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Dynamical analysis of an age-space structured malaria epidemic model

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Abstract. In this paper, we will revisit the model studied in Lou and Zhao (J Math Biol 62:543–568, 2011), where the model takes the form of a nonlocal and time-delayed reaction–diffusion model arising from the fixed incubation period. We consider the infection age to be a continuous variable but without the limitation of the fixed incubation period, leading to an age-space structured malaria model in a bounded domain. By performing the elementary analysis, we investigate the well-posedness of the model by proving the global existence of the solution, define the explicit formula of basic reproduction number when all parameters remain constant. By analyzing the characteristic equations and designing suitable Lyapunov functions, we also establish the threshold dynamics of the constant disease-free and positive equilibria. Our theoretical results are also validated by numerical simulations for 1-dimensional and 2-dimensional domains.

Mathematics Subject Classification. 35K57, 35B40, 92D30.

Keywords. Malaria model, Basic reproduction number, Threshold dynamics, Global asymptotic stability.

1. Introduction

Human malaria, caused by the genus *Plasmodium*, belongs to a mosquito-borne disease. The main intermediary vector is female anopheles mosquitoes. Rapid spread and global distribution (especially in Africa, Asia and South America) of human malaria causes public health problems and kills over a million people a year [20]. Since the classical work of [19], a lot of mathematical models based on vector-borne diseases framework (see, e.g., [3,7,9,13–17,27,30,31]) have been devoted to investigating the temporal and spatial patterns of disease burden and control strategies, which provides useful insights into the malaria transmission dynamics. In recent years, more and more biologically factors affecting vector-borne diseases are incorporated into mathematical models, such as immunity and clinical death [1,21], spatial heterogeneity [3,14,15,27,31], the mobility of human and mosquito populations, extrinsic incubation period (EIP), vector-bias mechanism and seasonality (see, e.g., [3,9,14,15,27,30,31]). Here, EIP is a time interval during which mosquitoes could not transmit the malaria parasite to humans, which varies from 10 to 14 days [12] and significantly affect the number of infected mosquitoes. Spatial heterogeneity reflects the distinct contact patterns in distinct geographic regions, demonstrating the diversity in habitats. It is widely accepted and well known that the environmental conditions vary spatially, affecting the biting patterns, so setting the disease transmission parameters depending the location variable is biologically reasonable. The reaction–diffusion model is one of the most common tool in describing the spatial evolution of an epidemic, generalizing the classical models [16,17,19].

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain equipped with a smooth boundary $\partial\Omega$. For $x \in \Omega$, we introduce the Laplacian operator $\Delta = \partial^2/\partial x^2$ to represent the random mobility of human and mosquito populations in the domain. At time t and location x, we denote by $S_m := S_m(t,x)$, $I_m := I_m(t,x)$ and $I_h := I_h(t,x)$ the density of susceptible mosquitoes, infected mosquitoes and infected humans, whose diffusion rates are given by D_m, D_m and D_h, respectively. The model studied in [14] takes the following form,

$$\begin{cases} \dfrac{\partial S_m}{\partial t} - D_m \Delta S_m = \mu(x) - \dfrac{b\beta(x)}{H(x)} S_m I_h - d_m S_m, \\ \dfrac{\partial I_m}{\partial t} - D_m \Delta I_m = -d_m I_m + e^{-d_m \tau} \displaystyle\int_\Omega \Gamma(D_m \tau, x, y) \dfrac{b\beta(y)}{H(y)} S_m(t-\tau, y) I_h(t-\tau, y) \mathrm{d}y, \\ \dfrac{\partial I_h}{\partial t} - D_h \Delta I_h = \dfrac{c\beta(x)}{H(x)} (H(x) - I_h) I_m - (d_h + \rho) I_h, \end{cases} \tag{1.1}$$

for $x \in \Omega, t > 0$ and

$$\frac{\partial S_m}{\partial n} = \frac{\partial I_m}{\partial n} = \frac{\partial I_h}{\partial n} = 0, \; x \in \partial\Omega, t > 0. \tag{1.2}$$

Here, the density of total human population is assumed to be remained at $H(x)$, $H(x) \in C^2(\Omega, (0,\infty)) \cap C^1(\overline{\Omega}, (0,\infty))$ and $H(x)$ satisfies

$$-D_h \Delta H(x) = d_h H(x) \left(1 - \frac{H(x)}{K(x)}\right), \; x \in \Omega; \quad \frac{\partial H(x)}{\partial n} = 0, \; x \in \partial\Omega, \tag{1.3}$$

where $K(x) \in C(\overline{\Omega}, (0,\infty))$ is the carrying capacity dependent of location x and d_h the birth rate of humans, so the density of susceptible humans is given by $H(x) - I_h$; The force of infection for human and mosquito populations is, respectively, characterized by $\frac{b\beta(x)}{H(x)} S_m I_h$ and $\frac{c\beta(x)}{H(x)}(H(x) - I_h) I_m$; $\mu(x)$ and $\beta(x)$, respectively, depict the space-dependent recruitment rate of adult female mosquitoes emerged from larval and biting rate; d_m and d_h, respectively, stand for the natural death rate of mosquito and human populations; b and c describe the transmission probabilities per bite from infected humans to susceptible female mosquitoes and from infected female mosquitoes to susceptible humans; ρ is the recovery rate of humans; $\Gamma(D_m \tau, x, y)$ is the Green function with respect to Laplace operator $D_m \Delta$ subject to (1.2); τ is a positive constant representing the fixed incubation period; $\frac{\partial}{\partial n}$ denotes the differentiation along the outward normal n to $\partial\Omega$. The basic assumptions on the parameters are as follows: $D_m, D_h, d_m, \rho, b, c \in (0,\infty)$; $\beta, \mu \in C(\overline{\Omega}, [0,\infty))$; $\beta_1 \in L^\infty(\mathbb{R}_+, [0,\infty))$.

The main feature of (1.1) is the nonlocal and time-delayed term appeared in I_m equation, which is obtained by the assumption that the incubation period is fixed at $\tau > 0$ and the standard method on characterizing age structured population with spatial diffusion [10]. Let $i_m := i_m(t, a, x)$ the density of the mosquitoes with infection age a at time t and location x, and $i_m(t, 0, x) = \frac{b\beta(x)}{H(x)} S_m I_h$ be the newly infected mosquitoes, which comes from the contact of susceptible mosquitoes and infectious humans. Then, $i_m(t, a, x)$ fulfills

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a} - D_m \Delta\right) i_m = -d_m i_m, \; x \in \Omega, \; t > 0, a \geq 0, \\ \dfrac{\partial i_m}{\partial n} = 0, \qquad x \in \partial\Omega, \; a \geq 0, \end{cases} \tag{1.4}$$

By the integration along characteristics, the nonlocal and time-delayed term in (1.1) is given by $i_m(t, \tau, x)$.

Unlike in [14] where spatial movements in EIP will cause nonlocal infection, here we plan to ignore the fixed incubation period and view the infection age as a continuous variable. In this paper, adopting the same notations used in [14], we directly investigate the following malaria model with age and space structure:

$$\begin{cases} \dfrac{\partial S_m}{\partial t} - D_m \Delta S_m = \mu(x) - \dfrac{b\beta(x)}{H(x)} S_m I_h - d_m S_m, & x \in \Omega,\ t > 0, \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a} - D_m \Delta\right) i_m = -d_m i_m, & x \in \Omega,\ t > 0,\ a > 0, \\ \dfrac{\partial I_h}{\partial t} - D_h \Delta I_h = \dfrac{c\beta(x)(H(x) - I_h)}{H(x)} \displaystyle\int_0^\infty \beta_1(a) i_m \mathrm{d}a - (d_h + \rho) I_h, & x \in \Omega,\ t > 0, \\ i_m(t, 0, x) = \dfrac{b\beta(x)}{H(x)} S_m I_h, & x \in \overline{\Omega},\ t > 0, \end{cases} \tag{1.5}$$

where the initial condition for (1.5) is given by, for all $x \in \overline{\Omega}$ and $a \geq 0$,

$$S_m(0, x) = \phi_1(x), \quad i_m(0, a, x) = \phi_2(a, x), \quad I_h(0, x) = \phi_3(x), \tag{1.6}$$

where $\phi_1, \phi_3 \in C_+(\Omega)$ and $\phi_2 \in L^1_+(\mathbb{R}_+, C(\Omega))$. The boundary condition for (1.5) is the homogeneous Neumann condition, that is, for all $x \in \partial\Omega$, $t > 0$ and $a > 0$,

$$\frac{\partial S_m}{\partial n} = \frac{\partial i_m}{\partial n} = \frac{\partial I_h}{\partial n} = 0. \tag{1.7}$$

We also note that some studies on spatial Zika models [9,15,27] could be viewed as an extension of the classical models in [16,17,19]. Specifically, in a recent paper [33], the authors implemented the global attractivity of a positive constant equilibrium of model (1.1) in a homogeneous case by designing a suitable Lyapunov functional, where the same problem was partially solved in [14], but requiring a sufficient condition through the fluctuation method. Very recently, Wang and Wang in [28] attempted to solve the global threshold dynamics of the problem (1.5)–(1.7) with mass-action mechanism and a stabilized density of susceptible humans $H(x)$, which is not altered by the epidemics as in [15].

Our main goal of this paper is to provide a rigorous analysis of (1.5), where (1.5) can be viewed as the one for the generalization version of model (1.1). Here, we use an age structured population with spatial diffusion reflecting the diffusion of the latent individuals. Following the main idea in [3,14] but using different analysis method, we address in Sect. 3, the basic questions on the existence, uniqueness and positivity of solutions to problem (1.5)–(1.7). We first treat the local existence of solution on $[0, T] \times \overline{\Omega}$ for small $T > 0$, where the method is different to that of [5,6,28,29,32]. The main reason is that we cannot construct a fixed point problem with one equation. To overcome this issue, we construct a fixed point problem with vector-valued functions. We also confirm that the solution never blows up in finite time and globally exists in a positive invariant set $\mathcal{D}$ for all $t > 0$. In Sect. 4, we derive the next-generation operator aiming to define the basic reproduction number $\Re_0$ through renewal equations. In general, $\Re_0$ cannot be directly calculated. However, in a spatially homogeneous case, the next-generation operator is compact. Thus, the Krein–Rutman theorem can be directly applied to get the explicit formula of $\Re_0$. Section 4 is devoted to investigating the local and global dynamics of the disease-free and positive steady states in a spatially homogeneous case. It should be highlighted here that it is not easy work to design suitable Lyapunov functions. The main results obtained in Sect. 4 are validated by numerical simulations in Sect. 5 for 1-dimensional and 2-dimensional domain.

2. Preliminaries

Throughout of the paper, for ease of notations, we set

$$\overline{\beta}_1 := \operatorname*{ess.sup}_{a \geq 0} \beta_1(a), \quad f^* := \sup_{x \in \Omega} f(x), \quad f_* := \inf_{x \in \Omega} f(x),$$

where $f \in \{\mu, \beta, H, \Lambda\}$.

Let $\mathbb{Y} := C(\overline{\Omega}, \mathbb{R})$ and $\mathbb{X} := L^1(\mathbb{R}_+, \mathbb{Y})$ equipped with norms

$$\|\varphi\|_{\mathbb{Y}} := \sup_{x\in\Omega} |\varphi(x)|, \varphi \in \mathbb{Y} \text{ and } \|\varphi\|_{\mathbb{X}} := \int_0^\infty \|\varphi(a)\|_{\mathbb{Y}} \mathrm{d}a,\ \varphi \in \mathbb{X},$$

, respectively. Denote the positive cones of $\mathbb{X}$ and $\mathbb{Y}$ by $\mathbb{X}_+$ and $\mathbb{Y}_+$, respectively. It is a classical fact that the diffusion operators $D_m\Delta$ and $D_h\Delta$ with (1.7) generate the strongly continuous semigroups $\{T_i(t)\}_{t\geq 0} : \mathbb{Y}_+ \to \mathbb{Y}_+$ $(i = 1, 2)$ defined by, for $t > 0$,

$$(T_i(t)\varphi)(x) = \int_\Omega \Gamma_i(t, x, y)\varphi(y)\mathrm{d}y, \text{and } T_i(0)\varphi = \varphi,\ \varphi \in \mathbb{Y}_+,$$

where $\Gamma_i(t, x, y)$ $(i = 1, 2)$ denote the associated Green functions. Note that, for any $\varphi \in \mathbb{Y}_+$, $i = 1, 2$ and $t > 0$,

$$\|T_i(t)\varphi\|_{\mathbb{Y}} \leq \int_\Omega \Gamma_i(t, x, y)\mathrm{d}y\|\varphi\|_{\mathbb{Y}} = \|\varphi\|_{\mathbb{Y}}, \tag{2.1}$$

because $\int_\Omega \Gamma_i(t, x, y)\mathrm{d}y = 1$.

