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A short remark on inviscid limit of the stochastic Navier–Stokes equations

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Abstract. In this article, we study the inviscid limit of the stochastic incompressible Navier–Stokes equations in three-dimensional space. We prove that a subsequence of weak martingale solutions of the stochastic incompressible Navier–Stokes equations converges strongly to a weak martingale solution of the stochastic incompressible Euler equations in the periodic domain under the well-accepted hypothesis, namely Kolmogorov hypothesis (**K41**).

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1. Introduction

The theory of turbulence revolutionized classical physics was put forward in the 1930 s and 1940 s by Kolmogorov, Richardson, Taylor, and others (see [26] and the references therein). It has been widely prominent in fluid mechanics, atmospheric and ocean sciences, and plasma physics. In the sequence of papers [34–36], usually referred to as **K41**, Kolmogorov took basically three basic axioms (homogeneity, isotropic and self-similarity) about the fluid flow and formally derived various universal predictions about the statistics of fully developed turbulence (see also [8] for reviews). On the mathematical side, the problem in the context of Navier–Stokes equations is very delicate and the rigorous understanding of these predictions is still in its infancy. Euler equations are the classical model for the motion of an incompressible, inviscid, homogeneous fluid. The addition of stochastic terms to the governing equations is commonly used to account for empirical, numerical and physical uncertainties in applications ranging from climatology to turbulence theory; see, for example, [23, Chapter 5]. In this article, we consider the stochastic Euler equations governing the time evolution of the velocity $\mathbf{u}$ and the scalar pressure field Π of an inviscid fluid on the three-dimensional torus $\mathbb{T}^3$. The system of equations reads

$$\begin{cases} \mathrm{d}\mathbf{u}(t,x) + [\operatorname{div}(\mathbf{u}(t,x)\otimes\mathbf{u}(t,x)) + \nabla_x\Pi(t,x)]\,\mathrm{d}t = \sigma(\mathbf{u}(t,x))\,\mathrm{d}W(t), & \text{in } (0,T)\times\mathbb{T}^3, \\ \operatorname{div}\mathbf{u}(t,x) = 0, & \text{in } (0,T)\times\mathbb{T}^3, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), & \text{in } \mathbb{T}^3 \end{cases} \tag{1.1}$$

where $T>0$ fixed, $\mathbf{u}_0$ is a given initial data. Let $\big(\Omega,\mathfrak{F},(\mathfrak{F}_t)_{t\geq 0},\mathbb{P}\big)$ be a stochastic basis, where $\big(\Omega,\mathfrak{F},\mathbb{P}\big)$ is a probability space and $(\mathfrak{F}_t)_{t\geq 0}$ is a complete filtration with the usual assumptions. We assume that W is an adapted cylindrical Wiener process defined on the probability space $(\Omega,\mathfrak{F},\mathbb{P})$, and the coefficient σ is generally nonlinear and satisfies suitable growth assumptions (see Sect. 2 for the complete list of assumptions). In particular, the map $\mathbf{u}\mapsto\sigma(\mathbf{u})$ is a Hilbert space-valued function signifying the *multiplicative* nature of the noise. In this article, we also consider the following Navier–Stokes equations subject to stochastic forcing,

$$\begin{cases} \mathrm{d}\mathbf{u}_\mu(t,x) + [\operatorname{div}(\mathbf{u}_\mu(t,x)\otimes\mathbf{u}_\mu(t,x)) + \nabla_x \Pi_\mu(t,x)]\,\mathrm{d}t \\ \qquad\qquad = \mu\Delta\mathbf{u}_\mu(t,x)\,\mathrm{d}t + \sigma(\mathbf{u}_\mu(t,x))\,\mathrm{d}W(t), & \text{in } (0,T)\times\mathbb{T}^3, \\ \operatorname{div}\mathbf{u}_\mu(t,x) = 0, & \text{in } (0,T)\times\mathbb{T}^3, \\ \mathbf{u}_\mu(0,x) = \mathbf{u}_0(x), & \text{in } \mathbb{T}^3. \end{cases} \tag{1.2}$$

1.1. Previous work

In the deterministic setup, for general initial data, global existence of a smooth solution remains a well-known open problem for Euler equations and also their dissipative counterpart, Navier–Stokes equations. Non-uniqueness of solutions for Euler equations was shown for the first time by Scheffer [38] who constructed a non-trivial weak solution of 2D incompressible Euler equations with compact support in time. Later, De Lellis, Székelyhidi [19,20] and Chiodaroli et al. [16] established groundbreaking results that confirms infinitely many weak solutions can be constructed for Euler equations in three dimensions. In these works, the method of so-called convex integration was used to prove the non-uniqueness of weak solutions to Euler equations. Furthermore, non-uniqueness results were established among weak solutions with dissipating energy, which is one of the well-accepted criteria for the selection of physically relevant solutions. In quest for a global-in-time solution, DiPerna [21] proposed a new concept of solution, known as measure-valued solution, for the nonlinear system of partial differential equations admitting uncontrollable oscillations. Moreover, Brenier et al. [7] proposed a new approach, seeing the measure-valued solutions as possibly the largest class, in which the family of smooth solutions is stable. In particular, they showed the so-called weak (measure-valued)–strong uniqueness principle for the incompressible Euler equations. On the other hand, the inviscid limit for Navier–Stokes equations has also been extensively studied in [2,14]. N. Masmoudi remarks about the inviscid limit of Navier–Stokes system in [37]. A weaker version of Kolmogorov hypothesis which was derived in [12,13,40] can provide the convergence of weak solutions of Navier–Stokes equations through a subsequence to a weak solution of Euler equations. In [25], under a weaker version of Kolmogorov's hypothesis, the authors also show that the limit of statistical solution of the incompressible Navier–Stokes equations is a statistical solution of the incompressible Euler equations. Recently, Hofmanová et. al [29] have identified a sufficient condition under which solutions to the 3D forced Navier–Stokes equations satisfy an L^p-in-time version of Kolmogorov 4/5 law for the behavior of the averaged third-order longitudinal structure function along the vanishing viscosity limit. In the stochastic setup, Glatt-Holtz and Vicol [27] obtained local well-posedness results for strong solutions of the stochastic incompressible Euler equations in two and three dimensions, and global well-posedness results in two dimensions for additive and linear multiplicative noise. Local well-posedness results for the three-dimensional stochastic compressible Euler equations were proved by Breit and Mensah [5]. Moreover, the convex integration method has already been applied in stochastic setting, namely to the isentropic Euler system by Breit, Feireisl and Hofmanova [4] and to the full Euler system by Chiodaroli, Feireisl and Flandoli [15]. There have been many attempts to define a suitable notion of measure-valued solutions for the stochastic incompressible Euler equations driven by *additive* noise, starting from the work of Kim [33], Breit & Moyo [3], and most recently by Hofmanová et al. [31], where the authors introduced a class of dissipative solutions which allowed them to demonstrate weak–strong uniqueness property and non-uniqueness of solutions in law. In the recent works [1,28], the authors also investigated the existence of global weak solutions of the stochastic Euler equations. In particular, by exploiting the structure of the noise, with a suitable radial symmetry and of transport type in Stratonovich form, respectively, the authors proved the existence of weak solutions which are strong from the probabilistic point of view. However, none of the above-mentioned frameworks can be applied to (1.1), since the driving noise is *multiplicative* in nature. We also mention recent works [10,11,30] on Euler equations driven by a *multiplicative noise*.

