x) = 0$. The last line in (6) refers to the homogeneous Neumann boundary condition. Throughout the paper, we will assume the previous conditions without further mention.

Let $\mathbb{R}_+ = [0, +\infty)$. We define $X = \{u : \overline{\Omega} \to \mathbb{R} : u \text{ continuous}\}$, $X_+ = \{u \in X : u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\}$ and $X_{++} = \{u \in X : u(x) > 0 \text{ for all } x \in \overline{\Omega}\}$. The space X is a Banach space equipped with the max norm $\|\cdot\|_\infty$. Next we consider $\mathcal{D} = \{v \in X : \nabla^2 v \in X \text{ and } n(x).\nabla v(x) = 0 \text{ on } x \in \partial\Omega\}$. Given I an interval, $\mathcal{C}(I, X_+)$ and $\mathcal{C}(I, X_{++})$ will denote the space of the continuous functions taking values on X_+ and X_{++}, respectively.

Consider the family of operators $A(t) = d\nabla^2 - a(t)id_X$ defined on $\mathcal{D}$ with id_X the identity in X. Let $\{U(t)\}$ be the evolution family on X generated by $\{A(t)\}_{t\in\mathbb{R}}$. By basic results on parabolic equations, see Refs. [15,29], we have the following properties:

(P1) $U(t)$ is an analytical and strongly continuous operator for all $t \geq 0$.

(P2) $U(t)(K) = Ke^{-\int_0^t a(s)ds}$ for any $K \in \mathbb{R}$ and $t \geq 0$.

(P3) $U(t)(X_+\backslash\{0\}) \subset X_{++}$ for all $t > 0$.

For any $\phi, \psi \in X$, we work with the next ordering relations:

- If $\phi - \psi \in X_+$, we write $\phi \geq \psi$.
- If $\phi \geq \psi$ and $\phi \neq \psi$, we write $\phi > \psi$.
- If $\phi - \psi \in X_{++}$, then $\phi \gg \psi$.

We note that **(P3)** implies that if $\phi > \psi$, then $U(t)\phi \gg U(t)\psi$ for all $t > 0$.

Given an initial data $\phi \in \mathcal{C}([-\tau, 0], X_+)$, we consider the integral equation

$$u^\phi(t,\cdot) = U(t)\phi(0,\cdot) + \int_0^t U(t-s)(b(s)h(u^\phi(s-\tau,\cdot)))\mathrm{d}s \tag{7}$$

with $u^\phi(t,\cdot) = \phi(t,\cdot)$ for all $t \in [-\tau, 0]$. It is well-known that if u is a solution of (6) then u is a solution of (7) as well. Conversely, if u is a solution of (7) and u is of class $\mathcal{C}^1$ in t and of class $\mathcal{C}^2$ in x, then u is a solution of (6). In general, we say that a solution of (7) is a mild solution of (6).

Using that $b(t) > 0$ and $h(u) \geq 0$ with $h(u) = 0$ if, and only if, $u = 0$, we deduce from (7) (see also **(P3)**) that any mild solution $u^\phi(t, x)$ with initial data $\phi \in \mathcal{C}([-\tau, 0], X_+)$ satisfies that $u^\phi(t, x) \geq 0$ for all $t \in [-\tau, T_{max})$, $x \in \overline{\Omega}$ with $[-\tau, T_{max})$ the maximal (right) interval of definition of u^ϕ. Even more, if

$\phi \in \mathcal{C}([-\tau,0],X_{++})$, then $u^{\phi}(t,x)>0$ for all $t\in[-\tau,T_{max})$ and $x\in\overline{\Omega}$. We refer to these last solutions as positive solutions of (6).

Next we check that the solutions of (7) cannot blow up. Take $\phi \in \mathcal{C}([-\tau,0],X_{+})$ and three positive constants β_1,β_2 and β_3 so that

$$\phi(t,x)\le\beta_1 \;\text{ for all } (t,x)\in[-\tau,0]\times\overline{\Omega} \tag{8}$$

and

$$b(t)h(u)\le\beta_2 \quad\text{and}\quad \frac{1}{a(t)}\le\beta_3 \tag{9}$$

for all $t,u\in[0,+\infty)$. Using **(P2)** and **(P3)**, expression (7) implies that

$$\begin{aligned}
u^{\phi}(t,x) &\le \beta_1 e^{-\int_0^t a(s)\mathrm{d}s}+\int_0^t \beta_2 e^{-\int_0^{t-s} a(r)\mathrm{d}r}\mathrm{d}s\\
&=\beta_1 e^{-\int_0^t a(s)\mathrm{d}s}+\int_0^t \frac{a(t-s)}{a(t-s)}\beta_2 e^{-\int_0^{t-s} a(r)\mathrm{d}r}\mathrm{d}s\\
&\le\beta_1 e^{-\int_0^t a(s)\mathrm{d}s}+\int_0^t \beta_2\beta_3 a(t-s) e^{-\int_0^{t-s} a(r)\mathrm{d}r}\mathrm{d}s\\
&=\beta_1 e^{-\int_0^t a(s)\mathrm{d}s}+\int_0^t \beta_2\beta_3 \frac{d}{\mathrm{d}s} e^{-\int_0^{t-s} a(r)\mathrm{d}r}\mathrm{d}s\\
&=\beta_1 e^{-\int_0^t a(s)\mathrm{d}s}+\beta_2\beta_3\left(1-e^{-\int_0^t a(s)\mathrm{d}s}\right).
\end{aligned} \tag{10}$$

