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Dynamics of a diffusion epidemic SIRI system in heterogeneous environment

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Abstract. This paper studies the dynamical behaviors of a diffusion epidemic SIRI system with distinct dispersal rates. The overall solution of the system is derived by using L^p theory and the Young's inequality. The uniformly boundedness of the solution is obtained for the system. The asymptotic smoothness of the semi-flow and the existence of the global attractor are discussed. Moreover, the basic reproduction number is defined in a spatially uniform environment and the threshold dynamical behaviors are obtained for extinction or continuous persistence of disease. When the spread rate of the susceptible individuals or the infected individuals is close to zero, the asymptotic profiles of the system are studied. This can help us to better understand the dynamic characteristics of the model in a bounded space domain with zero flux boundary conditions.

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1. Introduction

In recent years, the epidemic systems have attracted more and more attention from mathematical biologists. The epidemic systems can not only describe phenomena in real life, but also help people understand the natural world. Particularly, understanding of dynamic behavior can provide some guidance for some control of infectious diseases, such as SARS in 2003, H1N1 in 2010, Ebola in 2016, and COVID-19 in 2019 [1]. The established epidemic mathematical systems have played a significant role in the prediction and control of infectious diseases. Many control strategies have been developed for different control objectives, including harvesting control [2–4], threshold control [5,6] and impulsive control [7,8]. It is known that the disease transmission is often accompanied by a diffusion process in practical cases. According to different routes of disease transmission and related characteristics of phytopathology, some different forms of system have been established, such as hyper-infectivity diffusion epidemic system [9], age-structure diffusion epidemic system [10,11], multiple infection stages diffusion epidemic system [12,13], spatial heterogeneity diffusion epidemic system [14] and so on.

All the coefficients in most of the above-mentioned systems are constants. However, in real world, the spread of diseases are significantly affected by various environmental factors, for example, spatial position, water resource, temperature and so on. The host movement and spatial heterogeneity can also affect the spatial spreading of disease, and this requires a hybrid dynamic system. These two factors related to host–pathogen interactions seem to have received a little attention. Therefore, in recent years, by considering the above factors, many scholars have investigated the dynamic behavior of a reaction diffusion disease model with a environmental heterogeneity, to better control disease transmission.

Recently, by considering distinct dispersal rates of susceptible population and infected population, the dynamics of infectious disease system has been studied to show some interesting results. For example, Allen et al. [15] investigated the roles of diffusion system with spatial heterogeneity and the dynamics of the disease system. The research work in [15] showed that if the spatial environmental factors in the model need to be modified, low-risk locations can be selected, and further analysis can be achieved by limiting the range of spatial activities of susceptible people to eliminate the spread of infectious diseases. Subsequently, in [16,17], the asymptotic dynamical behaviors of the diffusion endemic system were studied. Especially, the results of [16] indicate that artificial control of the area of activity of infected individuals cannot effectively eradicate the spread of infectious diseases, and the diffusion rates of susceptible individuals and infectious individuals play different roles in determining the disease dynamics. Furthermore, by assuming that the disease transmission rate and recovery rate in the system are both spatiotemporal variables and continuous functions of time periodicity, it was shown in [18] that the spatial heterogeneity in the system and the temporal periodicity could improve the persistence of infectious diseases. Recently, Peng et al. [17] studied a linear source diffusion model with spatially heterogeneous environment, indicating that a varying total population can enhance persistence of infectious disease. Wu et al. [19] considered a diffusive host–pathogen model with heterogeneous parameters and distinct dispersal rates for the susceptible and infected hosts. Meanwhile, with considering the standard incidence infection mechanism, some diffusion epidemic models have been also studied. For example, it was found that controlling the diffusion rate of the susceptible individuals could help to control the disease, while controlling the diffusion rate of the infectious individuals could not eliminate the disease [20]. When considering total population varying in contrast, Li et al. [21] analyzed an spatial reaction diffusive SIS model under linear source. Zhu et al. [22] studied a spontaneous infection reaction diffusive SIS epidemic model under logistic source. Zhu et al. [23] discussed a spontaneous infection reaction diffusive SIS epidemic model with linear source in spatially heterogeneous environment. The asymptotic profiles of endemic steady state were investigated when the diffusion rate of susceptible population and infected population was small or large [21–23]. Recently, Li et al. [24] explored model in [21] to be one with logistic source. The main results in [21,24] showed that varying total population can enhance the persistence of infectious disease if the diffusion rate is large or small. When considering spontaneous social infection and disease transmission, Tong and Lei [25] extended the diffusive SIS model in [15] by adding the effect of spontaneous infection and investigated the asymptotic profiles of endemic steady state. Spontaneous social infection is an infection mechanism that differs from other disease transmission. When considering spontaneous social infection with linear source, a recent work [23] further investigated the effects of the movement and spatial heterogeneity on disease transmission. Wang et al. [26] studied a reaction diffusion cholera model with distinct dispersal rates in the human population. As far as we know, the research on a diffusion SIRI epidemic system with the $\frac{\beta_2(x)SR}{S+R+I}$ function in heterogeneous environment and linear source has not been found in existing works.

In real life, the rapid spread of the disease is mainly caused by the close contact with infected individuals and susceptible individuals, and by the large-scale flow of infected individuals and susceptible individuals in society. However, it is noted that the recovered individuals are those who are undergoing treatment or are still in the recovery phase, and their own vitality has been greatly reduced. At the same time, to minimize the impact of disease transmission, the control measures are taken for the recovered individuals, for example, centralized quarantine, home quarantine, etc. This implies the mobility of the recovered individuals in the society almost is zero. Therefore, based on this practical measure, we let the diffusion coefficient of the recovered individuals be zero.