1.2. Aim of the paper

In general, the vanishing viscosity limit of the stochastic Navier–Stokes equations (1.2) is not a solution to the corresponding Euler equations (1.1), since the uniform bounds cannot guarantee the convergence of the nonlinear term. In other words, when one considers a sequence of dissipative[1] weak martingale solutions (see Definition 3.1) to the incompressible Navier–Stokes equations (1.2) which are uniformly bounded in L^2 (see (4.1)), then a weak limit in L^2 obtained through vanishing viscosity does not satisfy the stochastic Euler equation (1.1) in the usual sense of distributions because of the appearance of oscillations and concentrations. The main issue that hinders convergence in the nonlinear term $\mathrm{div}_x(\mathbf{u}_\mu \otimes \mathbf{u}_\mu)$ requires enough regularity of the velocity to properly compute the limit $\mu \to 0$, e.g., see [9–11]. On this basis, we state that the existence of dissipative weak solutions for the stochastic Euler equations (1.1) are open in 3D, as in the convergence of the inviscid limit from the stochastic Navier–Stokes (1.2) for general initial data. The global existence theory of dissipative weak solutions to the stochastic Euler equations (1.1) in three-dimensional space has not been established. We make an effort to find a way of obtaining the existence result of a dissipative weak solution to the stochastic Euler equations (1.1) under special conditions. We observe that existence theories for the stochastic Euler equations are relevant to the mathematical theories of numerical analysis. Euler equations are fundamental for turbulence [17] and Kolmogorov assumption is well accepted in the field of fluid turbulence. Therefore, the main aim of this paper is to study the vanishing viscosity limit of the stochastic Navier–Stokes equations (1.2) under the well-known hypothesis by Kolmogorov [34–36]. Particularly, we are interested in investigating such limits of global weak martingale solutions from the stochastic incompressible Navier–Stokes equations (1.2) to the corresponding stochastic Euler equations (1.1) under the weaker Kolmogorov-type hypothesis, which was particularly motivated by [12,40] for incompressible fluids and [13] for compressible fluids. Compared to the previous work of Chen and Glimm [12], the special features of the stochastic Navier–Stokes equations (1.2) bring some difficulties to the mathematical analysis. Specifically, due to the additional probability variable, it is hard to obtain strong convergence of weak solutions to stochastic Navier–Stokes equations (1.2) in the given probability space. To overcome this difficulty, we need to use the so-called stochastic compactness. The main tool is Skorokhod's representation theorem in our analysis.

The manuscript is organized as follows. In Sect. 2, we first introduce mathematical setting, assumptions, and preliminary result. Then we give the definition of finite energy weak martingale solutions for the incompressible fluid flow equations driven by a multiplicative noise and state the main results of this article in Sect. 3. In Sect. 4, we give details of *a priori* estimates, space regularity estimates and time regularity estimates. In Sect. 5, we demonstrate the convergence of finite energy (dissipative) weak martingale solutions to the stochastic Navier–Stokes equations (1.1), using stochastic compactness, to show the existence of a finite energy weak martingale solution to the stochastic Euler equations (1.1) under the weak version of Kolmogorov hypothesis.

[1] Here a dissipative weak solution means a weak solution with finite energy inequality.

2. Mathematical setting

Function spaces: Let $C^\infty_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)$ be the space of infinitely differentiable three-dimensional vector fields $\mathbf{u}$ on $\mathbb{T}^3$, satisfying $\nabla\cdot\mathbf{u}=0$.

$$C^\infty_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)=\{\boldsymbol{\varphi}\in C^\infty(\mathbb{T}^3;\mathbb{R}^3):\nabla\cdot\boldsymbol{\varphi}=0\}.$$
$$L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)=\boldsymbol{cl}_{L^2(\mathbb{T}^3)}C^\infty_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)=\{\boldsymbol{\varphi}\in L^2(\mathbb{T}^3;\mathbb{R}^3):\nabla\cdot\boldsymbol{\varphi}=0\}.$$

In a similar fashion, we denote by $H^\alpha_{\text{div}}(\mathbb{T}^3)$ the closure of $C^\infty_{\text{div}}(\mathbb{T}^3)$ in $H^\alpha(\mathbb{T}^3;\mathbb{R}^3)$, for $\alpha\geq 0$. Identifying $L^2_{\text{div}}(\mathbb{T}^3)$ with its dual space $(L^2_{\text{div}}(\mathbb{T}^3))'$ and identifying $(L^2_{\text{div}}(\mathbb{T}^3))'$ with a subspace of $H^{-\alpha}(\mathbb{T}^3)$ (the dual space of $H^\alpha(\mathbb{T}^3)$), we have $H^\alpha_{\text{div}}(\mathbb{T}^3)\hookrightarrow L^2_{\text{div}}(\mathbb{T}^3)\equiv(L^2_{\text{div}}(\mathbb{T}^3))'\hookrightarrow H^{-\alpha}_{\text{div}}(\mathbb{T}^3)$, and we can denote the dual pairing between H^α_{div} and $H^{-\alpha}_{\text{div}}$ by $\langle\cdot,\cdot\rangle$ when no confusion may arise, see [24].

Moreover, we set $D(A):=H^2_{\text{div}}(\mathbb{T}^3)$ and define the linear operator $A:D(A)\subset L^2_{\text{div}}(\mathbb{T}^3)\to L^2_{\text{div}}(\mathbb{T}^3)$ by $A\mathbf{u}=-\Delta\mathbf{u}$. We then define the bilinear operator $B(\mathbf{u},\mathbf{v}):H^1_{\text{div}}\times H^1_{\text{div}}\to H^{-1}_{\text{div}}$ as

$$\langle B(\mathbf{u},\mathbf{v}),z\rangle:=\int_{\mathbb{T}^3}\mathbf{z}(x)\cdot(\mathbf{u}(x)\cdot\nabla)\mathbf{v}(x)\,\mathrm{d}x,\ \text{for all } \mathbf{z}\in H^1_{\text{div}}(\mathbb{T}^3).$$

Note that the bilinear operator B can be extended to a continuous operator

$$B:L^2_{\text{div}}(\mathbb{T}^3)\times L^2_{\text{div}}(\mathbb{T}^3)\to D(A^{-\alpha})=H^{-2\alpha}_{\text{div}}$$

for all $\alpha>\frac{5}{4}$, for details consult [24]. A straightforward computation using incompressibility condition reveals that

$$\langle B(\mathbf{u},\mathbf{v}),\mathbf{z}\rangle=-\langle B(\mathbf{u},\mathbf{z}),\mathbf{v}\rangle=-\langle\mathbf{u}\otimes\mathbf{v},\nabla\mathbf{z}\rangle \tag{2.1}$$

for all $\mathbf{u},\mathbf{v}\in H^1_{\text{div}}(\mathbb{T}^3)$ and $\mathbf{z}\in C^\infty_{\text{div}}(\mathbb{T}^3)$.