Now, it is clear that the solution $u^{\phi}(t,x)$ cannot blow up. In addition, for each $x\in\overline{\Omega}$,

$$\limsup_{t\to+\infty} u^{\phi}(t,x)\le\beta_2\beta_3.$$

On the other hand, by Corollary 2.2.5 in [29], we have that any mild solution of (6) is a classical solution for $t>\tau$. From now on, we will simply use the term solution. Collecting the above discussion, we obtain the following result:

Theorem 2.1. *For any $\phi\in\mathcal{C}([-\tau,0],X_+)$, Eq. (6) admits a unique classical solution $u^{\phi}\in\mathcal{C}([\tau,+\infty),X_+)\cap \mathcal{C}^1((\tau,+\infty),\mathcal{D})$. Moreover, for each $x\in\overline{\Omega}$, $\limsup_{t\to+\infty}u^{\phi}(t,x)\le\beta_2\beta_3$ with β_2 and β_3 satisfying (9).*

3. Permanence and global extinction

In this section, we prove that the positive solutions of (6) are separated from zero. To this goal, we introduce an additional condition:

(C1)

$$\frac{\min\{b(t):t\in[0,T]\}}{\max\{a(t):t\in[0,T]\}}f(0)>1.$$

Theorem 3.1. *Assume that* **(C1)** *holds. Given $u(t,x)$ a positive solution of* (6), there exists a constant $\Lambda>0$ so that

$$u(t,x)\ge\Lambda$$

for all $(t,x) \in [-\tau, +\infty) \times \overline{\Omega}$.

Proof. By Theorem 2.1, we know that $u(t,x)$ is a bounded function. Let M be an upper bound of u. On the other hand, by **(C1)**, we can find two constants $1 > c > 0$ and $\eta > 1$ so that

$$\frac{\min\{b(t) : t \in [0,T]\}}{\max\{a(t) : t \in [0,T]\}} f(u) > \eta$$

for all $u \in (0,c]$, or equivalently,

$$\frac{\min\{b(t) : t \in [0,T]\}}{\max\{a(t) : t \in [0,T]\}} h(u) > \eta u \tag{11}$$

for all $u \in (0,c]$. Using that f is continuous and strictly positive with $h(u) = uf(u)$, there exists $K > 0$ so that

$$\frac{\min\{b(t) : t \in [0,T]\}}{\max\{a(t) : t \in [0,T]\}} h(u) > K \tag{12}$$

for all $u \in [c, M]$. Now we define

$$\Lambda = \min\{\min\left\{\frac{u(t,x)}{2} : (t,x) \in [-\tau,\tau] \times \overline{\Omega}\right\}, \frac{c}{2}, K\}. \tag{13}$$

We stress that if $u \in [\Lambda, M]$, then

$$\min\{b(t) : t \in [0,T]\} h(u) > \Lambda \max\{a(t) : t \in [0,T]\} \tag{14}$$

for all $t \in [0,T]$. To see this claim, we distinguish between two cases:

Case 1 If $u \in [\Lambda, c]$, we know that $\min\{b(t) : t \in [0,T]\} uf(u) > \eta \max\{a(t) : t \in [0,T]\} u$ by (11). Since $\eta > 1$, we conclude that $\min\{b(t) : t \in [0,T]\} uf(u) > \max\{a(t) : t \in [0,T]\} \Lambda$.

Case 2 If $u \in [c, M]$, we have that $\min\{b(t) : t \in [0,T]\} h(u) > K \max\{a(t) : t \in [0,T]\}$ as a direct consequence of (12). We deduce that $\min\{b(t) : t \in [0,T]\} h(u) > \Lambda \max\{a(t) : t \in [0,T]\}$ by (13).

After these preliminary comments, we prove that $u(t,x)$ cannot reach the value Λ. Assume, by contradiction, that there is a first instant $t_0 > 0$ so that

$$u(t_0, x_0) = \Lambda$$

for some $x_0 \in \overline{\Omega}$. With the notation of Sect. 2,

$$\begin{aligned} u(t_0,\cdot) &= U(t_0)\left(u(0,\cdot)\right) + \int_0^{t_0} U(t_0-s)\left(b(s)h(u\left(s-\tau,\cdot\right))\right)\mathrm{d}s \\ &\geq U(t_0)(u(0,\cdot)) + \int_0^{t_0} U(t_0-s)(\min\{b(t): t\in[0,T]\}h(u(s-\tau,\cdot)))\mathrm{d}s \\ &\underbrace{\gg}_{(14)} U(t_0)(u(0,\cdot)) + \int_0^{t_0} U(t_0-s)(\max\{a(t): t\in[0,T]\}\Lambda))\mathrm{d}s \\ &\geq U(t_0)(u(0,\cdot)) + \int_0^{t_0} a(t_0-s)U(t_0-s)(\Lambda))\mathrm{d}s \\ &\underbrace{\gg}_{(13)} U(t_0)(\Lambda) + \int_0^{t_0} a(t_0-s)U(t_0-s)(\Lambda))\mathrm{d}s \\ &= \Lambda e^{-\int_0^{t_0} a(s)\mathrm{d}s} + \Lambda\left(1-e^{-\int_0^{t_0} a(s)\mathrm{d}s}\right). \end{aligned}$$

Thus, $u(t_0,\cdot) \gg \Lambda$. This is a contradiction. □

Remark 3.1. In the previous result, (13) is an explicit lower bound of $u(x,t)$. This bound has three elements: $\min\left\{\frac{u(t,x)}{2} : (t,x)\in[-\tau,\tau]\times\overline{\Omega}\right\}$, $\frac{c}{2}$ and K. Informally speaking, condition (11) says that $(0,c)$ is a repulsion region for the origin. We introduce K to avoid returns caused by the delay, see (12).