Motivated by the above discussions, the main purpose of this paper is to perform the dynamic analysis of a reaction–diffusion epidemic model with distinct dispersal rates. Currently, many forms of incidence functions for reaction–diffusion epidemic model have been developed, including [27–43]. However, these models did not include the class of recovered individuals and ignored the movement of recovered (latently infected) individuals. For some epidemic diseases, the infected individuals can recover incubation before showing symptoms. The track of the recovered individuals with no symptoms can spread the disease, which makes the disease harder to be controlled. Therefore, it seems imperative to include the recovered subclass and explore the influences of recovered individuals movement on disease spread. In addition, the host movement is taken into consideration. Thus, in this paper, we design a reaction–diffusion epidemic model (2.1) with linear source. The main contributions of this paper include three points:

1. Using L^p theory and the Young's inequality, we give the overall solution for the model and obtain the uniformly boundedness of the solution. Using linear differential operator, we discuss the asymptotic smoothness of the semi-flow and the existence of the global attractor.
2. We define the basic reproduction number $\mathcal{R}_0$ to spread the disease model in a spatially uniform environment and obtain the threshold dynamics of epidemic system for extinction or continuous persistence of disease.
3. If the susceptible diffusion rate or the infected diffusion rate is close to zero, we study the asymptotic profiles of the system using the principle eigenvalue method. We show that the recovered individuals can eliminate susceptible individuals by restricting movement, while limiting the mobility of infected hosts.

This paper is organized as follows. In Sect. 3, the well-posedness of (2.1) is established, and the existence of global compact attractor of (2.1) is discussed. The $\mathcal{R}_0$ of reaction–diffusion epidemic (2.1) is given in Sect. 4. In Sect. 5, the threshold dynamics of (2.1) are investigated. The asymptotic profiles of (2.1) are considered in Sect. 6, and the last section gives some discussions.

Notations: Denote $\mathscr{X}_+ := C(\bar{\Omega}, R_+^3)$ as a positive cone. For $1 < p < \infty$, $L_p(\Omega)$ is the Banach space of functions u, and the p^{th} power of the absolution value is integrable on domain Ω. $||u||_{L^p} = \left(\int_\Omega |u|^p \mathrm{d}u\right)^{\frac{1}{p}}, 1 < p < \infty; ||u||_{L^\infty} = ess\sup|u|, p = +\infty$. Young's inequality: $ab < \epsilon a^p + \epsilon^{\frac{-q}{p}} b^q$. In addition, for convenience, we use the following notations throughout this paper

$$\overline{h} = \max_{x\in\Omega} h(x) \text{ and } \underline{h} = \min_{x\in\Omega} h(x),$$

where $h(x) = A(x), \alpha(x), \beta_1(x), \beta_2(x), \gamma(x), c(x), m(x)$.

2. Model description

In this paper, we consider a reaction–diffusion epidemic model with distinct dispersal rates

$$\begin{cases} S_t = d_S\Delta S + A(x) - \alpha(x)S - \frac{\beta_1(x)SI}{S+I+R} - \frac{\beta_2(x)SR}{S+I+R}, & x\in\Omega, t>0, \\ I_t = d_I\Delta I + \frac{\beta_1(x)SI}{S+I+R} + \frac{\beta_2(x)SR}{S+I+R} - \gamma(x)I, & x\in\Omega, t>0, \\ R_t = c(x)I - m(x)R, & x\in\Omega, t>0, \\ \frac{\partial S}{\partial \mathbf{n}} = \frac{\partial I}{\partial \mathbf{n}} = 0, & x\in\partial\Omega, \\ S(x,0) = S_0(x), I(x,0) = I_0(x), R(x,0) = R_0(x), & x\in\Omega, \end{cases} \tag{2.1}$$

where S, I and R represent the density of the susceptible individuals, the infected individuals and the recovered individuals, respectively. $d_S > 0$ denotes measuring the mobility of the susceptible individuals. $d_I > 0$ is measuring the mobility of the infected individuals. $A(x)$ represents the recruitment rate of the susceptible individuals. $\alpha(x)$ and $m(x)$ denote the natural death rate of the susceptible individuals and recovered individuals, respectively. $\gamma(x)$ denotes the remove rate of the infected individuals. $c(x)$ represents the shedding rate of the recovered individuals from infected individuals. $\frac{\beta_1(x)SI}{S+I+R}$ is the function for indirect transmission between the susceptible individuals and infected individuals. $\frac{\beta_2(x)SR}{S+R+I}$ represents the function for indirect transmission between the susceptible individuals and recovered individuals.

In addition, the positive coefficients $A(x), \alpha(x), \beta_1(x), \beta_2(x), \gamma(x), c(x)$ and $m(x)$ in (2.1) are continuous, strictly and uniformly bounded on Ω. For the smooth boundary $\partial\Omega$, the habitat $\Omega \subset \mathbb{R}_n$ represents a bounded domain. $\frac{\partial}{\partial \mathbf{n}}$ denotes the derivative along the outward normal $\mathbf{n}$. $(S_0(x), I_0(x), R_0(x)) > 0$, $x \in \Omega$ represents the initial data of the system. More detailed explanations on the parameters can be found in [26,44,45], and the references therein.

3. Well-posedness of system (2.1).

In this section, a unique global positive solution of diffusion system (2.1) is given and a compact global attractor of system (2.1) is also obtained.

3.1. Existence analysis of the global solution for (2.1)

Now, we will give the existence of the global solution of reaction–diffusion system (2.1) by using some inequality techniques.

Lemma 3.1. *Let the solution $(S, I, R) \in \Omega \times [0, +\infty)$ of system (2.1) start from any initial data $u_0 = (S_0(x), I_0(x), R_0(x)) \in \mathscr{X}_+$. Then, system (2.1) has a unique positive global solution.*

Proof. We prove the validity of Lemma 3.1 in the following three steps.

Step 1. We prove that S of system (2.1) is a unique positive global solution.

Take the solution (S, I, R) of reaction–diffusion system (2.1) starting from the initial data u_0. From the S equation of reaction–diffusion system (2.1), we know that $\frac{\partial S}{\partial t} < d_S \Delta S + A(x) - \alpha(x)S$. Then, we can rewrite it as the following system

$$\begin{cases} z_t = d_S \Delta z + A(x) - \alpha(x)z, & (x,t) \in \Omega \times (0, +\infty), \\ \frac{\partial z}{\partial \mathbf{n}} = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Obviously, system (3.1) admits a unique globally asymptotically stable positive steady state. Using the Comparison Theorem, and choosing $U = \limsup_{t\to\infty} z$, we know that

$$\limsup_{t\to\infty} S \le U, \text{ uniformly for } x \in \Omega.$$

Therefore, there exists $\mathscr{D} > 0$ depending on initial data such that

$$||S|| < \mathscr{D}, \;\; t > 0 \tag{3.2}$$

holds, then, we know that S of reaction–diffusion system (2.1) is a unique positive global solution for time $t > 0$.