Let $\overline{\mathbb{X}} = \mathbb{Y} \times \mathbb{X} \times \mathbb{Y}$ and $\overline{\mathbb{X}}_+ = \mathbb{Y}_+ \times \mathbb{X}_+ \times \mathbb{Y}_+$, equipped with norm

$$\|(\varphi_1, \varphi_2, \varphi_3)\|_{\overline{\mathbb{X}}} := \|\varphi_1\|_{\overline{\mathbb{Y}}} + \|\varphi_2\|_{\overline{\mathbb{X}}} + \|\varphi_3\|_{\overline{\mathbb{Y}}}, \quad (\varphi_1, \varphi_2, \varphi_3) \in \overline{\mathbb{X}}.$$

The state space for our system is as follows:

$$\mathcal{D} := \left\{ (\varphi_1, \varphi_2, \varphi_3) \in \overline{\mathbb{X}}_+ : 0 \leq \varphi_1(x) + \int_0^\infty \varphi_2(a, x)\mathrm{d}a \leq \frac{\mu^*}{d_m},\ 0 \leq \varphi_3(x) \leq H(x),\ x \in \overline{\Omega} \right\}.$$

Our main result of this section reads as follows.

Theorem 2.1. *There exists a solution semiflow* $\{\Phi(t)\}_{t\geq 0} : \overline{\mathbb{X}}_+ \to \overline{\mathbb{X}}_+$ *such that, for any* $\phi := (\phi_1, \phi_2, \phi_3) \in \mathcal{D}$, $\Phi(0)\phi = \phi$ *and*

$$\Phi(t)\phi := (S_m(t, \cdot), i_m(t, a, \cdot), I_h(t, \cdot)) \in \mathcal{D}, \quad t > 0,$$

gives a unique global solution to problem (1.5)–(1.7).

Before proving Theorem 2.1, we first introduce a lemma. For convenience, let us denote the newly infected mosquitoes by

$$\mathcal{B}(S_m, I_h)(t, x) := i_m(t, 0, x) = \frac{b\beta(x)}{H(x)} S_m I_h, \quad t > 0,\ x \in \Omega. \tag{2.2}$$

By appealing to the method of characteristics, one can easily get that, for all $x \in \Omega$,

$$i_m(t, a, x) = \begin{cases} (T_1(a)(\mathcal{B}(S_m, I_h)(t - a, \cdot)))\Pi(a), & t > a, \\ (T_1(t)\phi_2(a - t, \cdot))\Pi(t), & a \geq t, \end{cases} \tag{2.3}$$

where $\Pi(a) := e^{-d_m a}$. Hence, we directly have

$$\int_0^\infty \beta_1(a) i_m(t, a, x)\mathrm{d}a = \int_0^t \beta_1(a)(T_1(a)(\mathcal{B}(S_m, I_h)(t - a, \cdot)))\Pi(a)\mathrm{d}a + \int_t^\infty \beta_1(a)(T_1(t)\phi_2(a - t, \cdot))\Pi(t)\mathrm{d}a.$$

We now show the local existence of the solution.

Lemma 2.2. *For each $\phi \in \overline{\mathbb{X}}$, there exists a $T > 0$ such that problem (1.5)–(1.7) has a unique solution for all $t \in (0, T)$.*

Proof. Solving the equations of S_m and I_h in (1.5), we directly obtain: for $t > 0$,

$$S_m(t,\cdot) = \mathbb{F}_1(t,\cdot) + \int_0^t e^{-d_m(t-s)} T_1(t-s)[\mu(\cdot) - \mathcal{B}(S_m, I_h)(s,\cdot)]\mathrm{d}s =: \mathcal{F}_1(S_m, I_h)(t,\cdot), \tag{2.4}$$

$$I_h(t,\cdot) = \mathbb{F}_2(t,\cdot) + \mathbb{F}_3(t,\cdot) + \int_0^t e^{-(d_h+\rho)(t-s)} T_2(t-s)[\mathcal{C}(S_m, I_h)(s,\cdot)]\mathrm{d}s =: \mathcal{F}_2(S_m, I_h)(t,\cdot), \tag{2.5}$$

where

$$\begin{aligned} &\mathbb{F}_1(t,\cdot) := e^{-d_m t} T_1(t)\phi_1, \quad \mathbb{F}_2(t,\cdot) := e^{-(d_h+\rho)t} T_2(t)\phi_3, \\ &\mathbb{F}_3(t,\cdot) := \int_0^t e^{-(d_h+\rho)(t-s)} T_2(t-s) \left[c\beta(\cdot) \int_s^\infty \beta_1(a) T_1(s) \phi_2(a-s,\cdot)\Pi(s)\mathrm{d}a \right] \mathrm{d}s, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \mathcal{C}(S_m, I_h)(s,\cdot) :=& c\beta(\cdot) \int_0^s \beta_1(a) T_1(a)[\mathcal{B}(S_m, I_h)(s-a,\cdot)]\Pi(a)\mathrm{d}a \\ &- \frac{c\beta(\cdot) I_h(s,\cdot)}{H(\cdot)} \int_0^s \beta_1(a) T_1(a)[\mathcal{B}(S_m, I_h)(s-a,\cdot)]\Pi(a)\mathrm{d}a \\ &- \frac{c\beta(\cdot) I_h(s,\cdot)}{H(\cdot)} \int_s^\infty \beta_1(a) T_1(s)[\phi_2(a-s,\cdot)]\Pi(s)\mathrm{d}a. \end{aligned}$$

For $T > 0$, we set

$$\begin{aligned} &\mathbb{Y}_T := C([0,T],\mathbb{Y}) \text{ with } \|\psi\|_{\mathbb{Y}_T} := \sup_{0 \le t \le T} \|\psi(t,\cdot)\|_{\mathbb{Y}}, \ \psi \in \mathbb{Y}_T, \\ &\mathbb{W}_T := \mathbb{Y}_T \times \mathbb{Y}_T \text{ with } \|(\psi_1,\psi_2)\|_{\mathbb{W}_T} := \|\psi_1\|_{\mathbb{Y}_T} + \|\psi_2\|_{\mathbb{Y}_T}, \ (\psi_1,\psi_2) \in \mathbb{W}_T. \end{aligned}$$

Let $\mathcal{F}$ be a nonlinear operator defined on $\mathbb{W}_T$ to itself,

$$\mathcal{F}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} := \begin{pmatrix} \mathcal{F}_1(\psi_1,\psi_2) \\ \mathcal{F}_2(\psi_1,\psi_2) \end{pmatrix}, \quad \psi_1,\psi_2 \in \mathbb{W}_T.$$

Next, we show that $\mathcal{F}$ has a fixed point on $\mathbb{W}_T$, i.e., (1.5)–(1.7) has a unique solution on $[0,T] \times \overline{\Omega}$. For any $(S'_m, I'_h), (S''_m, I''_h) \in \mathbb{W}_T$, we have

$$\begin{aligned} \|\mathcal{B}(S'_m, I'_h) - \mathcal{B}(S''_m, I''_h)\|_{\mathbb{Y}_T} \le& \frac{b\beta^*}{H_*} \|S'_m I'_h - S''_m I''_h\|_{\mathbb{Y}_T} \\ \le& \frac{b\beta^*}{H_*} \left(\|I'_h\|_{\mathbb{Y}_T} \|S'_m - S''_m\|_{\mathbb{Y}_T} + \|S''_m\|_{\mathbb{Y}_T} \|I'_h - I''_h\|_{\mathbb{Y}_T} \right). \end{aligned}$$

Hence, by virtue of (2.1), we obtain

$$\begin{aligned}\|\mathcal{F}_1(S'_m, I'_h) - \mathcal{F}_1(S''_m, I''_h)\|_{\mathbb{Y}_T} \leq & \int_0^t e^{-d_m(t-s)}\mathrm{d}s \|\mathcal{B}(S'_m, I'_h) - \mathcal{B}(S''_m, I''_h)\|_{\mathbb{Y}_T} \\ \leq & \frac{b\beta^*(1-e^{-d_m t})}{d_m H_*}\left(\|I'_h\|_{\mathbb{Y}_T}\|S'_m - S''_m\|_{\mathbb{Y}_T} + \|S''_m\|_{\mathbb{Y}_T}\|I'_h - I''_h\|_{\mathbb{Y}_T}\right) \\ \leq & h_1(T)\left\|\begin{pmatrix} S'_m \\ I'_h\end{pmatrix} - \begin{pmatrix} S''_m \\ I''_h\end{pmatrix}\right\|_{\mathbb{W}_T},\end{aligned}$$

where

$$h_1(T) := \frac{b\beta^*(1-e^{-d_m T})}{d_m H_*}\max\left(\|I'_h\|_{\mathbb{Y}_T}, \|S''_m\|_{\mathbb{Y}_T}\right).$$

Note that for any $0 < T_* < T$, we can regard $(S'_m, I'_h), (S''_m, I''_h)$ as functions in $\mathbb{W}_{T_*}$, and

$$\begin{aligned}h_1(T_*) = & \frac{b\beta^*(1-e^{-d_m T_*})}{d_m H_*}\max\left(\|I'_h\|_{\mathbb{Y}_{T_*}}, \|S''_m\|_{\mathbb{Y}_{T_*}}\right) \\ \leq & \frac{b\beta^*(1-e^{-d_m T_*})}{d_m H_*}\max\left(\|I'_h\|_{\mathbb{Y}_T}, \|S''_m\|_{\mathbb{Y}_T}\right) = \frac{1-e^{-d_m T_*}}{1-e^{-d_m T}}h_1(T),\end{aligned}$$

and thus, $h_1(T_*) \to 0$ as $T_* \to +0$. Hence, we let T being sufficiently small that $h_1(T) < 1$ (regarding T_* such that $h(T_*) < 1$ as a new T). Similarly, we obtain

$$\begin{aligned}\|\mathcal{C}(S'_m, I'_h) - \mathcal{C}(S''_m, I''_h)\|_{\mathbb{Y}_T} \leq & \sup_{0\leq s\leq T}\left\{c\beta^*\overline{\beta}_1\left(1+\frac{\|I'_h\|_{\mathbb{Y}_T}}{H_*}\right)\int_0^s \Pi(a)\mathrm{d}a\|\mathcal{B}(S'_m, I'_h) - \mathcal{B}(S''_m, I''_h)\|_{\mathbb{Y}_T}\right. \\ & + \frac{c\beta^*\|I'_h - I''_h\|_{\mathbb{Y}_T}}{H_*}\int_0^s \beta_1(a)\|T_1(a)[\mathcal{B}(S''_m, I''_h)(s-a,\cdot)]\|_{\mathbb{Y}}\Pi(a)\mathrm{d}a \\ & \left. + \frac{c\beta^*\|I'_h - I''_h\|_{\mathbb{Y}_T}}{H_*}\int_s^\infty \beta_1(a)\|T_1(s)[\phi_2(a-s,\cdot)]\|_{\mathbb{Y}}\Pi(s)\mathrm{d}a\right\} \\ \leq & \frac{c\beta^*\overline{\beta}_1}{d_m}\left(1+\frac{\|I'_h\|_{\mathbb{Y}_T}}{H_*}\right)\|\mathcal{B}(S'_m, I'_h) - \mathcal{B}(S''_m, I''_h)\|_{\mathbb{Y}_T} + \frac{c\beta^*\overline{\beta}_1}{H_*}\left(\frac{b\beta^*\|S''_m I''_h\|_{\mathbb{Y}_T}}{d_m H_*} + \|\phi_2\|_{\mathbb{X}}\right)\|I'_h - I''_h\|_{\mathbb{Y}_T},\end{aligned}$$

and hence,

$$\begin{aligned}\|\mathcal{F}_2(S'_m, I'_h) - \mathcal{F}_2(S''_m, I''_h)\|_{\mathbb{Y}_T} \leq & \int_0^t e^{-(d_h+\rho)(t-s)}\mathrm{d}s\|\mathcal{C}(S'_m, I'_h) - \mathcal{C}(S''_m, I''_h)\|_{\mathbb{Y}_T} \\ \leq h_2(T)\left\|\begin{pmatrix} S'_m \\ I'_h\end{pmatrix} - \begin{pmatrix} S''_m \\ I''_h\end{pmatrix}\right\|_{\mathbb{W}_T},\end{aligned}$$

where

$$\begin{aligned}h_2(T) := & \frac{c\beta^*\overline{\beta}_1(1-e^{-(d_h+\rho)T})}{(d_h+\rho)}\max(g_1, g_2), \\ g_1 := & \frac{b\beta^*}{d_m H_*}\left(1+\frac{\|I'_h\|_{\mathbb{Y}_T}}{H_*}\right)\|I'_h\|_{\mathbb{Y}_T}, \\ g_2 := & \frac{b\beta^*}{d_m H_*}\left(1+\frac{\|I'_h\|_{\mathbb{Y}_T}}{H_*}\right)\|S''_m\|_{\mathbb{Y}_T} + \frac{1}{H_*}\left(\frac{b\beta^*\|S''_m I''_h\|_{\mathbb{Y}_T}}{d_m H_*} + \|\phi_2\|_{\mathbb{X}}\right).\end{aligned}$$

Similar to the case of h_1, we let T being sufficiently small that $h_2(T) < 1$. Consequently, we obtain

$$\left\| \mathcal{F}\begin{pmatrix} S'_m \\ I'_h \end{pmatrix} - \mathcal{F}\begin{pmatrix} S''_m \\ I''_h \end{pmatrix} \right\|_{\mathbb{W}_T} \leq \max(h_1(T), h_2(T)) \left\| \begin{pmatrix} S'_m \\ I'_h \end{pmatrix} - \begin{pmatrix} S''_m \\ I''_h \end{pmatrix} \right\|_{\mathbb{W}_T}.$$

As $\max(h_1(T), h_2(T)) < 1$, the operator $\mathcal{F}$ is a strict contraction in $\mathbb{W}_T$. Consequently, $\mathcal{F}$ has a unique fixed point in $\mathbb{W}_T$. Hence, the local existence of S_m and I_h follows. The local existence of i_m then follows from (2.2) and (2.3). The regularity of the solution directly follows because the right-hand sides of (2.4) and (2.5) are continuously differentiable with respect to t and twice continuously differentiable with respect to x by virtue of the Green functions in $\{T_i(t)\}_{t\geq 0}$, $i = 1, 2$. This proves Lemma 2.2. □

Using Lemma 2.2, we continue to show Theorem 2.1.