The next result shows that condition **(C1)** is close to be a sufficient condition in Theorem 3.1. From a biological point of view, the following result says that if the natality rate is lower than the mortality rate, the population goes to extinction.

Theorem 3.2. *Assume that*

$$\frac{\max\{b(t): t\in[0,T]\}}{\min\{a(t): t\in[0,T]\}} f(0) < 1 \tag{15}$$

for all $t\in[0,T]$. Then, given $u(t,x)$ a positive solution of (6), for each $x\in\overline{\Omega}$, $\lim_{t\to+\infty} u(t,x) = 0$.

Proof. First, we pick two constants Γ_1, Γ_2 so that

$$\frac{\max\{b(t): t\in[0,T]\}}{\min\{a(t): t\in[0,T]\}} f(0) < \Gamma_1 \tag{16}$$

and $\Gamma_1 < \Gamma_2 < 1$. Fix $u(t,x)$ a positive solution of (6). By Theorem 2.1, we can take $M>0$ so that $u(t,x)\leq M$ for all $t\geq-\tau$ and $x\in\overline{\Omega}$. Let us prove that for each $n\in\mathbb{N}$, there is $t_n>0$ with $t_n\to+\infty$ so that $u(t,x)\leq M\Gamma_2^n$ for all $t\geq t_n$ and $x\in\overline{\Omega}$. By (7),

$$u(t,\cdot) = U(t)u(0,\cdot) + \int_0^{t} U(t-s)(b(s)h(u(s-\tau,\cdot)))\mathrm{d}s.$$

Using **(P2)**, **(P3)**, and f strictly decreasing, we have that

$$
\begin{aligned}
u(t,\cdot) &\le Me^{-\int_0^t a(s)\mathrm{d}s} + \int_0^t U(t-s)(\max\{b(t): t\in[0,T]\}u(s-\tau,\cdot)f(0))\mathrm{d}s \\
&\underbrace{\le}_{(16)} Me^{-\int_0^t a(s)\mathrm{d}s} + \int_0^t U(t-s)(M\min\{a(t): t\in[0,T]\}\Gamma_1)\mathrm{d}s \\
&\le Me^{-\int_0^t a(s)\mathrm{d}s} + \int_0^t a(t-s)U(t-s)(M\Gamma_1)\mathrm{d}s \\
&= Me^{-\int_0^t a(s)\mathrm{d}s} + \left(1-e^{-\int_0^t a(s)\mathrm{d}s}\right)M\Gamma_1.
\end{aligned}
$$

Notice that $\lim_{t\to+\infty} Me^{-\int_0^t a(s)\mathrm{d}s} + (1-e^{-\int_0^t a(s)\mathrm{d}s})M\Gamma_1 = M\Gamma_1$. Thus, there is $t_1>0$ so that $u(t,x)\le M\Gamma_2$ for all $t\ge t_1$ and $x\in\overline{\Omega}$. Recall that $\Gamma_1<\Gamma_2$. Arguing similarly and using this new information, we deduce that

$$
u(t,\cdot) \le M\Gamma_2 e^{-\int_{t_1+\tau}^t a(s)\mathrm{d}s} + \left(1-e^{-\int_{t_1+\tau}^t a(s)\mathrm{d}s}\right)M\Gamma_2\Gamma_1
$$

for all $t>t_1+\tau$. Since $\lim_{t\to+\infty} M\Gamma_2 e^{-\int_{t_1+\tau}^t a(s)\mathrm{d}s} + (1-e^{-\int_{t_1+\tau}^t a(s)\mathrm{d}s})M\Gamma_2\Gamma_1 = M\Gamma_2\Gamma_1$, we can find t_2 large enough so that $u(t,x)\le M\Gamma_2^2$ for all $t\ge t_2$ and $x\in\overline{\Omega}$. Reiterating this argument, we conclude the proof of the theorem. □

Remark 3.2. The theorems of this section can be applied in delay differential equations without diffusion. However, in this context, one can find better conditions for the permanence and extinction of the population in [3,17]. Specifically, (C1) and (15) can be replaced by $\min\{\frac{b(t)}{a(t)}: t\in[0,+\infty)\}f(0)>1$ and $\max\{\frac{b(t)}{a(t)}: t\in[0,+\infty)\}f(0)<1$, respectively.

4. Global attractivity

As a direct consequence of Corollary 3.1 in [3], the equation

$$
y'(t) = -a(t)y(t) + b(t)h(y(t-\tau)) \tag{17}
$$

admits a T-periodic solution $y_*(t)$ with $y_*(t)>0$ when **(C1)** is satisfied. To avoid misleading conclusions in the literature, we mention that in [3], the author considered functions satisfying $h(0)=0$, $h'(0)=1$ and $\frac{b(t)}{a(t)}>1$ for all $t\in[0,T]$. However, Corollary 3.1 in [3] also holds under **(C1)**. Notice that in Theorem 3.1 in [3], she really used a condition weaker than **(C1)**, [see second step (page 519 in [3]) and lines above (3.8) in page 521 in [3]].

In this section we analyze when $y_*(t)$ is globally attracting in (6). First we employ the change of variable

$$
v(t,x) = \frac{u(t,x)}{y_*(t)}.
$$

After some straightforward computations, we arrive at

$$\begin{cases} v_t & = d\nabla^2 v + b(t)\frac{y_*(t-\tau)}{y_*(t)}\left(v(t-\tau,x)f(y_*(t-\tau)v(t-\tau,x)) - v(t,x)f(y_*(t-\tau))\right) \\ n(x)\cdot\nabla v(t,x) = 0, \end{cases} \tag{18}$$

(in the first line $(t,x) \in [0,+\infty)\times\Omega$ and the second line $(t,x)\in[0,+\infty)\times\partial\Omega$).