Step 2. We prove that I of (2.1) is a unique positive global solution for $t > 0$.

By the second equation of (2.1), through analysis, it is difficult for us to directly find the solution I. Then, we denote $\mathcal{T}_2(t)$ as the semigroup generated by $d_I\Delta - \gamma(x)$ in $C(\Omega)$. Using Lemma 7.1 in "Appendix A", we have

$$I = \mathcal{T}_2(t)I_0(x) + \int_0^t \mathcal{T}_2(t-\mu)\left[\frac{\beta_1(x)S(x,\mu)I(x,\mu)}{S(x,\mu)+I(x,\mu)+R(x,\mu)} + \frac{\beta_2(x)S(x,\mu)R(x,\mu)}{R(x,\mu)+S(x,\mu)+I(x,\mu)}\right]\mathrm{d}\mu.$$

Taking norm computation, we know that there is a number $\beta = \max\{\overline{\beta_1}, \overline{\beta_2}\}$, such that

$$||I|| \le e^{-\lambda t}||I_0(x)|| + \beta \int_0^t e^{-\lambda(t-\mu)}(||I(x,\mu)|| + ||R(x,\mu)||)\mathrm{d}\mu, \tag{3.3}$$

where the positive number λ denotes the principal eigenvalue of $-d_I\Delta + \gamma(x)$.

In addition, from (2.1), by Lemma 7.1 in "Appendix A", we get

$$R = e^{-m(x)t}R_0(x) + c(x)\int_0^t \exp\{-m(x)(t-\mu)\}I(x,\mu)\mathrm{d}\mu.$$

We set a positive number $m = \min\left\{\frac{\lambda}{3}, \underline{m}\right\}$ to satisfy

$$||R|| \leq e^{-mt}||R_0|| + \overline{c}\int_0^t e^{-m(t-\mu)}||I(x,\mu)||\mathrm{d}\mu. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\begin{aligned}
||I|| &\leq e^{-\lambda t}||I_0|| + \beta\int_0^t e^{-\lambda(t-\mu)}[||I(x,\mu)|| + (e^{-m\mu}||R_0|| + \overline{c}\int_0^t e^{-m(\mu-\theta)}||I(x,\theta)||\mathrm{d}\theta)]\mathrm{d}\mu \\
&= e^{-\lambda t}||I_0|| + \beta\int_0^t e^{-\lambda(t-\mu)}||I(x,\mu)||\mathrm{d}\mu \\
&\quad + \beta||R_0||\int_0^t e^{-\lambda(t-\mu)}e^{-m\mu}\mathrm{d}\mu + \beta\overline{c}\int_0^t e^{-\lambda(t-\mu)}\int_0^t e^{-m(\mu-\theta)}||I(x,\theta)||\mathrm{d}\theta\mathrm{d}\mu \\
&\leq ||I_0|| + \beta\int_0^t e^{-\lambda(t-\mu)}||I(x,\mu)||\mathrm{d}\mu + \beta||R_0||\int_0^t e^{-m\mu}\mathrm{d}\mu \\
&\quad + \beta\overline{c}e^{-\lambda t}\int_0^t e^{m\theta}||I(x,\theta)||\int_0^t e^{-\mu m+\lambda\mu}\mathrm{d}\mu\mathrm{d}\theta.
\end{aligned}$$

Since

$$\int_0^t e^{-\lambda(t-\mu)}\int_0^t e^{-m(\mu-\theta)}||I(x,\theta)||\mathrm{d}\theta\mathrm{d}\mu = e^{-\lambda t}\int_0^t e^{m\theta}||I(x,\theta)||\int_0^t e^{\lambda\mu-m\mu}\mathrm{d}\mu\mathrm{d}\theta,$$

we have

$$\begin{aligned}
||I|| &\leq ||I_0|| + \beta\int_0^t e^{-\lambda(t-\mu)}||I(x,\mu)||\mathrm{d}\mu + \beta||R_0||\mathcal{K} + \frac{\beta\overline{c}e^{-mt}\int_0^t e^{m\theta}||I(x,\theta)||\mathrm{d}\theta}{\lambda - m} \\
&\leq ||I_0|| + \beta\int_0^t e^{-\lambda(t-\mu)}||I(x,\mu)||\mathrm{d}\mu + \frac{\beta||R_0||}{m} + \frac{\beta\overline{c}}{\lambda-m}\int_0^t e^{-m(t-\mu)}||I(x,\mu)||\mathrm{d}\mu \\
&< C_1 + C_2\int_0^t ||I(x,\mu)||\mathrm{d}\mu,
\end{aligned} \tag{3.5}$$

where $\mathcal{K} = \frac{1-e^{-mt}}{m}, C_1 = ||I_0|| + \frac{\beta||R_0||}{m}$ and $C_2 = \beta + \frac{\overline{c}\beta}{\lambda-m}$. Meanwhile, we know that $\lambda > m$ holds. Further, using Gronwall's inequality, we obtain

$$||I|| < C_1e^{C_2t}, t > 0. \tag{3.6}$$

Then, the solution I of (2.1) is a unique positive global solution for $t > 0$.

Step 3. We prove that the solution R of (2.1) is a unique positive global solution for $t > 0$.

From (3.4) and (3.5), we know that

$$||R|| \leq e^{-mt}||R_0|| + \frac{\overline{c}C_1e^{C_2t}(1-e^{-mt})}{m}, t > 0. \tag{3.7}$$

Then, for $t > 0$, we can obtain that R is a unique positive global solution. Consequently, based on (3.2), (3.6),(3.7) and Lemma 7.1 in "Appendix A", let $(S_0(x), I_0(x), R_0(x)) \in \mathscr{X}_+$ be any initial value of (2.1), then the solution $(S, I, R) \in \Omega \times [0, +\infty)$ of (2.1) is a unique positive global solution. This proof is finished. □

3.2. Uniform boundedness of solution

In order to better analyze the dynamics of system (2.1), we give the following Lemma.