Proof of Theorem 2.1. Let $\phi = (\phi_1, \phi_2, \phi_3) \in \mathcal{D}$ and $\tilde{T} \in (0, T)$. We first show the positivity of S_m on $(0, \tilde{T}) \times \overline{\Omega}$. Clearly, for $x \in \Omega,\ t \in (0, \tilde{T})$,

$$\frac{\partial S_m}{\partial t} - D_m \Delta S_m > -\left[\frac{b\beta(x)}{H(x)} I_h + d_m\right] S_m.$$

As $b\beta(x)I_h/H(x) + d_m$ is bounded and continuous on $(0, \tilde{T}) \times \overline{\Omega}$, a standard result for PDEs ensures that $S_m > 0$ for all $(t, x) \in (0, \tilde{T}) \times \overline{\Omega}$.

We next show that, for all $(t, x) \in (0, \tilde{T}) \times \overline{\Omega}$,

$$0 < M(t, x) := S_m + \int_0^\infty i_m \mathrm{d}a \leq \frac{\mu^*}{d_m}. \tag{2.7}$$

By (1.5)–(1.7), we have

$$\begin{cases} \dfrac{\partial M}{\partial t} - D_m \Delta M = \mu(x) - d_m M \leq \mu^* - d_m M, & x \in \Omega,\ t \in (0, \tilde{T}), \\ M(0, x) = \phi_1(x) + \displaystyle\int_0^\infty \phi_2(a, x) \mathrm{d}a \leq \dfrac{\mu^*}{d_m}, & x \in \overline{\Omega}, \\ \dfrac{\partial M}{\partial n} = 0, & x \in \partial\Omega,\ t \in (0, \tilde{T}). \end{cases}$$

One can then easily see from the maximum principle that the last inequality in (2.7) holds. In addition, we have

$$\frac{\partial M}{\partial t} - D_m \Delta M > -d_m M, \quad x \in \Omega,\ t \in (0, \tilde{T}).$$

Hence, similar to the above, one can easily see that $M(t, x) > 0$ for all $(t, x) \in (0, \tilde{T}) \times \overline{\Omega}$.

We then show that $I_h < H(x)$ for all $(t,x) \in (0,\tilde{T}) \times \overline{\Omega}$. Let $Y_h := H - I_h$. It then follows from (1.5)–(1.7) and (1.3) that

$$\begin{aligned}
&\frac{\partial Y_h}{\partial t} - D_h \Delta Y_h \\
&= -\frac{c\beta(x)Y_h}{H(x)} \int_0^\infty \beta_1(a) i_m \mathrm{d}a + (d_h+\rho)[H(x) - Y_h] - D_h \Delta H(x) \\
&= \Lambda(x) + \rho H(x) - \left[\frac{c\beta(x)}{H(x)} \int_0^\infty \beta_1(a) i_m \mathrm{d}a + d_h + \rho\right] Y_h \\
&> -\left[\frac{c\beta(x)\overline{\beta}_1}{H(x)} \int_0^\infty i_m \mathrm{d}a + d_h + \rho\right] Y_h, \quad x \in \Omega,\ t \in (0,\tilde{T}),
\end{aligned}$$

and

$$Y_h(0,x) = H(x) - \phi_3(x) \geq 0,\ x \in \overline{\Omega}; \quad \frac{\partial Y_h}{\partial n} = 0,\ x \in \partial\Omega,\ t \in (0,\tilde{T}).$$

Similar to the above, as $c\beta(x)\overline{\beta}_1 \int_0^\infty i_m \mathrm{d}a / H(x) + d_h + \rho$ is bounded and continuous on $(0,\tilde{T}) \times \overline{\Omega}$, the standard result for PDEs yields that $Y_h > 0$, $(t,x) \in (0,\tilde{T}) \times \overline{\Omega}$. We then directly have $I_h < H(x)$ for all $(t,x) \in (0,\tilde{T}) \times \overline{\Omega}$.

We continue to prove that $I_h \geq 0$ for all $(t,x) \in (0,\tilde{T}) \times \overline{\Omega}$. The abstract equation (2.5) in $\mathbb{Y}$ can be rewritten as follows: for $t \in (0,\tilde{T})$,

$$I_h(t,\cdot) = \mathbb{F}_2(t,\cdot) + \mathbb{F}_4(t,\cdot) + \int_0^t e^{-(d_h+\rho)(t-s)} T_2(t-s)[\tilde{\mathcal{C}}(S_m, I_h)(s,\cdot)]\mathrm{d}s, \tag{2.8}$$

where $\mathbb{F}_2$ is given as in (2.6) and

$$\begin{aligned}
\mathbb{F}_4(t,\cdot) &:= \int_0^t e^{-(d_h+\rho)(t-s)} T_2(t-s) \left[\frac{c\beta(\cdot)(H(\cdot) - I_h(s,\cdot))}{H(\cdot)} \int_s^\infty \beta_1(a) T_1(s) \phi_2(a-s,\cdot) \Pi(s) \mathrm{d}a\right] \mathrm{d}s, \\
\tilde{\mathcal{C}}(S_m, I_h)(s,\cdot) &:= \frac{c\beta(\cdot)(H(\cdot) - I_h(s,\cdot))}{H(\cdot)} \int_0^s \beta_1(a) T_1(a) [\mathcal{B}(S_m, I_h)(s-a,\cdot)] \Pi(a) \mathrm{d}a.
\end{aligned}$$

By (2.2), we get

$$\mathcal{B}(S_m, I_h)(t,\cdot) = \frac{b\beta}{H} S_m(t,\cdot) \left[\mathbb{F}_2(t,\cdot) + \mathbb{F}_4(t,\cdot) + \int_0^t e^{-(d_h+\rho)(t-s)} T_2(t-s)[\tilde{\mathcal{C}}(S_m, I_h)(s,\cdot)]\mathrm{d}s\right]. \tag{2.9}$$

This is a renewal equation and the solution can be written as $\mathcal{B} = \sum_{n=0}^\infty \mathcal{B}_n$, where

$$\begin{aligned}
&\mathcal{B}_0(t,\cdot) := \frac{b\beta}{H} S_m(t,\cdot) \left[\mathbb{F}_2(t,\cdot) + \mathbb{F}_4(t,\cdot)\right], \\
&\mathcal{B}_n(t,\cdot) := \frac{b\beta}{H} S_m(t,\cdot) \int_0^t e^{-(d_h+\rho)(t-s)} T_2(t-s) \left[\frac{c\beta(\cdot)(H(\cdot) - I_h(s,\cdot))}{H(\cdot)} \int_0^s \beta_1(a) T_1(a) [\mathcal{B}_{n-1}(s-a,\cdot)] \Pi(a) \mathrm{d}a\right] \mathrm{d}s, \\
&n = 1, 2, \ldots.
\end{aligned}$$

Since S_m and $H - I_h$ are positive, one can see that $\mathcal{B}_n$ is nonnegative for all $n \geq 0$. Hence, $\mathcal{B} = \sum_{n=0}^{\infty} \mathcal{B}_n$ is also nonnegative. From (2.8), one knows that $I_h \geq 0$ for all $(t, x) \in (0, \tilde{T}) \times \overline{\Omega}$. In addition, the nonnegativitiy of i_m also follows from (2.3).

In conclusion, the solution remains in the bounded set $\mathcal{D}$ for all $t \in (0, \tilde{T})$, that is, $\mathcal{D}$ is positively invariant for system (1.5)–(1.7). Thus, the solution never blows up in finite time and globally exists in $\mathcal{D}$ for all $t > 0$. The existence of the solution semiflow $\{\Phi(t)\}_{t\geq 0}$ is a simple consequence. This proves Theorem 2.1. □

3. The basic reproduction number

The disease-free equilibrium of (1.5) with boundary condition (1.7) can be written as $E_0 := (S_m^0(x), 0, 0) \in \mathcal{D}$, where $S_m^0(x)$ satisfies

$$-D_m \Delta S_m^0(x) = \mu(x) - d_m S_m^0(x), \ x \in \Omega; \quad \frac{\partial S_m^0(x)}{\partial n} = 0, \ x \in \partial\Omega.$$

More precisely, using the Green function Γ_1, we can obtain the following explicit formulation of $S_m^0(x)$:

$$S_m^0(x) = \int_0^\infty e^{-d_m s} \int_\Omega \Gamma_1(s, x, y)\mu(y) \mathrm{d}y \mathrm{d}s, \quad x \in \overline{\Omega}.$$

Note that, if $\mu(x) \equiv \mu$, then $S_m^0 \equiv \mu/d_m$.

By appealing to the standard procedures as those in [8,25], let us define the basic reproduction number $\Re_0$ of (1.5)–(1.7). The linearized system of (1.5)–(1.7) around E_0 is given by

$$\begin{cases} \dfrac{\partial i_m}{\partial t} + \dfrac{\partial i_m}{\partial a} - D_m \Delta i_m = -d_m i_m, \\ \dfrac{\partial I_h}{\partial t} - D_h \triangle I_h = c\beta(x) \displaystyle\int_0^\infty \beta_1(a) i_m \mathrm{d}a - (d_h + \rho) I_h, \\ i_m(t, 0, x) = \dfrac{b\beta(x)}{H(x)} S_m^0(x) I_h =: \widetilde{\mathcal{B}}(t, x), \\ i_m(0, a, x) = \phi_2(a, x), \ I_h(0, x) = \phi_3(x), \end{cases} \tag{3.1}$$

for $x \in \Omega, \ t > 0, \ a > 0$ and

$$\frac{\partial i_m}{\partial n} = \frac{\partial I_h}{\partial n} = 0, \ x \in \partial\Omega, \ t > 0, \ a > 0. \tag{3.2}$$

By integrating the equations of I_h and i_m in (3.1), we obtain the following abstract equations in $\mathbb{Y}$:

$$I_h(t, \cdot) = e^{-(d_h+\rho)t} T_2(t)\phi_3 + \int_0^t e^{-(d_h+\rho)(t-s)} T_2(t-s) \left(c\beta(\cdot) \int_0^\infty \beta_1(a) i_m(s, a, \cdot) \mathrm{d}a \right) \mathrm{d}s, \quad t > 0,$$

and

$$i_m(t, a, \cdot) = \begin{cases} T_1(a)\widetilde{\mathcal{B}}(t-a, \cdot)\Pi(a), & t > a, \\ T_1(t)\phi_2(a-t, \cdot)\Pi(t), & a \geq t, \end{cases} \quad t > 0, \ a > 0. \tag{3.3}$$

Hence, we get the following abstract equation in $\mathbb{Y}$: for $t > 0$,

$$\begin{aligned} \widetilde{\mathcal{B}}(t, \cdot) =& \frac{b\beta}{H} S_m^0(x) I_h(t, \cdot) \\ =& \mathbb{G}(t, \cdot) + \frac{b\beta}{H} S_m^0(x) \int_0^t e^{-(d_h+\rho)(t-s)} T_2(t-s) \left(c\beta(\cdot) \int_0^s \beta_1(a) T_1(a) \widetilde{\mathcal{B}}(s-a, \cdot) \Pi(a) \mathrm{d}a \right) \mathrm{d}s, \end{aligned}$$

where

$$\mathbb{G}(t,\cdot) := \frac{b\beta}{H} S_m^0(x)\Bigg[e^{-(d_h+\rho)t}T_2(t)\phi_3(\cdot) + \int_0^t e^{-(d_h+\rho)(t-s)}T_2(t-s)\left(c\beta(\cdot)\int_s^\infty \beta_1(a)T_1(s)\phi_2(a-s,\cdot)\Pi(s)\mathrm{d}a\right)\mathrm{d}s\Bigg].$$

Hence, the generational expression $\widetilde{\mathcal{B}} = \sum_{n=0}^\infty \widetilde{\mathcal{B}}_n$ can be obtained, where

$$\widetilde{\mathcal{B}}_0(t,\cdot) := \mathbb{G}(t,\cdot),$$

$$\widetilde{\mathcal{B}}_n(t,\cdot) := \frac{b\beta}{H} S_m^0(x)\int_0^t e^{-(d_h+\rho)(t-s)}T_2(t-s)\left(c\beta(\cdot)\int_0^s \beta_1(a)T_1(a)\widetilde{\mathcal{B}}_{n-1}(s-a,\cdot)\Pi(a)\mathrm{d}a\right)\mathrm{d}s,$$