The solutions of this problem are defined for all $t \geq 0$ and are bounded. Moreover, given an initial data $\varphi \in \mathcal{C}([-\tau,0]\times\overline{\Omega},(0,+\infty))$, by Theorem 3.1, there is $\widetilde{\Lambda} > 0$ with $v^{\varphi}(t,x) > \widetilde{\Lambda}$ for all $(t,x) \in [0,+\infty)\times\overline{\Omega}$. In terms of problem (18), given $v(t,x)$ a positive solution, we prove that for each $x\in\overline{\Omega}$, $\lim_{t\to+\infty} v(t,x) = 1$. Notice that this is equivalent to say that, given $u(t,x)$ a positive solution of (6), for each $x \in \overline{\Omega}$, $\lim_{t\to+\infty}[u(t,x) - y_*(t)] = 0$. Observe that

$$y_*(t)\left(\frac{u(t,x)}{y_*(t)} - 1\right) = u(t,x) - y_*(t)$$

and $y_*(t)$ is bounded and bounded away from zero.

Let $Y = \mathcal{C}([-\tau,0]\times\overline{\Omega},[0,+\infty)))$ be the Banach space with the maximum norm. For convenience, we assume that $\tau > T$. Otherwise we choose some $\overline{\tau} > \tau$ and insert Y in $\mathcal{C}([-\overline{\tau},0]\times\overline{\Omega},[0,+\infty))$ but we do not alter the delay in (18). We define the Poincaré operator

$$P : Y \longrightarrow Y$$
$$\varphi \mapsto v_T^{\varphi}$$

with $v_T^{\varphi} : [-\tau,0]\times\overline{\Omega} \to [0,+\infty)$ given by $v_T^{\varphi}(t,x) = v^{\varphi}(t+T,x)$ for all $(t,x)\in[-\tau,0]\times\overline{\Omega}$ and $v^{\varphi}(t,x)$ the solution of (18) with initial condition φ. For each $\varphi \in Y$,

$$\omega(\varphi) = \{\Psi \in Y : \exists n_k \to +\infty \text{ so that } P^{n_k}(\varphi) \to \Psi\}.$$

This set is compact and $P(\omega(\varphi)) = \omega(\varphi)$. Moreover, for every $\Psi\in\omega(\varphi)$, there is $v : \mathbb{R}\times\overline{\Omega} \longrightarrow [0,+\infty)$ with $v_0 = \Psi$ and $v_{nT} \in \omega(\varphi)$ for all $n\in\mathbb{Z}$, see Ref. [8,23]. In this section, we prove that for each $\varphi \in \mathcal{C}([-\tau,0]\times\overline{\Omega},(0,+\infty))$, $\omega(\varphi) = \{1\}$. Notice that if $\omega(\varphi)=\{1\}$, then, for each $x\in\overline{\Omega}$,

$$\lim_{t\to+\infty}[v^{\varphi}(t,x) - 1] = 0. \tag{19}$$

Indeed, take $t_n \to +\infty$. Since $\tau > T$, we can find a sequence $\{n_{t_n}\}\subset\mathbb{N}$ with $n_{t_n}\to+\infty$ so that

$$t_n = \widetilde{t}_n + n_{t_n}T$$

with $\widetilde{t}_n \in [-\tau,0]$. Using that $\omega(\varphi)=\{1\}$, we conclude that $P^{n_{t_n}}(\varphi)\to 1$. In particular, (19) holds.

Next we state a useful lemma in our analysis, (see Ref. [14]).

Lemma 4.1. *Let $T > 0$ and $W \subset \overline{\Omega}$ be an open domain with a smooth boundary ∂W. Let $u(t,x)$ be a continuous function on $[0,T]\times\overline{\Omega}$ with derivatives $\frac{\partial u}{\partial x_i}$, $\frac{\partial^2 u}{\partial x_j \partial x_i}$ and $\frac{\partial u}{\partial t}$ existing and being continuous on $(0,T]\times\Omega$. Let $Lu(t,x) = d\nabla u^2(t,x) - u_t(t,x)$. Then, the following claims are satisfied:*

- *(i) If $Lu(t,x) > 0$ (resp. < 0) for all $(t,x)\in(0,T)\times W$, then u cannot attain a local maximum in $(0,T)\times W$ (resp. local minimum).*
- *(ii) Suppose that the first derivatives of u with respect to the x_i variable exist and are continuous on $(0,T]\times\overline{\Omega}$. Let $Lu(t,x)\geq 0$ (resp. ≤ 0) for all $(t,x)\in(0,T)\times W$. If there exist $(t^*,x^*)\in(0,T)\times\partial W$ and $\varepsilon^*\in(0,T)$ and an open ball $S^*\subset W$ with $\partial S^*\cap\partial W = \{x^*\}$ and $u(t^*,x^*) > u(t,x)$ (resp. $u(t^*,x^*) < u(t,x)$) for all $(t,x)\in[t^*-\varepsilon,t^*+\varepsilon]\cap S^*$ then $n(x^*).\nabla u(t^*,x^*) > 0$, (resp. $n(x^*).\nabla u(t^*,x^*) < 0$).*

Inspired by Ref. [24,27], we introduce conditions on f to guarantee the global attraction to 1 in (18). With these assumptions, we are able to derive sharp criteria in considering the classical Nicholson's blowfly equation or the Mackey–Glass model.

Let θ be a positive constant with $y_*(t)\leq\theta$ for all $t\geq 0$.