Lemma 3.2. *For any solution $(S, I, R) \in \mathscr{X}_+$ of reaction–diffusion system (2.1) starting from the initial condition $u_0 = (S_0(x), I_0(x), R_0(x)) \in \mathscr{X}_+$, the following inequality*

$$||S||_{L^\infty(\Omega)} + ||I||_{L^\infty(\Omega)} + ||R||_{L^\infty(\Omega)} \leq \mathscr{M}^{**} \tag{3.8}$$

holds, where $\mathscr{M}^{**} > 0$ is independent of $u_0 \in \mathscr{X}_+$.

Proof. We will prove this Lemma in two steps.

Step 1. We prove that the solution of reaction–diffusion system (2.1) satisfies the L^1 bounded estimate. By using above Lemma 3.1, it is easy to know that there exists $\mathscr{M}_0 = ||U(x)||$ such that

$$||S||_{L^1(\Omega)} \leq \mathscr{M}_0, \tag{3.9}$$

where $\mathscr{M}_0$ is a positive function independent of the initial condition $(S_0(x), I_0(x), R_0(x)) \in \mathscr{X}_+$.

In addition, adding the first equation to the second equation of reaction–diffusion system (2.1), integrating all equations, and using the divergence theorem, we can get

$$\frac{\partial}{\partial t}\int_\Omega N\mathrm{d}x = \int_\Omega A(x)\mathrm{d}x - \int_\Omega \alpha(x)S\mathrm{d}x - \int_\Omega \gamma(x)I\mathrm{d}x \leq |\Omega|\overline{A} - \mathscr{H}\int_\Omega N\mathrm{d}x,$$

where $S + I = N$, $\mathscr{H}$ denotes $\min\{\alpha(x), \gamma(x) : x \in \Omega\}$ and $|\Omega|$ denotes the volume of Ω. Then, we have

$$||S||_{L^1(\Omega)} + ||I||_{L^1(\Omega)} \leq \frac{|\Omega|\overline{A}}{\mathscr{H}} + e^{-\mathscr{H}t}\int_\Omega N_0\mathrm{d}x,$$

where $S_0(x) + I_0(x) = N_0$. For the convenience of calculation, we set $\mathscr{M}_{11} = \frac{|\Omega|\overline{A}}{\mathscr{H}} + e^{-\mathscr{H}t}\int_\Omega N_0\mathrm{d}x$, which is a positive function independent of $(S_0(x), I_0(x), R_0(x)) \in \mathscr{X}_+$. Thus, we know that (S, I) of (2.1) fulfills the L^1 bounded estimate.

Now, we will prove that R of (2.1) satisfies the L^1 bounded estimate.

From the third equation of reaction–diffusion system (2.1), we can obtain

$$\frac{\partial}{\partial t}\int_\Omega R\mathrm{d}x = \int_\Omega [c(x)I - m(x)R]\mathrm{d}x < \overline{c}\mathscr{M}_{11} - \underline{m}\int_\Omega R\mathrm{d}x.$$

Then, there exists a positive function $\mathscr{M}_{12} := \frac{\overline{c}\mathscr{M}_{11}}{\underline{m}} + e^{-\underline{m}t}\int_\Omega R_0\mathrm{d}x$ such that

$$||R||_{L^1(\Omega)} < \mathscr{M}_{12}. \tag{3.10}$$

Hence, we set $\mathscr{M}_1 = \max\{\mathscr{M}_{11}, \mathscr{M}_{22}\}$, which is a positive function independent of $(S_0(x), I_0(x), R_0(x)) \in \mathscr{X}_+$, then we have

$$||S||_{L^1(\Omega)} + ||I||_{L^1(\Omega)} + ||R||_{L^1(\Omega)} < \mathscr{M}_1,$$

thus, the solution of (2.1) satisfies the L^1 bounded estimate.

Step 2. We prove that (S, R, I) of (2.1) fulfills the L^{2^i} bounded estimate.

For $i > 0$, there exists $\mathscr{M}_{2^i} > 0$ such that the solution (S, I, R) satisfies

$$||I||_{L^{2^i}(\Omega)} + ||R||_{L^{2^i}(\Omega)} \le \mathscr{M}_{2^i}, \forall t > T \tag{3.11}$$

for $T > 0$, where $\mathscr{M}_{2^i} > 0$ independent of $(S_0(x), I_0(x), R_0(x)) \in \mathscr{X}_+$.

We will prove that (3.11) holds by using the method of Mathematical Induction.

(1). If $i = 0$, it is easy to know that (3.11) holds.

(2). If $i = k-1$, (3.11) holds. Then, there are T and $\mathscr{M}_{2^{k-1}} > 0$ such that

$$||I||_{L^{2^{k-1}}(\Omega)} + ||R||_{L^{2^{k-1}}(\Omega)} < \mathscr{M}_{2^{k-1}}, \forall t > T. \tag{3.12}$$

Next, we need to verify (3.11) holds when $i = k$. Multiplying the second equation of reaction–diffusion system (2.1) by I^{2^k-1} and integrating over region Ω, we can get

$$\begin{aligned}\frac{1}{2^k}\frac{\partial}{\partial t}\int\limits_\Omega I^{2^k}\mathrm{d}x = d_I\int\limits_\Omega I^{2^k-1}\Delta I\mathrm{d}x + \int\limits_\Omega \beta_1(x)SI\frac{I^{2^k-1}}{S+I+R}\mathrm{d}x\\ + \int\limits_\Omega \beta_2(x)SR\frac{I^{2^k-1}}{R+S+I}\mathrm{d}x - \int\limits_\Omega \gamma(x)I^{2^k}\mathrm{d}x.\end{aligned}$$

In addition, based on Ref. [39], it is easy to show that

$$\begin{aligned}d_I\int\limits_\Omega I^{2^k-1}\Delta I\mathrm{d}x &= -d_I\int\limits_\Omega(\nabla I)(\nabla I^{2^k-1})\mathrm{d}x = -(2^k-1)d_I\int\limits_\Omega(\nabla I\nabla I)I^{2^k-2}\mathrm{d}x\\ &= -\left(\frac{2^k-1}{2^{2^k-2}}d_I\right)\int\limits_\Omega|\nabla I^{2^k-1}|^2\mathrm{d}x.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{2^k}\frac{\partial}{\partial t}\int\limits_\Omega I^{2^k}\mathrm{d}x = -\mathscr{D}_k\int\limits_\Omega|\nabla I^{2^k-1}|^2\mathrm{d}x + \int\limits_\Omega\beta_1(x)\frac{S}{S+I+R}I^{2^k}\mathrm{d}x\\ + \int\limits_\Omega\beta_2(x)\frac{S}{S+R+I}RI^{2^k-1}\mathrm{d}x - \int\limits_\Omega\gamma(x)I^{2^k}\mathrm{d}x,\end{aligned} \tag{3.13}$$

where $\mathscr{D}_k = \frac{2^k-1}{2^{2^k-2}}d_I$.