$n = 1, 2, \ldots.$

Note that $\widetilde{\mathcal{B}}_n$ denotes the newly infected population in the n-th generation. Let $\widehat{\mathcal{B}}_n := \int_0^\infty \widetilde{\mathcal{B}}_n(t,\cdot)\mathrm{d}t$. We then have, by changing the order of integration,

$$\begin{aligned}
\widehat{\mathcal{B}}_n =& \int_0^\infty \frac{b\beta}{H} S_m^0(x)\int_0^t e^{-(d_h+\rho)(t-s)}T_2(t-s)\left(c\beta(\cdot)\int_0^s \beta_1(a)T_1(a)\widetilde{\mathcal{B}}_{n-1}(s-a,\cdot)\Pi(a)\mathrm{d}a\right)\mathrm{d}s\mathrm{d}t\\
=&\frac{b\beta}{H} S_m^0(x)\int_0^\infty\int_s^\infty e^{-(d_h+\rho)(t-s)}T_2(t-s)\mathrm{d}t\left(c\beta(\cdot)\int_0^s \beta_1(a)T_1(a)\widetilde{\mathcal{B}}_{n-1}(s-a,\cdot)\Pi(a)\mathrm{d}a\right)\mathrm{d}s\\
=&\frac{b\beta}{H} S_m^0(x)\int_0^\infty e^{-(d_h+\rho)u}T_2(u)\mathrm{d}u\left(c\beta(\cdot)\int_0^\infty\int_0^s \beta_1(a)T_1(a)\widetilde{\mathcal{B}}_{n-1}(s-a,\cdot)\Pi(a)\mathrm{d}a ds\right)\\
=&\frac{b\beta}{H} S_m^0(x)\int_0^\infty e^{-(d_h+\rho)u}T_2(u)\mathrm{d}u\left(c\beta(\cdot)\int_0^\infty \beta_1(a)\Pi(a)T_1(a)\int_a^\infty \widetilde{\mathcal{B}}_{n-1}(s-a,\cdot)\mathrm{d}s\mathrm{d}a\right)\\
=&\frac{b\beta}{H} S_m^0(x)\int_0^\infty e^{-(d_h+\rho)u}T_2(u)\mathrm{d}u\left(c\beta(\cdot)\int_0^\infty \beta_1(a)\Pi(a)T_1(a)\widehat{\mathcal{B}}_{n-1}\mathrm{d}a\right).
\end{aligned}$$

Thus, the next-generation operator $\mathcal{K} : \mathbb{Y}_+ \to \mathbb{Y}_+$ can be defined by

$$\mathcal{K}\psi := \frac{b\beta}{H} S_m^0(x)\int_0^\infty e^{-(d_h+\rho)u}T_2(u)\mathrm{d}u\left(c\beta(\cdot)\int_0^\infty \beta_1(a)\Pi(a)T_1(a)\psi\right), \quad \psi \in \mathbb{Y}_+.$$

More precisely, for $\psi \in \mathbb{Y}_+$ and $x \in \overline{\Omega}$,

$$\mathcal{K}\psi(x) = \frac{b\beta(x)S_m^0(x)}{H(x)}\int_0^\infty e^{-(d_h+\rho)u}\int_\Omega \Gamma_2(u,x,y)c\beta(y)\int_0^\infty \beta_1(a)\Pi(a)\int_\Omega \Gamma_1(a,y,z)\psi(z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}u.$$

One can easily see that $\mathcal{K}$ is strictly positive, i.e., if $\psi \in \mathbb{Y}_+\setminus\{0\}$, then $\mathcal{K}\psi(x) > 0$ for all $x \in \overline{\Omega}$. According to [8,25], $\Re_0 := r(\mathcal{K})$, the spectral radius of $\mathcal{K}$. In general, $\Re_0$ cannot be explicitly calculated. However, in a spatially homogeneous case that

$$\mu(x) \equiv \mu, \beta(x) \equiv \beta \text{ and } H(x) \equiv H,$$

we can get that $S_m^0(x) \equiv \mu/d_m$ and $\mathcal{K}$ is compact. The Krein–Rutman theorem [2, Theorem 3.2] guarantees that $\Re_0$ is the only positive eigenvalue of $\mathcal{K}$ associated with a positive eigenvector. More precisely, we obtain

$$[\Re_0] = \frac{bc\beta^2\mu}{Hd_m(d_h+\rho)} \int_0^\infty \beta_1(a)\Pi(a)\mathrm{d}a. \tag{3.4}$$

4. Dynamical analysis in the spatially homogeneous case

In the spatially homogeneous case, problem (1.5)–(1.7) reduces to

$$\begin{cases} \dfrac{\partial S_m}{\partial t} - D_m\Delta S_m = \mu - \dfrac{b\beta}{H}S_mI_h - d_mS_m, & x\in\Omega,\ t>0,\\ \dfrac{\partial i_m}{\partial t} + \dfrac{\partial i_m}{\partial a} - D_m\Delta i_m = -d_mi_m, & x\in\Omega,\ t>0,\ a>0,\\ \dfrac{\partial I_h}{\partial t} - D_h\Delta I_h = \dfrac{c\beta(H-I_h)}{H}\displaystyle\int_0^\infty \beta_1(a)i_m\mathrm{d}a - (d_h+\rho)I_h, & x\in\Omega,\ t>0,\\ i_m(t,0,x) = \dfrac{b\beta}{H}S_mI_h, & x\in\overline{\Omega},\ t>0, \end{cases} \tag{4.1}$$

with the same initial and boundary conditions (1.6) and (1.7).

Corollary 4.1. *The solution semiflow $\{\Phi\}_{t\geq 0}$ of (4.1) admits a global attractor in $\mathcal{D}$.*

Proof. With the help of Theorem 2.1, one knows that Φ is point dissipative and eventually bounded on bounded sets of $\mathcal{D}$. Moreover, one can easily confirm that Φ is asymptotically smooth in the spatially homogeneous case by using the method as in [18]. An application of [22, Theorem 2.33] confirms that (4.1) admits a global attractor. This proves Corollary 4.1. □

System (4.1) has constant equilibria which are solutions to the following equations:

$$\begin{cases} 0 = \mu - \dfrac{b\beta}{H}S_mI_h - d_mS_m, \quad (d_h+\rho)I_h = \dfrac{c\beta(H-I_h)}{H}\displaystyle\int_0^\infty \beta_1(a)i_m(a)\mathrm{d}a,\\ \dfrac{\partial i_m}{\partial a} = -d_mi_m,\ a>0, \quad i_m(0) = \dfrac{b\beta}{H}S_mI_h. \end{cases} \tag{4.2}$$

Obviously, there exists a constant disease-free equilibrium $\tilde{E}_0 := (S_m^0, 0, 0) \in \mathcal{D}$, where $S_m^0 = \mu/d_m$. Moreover, rearranging (4.2), we have

$$S_m = \frac{H\mu}{b\beta I_h + Hd_m}, \quad I_h = \frac{c\beta HKi_m(0)}{c\beta Ki_m(0) + H(d_h+\rho)}, \quad i_m(a) = i_m(0)\Pi(a),\ a>0,$$

where $K := \int_0^\infty \beta_1(a)\Pi(a)\mathrm{d}a$. By the equation of $i_m(0)$ in (4.2), we have

$$\begin{aligned} i_m(0) = \frac{\mu b\beta I_h}{b\beta I_h + Hd_m} \quad &\Leftrightarrow \quad [i_m(0)-\mu]b\beta I_h + Hd_mi_m(0) = 0\\ &\Leftrightarrow \quad \frac{[i_m(0)-\mu]bc\beta^2HKi_m(0)}{c\beta Ki_m(0) + H(d_h+\rho)} + Hd_mi_m(0) = 0 \end{aligned}$$

$$\Leftrightarrow \quad c\beta K(b\beta+d_m)i_m(0)\left[i_m(0)-\frac{H(d_h+\rho)d_m}{c\beta K(b\beta+d_m)}\left([\Re_0]-1\right)\right]=0.$$

Thus, we have the following proposition on the existence of constant equilibrium.

Proposition 4.2. *Let $[\Re_0]$ is defined in (3.4). If $[\Re_0]>1$, then (4.1) admits a constant equilibrium $E^*=(S_m^*, i_m^*(a), I_h^*)\in\mathcal{D}$, where*

$$S_m^*=\frac{H\mu}{b\beta I_h^*+Hd_m},\quad I_h^*=\frac{c\beta HKi_m^*(0)}{c\beta Ki_m^*(0)+H(d_h+\rho)},\quad i_m^*(a)=i_m^*(0)\Pi(a),\ a>0,$$

and

$$i_m^*(0)=\frac{H(d_h+\rho)d_m}{c\beta K(b\beta+d_m)}\left([\Re_0]-1\right)>0.$$

4.1. Local asymptotic stability of equilibria

We shall prove that both $\tilde{E}_0$ and E^* are locally asymptotically stable (LAS).

Theorem 4.3. *Let $[\Re_0]$ be defined by (3.4).*

(i) If $[\Re_0]<1$, then $\tilde{E}_0$ is LAS;
(ii) If $[\Re_0]>1$, then E^* is LAS.

Proof. We first prove (i). The linearized system of (4.1) around $\tilde{E}_0$ is as follows:

$$\begin{cases}\dfrac{\partial S_m}{\partial t}-D_m\Delta S_m=-\dfrac{b\beta}{H}S_m^0I_h-d_mS_m, & x\in\Omega,\ t>0,\\ \dfrac{\partial i_m}{\partial t}+\dfrac{\partial i_m}{\partial a}-D_m\Delta i_m=-d_mi_m, & x\in\Omega,\ t>0,\ a>0,\\ \dfrac{\partial I_h}{\partial t}-D_h\Delta I_h=c\beta\displaystyle\int_0^\infty\beta_1(a)i_m\mathrm{d}a-(d_h+\rho)I_h, & x\in\Omega,\ t>0,\\ i_m(t,0,x)=\dfrac{b\beta}{H}S_m^0I_h, & x\in\overline{\Omega},\ t>0,\end{cases}\tag{4.3}$$

with boundary condition (1.7). Let μ_j $(j=1,2,\ldots)$ be the eigenvalues of linear operator $-\Delta$ on Ω with homogeneous Neumann boundary condition corresponding to the eigenvectors $v_j\in C^2(\Omega)\cap C^1(\overline{\Omega})$:

$$\Delta v_j=-\mu_jv_j,\ x\in\Omega;\quad \frac{\partial v_j}{\partial n}=0,\ x\in\partial\Omega.$$

From a well-known fact, we can assume that $0=\mu_0<\mu_1<\mu_2<\cdots$. Substituting $(S_m,i_m,I_h)=e^{\eta t}v_i(x)(u_1,u_2(a),u_3)$ $(\eta\in\mathbb{C})$ into (4.3) and dividing each side by $e^{\eta t}v_i(x)$, we have

$$\begin{cases}\eta u_1+D_m\mu_iu_1=-\dfrac{b\beta}{H}S_m^0u_3-d_mu_1,\\ \eta u_2(a)+\dfrac{\partial u_2(a)}{\partial a}+D_m\mu_iu_2(a)=-d_mu_2(a),\\ \eta u_3+D_h\mu_iu_3=c\beta\displaystyle\int_0^\infty\beta_1(a)u_2(a)\mathrm{d}a-(d_h+\rho)u_3,\\ u_2(0)=\dfrac{b\beta}{H}S_m^0u_3.\end{cases}\tag{4.4}$$

It is easy checked from the second and fourth equations of (4.4) that

$$u_2(a)=\frac{b\beta}{H}S_m^0u_3e^{-\eta a}\tilde{\Pi}(a),$$

where $\tilde{\Pi}(a) = e^{-D_m\mu_i a}\Pi(a)$. Rewriting (4.4) in terms of (u_1, u_3), we obtain the following characteristic equation:

$$\begin{vmatrix} \eta + D_m\mu_i + d_m & \frac{b\beta}{H}S_m^0 \\ 0 & \eta + D_h\mu_i + d_h + \rho - \frac{bc\beta^2}{H}S_m^0\int_0^\infty \beta_1(a)e^{-\eta a}\tilde{\Pi}(a)\mathrm{d}a \end{vmatrix} = \mathcal{K}(\eta)(\eta + D_m\mu_i + d_m) = 0,$$

where $\mathcal{K}(\eta) = \eta + D_h\mu_i + d_h + \rho - \frac{bc\beta^2}{H}S_m^0\int_0^\infty \beta_1(a)e^{-\eta a}\tilde{\Pi}(a)\mathrm{d}a$. To show that $\tilde{E}_0$ is LAS, we suppose on the contrary that $\eta = m + ni$ $(m, n \in \mathbb{R},\ i^2 = -1)$ with $m \geq 0$. We then see that $\eta + D_m\mu_i + d_m \neq 0$, and thus, we can only pay attention to the roots of $\mathcal{K}(\eta) = 0$. This equation can be rewritten as

$$\frac{\frac{bc\beta^2}{H}S_m^0\int_0^\infty \beta_1(a)e^{-\eta a}\tilde{\Pi}(a)\mathrm{d}a}{\eta + D_h\mu_i + d_h + \rho} = 1.$$

Taking the absolute value of both sides, we have

$$\begin{aligned} 1 &= \left|\frac{\frac{bc\beta^2}{H}S_m^0\int_0^\infty \beta_1(a)e^{-\eta a}\tilde{\Pi}(a)\mathrm{d}a}{\eta + D_h\mu_i + d_h + \rho}\right| \\ &\leq \frac{\frac{bc\beta^2}{H}S_m^0\int_0^\infty \beta_1(a)\Pi(a)\mathrm{d}a}{d_h + \rho} = [\Re_0], \end{aligned}$$

which leads to a contradiction with $[\Re_0] < 1$. Hence, $m \leq 0$. This proves (i).