(C2) If $a, b \geq 0$ satisfy that $a - 1 \geq |b - 1|$ then,

$$\frac{bf(\lambda b)}{f(\lambda)} \leq a$$

for all $\lambda \in (0, \theta]$. Moreover, $\frac{bf(\lambda b)}{f(\lambda)} = a$ for some $\lambda \in (0, \theta] \iff a = b = 1$.

(C3) If $a, b \geq 0$ satisfy that $1 - a \geq |b - 1|$ then,

$$\frac{bf(\lambda b)}{f(\lambda)} \geq a$$

for all $\lambda \in (0, \theta]$. Moreover, $\frac{bf(\lambda b)}{f(\lambda)} = a$ for some $\lambda \in (0, \theta] \iff a = b = 1$ or $a = b = 0$.

Now we are ready to give the main result of this section:

Theorem 4.1. *Assume that* **(C1)-(C3)** *hold. Given* $\phi \in \mathcal{C}([-\tau, 0] \times \overline{\Omega}, (0, +\infty))$ *and* $x \in \overline{\Omega}$, $\lim_{t \to +\infty} v^{\phi}(t, x) - 1 = 0$.

Proof. Let

$$\Delta = \sup\{|\Psi(t, x) - 1| : (t, x) \in [-\tau, 0] \times \overline{\Omega} \text{ and } \Psi \in \omega(\phi)\}.$$

Since $\omega(\phi)$ is compact, there are $(t_0, x_0) \in [-\tau, 0] \times \overline{\Omega}$ and $\widetilde{\Psi} \in \omega(\phi)$ so that

$$\Delta = |\widetilde{\Psi}(t_0, x_0) - 1|.$$

We recall that $v^{\widetilde{\Psi}}$ is a solution of (18) in the classical sense defined on $\mathbb{R} \times \overline{\Omega}$. Moreover, by Theorem 3.1, there exists $\widetilde{\Gamma} > 0$ so that

$$v^{\widetilde{\Psi}}(t, x) \geq \widetilde{\Gamma}$$

for all $(t, x) \in \mathbb{R} \times \overline{\Omega}$. Our goal is to prove that $\Delta = 0$. Assume, by contradiction, that $\Delta > 0$. We distinguish among four cases:

Case 1 $\widetilde{\Psi}(t_0, x_0) - 1 > 0$ and $(t_0, x_0) \in [-\tau, 0] \times \Omega$.

Notice that $\widetilde{\Psi}$ attains at (t_0, x_0) a local maximum. By the definition of Δ,

$$|\widetilde{\Psi}(t_0, x_0) - 1| \geq |v^{\widetilde{\Psi}}(t_0 - \tau, x_0) - 1|.$$

Observe that by $\tau > T$, $v^{\widetilde{\Psi}}(t_0 - \tau, x_0) = v^{\widetilde{\Psi}}(t^* - mT, x_0)$ for suitable $t^* \in [-\tau, 0]$ and $m \in \mathbb{N}$. We stress that $v^{\widetilde{\Psi}}_{-mT} \in \omega(\phi)$ with $v^{\widetilde{\Psi}}_{-mT}(t, x) = v^{\widetilde{\Psi}}(t - mT, x)$ for $(t, x) \in [-\tau, 0] \times \overline{\Omega}$. Using that $\widetilde{\Psi}(t_0, x_0) \neq 1$ and taking $a = \widetilde{\Psi}(t_0, x_0)$, $b = v^{\widetilde{\Psi}}(t_0 - \tau, x_0)$ and $\lambda = y_*(t_0 - \tau)$, we deduce by **(C2)** that

$$v^{\widetilde{\Psi}}(t_0 - \tau, x) f(y_*(t_0 - \tau) v^{\widetilde{\Psi}}(t_0 - \tau, x_0)) < f(y_*(t_0 - \tau)) \widetilde{\Psi}(t_0, x_0.)$$

Using this inequality in (18), we have that

$$L v^{\widetilde{\Psi}}(t_0, x_0) = -\frac{\partial v^{\widetilde{\Psi}}}{\partial t}(t_0, x_0) + d\nabla^2 v^{\widetilde{\Psi}}(t_0, x_0) > 0.$$

In particular, we can achieve this inequality in a neighborhood of (t_0, x_0) by continuity. This is a contradiction with Lemma 4.1 because $v^{\widetilde{\Psi}}(t, x)$ attains a local maximum at (t_0, x_0).

Case 2 $\widetilde{\Psi}(t_0, x_0) - 1 < 0$ and $(t_0, x_0) \in [-\tau, 0] \times \Omega$.

The argument is the same as in Case 1 replacing local maximum by local minimum.

Case 3 $\widetilde{\Psi}(t_0, x_0) - 1 > 0$ and $(t_0, x_0) \in [-\tau, 0] \times \partial\Omega$.