By system (2.1), there exists time $t_0 > 0$ such that

$$\int\limits_\Omega\beta_1(x)\frac{S}{S+I+R}I^{2^k}\mathrm{d}x \le \overline{\beta_1}\int\limits_\Omega I^{2^k}\mathrm{d}x, \text{ for } t > t_0$$

and

$$\int\limits_\Omega\beta_2(x)\frac{S}{S+R+I}I^{2^k-1}R\mathrm{d}x \le \overline{\beta_2}\int\limits_\Omega RI^{2^k-1}\mathrm{d}x, \text{ for } t > t_0. \tag{3.14}$$

Using Young's inequality: $ab < \epsilon a^p + \epsilon^{\frac{-q}{p}}b^q$, where $a, b, \epsilon > 0$, $\frac{1}{p} + \frac{1}{q} = 1$. In this paper, we choose $p = 2^k$ and $q = \frac{2^k}{2^k-1}$, then we have

$$\int\limits_\Omega RI^{2^k-1}\mathrm{d}x < \epsilon\int\limits_\Omega R^{2^k}\mathrm{d}x + C_{\epsilon_1}\int\limits_\Omega I^{2^k}\mathrm{d}x, \text{ for } t > t_0,$$

where $C_{\epsilon_1} := \epsilon^{\frac{-q}{p}} = \epsilon^{-\frac{1}{2^k-1}}$, we can estimate (3.14) by setting $\epsilon = \frac{m}{4\overline{\beta_2}}$.

Thus, (3.13) can be estimated by

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega I^{2^k}\mathrm{d}x < -\mathscr{D}_k\int_\Omega |\nabla I^{2^k-1}|^2\mathrm{d}x + \frac{m}{4}\int_\Omega R^{2^k}\mathrm{d}x + C_k\int_\Omega I^{2^k}\mathrm{d}x, \tag{3.15}$$

where $C_k = \overline{\beta_1} + \overline{\beta_2}C_{\epsilon_1}$.

Multiplying the R equation of reaction–diffusion model (2.1) by R^{2^k-1} and integrating over Ω, we can obtain

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega R^{2^k}\mathrm{d}x = \int_\Omega c(x)R^{2^k-1}I\mathrm{d}x - \int_\Omega m(x)R^{2^k}\mathrm{d}x < \overline{c}\int_\Omega R^{2^k-1}I\mathrm{d}x - \underline{m}\int_\Omega R^{2^k}\mathrm{d}x. \tag{3.16}$$

Again using Young's inequality, we have

$$\int_\Omega R^{2^k-1}I\mathrm{d}x < \frac{m}{4\overline{c}}\int_\Omega R^{2^k}\mathrm{d}x + C_{\epsilon_2}\int_\Omega I^{2^k}\mathrm{d}x,$$

where $C_{\epsilon_2} := \epsilon^{\frac{-q}{p}} = \epsilon^{-\frac{1}{2^k-1}}$. We set $\epsilon_2 = \frac{m}{4\overline{c}}, p = \frac{2^k}{2^k-1}$ and $q = 2^k$, then, (3.16) becomes

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega R^{2^k}\mathrm{d}x < \frac{m}{4}\int_\Omega R^{2^k}\mathrm{d}x + \overline{c}C_{\epsilon_2}\int_\Omega I^{2^k}\mathrm{d}x - \underline{m}\int_\Omega R^{2^k}\mathrm{d}x. \tag{3.17}$$

Combining (3.15) with (3.17) yields

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega (I^{2^k} + R^{2^k})\mathrm{d}x < -\mathscr{D}_k\int_\Omega |\nabla I^{2^k-1}|^2\mathrm{d}x + E_k\int_\Omega I^{2^k}\mathrm{d}x - \frac{m}{2}\int_\Omega R^{2^k}\mathrm{d}x \tag{3.18}$$

for $t > t_0$, where $E_k = C_k + \overline{c}C\epsilon_2$.

By using interpolation inequality, we know that for any $\epsilon > 0$, there is a positive number C_ϵ such that

$$||\xi||_2^2 < \epsilon||\nabla\xi||_2^2 + C_\epsilon||\xi||_1^2,$$

where $\xi \in W^{1,2}(\Omega)$. Let $\epsilon = \frac{\mathscr{D}_k}{2E_k}$, $\xi = I^{2^{k-1}}$, then

$$-\mathscr{D}_k\int_\Omega |\nabla I^{2^{k-1}}|^2\mathrm{d}x < -2E_k\int_\Omega I^{2^k}\mathrm{d}x + 2E_kC_{\epsilon_3}\left(\int_\Omega I^{2^k-1}\mathrm{d}x\right)^2.$$

Thus, inequality (3.18) becomes

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega (I^{2^k} + R^{2^k})\mathrm{d}x < \mathscr{L}\int_\Omega (I^{2^k} + R^{2^k})\mathrm{d}x + 2E_kC_{\epsilon_3}\left(\int_\Omega I^{2^k-1}\mathrm{d}x\right)^2 \text{ for } t > t_0,$$

where $\mathscr{L} = \min\{E_k, \frac{m}{2}\}$.