We next proceed to prove (ii). The linearized system of (4.1) around E^* is as follows:

$$\begin{cases} \dfrac{\partial S_m}{\partial t} - D_m\Delta S_m = -\dfrac{b\beta}{H}S_m^* I_h - \dfrac{b\beta}{H}I_h^* S_m - d_m S_m, & x \in \Omega,\ t > 0, \\ \dfrac{\partial i_m}{\partial t} + \dfrac{\partial i_m}{\partial a} - D_m\Delta i_m = -d_m i_m, & x \in \Omega,\ t > 0,\ a > 0, \\ \dfrac{\partial I_h}{\partial t} - D_h\Delta I_h = c\beta\displaystyle\int_0^\infty \beta_1(a)i_m\mathrm{d}a - \dfrac{c\beta}{H}I_h^*\displaystyle\int_0^\infty \beta_1(a)i_m\mathrm{d}a & \\ \qquad\qquad -\dfrac{c\beta}{H}I_h\displaystyle\int_0^\infty \beta_1(a)i_m^*(a)\mathrm{d}a - (d_h+\rho)I_h, & x \in \Omega,\ t > 0, \\ i_m(t,0,x) = \dfrac{b\beta}{H}S_m I_h^* + \dfrac{b\beta}{H}S_m^* I_h, & x \in \overline{\Omega},\ t > 0, \end{cases} \tag{4.5}$$

with boundary condition (1.7). Substituting $(S_m, i_m, I_h) = e^{\eta t} v_i(x)(u_1, u_2(a), u_3)$ into (4.5) and dividing each side by $e^{\eta t} v_i(x)$, we have

$$\begin{cases} \eta u_1 + D_m \mu_i u_1 = -\frac{b\beta}{H} S_m^* u_3 - \frac{b\beta}{H} I_h^* u_1 - d_m u_1, \\ \eta u_2(a) + \frac{\partial u_2(a)}{\partial a} + D_m \mu_i u_2(a) = -d_m u_2(a), \\ \eta u_3 + D_h \mu_i u_3 = c\beta \int_0^\infty \beta_1(a) u_2(a) \mathrm{d}a - \frac{c\beta}{H} I_h^* \int_0^\infty \beta_1(a) u_2(a) \mathrm{d}a \\ \qquad\qquad - \frac{c\beta}{H} u_3 \int_0^\infty \beta_1(a) i_m^*(a) \mathrm{d}a - (d_h + \rho) u_3, \\ u_2(0) = \frac{b\beta}{H} S_m^* u_3 + \frac{b\beta}{H} I_h^* u_1. \end{cases} \tag{4.6}$$

It is easily checked that

$$u_2(a) = \left(\frac{b\beta}{H} S_m^* u_3 + \frac{b\beta}{H} I_h^* u_1 \right) e^{-\eta a} \tilde{\Pi}(a).$$

Hence, rewriting (4.6) in terms of (u_1, u_3), we obtain the following characteristic equation:

$$\begin{vmatrix} \eta + D_m \mu_i + d_m + \frac{b\beta}{H} I_h^* & \frac{b\beta}{H} S_m^* \\ -\frac{bc\beta^2 (H - I_h^*)}{H^2} I_h^* P & \eta + D_h \mu_i - \frac{bc\beta^2 (H - I_h^*)}{H^2} S_m^* P + \frac{c\beta}{H} Q + (d_h + \rho) \end{vmatrix} = 0,$$

where $P = \int_0^\infty \beta_1(a) e^{-\eta a} \tilde{\Pi}(a) \mathrm{d}a$ and $Q = \int_0^\infty \beta_1(a) i_m^*(a) \mathrm{d}a$. Rearranging this equation, we obtain

$$\eta + D_h \mu_i + d_h + \rho + \frac{c\beta}{H} Q = \frac{\eta + D_m \mu_i + d_m}{\eta + D_m \mu_i + d_m + \frac{b\beta}{H} I_h^*} \frac{bc\beta^2 (H - I_h^*)}{H^2} S_m^* P. \tag{4.7}$$

To show that E^* is LAS, we suppose on the contrary that $\eta = m + ni$ $(m, n \in \mathbb{R},\ i^2 = -1)$ with $m \geq 0$. By taking the absolute value of both sides of (4.7), we obtain

$$\begin{aligned} d_h + \rho < \left| \eta + D_h \mu_i + d_h + \rho + \frac{c\beta}{H} Q \right| = \left| \frac{\eta + D_m \mu_i + d_m}{\eta + D_m \mu_i + d_m + \frac{b\beta}{H} I_h^*} \frac{bc\beta^2 (H - I_h^*)}{H^2} S_m^* P \right| \\ < \frac{bc\beta^2 (H - I_h^*)}{H^2} S_m^* K. \end{aligned}$$

It then follows from the equilibrium equations that

$$0 < \frac{bc\beta^2 (H - I_h^*)}{H^2} S_m^* K - (d_h + \rho) = 0,$$

a contradiction. Hence, $m \leq 0$. This proves (ii). □

4.2. Global dynamics

We shall investigate the threshold dynamics of (4.1) in terms of $[\Re_0]$, that is, both $\tilde{E}_0$ and E^* are globally attractive. This together with the related results in above subsection tells us that both $\tilde{E}_0$ and E^* are globally asymptotically stable (GAS).

Theorem 4.4. *Suppose that* $[\Re_0] < 1$. *Then,* $\tilde{E}_0$ *is GAS in* $\mathcal{D}$.

Proof. Note that, in $\mathcal{D}$, $S_m \leq \mu/d_m = S_m^0$ for all $t > 0$ and $x \in \overline{\Omega}$. Hence, an application of the comparison principle gives $0 \leq i_m \leq \bar{i}_m$ and $0 \leq I_h \leq \overline{I}_h$, where $(\bar{i}_m, \overline{I}_h)$ is the solution to the following auxiliary system:

$$\begin{cases} \dfrac{\partial \bar{i}_m}{\partial t} + \dfrac{\partial \bar{i}_m}{\partial a} - D_m \Delta \bar{i}_m = -d_m \bar{i}_m, \\ \dfrac{\partial \overline{I}_h}{\partial t} - D_h \Delta \overline{I}_h = \dfrac{c\beta(H - \overline{I}_h)}{H} \displaystyle\int_0^\infty \beta_1(a) \bar{i}_m \mathrm{d}a - (d_h + \rho)\overline{I}_h, \\ \bar{i}_m(t, 0, x) = \dfrac{b\beta}{H} S_m^0 \overline{I}_h, \\ \bar{i}_m(0, a, x) = \phi_2(a, x), \quad \overline{I}_h(0, x) = \phi_3(x), \end{cases}$$

for $x \in \Omega,\ t > 0,\ a > 0$ and

$$\frac{\partial \bar{i}_m}{\partial n} = \frac{\partial \overline{I}_h}{\partial n} = 0,\ x \in \partial\Omega,\ t > 0,\ a > 0.$$

It then suffices to show that $(\bar{i}_m, \overline{I}_h)$ converges to $(0, 0)$ as time goes to infinity, which implies that (i_m, I_h) convereges to $(0, 0)$, and thus, S_m converges to S_m^0 as time goes to infinity.

Let

$$\Psi_1(a) := \frac{c\beta}{\Pi(a)} \int_a^\infty \beta_1(\theta)\Pi(\theta)\mathrm{d}\theta.$$

One can then easily check that

$$\begin{cases} \Psi_1'(a) = -c\beta\beta_1(a) + d_m \Psi_1(a), \\ \Psi_1(0) = c\beta K. \end{cases}$$

Let $V(t) := V_1(t) + V_2(t)$ be a Lyapunov function, where

$$V_1(t) := \int_\Omega \int_0^\infty \Psi_1(a) \bar{i}_m \mathrm{d}a\mathrm{d}x, \quad V_2(t) := \int_\Omega \overline{I}_h \mathrm{d}x.$$

We then have that

$$\begin{aligned} V_1'(t) =& \frac{\partial}{\partial t} \int_\Omega \int_0^\infty \Psi_1(a) \bar{i}_m \mathrm{d}a\mathrm{d}x = \int_\Omega \int_0^\infty \Psi_1(a) \left[\frac{\partial \bar{i}_m}{\partial t}\right] \mathrm{d}a\mathrm{d}x \\ =& \int_\Omega \int_0^\infty \Psi_1(a) \left[-\frac{\partial \bar{i}_m}{\partial a} + D_m \Delta \bar{i}_m - d_m \bar{i}_m\right] \mathrm{d}a\mathrm{d}x \\ =& \int_\Omega \left\{ \Psi_1(0)\bar{i}_m(t, 0, x) + \int_0^\infty \left[\Psi_1'(a) - d_m \Psi_1(a)\right] \bar{i}_m \mathrm{d}a \right\} dx \\ =& \int_\Omega \left[\frac{bc\beta^2 K}{H} S_m^0 \overline{I}_h - c\beta \int_0^\infty \beta_1(a) \bar{i}_m \mathrm{d}a \right] dx, \end{aligned}$$

and

$$V_2'(t) = \int\limits_{\Omega} \left[D_h \Delta \overline{I}_h + \frac{c\beta(H - \overline{I}_h)}{H} \int\limits_0^\infty \beta_1(a)\overline{i}_m \mathrm{d}a - (d_h + \rho)\overline{I}_h \right] \mathrm{d}x$$

$$= \int\limits_{\Omega} \left[\left(c\beta - \frac{\overline{I}_h}{H} \right) \int\limits_0^\infty \beta_1(a)\overline{i}_m \mathrm{d}a - (d_h + \rho)\overline{I}_h \right] \mathrm{d}x.$$

Hence, the derivative of V gives

$$V'(t) = \int\limits_{\Omega} \left[\frac{bc\beta^2 K}{H} S_m^0 \overline{I}_h - (d_h + \rho)\overline{I}_h - \frac{\overline{I}_h}{H} \int\limits_0^\infty \beta_1(a)\overline{i}_m da \right] dx$$

$$= (d_h + \rho)\left([\Re_0] - 1 \right) \int\limits_{\Omega} \overline{I}_h dx - \frac{1}{H} \int\limits_{\Omega} \overline{I}_h \int\limits_0^\infty \beta_1(a)\overline{i}_m \mathrm{d}a\mathrm{d}x \leq 0.$$

Consequently, $\tilde{E}_0$ is globally attractive in $\mathcal{D}$ when $[\Re_0] < 1$ (see, for instance, [26, Theorem 4.2]). Combined with the results in Theorem 4.3, one knows that $\tilde{E}_0$ is GAS. This completes the proof of Theorem 4.4. □

To define a Lyapunov function for E^* when $[\Re_0] > 1$, we need a uniform persistence result. The following estimation for S_m immediately follows.

Proposition 4.5. *There exists an $\epsilon_0 > 0$ such that, for any $\phi \in \mathcal{D}$ and $x \in \overline{\Omega}$,*

$$\liminf_{t\to\infty} S_m > \epsilon_0. \tag{4.8}$$

Proof. By Theorem 2.1, and $I_h \leq H$, one can get that $\frac{\partial S_m}{\partial t} \geq D_m \Delta S_m + \mu - (b\beta + d_m)S_m$. Again from the comparison principle, one can get that for any $x \in \overline{\Omega}$,

$$\liminf_{t\to\infty} S_m > \frac{\mu}{b\beta + d_m} =: \epsilon_0.$$

This proves Proposition 4.5. □

We next define the following subset of $\mathcal{D}$:

$$\mathcal{D}_0 := \{\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathcal{D} : \varphi_3 \not\equiv 0\}. \tag{4.9}$$

Epidemiologically, $\mathcal{D}_0$ is the set where the disease persists. The forthcoming lemma immediately follows.

Lemma 4.6. *If $\phi \in \mathcal{D}_0$, then $I_h > 0$ for all $t > 0$ and $x \in \overline{\Omega}$.*

The following result indicates that $\{\tilde{E}_0\}$ is a uniform weak repeller in $\mathcal{D}$.

Lemma 4.7. *If $[\Re_0] > 1$, then there exists an $\epsilon_1 > 0$ such that, for any $\phi \in \mathcal{D}_0$,*

$$\limsup_{t\to\infty} \|\Phi(t)\phi - \tilde{E}_0\|_{\mathbb{X}} > \epsilon_1.$$

Proof. We proceed it indirectly and assume that for any $\epsilon_1 > 0$, there exists a $\phi \in \mathcal{D}_0$ that

$$\limsup_{t\to\infty} \|\Phi(t)\phi - \tilde{E}_0\|_{\mathbb{X}} \leq \epsilon_1.$$

This inequality implies that there exists a $t_1 > 0$ such that, for any $t > t_1$ and $x \in \overline{\Omega}$,

$$S_m \geq S_m^0 - \epsilon_1, \quad \int\limits_0^\infty i_m \mathrm{d}a \leq \epsilon_1, \quad I_h \leq \epsilon_1. \tag{4.10}$$

Without loss of generality, taking $\Phi(t_1)\phi$ be the new initial condition, we can assume that inequalities (4.10) hold for all $t > 0$ and $x \in \overline{\Omega}$.