Arguing as in Case 1, we have that $\widetilde{\Psi}(t_0, x_0) \geq v^{\widetilde{\Psi}}(t, x)$ for all $(t, x) \in \mathbb{R} \times \overline{\Omega}$. Moreover,

$$v^{\widetilde{\Psi}}(t_0 - \tau, x_0) f(y_*(t_0 - \tau) v^{\widetilde{\Psi}}(t_0 - \tau, x_0)) < f(y_*(t_0 - \tau)) \widetilde{\Psi}(t_0, x_0).$$

By continuity and the smoothness of $\partial\Omega$, there exist $\varepsilon > 0$ and an open ball $S^* \subset \Omega$ such that $\partial S^* \cap \partial\Omega = \{x_0\}$ and

$$v^{\widetilde{\Psi}}(t-\tau,x)f(y_*(t-\tau)v^{\widetilde{\Psi}}(t-\tau,x)) < f(y_*(t-\tau))v^{\widetilde{\Psi}}(t,x)$$

for all $(t,x) \in [t_0-\varepsilon, t_0+\varepsilon] \times S^*$. From the expression of (18), we obtain that

$$Lv^{\widetilde{\Psi}}(t_0,x_0) = -\frac{\partial v^{\widetilde{\Psi}}}{\partial t}(t_0,x_0) + d\nabla^2 v^{\widetilde{\Psi}}(t_0,x_0) > 0.$$

In particular, we can achieve this inequality in a neighborhood of (t_0,x_0) by continuity. This is a contradiction because, by Lemma 4.1, $n(x_0)\nabla v^{\widetilde{\Psi}}(t_0,x_0) > 0$.

Case 4 $\widetilde{\Psi}(t_0,x_0) - 1 < 0$ and $(t_0,x_0) \in [-\tau,0] \times \partial\Omega$.

The argument is the same as Case 3 replacing local maximum by local minimum. □

The previous criterion was obtained in periodic delay differential equations without diffusion in Ref. [17]. It is worth noting that the arguments developed in that paper cannot be adapted to (6).

5. Applications

The main goal of this section is to translate the abstract framework of Theorem 4.1 into a more applied one. The next analysis will suggest two important facts:

- The conditions required in Theorem 4.1 are normally satisfied in classical models.
- Our approach typically leads to sharp criteria of global attractivity.

In Sect. 5.1, we show that Theorem 4.1 always works when $h(x) = xf(x)$ is strictly increasing. In particular, our approach includes some powerful results developed from the theory of monotone flows. In Sects. 5.2 and 5.3, we apply Theorem 4.1 when $h(x) = xe^{-x}$ and $h(x) = \frac{x}{1+x^2}$, respectively. In our arguments, the derivation of a sharp upper bound for $y_*(t)$ in (17) plays a critical role.

5.1. Monotone growth rates and global attraction

Consider

$$\begin{cases} u_t(t,x) = d\nabla^2 u(t,x) - a(t)u(t,x) + b(t)h(u(t-\tau,x)) & x \in \Omega \\ n(x).\nabla u(t,x) = 0 & x \in \partial\Omega \end{cases} \tag{20}$$

where $\Omega \subset \mathbb{R}^N$ is a connected and bounded domain with smooth boundary. The constants d and τ are strictly positive. The functions $a,b : \mathbb{R} \longrightarrow (0,+\infty)$ are continuous, T-periodic and strictly positive. The next result shows that when $h(u) = uf(u)$ is strictly increasing, we expect simple dynamics in (20).

Theorem 5.1. *Consider $u^\phi(t,x)$ a solution of (20) with $\phi \in \mathcal{C}([-\tau,0],X_{++})$. Assume that h is strictly increasing.*

(i) *If* **(C1)** *holds, then there is a T-periodic solution $y_*(t) > 0$ of*

$$y'(t) = -a(t)y(t) + b(t)h(y(t-\tau)) \tag{21}$$

so that, for each $x \in \overline{\Omega}$, $\lim_{t\to+\infty}[u^\phi(t,x) - y_*(t)] = 0$.

(ii) *If (15) holds, then, for each $x \in \overline{\Omega}$, $\lim_{t\to+\infty} u^\phi(t,x) = 0$.*

Proof. To prove this theorem, we realize that $f(x)$ always satisfies conditions **(C2)** and **(C3)** for any value of θ when $h(x) = xf(x)$ is strictly increasing. Let us check property **(C2)**. The proof of **(C3)** is analogous and we omit the details. Take $a,b \geq 0$ and $\lambda > 0$ with $a - 1 \geq |b-1|$. We distinguish between two cases:

Case 1 $b \geq 1$.

In this case, $a-1 \geq |b-1|$ implies that $a \geq b \geq 1$. It is clear that

$$b\frac{f(\lambda b)}{f(\lambda)} \leq b$$

because f is strictly decreasing. Since $b \leq a$, we conclude that

$$b\frac{f(\lambda b)}{f(\lambda)} \leq a.$$

Let us prove the property regarding the equality. Assume that $bf(\lambda b) = af(\lambda)$ for some $\lambda > 0$. Let us prove that $a = b$. Suppose, by contradiction, that $a > b$. Then, $bf(\lambda b) = af(\lambda)$ implies that $f(\lambda b) > f(\lambda)$. Using that f is strictly decreasing, we deduce that $\lambda b < \lambda$. This is a contradiction with $b \geq 1$. Since $a = b$, we have that $bf(\lambda b) = bf(\lambda)$. Thus, $b = 1$.
Case 2 $b \leq 1$.
In this case, the condition $a - 1 \geq |b-1|$ implies that $a \geq 2-b$. Let us prove that

$$b\frac{f(\lambda b)}{f(\lambda)} \leq 2-b$$

or, equivalently,

$$b(f(\lambda b) + f(\lambda)) \leq 2f(\lambda).$$

Using that $h(x) = xf(x)$ is strictly increasing and $b \leq 1$, we have that $\lambda bf(\lambda b) \leq \lambda f(\lambda)$. This implies that $bf(\lambda b) \leq f(\lambda)$. Thus,

$$b(f(\lambda b) + f(\lambda)) \leq f(\lambda) + bf(\lambda) \leq (1+b)f(\lambda) \leq 2f(\lambda).$$

Now, we focus on the property regarding the equality. Assume that $bf(\lambda b) = af(\lambda)$ for some $\lambda \in (0, \theta]$. Let us prove that $a = 1$ and $b = 1$. Suppose, by contradiction, that either $a \neq 1$ or $b \neq 1$. Notice that if $a = 1$, then $b = 1$. Analogously, if $b = 1$, $a = 1$. Thus, it is not restrictive to suppose that $b < 1 < a$. Since $xf(x)$ is strictly increasing, we have that $bf(\lambda b) \leq af(\lambda a)$. Hence, $af(\lambda) \leq af(\lambda a)$. Using that f is strictly decreasing, we obtain that $a \leq 1$, a contradiction.
After the check of **(C2)** and **(C3)**, the proof of the theorem is a direct consequence of Theorems 3.2 and 4.1. □

Remark 5.1. The property regarding the equality discussed in case 1 is valid for any strictly decreasing function f.