Helmholtz projection: An important consequence of elliptic theory is the existence of Helmholtz's decomposition. It allows to decompose any vector-valued function in $L^2(\mathbb{T}^3;\mathbb{R}^3)$ into a divergence free part and a gradient part. Set

$$(L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3))^\perp:=\{\mathbf{u}\in L^2(\mathbb{T}^3;\mathbb{R}^3)|\mathbf{u}=\nabla\psi,\ \psi\in H^1_{\text{div}}(\mathbb{T}^3;\mathbb{R})\}.$$

Helmholtz's decomposition is defined by

$$\mathbf{u}=\mathcal{P}_H\mathbf{u}+\mathcal{Q}_H\mathbf{u},\quad\text{for any}\quad\mathbf{u}\in L^2(\mathbb{T}^3;\mathbb{R}^3),$$

where $\mathcal{P}_H$ is the projection from $L^2(\mathbb{T}^3;\mathbb{R}^3)$ to $L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)$ and $\mathcal{Q}_H=\mathbb{I}-\mathcal{P}_H$ is also a projection from $L^2(\mathbb{T}^3;\mathbb{R}^3)$ to $(L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3))^\perp$. Note that $L^2(\mathbb{T}^3;\mathbb{R}^3)$ admits a decomposition

$$L^2(\mathbb{T}^3;\mathbb{R}^3)=L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)\oplus(L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3))^\perp.$$

This decomposition is orthogonal with respect to $L^2(\mathbb{T}^3;\mathbb{R}^3)$-inner product. By property of projection $\mathcal{P}_H$, we have for $\mathbf{u}\in L^2(\mathbb{T}^3;\mathbb{R}^3)$

$$\langle\mathcal{P}_H\mathbf{u},\psi\rangle=\langle\mathbf{u},\psi\rangle,\qquad\text{for all }\psi\in L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3). \tag{2.2}$$

2.1. Stochastic framework

Here we specify details of the stochastic forcing term.

Brownian motions: Let $(\Omega,\mathfrak{F},(\mathfrak{F}_t)_{t\geq0},\mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration. The stochastic process W is a cylindrical $(\mathfrak{F}_t)$-Wiener process in a separable Hilbert space $\mathfrak{U}$. It is formally given by the expansion

$$W(t)=\sum_{k\geq1}e_k\beta_k(t),$$

where $\{\beta_k\}_{k\geq 1}$ is a sequence of mutually independent real-valued Brownian motions relative to $(\mathfrak{F}_t)_{t\geq 0}$ and $\{e_k\}_{k\geq 1}$ is an orthonormal basis of $\mathfrak{U}$. Finally, we define the auxiliary space $\mathfrak{U}_0 \supset \mathfrak{U}$ via

$$\mathfrak{U}_0 := \left\{ \mathbf{u} = \sum_{k\geq 1} \gamma_k e_k;\ \sum_{k\geq 1} \frac{\gamma_k^2}{k^2} < \infty \right\},$$

endowed with the norm

$$\|\mathbf{u}\|_{\mathfrak{U}_0}^2 = \sum_{k\geq 1} \frac{\gamma_k^2}{k^2}, \quad \mathbf{u} = \sum_{k\geq 1} \gamma_k e_k.$$

Note that the embedding $\mathfrak{U} \hookrightarrow \mathfrak{U}_0$ is Hilbert–Schmidt. Moreover, $\mathbb{P}$-a.s., trajectories of W are in $C([0,T];\mathfrak{U}_0)$.
Multiplicative noise: For each $\mathbf{u} \in L^2(\mathbb{T}^3;\mathbb{R}^3)$, we introduce a mapping $\sigma(\mathbf{u}) : \mathfrak{U} \to L^2(\mathbb{T}^3;\mathbb{R}^3)$ given by

$$\sigma(\mathbf{u})e_k = \sigma_k(\mathbf{u}(\cdot)).$$

In particular, we suppose that the coefficients $\sigma_k : \mathbb{R}^3 \to \mathbb{R}^3$ are C^1-functions that satisfy the following conditions, for every $\xi, \zeta \in \mathbb{R}^3$,

$$\sum_{k\geq 1} |\sigma_k(\xi)|^2 \leq D_0(1+|\xi|^2), \tag{2.3}$$

$$\sum_{k\geq 1} |\sigma_k(\xi) - \sigma_k(\zeta)|^2 \leq D_1|\xi-\zeta|^2. \tag{2.4}$$

The assumption (2.3) imposed on σ implies that

$$\sigma : L^2(\mathbb{T}^3;\mathbb{R}^3) \to L_2(\mathfrak{U}; L^2(\mathbb{T}^3;\mathbb{R}^3)),$$

where $L_2(\mathfrak{U}; L^2(\mathbb{T}^3;\mathbb{R}^3))$ denotes the space of Hilbert–Schmidt operators from $\mathfrak{U}$ to $L^2(\mathbb{T}^3;\mathbb{R}^3)$. Thus, given a predictable process $\mathbf{u} \in L^2(\Omega; L^2(0,T; L^2(\mathbb{T}^3;\mathbb{R}^3)))$, the stochastic integral

$$\int_0^t \sigma(\mathbf{u})\ \mathrm{d}W = \sum_{k\geq 1} \int_0^t \sigma_k(\mathbf{u})\ \mathrm{d}W_k$$

is a well-defined $(\mathfrak{F}_t)$-martingale taking values in $L^2(\mathbb{T}^3;\mathbb{R}^3)$; see [6, Sect. 2.3] for a detailed construction.

2.2. Preliminary result

In this subsection, we state the convergence theorem for stochastic integrals. This result will be used below to facilitate the passage to the limit in the Navier–Stokes equations. The proof of the following result can be found in [18, Lemma 2.1]. We also refer to [1, Lemma 4.1] for a improved version of the following result.

Lemma 2.1. *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space and Y a separable Hilbert space. For $n \in \mathbb{N}$, let W_n be a $(\mathfrak{F}_t)$-cylindrical Wiener process and let $\mathcal{G}_n$ be a $(\mathfrak{F}_t)$-predictable measurable stochastic process ranging in $L_2(\mathfrak{U};Y)$. Suppose that*

$$W_n \to W \ \textit{in probability in}\ C([0,T];\mathfrak{U}_0),$$
$$\mathcal{G}_n \to \mathcal{G} \ \textit{in probability in}\ L^2(0,T;L_2(\mathfrak{U},Y)),$$

where W is a cylindrical Wiener process adapted to a filtration $(\mathfrak{F}_t)_{t\geq 0}$, and $\mathcal{G}$ is $(\mathfrak{F}_t)$-progressively measurable. Then

$$\int_0^{\cdot} \mathcal{G}_n \mathrm{d}W_n \to \int_0^{\cdot} \mathcal{G}\mathrm{d}W \ \textit{in probability in}\ L^2(0,T;Y).$$