For simplicity, we write $\mathcal{B}(t) = \mathcal{B}(S_m, I_h)(t)$. By (2.9), we obtain the following abstract inequality in $\mathbb{Y}$:

$$\begin{aligned}\mathcal{B}(t) \geq & \frac{b\beta}{H}(S_m^0 - \epsilon_1)\int_0^t e^{-(d_h+\rho)(t-s)}T_2(t-s)\left[\frac{c\beta(H-\epsilon_1)}{H}\int_0^s \beta_1(a)T_1(a)[\mathcal{B}(s-a)]\Pi(a)\mathrm{d}a\right]\mathrm{d}s \\ = & \frac{bc\beta^2}{H}(S_m^0 - \epsilon_1)\left(1 - \frac{\epsilon_1}{H}\right)\int_0^t e^{-(d_h+\rho)(t-s)}T_2(t-s)\left[\int_0^s \beta_1(a)T_1(a)[\mathcal{B}(s-a)]\Pi(a)\mathrm{d}a\right]\mathrm{d}s.\end{aligned} \tag{4.11}$$

For any $\lambda > 0$, let $\hat{\mathcal{B}}(\lambda) := \int_0^\infty e^{-\lambda t}\int_\Omega \mathcal{B}(t,x)\mathrm{d}x\mathrm{d}t$. By Lemma 4.6 and (2.2), we can easily confirm that $0 < \hat{\mathcal{B}}(\lambda) < +\infty$. Moreover, from (4.11), we have

$$\begin{aligned}\hat{\mathcal{B}}(\lambda) \geq & \frac{bc\beta^2}{H}(S_m^0 - \epsilon_1)\left(1 - \frac{\epsilon_1}{H}\right)\int_0^\infty e^{-\lambda t}\int_0^t e^{-(d_h+\rho)(t-s)}\int_0^s \beta_1(a)\int_\Omega \mathcal{B}(s-a,x)\mathrm{d}x\Pi(a)\mathrm{d}a\mathrm{d}s\mathrm{d}t \\ = & \frac{bc\beta^2}{H}(S_m^0 - \epsilon_1)\left(1 - \frac{\epsilon_1}{H}\right)\frac{1}{\lambda + d_h + \rho}\int_0^\infty e^{-\lambda a}\beta_1(a)\Pi(a)\mathrm{d}a\hat{\mathcal{B}}(\lambda) \\ = & [\Re_{\epsilon_1,\lambda}]\hat{\mathcal{B}}(\lambda),\end{aligned} \tag{4.12}$$

where

$$[\Re_{\epsilon_1,\lambda}] := \frac{bc\beta^2}{H}(S_m^0 - \epsilon_1)\left(1 - \frac{\epsilon_1}{H}\right)\frac{1}{\lambda + d_h + \rho}\int_0^\infty e^{-\lambda a}\beta_1(a)\Pi(a)\mathrm{d}a.$$

One can easily see that $[\Re_{\epsilon_1,\lambda}] \to [\Re_0] > 1$ as $(\epsilon_1, \lambda) \to (0, 0)$, which allow us to choose $\epsilon_1 > 0$ and $\lambda > 0$ small enough such that $[\Re_{\epsilon_1,\lambda}] > 1$. We then have from (4.12) that $\hat{\mathcal{B}}(\lambda) > \hat{\mathcal{B}}(\lambda)$, a contradiction. This proves Lemma 4.7. □

Using Lemma 4.7, we now prove the following result.

Proposition 4.8. *If $[\Re_0] > 1$, then there exists an $\epsilon_2 > 0$ such that, for any $\phi \in \mathcal{D}_0$ and $x \in \overline{\Omega}$,*

$$\liminf_{t\to\infty} I_h(t,x) > \epsilon_2. \tag{4.13}$$

Before the proof, we prepare some notations:

- $\partial\mathcal{D}_0 := \mathcal{D} \setminus \mathcal{D}_0 = \{\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathcal{D} : \varphi_3 \equiv 0\}$.
- $M_\partial := \{\varphi \in \partial\mathcal{D}_0 : \Phi(t)\varphi \in \partial\mathcal{D}_0 \text{ for all } t > 0\}$.
- $\omega(\varphi) := \cap_{t\geq 0}\overline{\cup_{s\geq t}\Phi(s)\varphi}$: the omega limit set.
- $\delta(\varphi) := \inf_{x\in\Omega}\varphi_3(x)$, $\rho : \mathcal{D} \to \mathbb{R}_+$: a generalized distance function.
- $W^s(\tilde{E}_0) := \{\varphi \in \mathcal{D} : \lim_{t\to\infty}\|\Phi(t)\varphi - \tilde{E}_0\|_{\mathbb{X}} = 0\}$: the stable set of $\tilde{E}_0$.

Proof. One can easily confirm that

1. $\cup_{\varphi\in M_\partial}\omega(\varphi) = \{\tilde{E}_0\}$.
2. No subset of $\{\tilde{E}_0\}$ forms a cycle in $\partial\mathcal{D}_0$.
3. $\{\tilde{E}_0\}$ is isolated in $\mathcal{D}$.
4. $W^s(\tilde{E}_0) \cap \delta^{-1}(0, \infty) = \emptyset$.

Moreover, by Corollary 4.1, Φ has a global attractor in $\mathcal{D}$. Thus, conditions in [23, Theorem 3] are satisfied and there exists an $\epsilon_2 > 0$ such that

$$\min_{\varphi \in \mathcal{L}} \delta(\varphi) > \epsilon_2,$$

where $\mathcal{L}$ is an arbitrary compact chain transitive set in $\mathcal{D} \setminus \{\tilde{E}_0\}$. Hence, for any $\phi \in \mathcal{D}_0$ and $x \in \overline{\Omega}$, (4.13) holds. This proves Proposition 4.8. □

By Propositions 4.5 and 4.8, there exists $c_i > 0, i = 1, 2$, such that, for any total trajectory in a persistence attractor (see, e.g., [18, Theorem 8.3]), the following inequalities hold:

$$c_1 < \frac{i_m}{i_m^*(a)} = \frac{T_1(a)(\mathcal{B}(t-a,\cdot))(x)}{i_m^*(0)} < c_2, \quad x \in \overline{\Omega},\ t \in \mathbb{R},\ a \geq 0.$$

Thus, for any total trajectory in a persistence attractor, the following functions are finite for all $t \in \mathbb{R}$:

$$\begin{aligned} W_1(t) &:= \int_\Omega S_m^* g\left(\frac{S_m}{S_m^*}\right) \mathrm{d}x, \quad W_2(t) := \int_\Omega \int_0^\infty \Psi_2(a) i_m^*(a) g\left(\frac{i_m}{i_m^*(a)}\right) \mathrm{d}a \mathrm{d}x, \\ W_3(t) &:= \int_\Omega I_h^* g\left(\frac{I_h}{I_h^*}\right) \mathrm{d}x, \end{aligned}$$

where $g(u) := u - 1 - \ln u$ and

$$\Psi_2(a) := \frac{c\beta}{\Pi(a)} \left(1 - \frac{I_h^*}{H}\right) \int_a^\infty \beta_1(\theta)\Pi(\theta) d\theta.$$

Clearly, $g(u) > 0$ for each $u \in (0, \infty) \setminus \{1\}$ and $g(1) = 0$. Using a Lyapunov function $W := \kappa W_1 + W_2 + W_3$ with $\kappa > 0$ to be determined below, we can obtain the following result.

Theorem 4.9. *Suppose that* $[\Re_0] > 1$*. Then,* E^* *is GAS in* $\mathcal{D}_0$*.*

Proof. By appealing to [18, Theorem 9.5], we consider a total trajectory in the persistence attractor. Then, $W(t) = \kappa W_1(t) + W_2(t) + W_3(t)$ is finite for all $t \in \mathbb{R}$. Direct calculation gives

$$
\begin{aligned}
W_1'(t) = & \int_\Omega \left[\left(1 - \frac{S_m^*}{S_m}\right)\left(D_m \Delta S_m + \mu - \frac{b\beta}{H} S_m I_h - d_m S_m\right)\right] \mathrm{d}x \\
= & - D_m S_m^* \int_\Omega \frac{|\nabla S_m|^2}{S_m^2} \mathrm{d}x + d_m S_m^* \int_\Omega \left(2 - \frac{S_m^*}{S_m} - \frac{S_m}{S_m^*}\right) \mathrm{d}x \\
& + \frac{b\beta}{H} S_m^* I_h^* \int_\Omega \left(1 - \frac{S_m^*}{S_m} + \frac{I_h}{I_h^*} - \frac{S_m I_h}{S_m^* I_h^*}\right) \mathrm{d}x \\
= & - D_m S_m^* \int_\Omega \frac{|\nabla S_m|^2}{S_m^2} \mathrm{d}x - d_m S_m^* \int_\Omega \left[g\left(\frac{S_m^*}{S_m}\right) + g\left(\frac{S_m}{S_m^*}\right)\right] \mathrm{d}x \\
& + i_m^*(0) \int_\Omega \left[-g\left(\frac{S_m^*}{S_m}\right) + g\left(\frac{I_h}{I_h^*}\right) - g\left(\frac{S_m I_h}{S_m^* I_h^*}\right)\right] \mathrm{d}x, \\
W_3'(t) = & \int_\Omega \left[\left(1 - \frac{I_h^*}{I_h}\right)\left(D_h \Delta I_h + c\beta\left(1 - \frac{I_h}{H}\right)\int_0^\infty \beta_1(a) i_m \mathrm{d}a - (d_h + \rho) I_h\right)\right] \mathrm{d}x \\
= & - D_h I_h^* \int_\Omega \frac{|\nabla I_h|^2}{I_h^2} \mathrm{d}x + c\beta \int_\Omega \left(1 - \frac{I_h^*}{I_h}\right)\left(1 - \frac{I_h}{H}\right)\int_0^\infty \beta_1(a) i_m \mathrm{d}a\mathrm{d}x \\
& + (d_h + \rho) I_h^* \int_\Omega \left(1 - \frac{I_h}{I_h^*}\right) \mathrm{d}x \\
= & - D_h I_h^* \int_\Omega \frac{|\nabla I_h|^2}{I_h^2} \mathrm{d}x - c\beta \int_\Omega \frac{(I_h - I_h^*)^2}{H I_h} \int_0^\infty \beta_1(a) i_m \mathrm{d}a\mathrm{d}x \\
& + c\beta\left(1 - \frac{I_h^*}{H}\right) \int_\Omega \int_0^\infty \beta_1(a) i_m^* \left(1 - \frac{I_h}{I_h^*} + \frac{i_m}{i_m^*} - \frac{I_h^* i_m}{I_h i_m^*}\right) \mathrm{d}a\mathrm{d}x \\
= & - D_h I_h^* \int_\Omega \frac{|\nabla I_h|^2}{I_h^2} \mathrm{d}x - c\beta \int_\Omega \frac{(I_h - I_h^*)^2}{H I_h} \int_0^\infty \beta_1(a) i_m \mathrm{d}a\mathrm{d}x \\
& - c\beta\left(1 - \frac{I_h^*}{H}\right) K i_m^*(0) \int_\Omega g\left(\frac{I_h}{I_h^*}\right) \mathrm{d}x \\
& + c\beta\left(1 - \frac{I_h^*}{H}\right) \int_\Omega \int_0^\infty \beta_1(a) i_m^* \left[g\left(\frac{i_m}{i_m^*}\right) - g\left(\frac{I_h^* i_m}{I_h i_m^*}\right)\right] \mathrm{d}a\mathrm{d}x,
\end{aligned}
$$

and

$$
W_2'(t) = \int_\Omega \int_0^\infty \Psi_2(a) \left(1 - \frac{i_m^*}{i_m}\right)\left(D_m \Delta i_m - \frac{\partial i_m}{\partial a} - d_m i_m\right) \mathrm{d}a\mathrm{d}x
$$

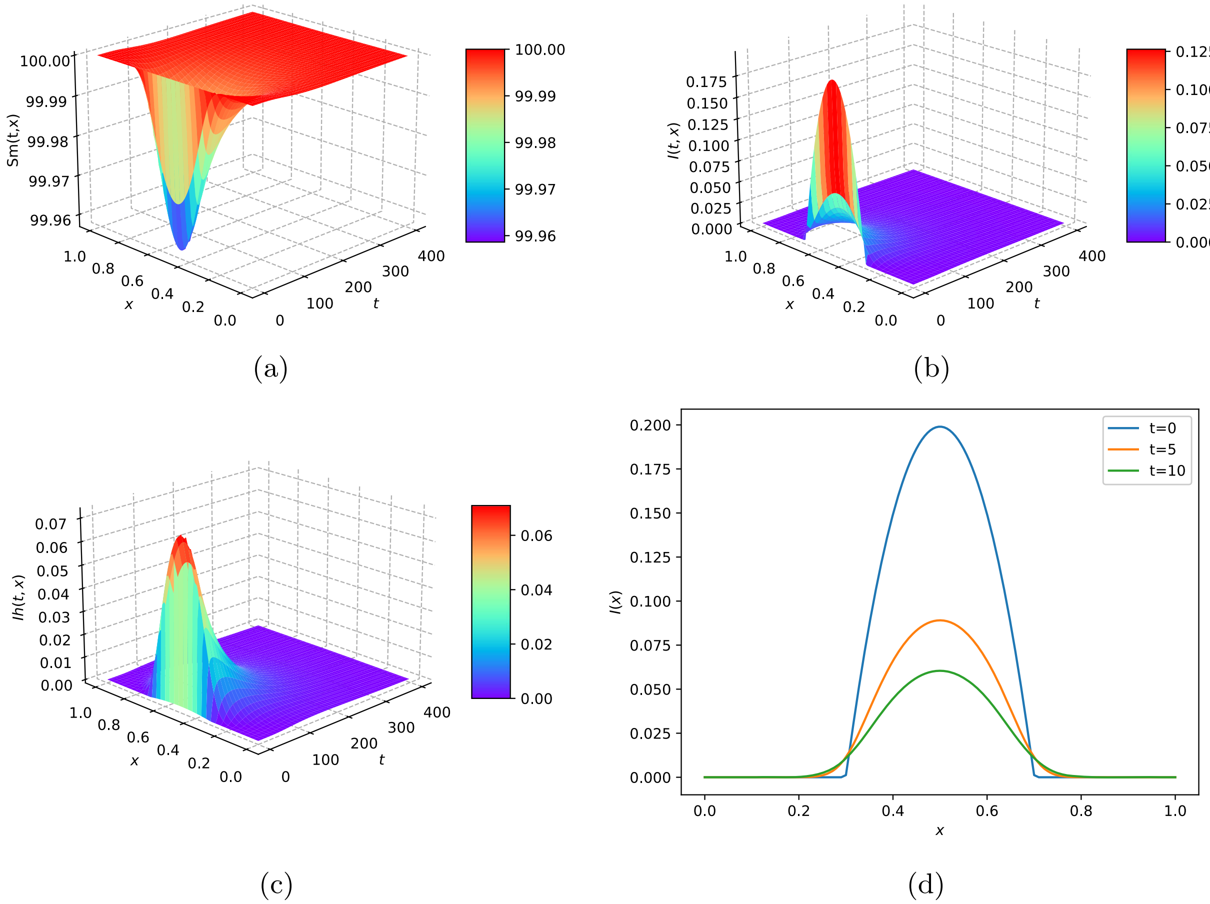

FIG. 1. $\Omega=(0,1)$. The time evolution of susceptible mosquitoes $S_m(t,x)$, infected mosquitoes with $I(t,x)=\int_0^\infty i_m(t,a,x)\mathrm{d}a)$ and infected humans $I_h(t,x)$ of system (4.1) with (5.1), $\beta_=0.026$ and $\Omega=(0,1)$. The initial data is $\phi_1(x)=100$, $\phi_2(a,x)=e^{-d_m*a}(x-0.3)(0.7-x)$ and $\phi_3(x)=0$.