A prototype of growth rate for which Theorem 5.1 can be applied is the classical Beverton-Holt function $h(y) = \frac{y}{k+y}$ with $k > 0$.

5.2. Nicholson blowfly equation with periodic coefficients and diffusion

Consider

$$\begin{cases} u_t(t,x) = d\nabla^2 u(t,x) - a(t)u(t,x) + b(t)u(t-\tau,x)e^{-u(t-\tau,x)} & x \in \Omega \\ n(x)\cdot\nabla u(t,x) = 0 & x \in \partial\Omega \end{cases} \tag{22}$$

where $\Omega \subset \mathbb{R}^N$ is a connected and bounded domain with smooth boundary. The constants d and τ are strictly positive. The functions $a, b : \mathbb{R} \longrightarrow (0, +\infty)$ are continuous, T-periodic and strictly positive. Model (22) can be perceived as the natural extension of the classical Nicholson blowfly equation

$$y'(t) = -ay(t) + by(t-\tau)e^{-y(t-\tau)}$$

when we have into account diffusion and seasonal fluctuations of the environment, see Ref. [3,17].

Lemma 5.1. *For $\theta = 2$, $f(x) = e^{-x}$ satisfies conditions* **(C2)** *and* **(C3)**.

Proof. We focus on the proof of **(C2)**. To deduce **(C3)**, we have to repeat the same arguments. Take $a, b \geq 0$ with $a - 1 \geq |b - 1|$ and $\lambda \in (0, 2]$. We distinguish between two cases:
Case 1 $b \geq 1$.
In this case, $a - 1 \geq |b - 1|$ implies that $a \geq b \geq 1$. It is clear that

$$b\frac{f(\lambda b)}{f(\lambda)} = be^{\lambda(1-b)} \leq a$$

because $be^{\lambda(1-b)} \leq b$ for all $b \geq 1$.
To check the property regarding the equality for this case, we invoke to Remark 5.1.
Case 2 $b \leq 1$.
In this case, the condition $a - 1 \geq |b - 1|$ implies that $a \geq 2 - b$. On the other hand,

$$b\frac{f(\lambda b)}{f(\lambda)} = be^{\lambda(1-b)} \leq a$$

is equivalent to $\lambda(1 - b) + \ln b \leq \ln a$. To guarantee this last inequality, we observe that $\ln b + \lambda(1 - b) \leq \ln(2 - b)$. Note that $\varphi(x) = \lambda(1 - x) + \ln x - \ln(2 - x)$ is strictly increasing in $(0, 1)$ with $\varphi(1) = 0$.
Let us check the property regarding the equality. Assume that $be^{-\lambda b} = ae^{-\lambda}$ for some $\lambda \in (0, 2]$ with $a \geq 2 - b$. We have to prove that $a = b = 1$. By the previous analysis, if $be^{-\lambda b} \geq (2 - b)e^{-\lambda}$ with $b \in (0, 1]$ then $b = 1$. Now, from $be^{-\lambda b} = ae^{-\lambda}$ and $b = 1$, we conclude that $a = 1$. □

Now we are ready to give the main result of this section:

Theorem 5.2. *Consider $u^{\phi}(t, x)$ a solution of (22) with $\phi \in \mathcal{C}([-\tau, 0], X_{++})$. Assume that*

$$\frac{b(t)}{a(t)} \leq 2e \tag{23}$$

for all $t \in [0, T]$.

(i) *If* **(C1)** *holds, then there is a T-periodic solution $y_*(t) > 0$ of*

$$y'(t) = -a(t)y(t) + b(t)y(t - \tau)e^{-y(t-\tau)} \tag{24}$$

so that, for each $x \in \overline{\Omega}$, $\lim_{t\to+\infty}[u^{\phi}(t, x) - y_*(t)] = 0$.

(ii) *If (15) holds, then, for each $x \in \overline{\Omega}$, $\lim_{t\to+\infty} u^{\phi}(t, x) = 0$.*

Proof. Let us prove that under (23), $y_*(t) \leq 2$ for all $t \in [0, T]$. Indeed, take $t_0 \in [0, T]$ so that

$$y_*(t_0) = \max\{y(t) : t \in \mathbb{R}\}.$$

Then,

$$0 = y_*'(t_0) = -a(t_0)y_*(t_0) + b(t_0)y(t_0 - \tau)e^{-y(t_0-\tau)}.$$

In particular,

$$y_*(t_0) = \frac{b(t_0)}{a(t_0)}y(t_0 - \tau)e^{-y(t_0-\tau)} \leq \frac{b(t_0)}{a(t_0)}e^{-1}$$

because $h(y) = ye^{-y} \leq e^{-1}$ for all $y \in [0, +\infty)$. By condition (23), we conclude that $y_*(t) \leq 2$ for all $t \in \mathbb{R}$. Now, the proof of the theorem is a direct consequence of Theorem 3.2, Lemma 5.1 and Theorem 4.1. □

The next result shows that a better estimate for an upper bound of $y_*(t)$ leads to a better criterion.