From (3.12), we have

$$\limsup_{t\to\infty}\int_\Omega I^{2^k-1}\mathrm{d}x < \mathscr{M}_{2^k-1}.$$

Then, we can further obtain

$$||I||_{L^{2^k}(\Omega)} + ||R||_{L^{2^k}(\Omega)} \le \mathscr{M}_{2^k},$$

where $\mathscr{M}_{2^k} = \sqrt[2^k]{\frac{2E_kC\epsilon_3\mathscr{M}_{2^k-1}}{\mathscr{L}}}$.

According to the continuous embedding $L^q(\Omega) \subset L^p(\Omega), q > p > 1$, for $p > 1$, there exists $\mathscr{M}_p > 0$ independent of the initial condition $(S_0(x), I_0(x), R_0(x)) \in \mathscr{X}_+$, such that

$$||I||_{L^p(\Omega)} + ||R||_{L^p(\Omega)} \le \mathscr{M}_p. \tag{3.19}$$

By using Lemma 2.4 in [38], there exists $\mathscr{M}^* > 0$ independent of $(S_0(x), I_0(x), R_0(x)) \in \mathscr{X}_+$ such that $||I||_{L^\infty(\Omega)} < \mathscr{M}^*$. Then, by (3.10), we have

$$||R||_{L^\infty(\Omega)} \le \frac{\overline{c}\mathscr{M}^*}{\underline{m}} + e^{-\underline{m}t}\overline{R_0}|\Omega| \tag{3.20}$$

and

$$||S||_{L^\infty(\Omega)} \le \mathscr{M}_0. \tag{3.21}$$

From Eq. (3.19), Eq.(3.20) and Eq.(3.21), there exists $\mathscr{M}^{**} > 0$ independent of $(S_0(x), I_0(x), R_0(x)) \in \mathscr{X}_+$, for (S, I, R) of reaction–diffusion system (2.1), we can obtain the following inequality

$$||S||_{L^\infty(\Omega)} + ||I||_{L^\infty(\Omega)} + ||R||_{L^\infty(\Omega)} \le \mathscr{M}^{**}.$$

Thus, Lemma 3.2 holds. This proof is completed. □

3.3. Asymptotic smoothness of solution semiflow

Note that there is no diffusion term with the third equation in reaction–diffusion system (2.1). This means that the solution map $\phi(t)$ is not compact. The compact nature of many solutions cannot be directly applied to system (2.1). To overcome this problem, we here discuss the asymptotic smoothness of solution semiflow of (2.1) in "Appendix B". Similar to Lemma 2.5 in [38], we have the following Lemmas.

Lemma 3.3. [38] *For time $t > 0$, any bounded set $\mathscr{B}$ belongs to $\mathscr{X}_+$, that is, $\mathscr{B} \subset \mathscr{X}_+$ and set $S := \left\{\int_0^t \exp\{-m(x)(t-\mu)\}c(x)I(x,\mu;u_0)\mathrm{d}\mu : u_0 \in \mathscr{B}\right\}$, then set S is precompact in $C(\Omega)$ where $u_0 = (S_0(x), I_0(x), R_0(x))$.*

Lemma 3.4. *For $t > 0$, there is the semiflow generated $\phi(t)$ of the solution of reaction–diffusion system (2.1) such that $\phi(t) : \mathscr{X}_+ \to \mathscr{X}_+$. Then, $\phi(t)$ is a κ-contraction.*

Proof. For $t > 0$, we consider the following the semiflow generated $\phi(t)$:

$$\phi(t) = \phi_1(t) + \phi_2(t),$$

where

$$\phi_1(t)u_0 = \left\{S(x,t;S_0), I(x,t;I_0), \int_0^t e^{-m(x)(t-\mu)}c(x)I(x,\mu;I_0)\mathrm{d}\mu\right\}$$

and

$$\phi_2(t)u_0 = \{0, 0, \exp\{-m(x)t\}R_0(x)\}.$$

Then, using Lemma 3.3, for any $t > 0$, we can show that $\phi_1(t)\mathscr{B}$ is precompact. Hence, $\kappa(\phi_1(t)\mathscr{B}) = 0$.

Moreover, we consider the following the operator norm of the semiflow generated $\phi_2(t)$, which can be estimated as

$$||\phi_2(t)|| = \sup_{\psi\in\mathscr{X}_+} \frac{||\phi_2(t)\psi||\mathscr{X}_+}{||\psi||\mathscr{X}_+} < \exp\{-m(x)t\} \sup_{\psi\in\mathscr{X}_+} \frac{||\psi||\mathscr{X}_+}{||\psi||\mathscr{X}_+} < \exp\{-\underline{m}t\}.$$

For $t > 0$, we have

$$\kappa(\phi(t)\mathscr{B}) < \exp\{-\underline{m}t\}\kappa(\mathscr{B}).$$

Hence, the semi-flow generated $\phi(t)$ is a κ-contraction on the domain $\mathscr{X}_+$, which is the contraction function $\exp\{-\underline{m}t\}$. This proof is completed. □

Similar to Lemma 2.7 in [38], we have

Lemma 3.5. *System* (2.1) *admits a connected global attractor in the domain* $\mathscr{X}_+$.

4. Steady state and $\mathcal{R}_0$ of (2.1)

Next, we derive $\mathcal{R}_0$ and the steady state of (2.1) with distinct dispersal rates. From (2.1), we know that the steady state fulfills the following system

$$\begin{cases} d_S\Delta S + A(x) - \alpha(x)S - \frac{\beta_1(x)SI}{S+I+R} - \frac{\beta_2(x)SR}{S+R+I} = 0, & x\in\Omega, t>0,\\ d_I\Delta I + \frac{\beta_1(x)SI}{S+I+R} + \frac{\beta_2(x)SR}{S+R+I} - \gamma(x)I = 0, & x\in\Omega, t>0,\\ c(x)I - m(x)R = 0, & x\in\Omega, t>0,\\ \frac{\partial S}{\partial \mathbf{n}} = \frac{\partial I}{\partial \mathbf{n}} = 0, & x\in\partial\Omega. \end{cases} \tag{4.1}$$

From (4.1), we can know that (2.1) has a unique positive disease-free steady state (DFSS)$Q_0 = (U, 0, 0)$. In addition, (2.1) has a positive endemic steady state (PESS) $Q_1 = (S_{**}, I_{**}, R_{**})$.