$$=-D_m\int_\Omega\int_0^\infty \Psi_2(a)\frac{|\nabla i_m|^2}{i_m^2}\mathrm{d}a\mathrm{d}x-\int_\Omega\int_0^\infty \Psi_2(a)\left(1-\frac{i_m^*}{i_m}\right)\left(\frac{\partial i_m}{\partial a}+d_m i_m\right)\mathrm{d}a\mathrm{d}x.$$

Here, note that

$$i_m^*\frac{\partial}{\partial a}g\left(\frac{i_m}{i_m^*}\right)=i_m^*\left(1-\frac{i_m^*}{i_m}\right)\frac{\partial}{\partial a}\left(\frac{i_m}{i_m^*}\right)=\left(1-\frac{i_m^*}{i_m}\right)\left(\frac{\partial i_m}{\partial a}+d_m i_m\right).$$

Thus, we have

$$\begin{aligned}W_2'(t)=&-D_m\int_\Omega\int_0^\infty \Psi_2(a)\frac{|\nabla i_m|^2}{i_m^2}\mathrm{d}a\mathrm{d}x-\int_\Omega\int_0^\infty \Psi_2(a)i_m^*\frac{\partial}{\partial a}g\left(\frac{i_m}{i_m^*}\right)\mathrm{d}a\mathrm{d}x\\ =&-D_m\int_\Omega\int_0^\infty \Psi_2(a)\frac{|\nabla i_m|^2}{i_m^2}\mathrm{d}a\mathrm{d}x+\Psi_2(0)i_m^*(0)\int_\Omega g\left(\frac{i_m(t,0,x)}{i_m^*(0)}\right)\mathrm{d}x\end{aligned}$$

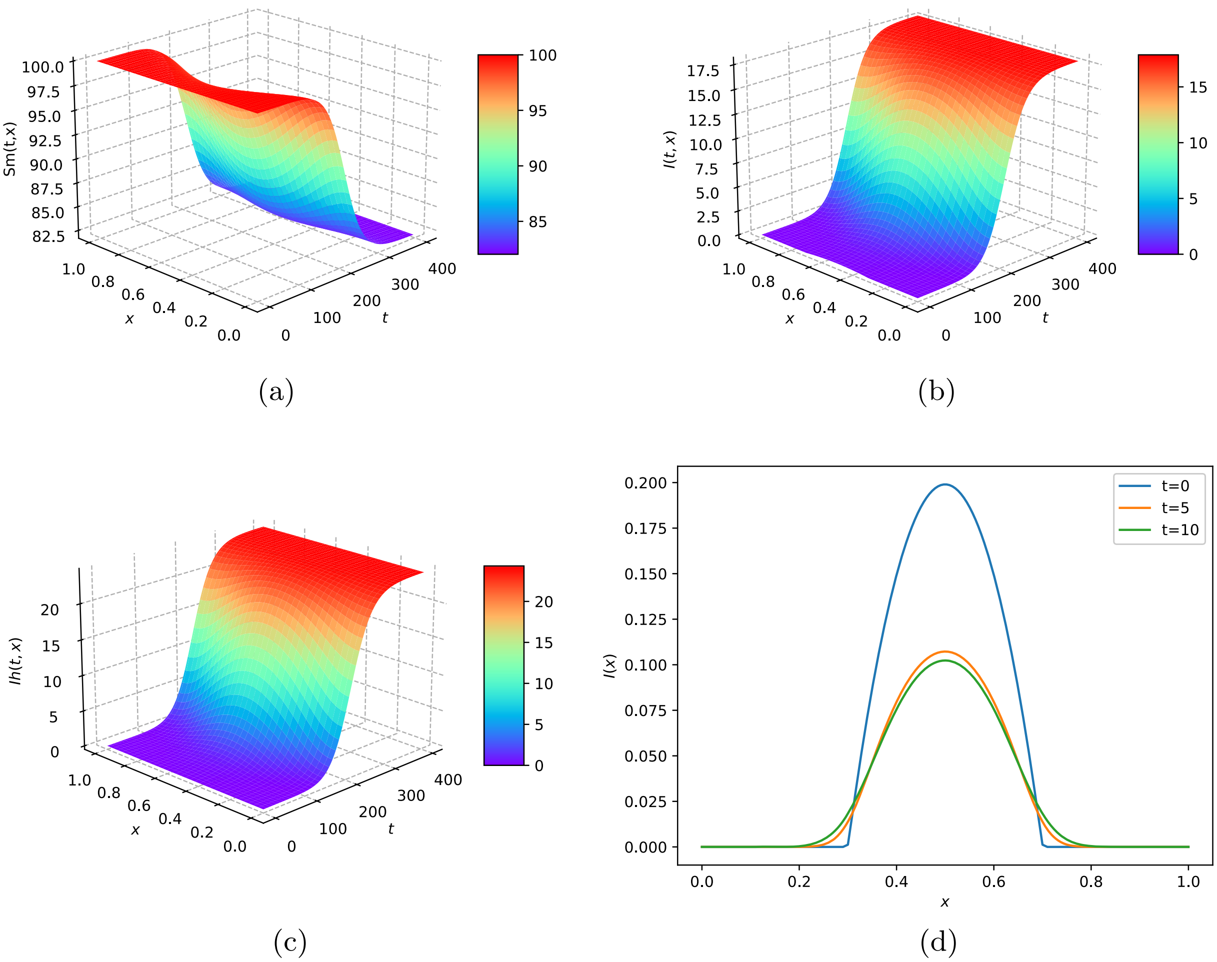

FIG. 2. The time evolution of susceptible mosquitoes $S_m(t,x)$, infected mosquitoes with $I(t,x) = \int_0^\infty i_m(t,a,x)\mathrm{d}a)$ and infected humans $I_h(t,x)$ of system (4.1) with (5.1), $\beta = 0.036$ and $\Omega = (0,1)$. The initial data is $\phi_1(x) = 100$, $\phi_2(a,x) = e^{-d_m * a}(x-0.3)(0.7-x)$ and $\phi_3(x) = 0$.

$$- c\beta\left(1 - \frac{I_h^*}{H}\right)\int_\Omega \int_0^\infty \beta_1(a) i_m^* g\left(\frac{i_m}{i_m^*}\right)\mathrm{d}a\mathrm{d}x.$$

Hence, letting $\kappa := c\beta(1 - I_h^*/H)K$, we obtain

$$\begin{aligned} W'(t) =& \kappa W_1'(t) + W_2'(t) + W_3'(t) \\ =& - \kappa D_m S_m^* \int_\Omega \frac{|\nabla S_m|^2}{S_m^2}\mathrm{d}x - \kappa d_m S_m^* \int_\Omega \left[g\left(\frac{S_m^*}{S_m}\right) + g\left(\frac{S_m}{S_m^*}\right)\right]\mathrm{d}x - \kappa i_m^*(0)\int_\Omega g\left(\frac{S_m^*}{S_m}\right)\mathrm{d}x \\ & - D_h I_h^* \int_\Omega \frac{|\nabla I_h|^2}{I_h^2}\mathrm{d}x - c\beta\int_\Omega \frac{(I_h - I_h^*)^2}{H I_h}\int_0^\infty \beta_1(a) i_m \mathrm{d}a\mathrm{d}x \end{aligned}$$

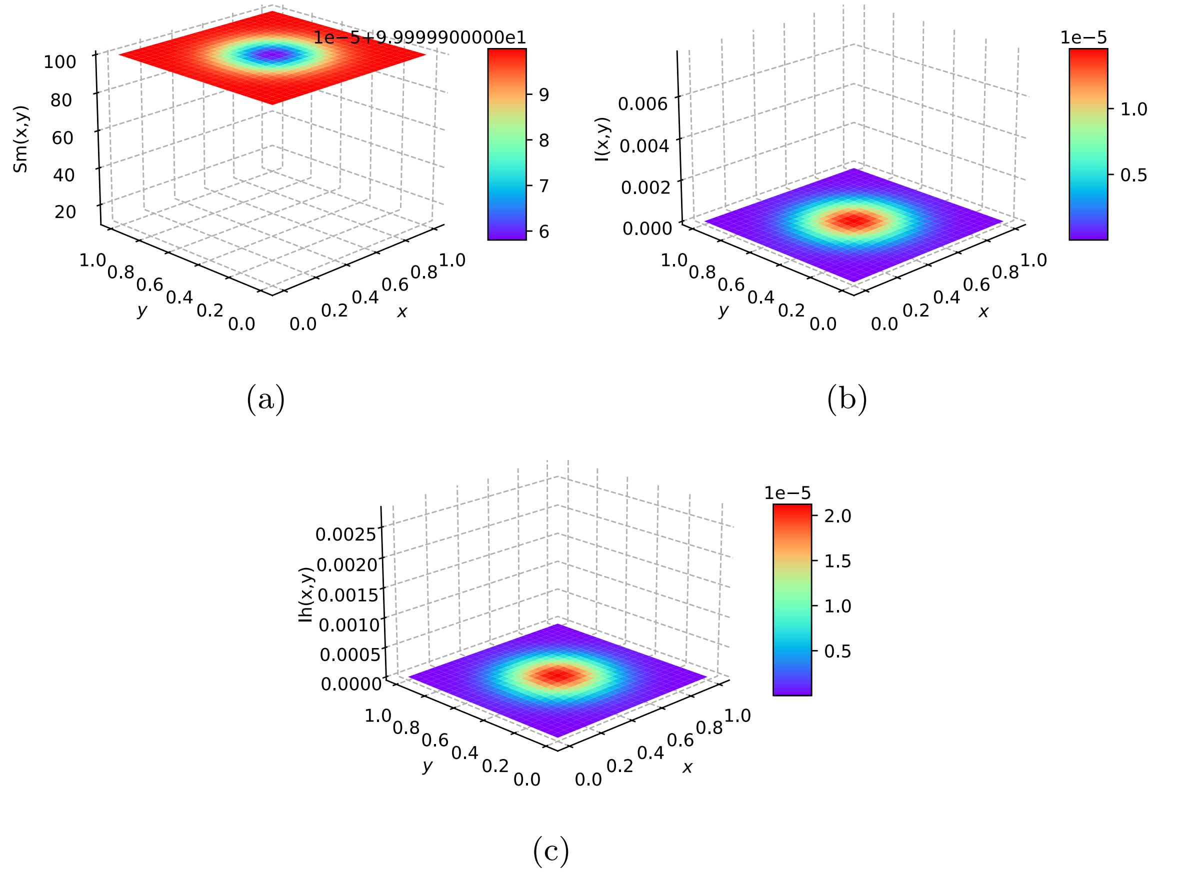

Fig. 3. $t = 280$. The time evolution of susceptible mosquitoes $S_m(t,x)$, infected mosquitoes with $I(t,x) = \int\limits_0^\infty i_m(t,a,x)\mathrm{d}a)$ and infected humans $I_h(t,x)$ of system (4.1) with (5.1), $\beta = 0.26$ and $\Omega = (0,1)\times(0,1)$. The initial data is $\phi_1(x,y) = 100$, $\phi_2(a,x,y) = e^{-d_m * a}(x-0.3)(0.7-x)(y-0.3)(0.7-y)$ and $\phi_3(x,y) = 0$.

$$
\begin{aligned}
&- c\beta\left(1-\frac{I_h^*}{H}\right)\int\limits_\Omega\int\limits_0^\infty \beta_1(a)i_m^* g\left(\frac{I_h^* i_m}{I_h i_m^*}\right)\mathrm{d}a\mathrm{d}x - D_m\int\limits_\Omega\int\limits_0^\infty \Psi_2(a)\frac{|\nabla i_m|^2}{i_m^2}\mathrm{d}a\mathrm{d}x\\
\leq& 0.
\end{aligned}
$$

One can easily see that $W'(t) = 0$ iff $(S_m, i_m, I_h) = E^*$. As in the proof of [18, Theorem 9.5], we see that the singleton $\{E^*\}$ is indeed the persistence attractor. This gives the global attractivity of E^*. Together with Theorem 4.3, one can get E^* is GAS. This proves Theorem 4.9. □

5. Numerical simulations

5.1. Dynamical behaviors of system (4.1)

We perform numerical simulations to support the main results obtained in Sect. 4. Specifically, we shall carry out the simulations for 1-dimensional and 2-dimensional domain to validate Theorems 4.4 and 4.9, that is, both $\tilde{E}_0$ and E^* are GAS.