2.3. Kolmogorov hypothesis

Here we give the details of the Kolmogorov hypotheses for incompressible fluids and the corresponding Kolmogorov-type hypothesis in mathematical terms. This is a standard criterion to ensure the convergence of weak solutions of Navier–Stokes equations to a weak solution of Euler equations. This criterion is inspired from well-known experimental and theoretical concepts in the study of turbulent flows (see [32] for review). It is based on the energy spectrum $E_\mu(t,j)$ associated with a vector field u_μ, defined as

$$E_\mu(t,j) = \frac{1}{2}\mathbb{E}\left(\sum_{|\hat{j}|=j, \hat{j}\in\mathbb{Z}^3} |\hat{\mathbf{u}}_\mu(t,\hat{j})|^2 \right), \quad \forall\ j\in\mathbb{N},$$

where $\hat{\mathbf{u}}_\mu(t,\hat{j})$ is $\hat{j}$th Fourier coefficient of $\mathbf{u}$, defined as

$$\hat{\mathbf{u}}_\mu(t,\hat{j}) = \int_{\mathbb{T}^3} \mathbf{u}_\mu(x,t)\exp^{-ix\cdot\hat{j}}\,\mathrm{d}x.$$

Note that the kinetic energy is obtained as a sum

$$\frac{1}{2}\mathbb{E}\int_{\mathbb{T}^3} |\mathbf{u}_\mu(t)|^2\,\mathrm{d}x = \sum_{j\in\mathbb{N}} E_\mu(t,j).$$

For this subsection, we closely follow [12]. Note that two fundamental assumptions for the isotropic incompressible turbulence were proposed by Kolmogorov [34–36]:

(a). At sufficiently high wave numbers, the energy spectrum $E_\mu(t,j)$ can depend only on the fluid viscosity μ, the dissipation rate ε and the wave number j.

(b). $E_\mu(t,j)$ should be independent of the viscosity μ as the Reynolds number tends to infinity:

$$E_\mu(t,j) \approx \alpha\,\varepsilon^{2/3} j^{-5/3}, \tag{2.5}$$

in the limit of infinite Reynolds number, where α depends on t, but is independent of dissipation rate ε and wave number j.

Under the above Kolmogorov's two hypotheses, Chen–Glimm [12,13] interpreted in mathematical terms for the incompressible and compressible Kolmogorov-type hypothesis in deterministic setting. Similarly, here we generalize them for the stochastic fluid equations as follows.

Assumptions (K41)

For any $T>0$, there exist constants $C_T>0$ and $M\in\mathbb{N}$, depending on initial data but independent of viscosity μ, such that

$$\int_0^T E_\mu(t,j)\,\mathrm{d}t \le C_T\, j^{-5/3}, \quad \forall\ \ j\ge M. \tag{2.6}$$

For our analysis, the following weaker version of Assumption (**wK41**) is sufficient.
Weaker version of Kolmogorov hypothesis (wK41) [2]

[2] In [12,13,40], the authors refer this type of condition as a weaker version of Kolmogorov hypothesis.

For any $T > 0$, there exist positive constants $C_T > 0$ and $M \in \mathbb{N}$, depending on initial data $\mathbf{u}_0$ but independent of the viscosity μ, such that for $j = |\hat{j}| \geq M$,

$$\sup_{|\hat{j}|=j \geq M} \left(j^{3+\gamma} \mathbb{E} \int_0^T |\hat{\mathbf{u}}_\mu(t,\hat{j})|^2 \mathrm{d}t \right) \leq C_T, \tag{2.7}$$

for some $\gamma > 0$.

3. Definitions and main results

3.1. Incompressible Navier–Stokes equations

The incompressible Navier–Stokes equations driven by noise is studied by Flandoli & Gatarek in [24], [11, Theorem 2.13] where the authors proved the existence of a dissipative weak martingale solutions to the stochastic Navier–Stokes equations (1.2).

Definition 3.1. (*Dissipative weak martingale solution*) Let $\alpha > \frac{5}{4}$ and Λ_μ be a Borel probability measure on $L^2_{\mathrm{div}}(\mathbb{T}^3)$. Then $\left[\left(\Omega_\mu, \mathfrak{F}_\mu, (\mathfrak{F}_{\mu,t})_{t\geq 0}, \mathbb{P}_\mu\right); \mathbf{u}_\mu, W_\mu\right]$ is a dissipative weak martingale solution of (1.2) if

(a) $\left(\Omega_\mu, \mathfrak{F}_\mu, (\mathfrak{F}_{\mu,t})_{t\geq 0}, \mathbb{P}_\mu\right)$ is a stochastic basis with a complete right-continuous filtration,
(b) W_μ is a $(\mathfrak{F}_{\mu,t})$-adapted cylindrical Wiener process,
(c) the velocity field $\mathbf{u}_\mu$ is a $L^2_{\mathrm{div}}(\mathbb{T}^3)$-valued progressively measurable process and $\mathbb{P}-$a.s.

$$\mathbf{u}_\mu(\cdot,\omega) \in C([0,T]; H^{-2\alpha}_{\mathrm{div}}(\mathbb{T}^3)) \cap L^\infty(0,T; L^2_{\mathrm{div}}(\mathbb{T}^3)) \cap L^2(0,T; H^1_{\mathrm{div}}(\mathbb{T}^3))$$

(d) $\Lambda_\mu = \mathbb{P}_\mu \circ \left[\mathbf{u}_\mu(0)\right]^{-1}$,
(e) for all $\boldsymbol{\varphi} \in H^{2\alpha}_{\mathrm{div}}(\mathbb{T}^3)$, $\mathbb{P}$-a.s., for all $t \in [0,T]$,

$$\begin{aligned} \langle \mathbf{u}_\mu(t), \boldsymbol{\varphi}\rangle = \langle \mathbf{u}_\mu(0), \boldsymbol{\varphi}\rangle - \int_0^t \langle B(\mathbf{u}_\mu(s), \mathbf{u}_\mu(s)), \boldsymbol{\varphi}\rangle \, \mathrm{d}s + \mu \int_0^t \langle \Delta \mathbf{u}_\mu(s), \boldsymbol{\varphi}\rangle \mathrm{d}s \\ + \int_0^t \langle \mathcal{P}_H \sigma(\mathbf{u}_\mu(s)), \boldsymbol{\varphi}\rangle \, \mathrm{d}W(s), \end{aligned} \tag{3.1}$$

(f) for all $\phi \in C_c^\infty([0,T))$, $\phi \geq 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned} -\int_0^T \partial_t \phi \int_{\mathbb{T}^3} \frac{1}{2} |\mathbf{u}_\mu(t)|^2 \, \mathrm{d}x \, \mathrm{d}t + \mu \int_0^T \phi \int_{\mathbb{T}^3} |\nabla_x \mathbf{u}_\mu(t)|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \phi(0) \int_{\mathbb{T}^3} \frac{1}{2} |\mathbf{u}_\mu(0)|^2 \\ + \sum_{k=1}^\infty \int_0^T \phi \left(\int_{\mathbb{T}^3} \mathcal{P}_H \sigma_k(\mathbf{u}_\mu(t)) \cdot \mathbf{u}_\mu(t) \, \mathrm{d}x \right) \mathrm{d}W_k(t) + \frac{1}{2} \sum_{k=1}^\infty \int_0^T \phi \int_{\mathbb{T}^3} |\mathcal{P}_H \sigma_k(\mathbf{u}_\mu(t))|^2 \, \mathrm{d}t \end{aligned} \tag{3.2}$$

holds.