Based on Ref. [31], we linearize (2.1) at the disease free steady state Q_0, then

$$\begin{cases} S_t = d_S\Delta S - \alpha(x)S - \beta_1(x)I - \beta_2(x)R, & x\in\Omega, t>0,\\ I_t = d_I\Delta I + \beta_1(x)I + \beta_2(x)R - \gamma(x)I, & x\in\Omega, t>0,\\ R_t = c(x)I - m(x)R, & x\in\Omega, t>0,\\ \frac{\partial S}{\partial \mathbf{n}} = \frac{\partial I}{\partial \mathbf{n}} = 0, & x\in\partial\Omega. \end{cases} \tag{4.2}$$

Observe that the second and third equations (4.2) do not contain variable S, then a system is considered as follows

$$\begin{cases} I_t = d_I\Delta I + \beta_1(x)I + \beta_2(x)R - \gamma(x)I, & x\in\Omega, t>0,\\ R_t = c(x)I - m(x)R, & x\in\Omega, t>0,\\ \frac{\partial I}{\partial \mathbf{n}} = 0, & x\in\partial\Omega. \end{cases} \tag{4.3}$$

Now, denote $\mathcal{T}(t)\phi = (I(x,t,\varphi), R(x,t,\varphi))$ as the solution semiflow of (4.3) for $\varphi\in C(\Omega,\mathbb{R}^2)$. Since (4.3) and Ref. [33], $\mathcal{T}(t)$ is a positive C_0-semigroup with generator. Then, we choose a semigroup generated $\mathcal{T}(t)$ in $\mathscr{B}$, where

$$\mathscr{B} := \begin{pmatrix} d_I\Delta + \beta_1(x) - \gamma(x) & \beta_2(x)\\ c(x) & -m(x)\end{pmatrix} = \begin{pmatrix} d_I\Delta - \gamma(x) & 0\\ c(x) & -m(x)\end{pmatrix} + \begin{pmatrix} \beta_1(x) & \beta_2(x)\\ 0 & 0\end{pmatrix} = B + F.$$

Let $\mathcal{L} := -FB^{-1}$ be the generation operator, we can obtain

$$\mathcal{L}\phi(x) = \int_0^\infty F(x)K\mathrm{d}t = F(x)\int_0^\infty K\mathrm{d}t,$$

where $K = \mathcal{T}(t)\phi(x)$, $\phi\in C(\Omega,\mathbb{R}^2), x\in\Omega$. Then, the $\mathcal{R}_0$ of (2.1) is defined as

$$\mathcal{R}_0 := r(\mathcal{L}),$$

where $r(\mathcal{L}) = \sup\{|\lambda|\}$, λ is a part of $\sigma(\mathcal{L})$, $\sigma(\mathcal{L})$ denotes the spectrum of $\mathcal{L}$. Obviously, $\mathscr{B}$ is a resolvent-positive operator. Using Refs. [33,36], we have the follow Lemmas.

Lemma 4.1. *Let $s(\mathscr{B}) = \sup\{\mathrm{Re}\lambda\}$ be the spectral bound of the operators $\mathscr{B}$ for $\lambda \in \sigma(\mathscr{B})$. Then, $\mathcal{R}_0 - 1$ has the same sign as that of $s(\mathscr{B})$.*

Lemma 4.2. *Let $\overline{\lambda_0}$ be the principal eigenvalue of the problem*

$$\begin{cases} d_I \Delta \varphi - \gamma(x)\varphi + \overline{\lambda_0}\left(\beta_1(x) + \frac{\beta_2(x)c(x)}{m(x)}\right)\varphi = 0, & x \in \Omega, \\ \frac{\partial \varphi}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, \end{cases} \tag{4.4}$$

then $\mathcal{R}_0 = \frac{1}{\overline{\lambda_0}}$.

Proof. According to Theorem 3.3 in [36], express the following matrices

$$\mathcal{F} := \begin{pmatrix} \mathcal{F}_{11}, \mathcal{F}_{12} \\ \mathcal{F}_{21}, \mathcal{F}_{22} \end{pmatrix} \text{ and } \mathcal{V} := \begin{pmatrix} \mathcal{V}_{11}, \mathcal{V}_{12} \\ \mathcal{V}_{21}, \mathcal{V}_{22} \end{pmatrix},$$

where

$$\mathcal{F}_{11} := \beta_1(x), \mathcal{F}_{12} := \beta_2(x), \mathcal{F}_{21} := 0, \mathcal{F}_{22} := 0,$$
$$\mathcal{V}_{11} := \gamma(x), \mathcal{V}_{12} := 0, \mathcal{V}_{21} := -c(x), \mathcal{V}_{22} := m(x).$$

Due to $\mathcal{F}_{21} = 0, \mathcal{F}_{22} = 0$, based on Ref. [38], we can get that

$$\mathcal{R}_0 = r(-B^{-1}F) = r(-B_1^{-1}F_2),$$

where

$$B_1 := d_I\Delta - (\mathcal{V}_{11} - \mathcal{V}_{12}\mathcal{V}_{22}^{-1}\mathcal{V}_{21}) = d_I\Delta - \gamma(x)$$

and

$$F_2 := \mathcal{F}_{11} - \mathcal{F}_{12}\mathcal{V}_{22}^{-1}\mathcal{V}_{21} = \beta_1(x) + \frac{\beta_2(x)c(x)}{m(x)}.$$

Thus, for $\forall\varphi \in C(\Omega, \mathbb{R}^2)$, we can obtain

$$-B_1^{-1}F_2\varphi = -[d_I\Delta - \gamma(x)]^{-1}\left(\beta_1(x) + \frac{c(x)\beta_2(x)}{m(x)}\right)\varphi$$

and

$$\mathcal{R}_0 := r(\mathcal{L}) = r\left(-(d_I\Delta - \gamma(x))^{-1}(\beta_1(x) + \frac{c(x)\beta_2(x)}{m(x)})\right).$$