Theorem 5.3. *Assume that $\tau = nT$ with $n \in \mathbb{N}$ and*

$$\frac{b(t)}{a(t)} \leq e^2 \tag{25}$$

for all $t \in [0, T]$. If **(C1)** *holds, then there is a T-periodic solution $y_*(t) > 0$ of*

$$y'(t) = -a(t)y(t) + b(t)y(t-\tau)e^{-y(t-\tau)} \tag{26}$$

so that, for each $x \in \overline{\Omega}$, $\lim_{t\to+\infty}[u^\phi(t,x) - y_*(t)] = 0$.

Proof. Let us prove that $y_*(t) \leq 2$ for all $t \in [0, T]$. Take $t_0 \in [0, T]$ so that

$$y_*(t_0) = \max\{y(t) : t \in \mathbb{R}\}.$$

Then,

$$0 = y_*'(t_0) = -a(t_0)y_*(t_0) + b(t_0)y(t_0 - \tau)e^{-y(t_0-\tau)}$$

Using that $y_*(t)$ is T-periodic and $\tau = nT$, we conclude that

$$e^{y_*(t_0)} = \frac{b(t_0)}{a(t_0)}.$$

Thus, $y_*(t_0) = \ln \frac{b(t_0)}{a(t_0)}$. Using (25), we conclude that $y_*(t_0) \leq 2$. The rest of the proof is the same as that in Theorem 5.2 □

It is worth mentioning that the optimal delay-independent condition of global attraction toward a positive equilibrium in the classical Nicholson's blowfly equation

$$y'(t) = -ay(t) + by(t-\tau)e^{-y(t-\tau)}$$

is $a, b > 0$ and

$$1 < \frac{b}{a} \leq e^2.$$

Informally speaking, Theorem 5.3 recovers the optimal results of global attraction in (22) in the absence of seasonal fluctuations and diffusion.

5.3. Mackey Glass equations with periodic coefficients and diffusion

Consider

$$\begin{cases} u_t(t,x) = d\nabla^2 u(t,x) - a(t)u(t,x) + b(t)\frac{u(t-\tau,x)}{1+u(t-\tau,x)^2} & x \in \Omega \\ n(x).\nabla u(t,x) = 0 & x \in \partial\Omega \end{cases} \tag{27}$$

where $\Omega \subset \mathbb{R}^N$ is a connected and bounded domain with smooth boundary. The constants d and τ are strictly positive. The functions $a, b : \mathbb{R} \longrightarrow (0, +\infty)$ are continuous, T-periodic and strictly positive.

Theorem 5.4. *Consider $u^\phi(t,x)$ a solution of (27) with $\phi \in \mathcal{C}([-\tau, 0], X_{++})$.*

(i) *If* **(C1)** *holds, then there is a T-periodic solution $y_*(t) > 0$ of*

$$y'(t) = -a(t)y(t) + b(t)\frac{y(t-\tau)}{1 + y(t-\tau)^2} \tag{28}$$

so that, for each $x \in \overline{\Omega}$, $\lim_{t\to+\infty}[u^\phi(t,x) - y_*(t)] = 0$.

(ii) *If (15) holds, then, for each $x \in \overline{\Omega}$, $\lim_{t\to+\infty} u^\phi(t,x) = 0$.*

Proof. To prove this theorem we simply observe that $f(x) = \frac{1}{1+x^2}$ satisfies conditions **(C2)** and **(C3)** for any value of θ. Let us check property **(C2)**. The proof of **(C3)** is analogous and we omit the details. Take $a, b \geq 0$ and $\lambda > 0$ with $a - 1 \geq |b - 1|$. We distinguish between two cases:
Case 1 $b \geq 1$.
In this case, $a - 1 \geq |b - 1|$ implies that $a \geq b \geq 1$. It is clear that

$$b\frac{f(\lambda b)}{f(\lambda)} \leq b$$

because f is strictly decreasing. Since $b \leq a$, we conclude that

$$b\frac{f(\lambda b)}{f(\lambda)} \leq a.$$

To check the property regarding the equality for this case, we invoke to Remark 5.1.
Case 2 $b \leq 1$.
In this case, the condition $a - 1 \geq |b - 1|$ implies that $a \geq 2 - b$. Let us prove that

$$b\frac{f(\lambda b)}{f(\lambda)} \leq 2 - b$$

or, equivalently,

$$\frac{b}{1 + \lambda^2 b^2} - \frac{2 - b}{1 + \lambda^2} \leq 0.$$

To see this claim, we define the function $g(x) = \frac{x}{1+\lambda^2 x^2} - \frac{(2-x)}{1+\lambda^2}$. Notice that $g(1) = 0$ and g is increasing in $(0, 1)$ for any value of $\lambda > 0$. The proof of the property of the equality is a direct consequence of this analysis. Since $a \geq 2 - b$ and $b\frac{f(\lambda b)}{f(\lambda)} = 2 - b$ is satisfied if and, only if, $b = 1$. We deduce that $b\frac{f(\lambda b)}{f(\lambda)} = a$ if, and only, if $a = b = 1$. □

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Alfonso Ruiz-Herrera
Departamento de Matemáticas
Universidad de Oviedo
Oviedo
Spain
e-mail: ruizalfonso@uniovi.es

Tarik Mohammed Touaoula
Departament of Mathematics
University of Tlemcen
Tlemcen
Algeria
e-mail: mohammedtarik.touaoula@aast.dz

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