Remark 3.2. Note that, in the view of Skorokhod [39], it is possible to consider,

$$\left(\Omega_\mu, \mathfrak{F}_\mu, \mathbb{P}_\mu\right) = \left([0,1], \mathcal{B}([0,1]), \mathcal{L}_{\mathbb{R}}\right),$$

for every μ. However, it is worth noticing that it may not be possible to obtain a filtration that is independent of μ, due to lack of pathwise uniqueness for the underlying system.

3.2. Incompressible Euler equations

Definition 3.3. (*Dissipative weak martingale solution*) Let $\alpha > \frac{5}{4}$ and Λ be a Borel probability measure on $L^2_{\mathrm{div}}(\mathbb{T}^3)$. Then $\big[\big(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P}\big); \mathbf{u}, W\big]$ is a dissipative weak martingale solution of (1.1) if

(a) $\big(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P}\big)$ is a stochastic basis with a complete right-continuous filtration,
(b) W is a $(\mathfrak{F}_t)$-adapted cylindrical Wiener process,
(c) the velocity field $\mathbf{u}$ is a $L^2_{\mathrm{div}}(\mathbb{T}^3)$-valued predictable measurable process and $\mathbb{P}-$a.s.

$$\mathbf{u}(\cdot,\omega) \in C([0,T]; H^{-2\alpha}_{\mathrm{div}}(\mathbb{T}^3)) \cap L^\infty(0,T; L^2_{\mathrm{div}}(\mathbb{T}^3))$$

(d) $\Lambda = \mathbb{P} \circ \big[\mathbf{u}(0)\big]^{-1}$,
(e) for all $\boldsymbol{\varphi} \in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$, $\mathbb{P}$-a.s., for all $t \in [0,T]$,

$$\langle \mathbf{u}(t), \boldsymbol{\varphi}\rangle = \langle \mathbf{u}(0), \boldsymbol{\varphi}\rangle - \int_0^t \langle B(\mathbf{u}(s), \mathbf{u}(s)), \boldsymbol{\varphi}\rangle \,\mathrm{d}s + \int_0^t \langle \mathcal{P}_H \sigma(\mathbf{u}(s)), \boldsymbol{\varphi}\rangle \,\mathrm{d}W(s), \tag{3.3}$$

(f) for all $\phi \in C^\infty_c([0,T))$, $\phi \geq 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned} -\int_0^T \partial_t \phi \int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}(t)|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \phi(0) \int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}(0)|^2 \\ + \sum_{k=1}^\infty \int_0^T \phi\Big(\int_{\mathbb{T}^3} \mathcal{P}_H \sigma_k(\mathbf{u}(t)) \cdot \mathbf{u}(t)\,\mathrm{d}x\Big)\mathrm{d}W_k(t) + \frac{1}{2}\sum_{k=1}^\infty \int_0^T \phi \int_{\mathbb{T}^3} |\mathcal{P}_H \sigma_k(\mathbf{u}(t))|^2 \,\mathrm{d}t \end{aligned} \tag{3.4}$$

holds.

3.3. Statements of main results

We now state the main results of this paper. The result states the inviscid limit of the stochastic Navier–Stokes equations.

Theorem 3.4. *Let* $\mathbf{u}_0 \in L^p(\Omega; L^2_{\mathrm{div}}(\mathbb{T}^3))$ *for all* $p \geq 1$. *Let* $\big[\big(\Omega, \mathfrak{F}, (\mathfrak{F}_{\mu,t})_{t\geq 0}, \mathbb{P}\big); \mathbf{u}_\mu, W_\mu\big]$ *be a family of dissipative weak martingale solutions to the stochastic Navier–Stokes equations* (1.2) *in the sense of Definition* 3.1 *and satisfying the weaker version of Kolmogorov hypothesis* ***(wK41)*** (2.7)*. Then there exists a sequence* $\big[\big(\widetilde{\Omega}, \widetilde{\mathfrak{F}}, (\widetilde{\mathfrak{F}}_{\mu_n,t})_{t\geq 0}, \widetilde{\mathbb{P}}\big); \widetilde{\mathbf{u}}_{\mu_n}, \widetilde{W}_{\mu_n}\big]$ *of dissipative weak martingale solutions to the Navier–Stokes equations and a dissipative martingale solution* $\big[\big(\widetilde{\Omega}, \widetilde{\mathfrak{F}}, (\widetilde{\mathfrak{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}}\big); \widetilde{\mathbf{u}}, \widetilde{W}\big]$ *to the stochastic Euler equations* (1.1) *in the sense of Definition* 3.3 *such that* $\widetilde{\mathbb{P}}$*-almost surely,*

$$\widetilde{\mathbf{u}}_{\mu_n} \to \widetilde{\mathbf{u}} \;\; in \;\; L^2(0,T; L^2_{\mathrm{div}}(\mathbb{T}^3)).$$

Remark 3.5. Please note that compared to deterministic work [12], due to the time regularity of Itô integral, one need bounded higher-order moment (i.e., $\mathbf{u}_0 \in L^p(\Omega; L^2_{\mathrm{div}}(\mathbb{T}^3))$) to get the uniform bound in the sufficient regular Sobolev space (see Proposition 4.2).

4. A priori estimates

4.1. Uniform L^2- estimate

In what follows, we can now derive *a priori* bounds from the above energy inequality (for details see [11]). Indeed, after taking pth power and expectation of both sides (1.1), making use of Gronwall's and BDG inequality, we immediately get the following uniform bounds in n, for all $p \geq 1$,

$$\mathbb{E}\big[\sup_{t\in[0,T]}\|\mathbf{u}_\mu(t)\|^p_{L^2_{\rm div}(\mathbb{T}^3;\mathbb{R}^3)}\big] + \mathbb{E}\Big|\int_0^T \|\sqrt{\mu}\nabla\mathbf{u}_\mu\|^2_{L^2(\mathbb{T}^3)}\mathrm{d}s\Big|^{p/2} \leq C\big(1+\mathbb{E}\|\mathbf{u}_0\|^p_{L^2_{\rm div}(\mathbb{T}^3;\mathbb{R}^3)}\big). \tag{4.1}$$

4.2. Space regularity estimate

Proposition 4.1. *Let $\mathbf{u}_0 \in L^2(\Omega; L^2_{\rm div}(\mathbb{T}^3))$. Let $\mathbf{u}_\mu$ be a weak martingale solution to the stochastic Navier–Stokes equations (1.2) with initial data $\mathbf{u}_0$. Under the assumption (2.7), for any $T > 0$ and any $\beta \in (0, \gamma/2)$, there exists a positive constant C, independent of $\mu > 0$, such that*

$$\mathbb{E}\|\mathbf{u}_\mu\|^2_{L^2(0,T;H^\beta_{\rm div}(\mathbb{T}^3))} \leq C\,(1+\mathbb{E}\|\mathbf{u}_0\|^2_{L^2_{\rm div}(\mathbb{T}^3)}). \tag{4.2}$$