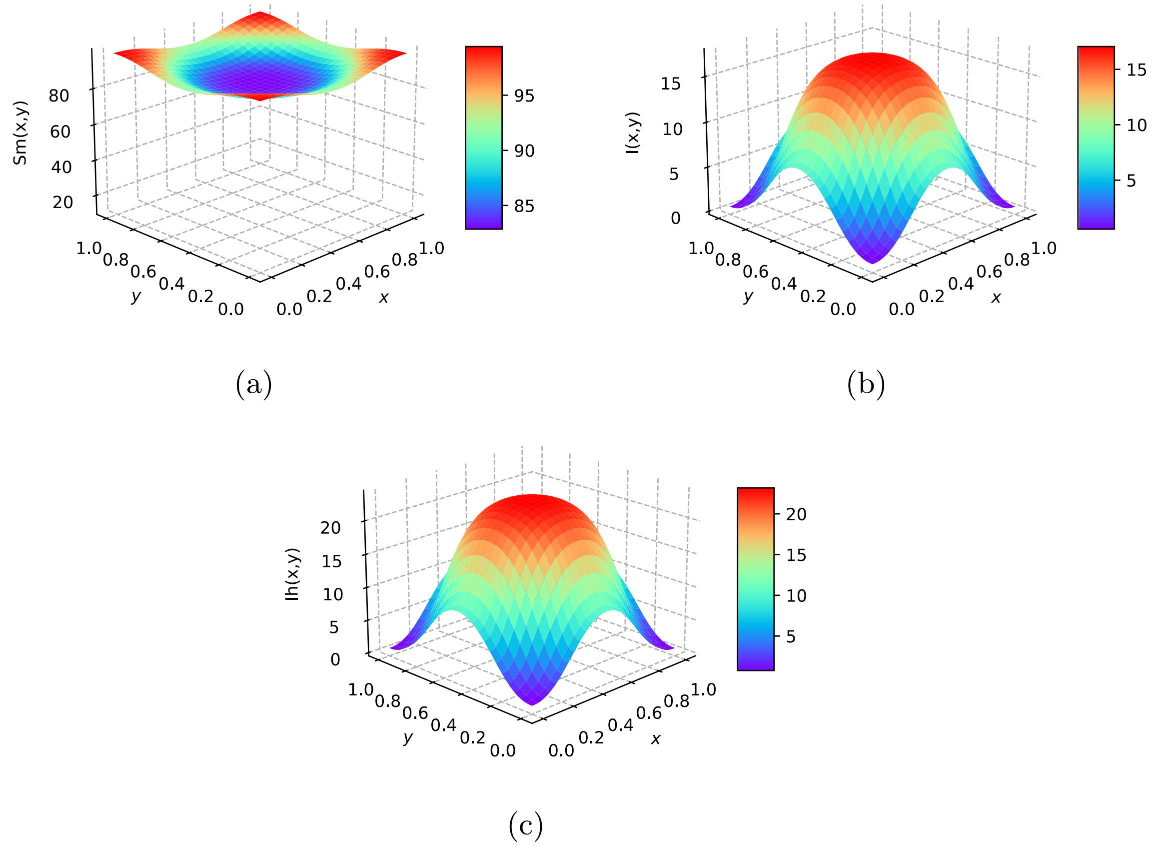

FIG. 4. $t = 350$. The time evolution of susceptible mosquitoes $S_m(t,x)$, infected mosquitoes with $I(t,x) = \int_0^\infty i_m(t,a,x)\mathrm{d}a)$ and infected humans $I_h(t,x)$ of system (4.1) with (5.1), $\beta = 0.36$ and $\Omega = (0,1)\times(0,1)$. The initial data is $\phi_1(x,y) = 100,\ \phi_2(a,x,y) = e^{-d_m * a}(x-0.3)(0.7-x)(y-0.3)(0.7-y)$ and $\phi_3(x,y) = 0$.

For the case that $\Omega = (0,1)$, we set the following parameters:

$$\begin{aligned} &\mu = 20,\ d_m = 0.2,\ b = 0.5,\ c = 0.5,\ H = 100, d_h = 0.00004,\ \rho = 0.1, \\ &\quad D_m = D_h = 0.000125,\ \beta_1 = 1. \end{aligned} \tag{5.1}$$

If we take $\beta = 0.26$, we can compute $[\Re_0] = 0.844669$. From Theorem 4.4, we know that $\tilde{E}_0$ is GAS in $\mathcal{D}$. Figure 1a, b and c illustrates that the density of susceptible mosquitoes will attain a positive level and infected mosquitoes and infected humans decay to zero. We also know from Fig. 1d that the spatial distribution of infected mosquitoes gradually enlarges with higher prevalence but decays to zero.

If we take $\beta = 0.36$ and the other parameters remain the same as in (5.1), then $[\Re_0] = 1.619366$. It is known from Theorem 4.9 that E^* is GAS in $\mathcal{D}_0$. Figure 2a, b and c illustrates that the densities of susceptible mosquitoes, infected mosquitoes and infected humans will attain a positive level as time evolves. Figure 2d illustrates that the spatial distribution of infected mosquitoes gradually enlarge with higher prevalence.

For the case that $\Omega = (0,1)\times(0,1)$. We set the same parameters as in Fig. 1 and 2. Figure 3a illustrates that the density of susceptible mosquitoes will attain a positive level. We can see from Fig. 3b, c that the

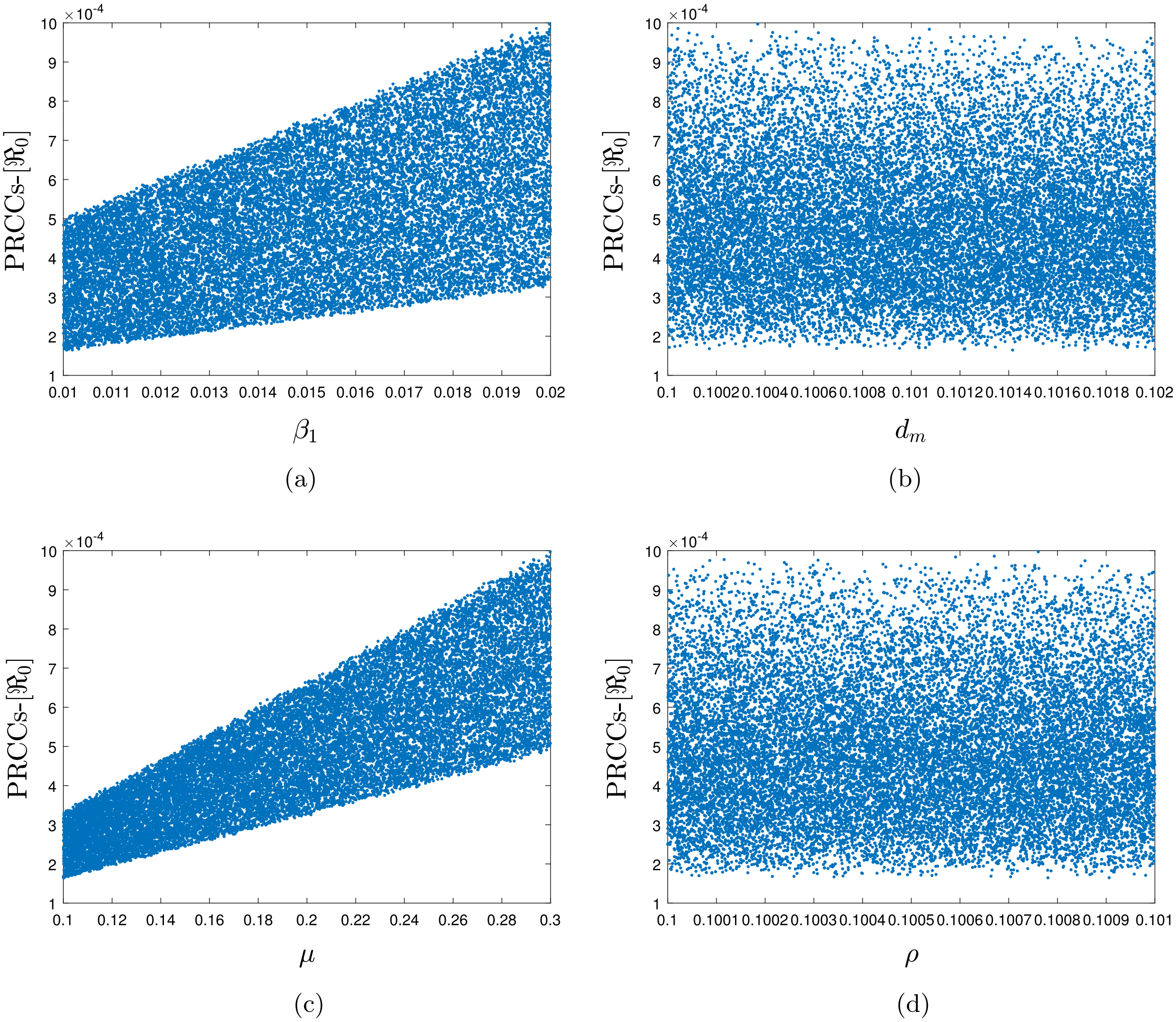

Fig. 5. PRCC for [$\Re_0$].

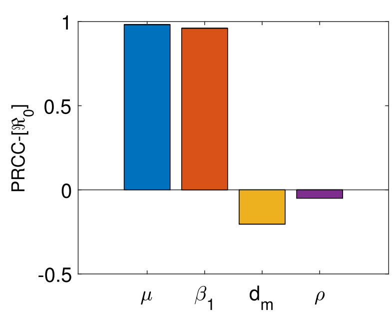

Fig. 6. Sensitive analysis of the [$\Re_0$] via parameters

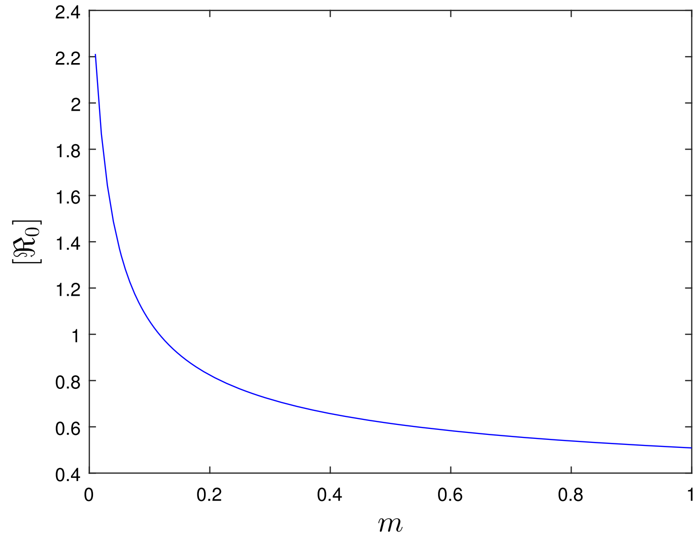

FIG. 7. The influence of n on $[\Re_0]$.

densities of infected mosquitoes and infected humans decay to zero. Figure 4 demonstrates the densities of susceptible mosquitoes, infected mosquitoes and infected humans will attain a positive level.

5.2. The influence of parameters on $[\Re_0]$

To analyze the effects of the parameter values on $[\Re_0]$, we perform sensitivity analysis to check the effects of the parameter values on $[\Re_0]$ by Latin Hypercube Sampling and partial rank correlation coefficient (PRCC) method (see, for example, [4,11]). Under the setting that μ, d_m, ρ and β_1 are changed concomitantly, we can observe the dependence of $[\Re_0]$ on parameters μ, d_m, ρ and β_1, respectively. Specifically, numeric plots in Fig. 5 indicate that $[\Re_0]$ is a monotonically increasing function with respect to μ and β_1, while $[\Re_0]$ is a monotonically decreasing function of d_m and ρ, respectively. Figure 6 demonstrates that $[\Re_0]$ is more sensitive to μ and β_1.

We next investigate the influence of $\beta_1(a)$ on $[\Re_0]$. As pointed in [24], the smaller the age of infection, the smaller transmission rate $\beta_1(a)$ of the disease. The rate of infection increases along with the infectious age. When the age of infection is very large, the infection rate is reduced to zero due to the loss of infectivity. Therefore, we artificially select the following form of β_1,

$$\beta_1(a) = 0.3 + 0.63ae^{-n(a-10)^2}, \ n \in (0, 1].$$

It can be observed from Fig. 7 that $[\Re_0]$ decreases monotonically as n increases.

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Data Availability The datasets generated and/or analyzed during the current study are not publicly available but are available from the authors on reasonable request.

Declarations

Conflict of interest The authors declare that they have no competing interests.

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