Proof. Using the definition of fractional derivatives via Fourier transform, Parseval identity, and assumption (2.7), we have for all $\beta \in (0, \gamma/2)$,

$$\begin{aligned}
\mathbb{E}\|\mathbf{u}_\mu\|^2_{L^2(0,T;H^\beta_{\rm div}(\mathbb{T}^3))} &\leq \mathbb{E}\left(\int_0^T\left(\sum_{\hat{\jmath}\in\mathbb{Z}^3}|\hat{\jmath}|^{2\beta}|\hat{\mathbf{u}}_\mu(\hat{\jmath},k)|^2\right)\mathrm{d}t\right)\\
&= \mathbb{E}\left(\int_0^T\left(\sum_{0\leq|\hat{\jmath}|\leq M}|\hat{\jmath}|^{2\beta}|\hat{\mathbf{u}}_\mu(\hat{\jmath},k)|^2\right)\mathrm{d}t\right) + \mathbb{E}\left(\int_0^T\left(\sum_{|\hat{\jmath}|> M}|\hat{\jmath}|^{2\beta}|\hat{\mathbf{u}}_\mu(\hat{\jmath},k)|^2\right)\mathrm{d}t\right)\\
&\leq M^{2\beta}\mathbb{E}\|\mathbf{u}_\mu\|^2_{L^2_{\rm div}(0,T;L^2(\mathbb{T}^3))} + C\sum_{|\hat{\jmath}|>M}|\hat{\jmath}|^{2\beta-3-\gamma}\\
&\leq C\,M^{2\beta}(1+\mathbb{E}\|\mathbf{u}_0\|^2_{L^2_{\rm div}(\mathbb{T}^3)}) + C\sum_{j>M}j^2\cdot j^{2\beta-3-\gamma}\\
&\leq C\,M^{2\beta}(1+\mathbb{E}\|\mathbf{u}_0\|^2_{L^2_{\rm div}(\mathbb{T}^3)}) + C\sum_{j\in\mathbb{N}}j^{2\beta-1-\gamma}\\
&\leq C\,M^{2\beta}(1+\mathbb{E}\|\mathbf{u}_0\|^2_{L^2_{\rm div}(\mathbb{T}^3)}) + C_\beta \qquad\qquad (\text{where } C_\beta = \sum_{j\in\mathbb{N}}j^{2\beta-1-\gamma}<\infty)\\
&\leq C(T,M,\beta)<\infty.
\end{aligned}$$

This completes the proof. □

4.3. Time regularity estimate

Proposition 4.2. *Let $\mathbf{u}_0 \in L^p(\Omega; L^2_{\rm div}(\mathbb{T}^3))$ for all $p > 2$. Let $\mathbf{u}_\mu$ be a finite energy weak martingale solution to the stochastic Navier–Stokes equations (1.2) with initial data $\mathbf{u}_0$. Then for every $\nu < \frac{1}{2}$, and $m > \frac{5}{2}$, there exists a positive constant C such that for all $p > 2$,*

$$\mathbb{E}\|\mathbf{u}_\mu\|^p_{W^{\nu,p}\left([0,T];H^{-m}_{\rm div}(\mathbb{T}^3)\right)} \leq C(1+\mathbb{E}\|\mathbf{u}_0\|^p_{L^2_{\rm div}(\mathbb{T}^3)}). \tag{4.3}$$

Proof. For convenience, we rewrite the equation (1.2) as

$$\int_{\mathbb{T}^3}\mathbf{u}_\mu(t)\cdot\boldsymbol{\varphi}\,\mathrm{d}x = \int_{\mathbb{T}^3}\mathbf{u}_\mu(0)\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^3}I_\mu(s):\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3}\mathcal{P}_H\sigma(\mathbf{u}_\mu)\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}W$$

for all $t \in [0,T]$, for all $\boldsymbol{\varphi} \in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$, where

$$I_\mu := -\mu \nabla \mathbf{u}_\mu + \mathbf{u}_\mu \otimes \mathbf{u}_\mu.$$

Let us consider the functional

$$\langle \mathcal{I}_\mu(t), \boldsymbol{\varphi} \rangle := \int_0^t \int_{\mathbb{T}^3} I_\mu(s) : \nabla \boldsymbol{\varphi} \,\mathrm{d}x \,\mathrm{d}s,$$

which is related to the deterministic part of the equation. From the embedding $L^1(\mathbb{T}^3) \hookrightarrow H^{-m}(\mathbb{T}^3)$ for $m > \frac{3}{2}$, we obtain for $m > \frac{3}{2}$, for all $t \in [0,T]$

$$\|\mathbf{u}_\mu(t) \otimes \mathbf{u}_\mu(t)\|_{H^{-m}_{\mathrm{div}}(\mathbb{T}^3)} \leq C \, \|\mathbf{u}_\mu(t)\|^2_{L^2(\mathbb{T}^3)},$$

and

$$\|\nabla_x \mathbf{u}_\mu(t)\|_{H^{-m}_{\mathrm{div}}(\mathbb{T}^3)} \leq \|\mathbf{u}_\mu(t)\|_{L^2(\mathbb{T}^3)}.$$

Then from the energy estimate (4.1) and the above estimate, we deduce for all $\nu \in [0,1]$, $p \geq 1$, and $m > \frac{5}{2}$,

$$\mathbb{E}\left[\|\mathcal{I}_\mu\|^p_{W^{\nu,p}(0,T;H^{-m}_{\mathrm{div}}(\mathbb{T}^3))} \right] \leq C(1 + \mathbb{E}\|\mathbf{u}_0\|^p_{L^2_{\mathrm{div}}(\mathbb{T}^3)}).$$

For the stochastic term from [24, Lemma 2.1], we have, for $p > 2$, for $\nu \in (0, 1/2)$

$$\mathbb{E}\left[\left\| \int_0^{\cdot} \mathcal{P}_H \sigma(\mathbf{u}_\mu) dW \right\|^p_{W^{\nu,p}([0,T];L^2_{\mathrm{div}}(\mathbb{T}^3))} \right] \leq C(1 + \mathbb{E}\|\mathbf{u}_0\|^p_{L^2_{\mathrm{div}}(\mathbb{T}^3)})$$

Combining the previous estimates, we conclude for all $p \geq 2$, for all $m > \frac{5}{2}$, for all $\nu < \frac{1}{2}$,

$$\mathbb{E}\left[\|\mathbf{u}_\mu\|^p_{W^{\nu,p}([0,T];H^{-m}_{\mathrm{div}}(\mathbb{T}^3))} \right] \leq C(1 + \mathbb{E}\|\mathbf{u}_0\|^p_{L^2_{\mathrm{div}}(\mathbb{T}^3)}).$$

It completes the proof. □

5. Stochastic compactness

Now, we have all in hand to conclude our compactness argument by showing the tightness of a certain collection of laws. First, let us introduce some notations which will be used later on. If E is a Banach space and $t \in [0,T]$, we consider the space of continuous E-valued functions and denote by $\mathbf{s}_t$ the operator of restriction of the interval $[0,t]$. To be more precise, we define

$$\mathbf{s}_t : C([0,T]; E) \to C([0,t] : E)$$

given by

$$\mathbf{s}_t : \mathbf{g} \mapsto \mathbf{g}_{|[0,t]}.$$

Clearly, $\mathbf{s}_t$ is a continuous mapping. Let us define the path space

$$\mathcal{X}_\mathbf{u} := C([0,T]; H^{-2\alpha}_{\mathrm{div}}(\mathbb{T}^3)) \cap L^2([0,T]; L^2_{\mathrm{div}}(\mathbb{T}^3)), \quad \text{and} \quad \mathcal{X}_W = C([0,T]; \mathfrak{U}_0)$$

equipped with the norms

$$\|\cdot\|_\mathbf{u} = \|\cdot\|_{L^2(0,T;L^2_{\mathrm{div}}(\mathbb{T}^3))} + \|\cdot\|_{C([0,T];H^{-2\alpha}_{\mathrm{div}}(\mathbb{T}^3))}, \text{ and } \|\cdot\|_W = \|\cdot\|_{C([0,T];\mathfrak{U}_0)}, \tag{5.1}$$

where $\alpha > \frac{5}{4}$. Next, we set $\mathcal{X} = \mathcal{X}_\mathbf{u} \times \mathcal{X}_W$. Let $\lambda_{\mathbf{u}_\mu}$ denote the law of $\mathbf{u}_\mu$ on $\mathcal{X}_\mathbf{u}$ and λ_W denote the law of W_μ on $\mathcal{X}_W$. Their joint law on $\mathcal{X}$ is then denoted by λ_μ.

5.1. Tightness of laws

Lemma 5.1. *The set $\{\lambda_\mu; \mu \in (0,1)\}$ is tight on $\mathcal{X}$.*

Proof. First, we employ an Aubin-Dubinskii-type compact embedding theorem which, in our setting, reads [24, Theorem 2.1]:

$$L^2(0,T;H^\beta_{\mathrm{div}}(\mathbb{T}^3)) \cap H^\nu(0,T;H^{-m}_{\mathrm{div}}(\mathbb{T}^3)) \hookrightarrow\hookrightarrow L^2(0,T;L^2_{\mathrm{div}}(\mathbb{T}^3)).$$

For $M > 0$, we define the set for any $m > 5/2$

$$B_{1,M} = \left\{\mathbf{u} \in L^2(0,T;H^\beta_{\mathrm{div}}(\mathbb{T}^3)) \cap H^\nu(0,T;H^{-m}_{\mathrm{div}}(\mathbb{T}^3)); \|\mathbf{u}\|_{L^2(0,T;H^\beta_{\mathrm{div}}(\mathbb{T}^3))} + \|\mathbf{u}\|_{H^\nu(0,T;H^{-m}_{\mathrm{div}}(\mathbb{T}^3))} \leq M\right\}$$

which is thus relatively compact in $L^2(0,T;L^2_{\mathrm{div}}(\mathbb{T}^3))$. Moreover, by estimates (4.2)–(4.3), we have

$$\lambda_{\mathbf{u}_\mu}(B^C_{1,M}) \leq \frac{C}{M}.$$

In order to prove the tightness in $C([0,T];H^{-2\alpha}_{\mathrm{div}}(\mathbb{T}^3))$, we employ the compact embedding [22, Theorem 2.2],

$$W^{\nu,p}(0,T;H^{-m}_{\mathrm{div}}(\mathbb{T}^3)) \hookrightarrow\hookrightarrow C([0,T];H^{-2\alpha}_{\mathrm{div}}(\mathbb{T}^3)) \tag{5.2}$$

where $\nu p > 1$ and choose m at the beginning such that $\frac{m}{2} < \alpha$. Define

$$B_{2,M} = \left\{\mathbf{u} \in W^{\nu,p}(0,T;H^{-m}_{\mathrm{div}}(\mathbb{T}^3)); \|\mathbf{u}\|_{W^{\nu,p}(0,T;H^{-m}_{\mathrm{div}}(\mathbb{T}^3))} \leq M\right\}$$

then by (4.3), we have

$$\lambda_{\mathbf{u}_\mu}(B^C_{2,M}) \leq \frac{C}{M^p}.$$

As a consequence, the set $B_M = B_{1,M} \cap B_{2,M}$ is relatively compact in $\mathcal{X}_{\mathbf{u}}$. If $\delta > 0$ is given, then for some suitably chosen $M > 0$, it holds true that

$$\lambda_\mu(B_M) \geq 1 - \delta,$$

and we obtain the tightness of $\{\lambda_{\mathbf{u}_\mu}; \mu \in (0,1)\}$. Since also the law λ_W is tight as being Radon measure on the Polish spaces $\mathcal{X}_W$, we conclude that also the set of their joint laws $\{\lambda_\mu; \mu \in (0,1)\}$ is tight and Prokhov's theorem therefore implies that it is also relatively weakly compact. □

Having secured all necessary tightness results, we can now apply Skorokhod representation theorem to extract an almost sure convergence on a new probability space. To that context, we infer the following result:

Proposition 5.2. *There exists a subsequence $\lambda_{\mathbf{u}_{\mu_n}}$, a probability space $(\widetilde{\Omega}, \widetilde{\mathfrak{F}}, \widetilde{\mathbb{P}})$ with $\mathcal{X}$-valued Borel measurable random variables $(\widetilde{\mathbf{u}}_{\mu_n}, \widetilde{W}_{\mu_n})$ and $(\widetilde{\mathbf{u}}, \widetilde{W})$ such that*

(1) the law of $(\widetilde{\mathbf{u}}_{\mu_n}, \widetilde{W}_{\mu_n})$ is given by $\lambda_{\mathbf{u}_{\mu_n}}$,
(2) the law of $(\widetilde{\mathbf{u}}, \widetilde{W})$, denoted by λ, is a Radon measure,
(3) $(\widetilde{\mathbf{u}}_{\mu_n}, \widetilde{W}_{\mu_n})$ converges $\widetilde{\mathbb{P}}$-almost surely to $(\widetilde{\mathbf{u}}, \widetilde{W})$ in the topology of $\mathcal{X}$, i.e.,

$$\widetilde{\mathbf{u}}_{\mu_n} \to \bar{\mathbf{u}} \text{ in } C([0,T];H^{-2\alpha}_{\mathrm{div}}(\mathbb{T}^3)) \cap L^2(0,T;L^2_{\mathrm{div}}(\mathbb{T}^3)), \qquad \widetilde{W}_{\mu_n} \to \widetilde{W} \text{ in } C([0,T];\mathfrak{U}_0)).$$

5.2. Passing to the limit

Note that, in view of the equality of joint laws, the energy inequality (3.2) and the *a priori* estimate (4.1) for the new random variables hold on the new probability space. Making use of convergence results given by Proposition 5.2, we can now pass to the limit in the approximate equation (1.2) and the energy inequality (3.2). First we show that the approximations $\widetilde{\mathbf{u}}_{\mu_n}$ solve the equation given by (1.2) on the new probability space $(\widetilde{\Omega}, \widetilde{\mathfrak{F}}, \widetilde{\mathbb{P}})$. For that purpose, let us denote by $(\widetilde{\mathfrak{F}}_t^{\mu_n})$ and $(\widetilde{\mathfrak{F}}_t)$, $\widetilde{\mathbb{P}}$-augmented canonical filtration of the process $(\widetilde{\mathbf{u}}_{\mu_n}, \widetilde{W}_{\mu_n})$ and $(\widetilde{\mathbf{u}}, \widetilde{W})$, respectively. This means

$$\widetilde{\mathfrak{F}}_t^{\mu_n} = \sigma\big(\sigma\big(\mathbf{s}_t\widetilde{\mathbf{u}}_{\mu_n}, \mathbf{s}_t\widetilde{W}_{\mu_n}\big) \cup \big\{N \in \widetilde{\mathfrak{F}};\ \widetilde{\mathbb{P}}(N) = 0\big\}\big), \quad t \in [0, T],$$
$$\widetilde{\mathfrak{F}}_t = \sigma\big(\sigma\big(\mathbf{s}_t\widetilde{\mathbf{u}}, \mathbf{s}_t\widetilde{W}\big) \cup \big\{N \in \widetilde{\mathfrak{F}};\ \widetilde{\mathbb{P}}(N) = 0\big\}\big), \quad t \in [0, T],$$

$\widetilde{\mathbf{u}}$ is a $(\widetilde{\mathfrak{F}}_t)$-predictable $H^{-2\alpha}_{\mathrm{div}}(\mathbb{T}^3)$-valued process since it has continuous trajectories. Furthermore, by the embedding $L^2(\mathbb{T}^3) \hookrightarrow H^{-2\alpha}_{\mathrm{div}}(\mathbb{T}^3)$, we conclude that

$$\widetilde{\mathbf{u}} \in L^2(\widetilde{\Omega} \times [0, T], \widetilde{\mathcal{P}}, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; L^2_{\mathrm{div}}(\mathbb{T}^3)),$$

where $\widetilde{\mathcal{P}}$ denotes the predictable σ-algebra associated with $(\widetilde{\mathfrak{F}}_t)_{t\geq 0}$.

Proposition 5.3. *For every $n \in \mathbb{N}$, $\big((\widetilde{\Omega}, \widetilde{\mathfrak{F}}, (\widetilde{\mathfrak{F}}_{\mu_n,t})_{t\geq 0}, \widetilde{\mathbb{P}}), \widetilde{\mathbf{u}}_{\mu_n}, \widetilde{W}\big)$ is a finite energy weak martingale solution to (1.2) with the initial law Λ_{μ_n}.*

Proof. The proof of the above proposition is standard, and one can furnish the proof following the same line of argument, as in the monograph by Breit et. al. [6, Theorem 2.9.1]. For brevity, we skip all the details. □

We remark that, in light of the above proposition, the new random variables satisfy the following equations and the energy inequality on the new probability space

- for all $\varphi \in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$, $\widetilde{\mathbb{P}}$-a.s., for all $t \in [0, T]$

$$\begin{aligned}\langle \widetilde{\mathbf{u}}_{\mu_n}(t), \varphi\rangle = \langle \widetilde{\mathbf{u}}_{\mu_n}(0), \varphi\rangle - \int_0^t \langle \widetilde{\mathbf{u}}_{\mu_n}(s) \otimes \widetilde{\mathbf{u}}_{\mu_n}(s), \nabla_x\varphi\rangle\,\mathrm{d}s - \mu_n \int_0^t \langle \nabla_x\widetilde{\mathbf{u}}_{\mu_n}(s), \nabla_x\varphi\rangle\,\mathrm{d}s \\ + \int_0^t \langle \sigma(\widetilde{\mathbf{u}}_{\mu_n}(s)), \varphi\rangle\,\mathrm{d}\widetilde{W}_{\mu_n}(s),\end{aligned} \tag{5.3}$$

- the energy inequality, for all $\phi \in C^\infty_c([0, T))$, $\phi \geq 0$, $\widetilde{\mathbb{P}}$-a.s.,

$$\begin{aligned}&-\int_0^T \partial_t\phi \int_{\mathbb{T}^3} \frac{1}{2}|\widetilde{\mathbf{u}}_{\mu_n}(t)|^2\,\mathrm{d}x\,\mathrm{d}t + \mu_n \int_0^T \phi \int_{\mathbb{T}^3} |\nabla_x\widetilde{\mathbf{u}}_{\mu_n}(t)|^2\,\mathrm{d}x\,\mathrm{d}t \leq \phi(0)\int_{\mathbb{T}^3} \frac{1}{2}|\widetilde{\mathbf{u}}_{\mu_n}(0)|^2\,\mathrm{d}x \\ &+ \sum_{k=1}^\infty \int_0^T \phi\Big(\int_{\mathbb{T}^3} \sigma_k(\widetilde{\mathbf{u}}_{\mu_n}(t)) \cdot \widetilde{\mathbf{u}}_{\mu_n}(t)\,\mathrm{d}x\Big)\mathrm{d}\widetilde{W}_{\mu_n,k}(t) + \frac{1}{2}\sum_{k=1}^\infty \int_0^T \phi \int_{\mathbb{T}^3} |\mathcal{P}_H\sigma_k(\widetilde{\mathbf{u}}_{\mu_n}(t))|^2\,\mathrm{d}t,\end{aligned} \tag{5.4}$$

  holds.

Now we are in a position to pass to the limit in μ_n in (5.3) and (5.4). To see this, note that we have *a priori* estimate (4.1) for the new random variable. With the help of strong convergence of $\widetilde{\mathbf{u}}_{\mu_n}$, Lemma 2.1 and the assumptions (2.3) on the coefficients σ_k, we can pass to the limit in all the terms of the formulation and the energy inequality to conclude that

- for all $\varphi \in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$, $\widetilde{\mathbb{P}}$-a.s., for all $t \in [0,T]$,

$$\langle \widetilde{\mathbf{u}}(t), \varphi \rangle = \langle \widetilde{\mathbf{u}}(0), \varphi \rangle - \int_0^t \langle \widetilde{\mathbf{u}}(s) \otimes \widetilde{\mathbf{u}}(s), \nabla_x \varphi \rangle \,\mathrm{d}s + \int_0^t \langle \sigma(\widetilde{\mathbf{u}}(s)), \varphi \rangle \,\mathrm{d}\widetilde{W}(s), \tag{5.5}$$

- the energy inequality, for all $\phi \in C_c^\infty([0,T))$, $\phi \geq 0$, $\widetilde{\mathbb{P}}$-a.s.,

$$\begin{aligned} &- \int_0^T \partial_t \phi \int_{\mathbb{T}^3} \frac{1}{2} |\widetilde{\mathbf{u}}(t)|^2 \,\mathrm{d}x \,\mathrm{d}t \leq \phi(0) \int_{\mathbb{T}^3} \frac{1}{2} |\widetilde{\mathbf{u}}(0)|^2 \,\mathrm{d}x \\ &\quad + \sum_{k=1}^\infty \int_0^T \phi \left(\int_{\mathbb{T}^3} \sigma_k(\widetilde{\mathbf{u}}(t)) \cdot \widetilde{\mathbf{u}}(t) \,\mathrm{d}x \right) \mathrm{d}\widetilde{W}_k(t) + \frac{1}{2} \sum_{k=1}^\infty \int_0^T \phi \int_{\mathbb{T}^3} |\mathcal{P}_H \sigma_k(\widetilde{\mathbf{u}}(t))|^2 \,\mathrm{d}t \end{aligned} \tag{5.6}$$

holds.

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