Furthermore, $\mathcal{R}_0$ satisfies

$$-[d_I\Delta - \gamma(x)]^{-1}\left(\beta_1(x) + \frac{c(x)\beta_2(x)}{m(x)}\right)\varphi = \mathcal{R}_0\varphi, \varphi \in C(\Omega, \mathbb{R}^2),$$

that is,

$$d_I\Delta\varphi - \gamma(x)\varphi + \left(\frac{\beta_1(x)m(x) + c(x)\beta_2(x)}{m(x)}\right)\frac{1}{\mathcal{R}_0}\varphi = 0, \varphi \in C(\Omega, \mathbb{R}^2).$$

Thus, Lemma 4.2 holds. □

From Lemma 4.2, based on Ref. [38], $\mathcal{R}_0$ is defined as

$$\mathcal{R}_0 = \frac{1}{\overline{\lambda_0}} = \sup_{\varphi \in H^1(\Omega), \varphi \neq 0}\left\{\frac{\int_\Omega \left(\beta_1(x) + \frac{c(x)\beta_2(x)}{m(x)}\right)\varphi^2 \mathrm{d}x}{\int_\Omega d_I|\nabla\varphi|^2 + \gamma\varphi^2 \mathrm{d}x}\right\}. \tag{4.5}$$

Equation (4.5) implies that $\mathcal{R}_0$ depends on the positive diffusion coefficient d_I (see Theorem 2 in [15] and [38]). Then, we are now ready to state the following main results.

Theorem 4.1. *Let* $\Theta(x) = \beta_1(x) + \frac{c(x)\beta_2(x)}{m(x)}$, *the following properties are satisfied.*

1. $\mathcal{R}_0$ *is decreasing in measuring the mobility* d_I *of the infected individuals with*

$$\lim_{d_I \to 0} \mathcal{R}_0 = \max\left\{\frac{\beta_1(x)}{\gamma(x)} + \frac{c(x)\beta_2(x)}{\gamma(x)m(x)} : x \in \Omega\right\}$$

and

$$\lim_{d_I \to +\infty} \mathcal{R}_0 = \frac{\int\limits_\Omega \Theta(x)\varphi^2 \mathrm{d}x}{\int\limits_\Omega \gamma(x)\varphi^2 \mathrm{d}x}.$$

2. *If domain* Ω *is a favorable environment for the recovered individuals in the situation, that is,*

$$\frac{\int\limits_\Omega \Theta(x)\varphi^2 \mathrm{d}x}{\int\limits_\Omega \gamma(x)\varphi^2 \mathrm{d}x} > 1,$$

then $\mathcal{R}_0 > 1$ *for all* $d_I > 0$.

3. *If domain* Ω *is a non-favorable environment for the recovered individuals in the situation, that is,*

$$\frac{\int\limits_\Omega \Theta(x)\varphi^2 \mathrm{d}x}{\int\limits_\Omega \gamma(x)\varphi^2 \mathrm{d}x} < 1,$$

and the condition $\frac{\beta_1(x)}{\gamma(x)} + \frac{c(x)\beta_2(x)}{\gamma(x)m(x)} > 1$, *then there is* d_I^* *such that* $\mathcal{R}_0 > 1$ *when* $d_I < d_I^*$, *and* $\mathcal{R}_0 < 1$ *when* $d_I > d_I^*$.

Lemma 4.3. *Let* $\mathscr{K}_0$ *be the principal eigenvalue of the eigenvalue problem*

$$\begin{cases} d_I\Delta\phi - \gamma(x)\phi + \Theta(x)\phi = \eta\phi, & x \in \Omega, \\ \frac{\partial\phi}{\partial\mathbf{n}} = 0, & x \in \partial\Omega, \end{cases} \tag{4.6}$$

then $\mathcal{R}_0 - 1$ *and* $s(\mathscr{B})$ *have the same sign as that of* $\mathscr{K}_0$, *where* $\Theta(x) = \beta_1(x) + \frac{\beta_2(x)c(x)}{m(x)}$.

Proof. To establish this Lemma 4.3, we can easily verify that there admits a least eigenvalue $\mathscr{K}_0$ of (4.6). Its corresponding φ can be chosen in domain Ω, that is,

$$d_I\Delta\varphi - \gamma(x)\varphi + \Theta(x)\varphi = \mathscr{K}_0\varphi, \quad \text{for all } x \in \Omega \tag{4.7}$$

and the other condition $\frac{\partial\varphi}{\partial\mathbf{n}} = 0$ for all variable $x \in \partial\Omega$, where $\Theta(x) = \beta_1(x) + \frac{\beta_2(x)c(x)}{m(x)}$.

In the following, we consider

$$d_I\Delta\phi - \gamma(x)\phi + \Theta(x)\frac{1}{\mathcal{R}_0}\phi = 0 \tag{4.8}$$

for all variable $x \in \Omega$ and the condition $\frac{\partial\phi}{\partial\mathbf{n}} = 0$ for all variable $x \in \partial\Omega$.

Multiplying (4.7) by ϕ and (4.8) by φ, integrating by parts on Ω, and subtracting the equation, yields

$$\left(1 - \frac{1}{\mathcal{R}_0}\right)\int\limits_\Omega \Theta(x)\phi\varphi \mathrm{d}x = \mathscr{K}_0 \int\limits_\Omega \phi\varphi \mathrm{d}x.$$

Observe that $\int\limits_\Omega \Theta(x)\phi\varphi \mathrm{d}x$ and $\int\limits_\Omega \phi\varphi \mathrm{d}x$ are both positive, the condition $(1 - \frac{1}{\mathcal{R}_0})$ and $\mathscr{K}_0$ have the same sign, which ensures that $\mathcal{R}_0 > 1$ if $\mathscr{K}_0 > 0$ and $\mathcal{R}_0 < 1$ if $\mathscr{K}_0 < 0$. The proof is completed. □

Choosing $I = e^{\lambda t}\psi_2$, $R = e^{\lambda t}\psi_3$ and taking it into (4.2), we get the following equations

$$\begin{cases} \lambda\psi_2 = d_I\Delta\psi_2 + \beta_1(x)\psi_2 + \beta_2(x)\psi_3 - \gamma(x)\psi_2, & x \in \Omega, \\ \lambda\psi_3 = c(x)\psi_2 - m(x)\psi_3, & x \in \Omega, \\ \frac{\partial\psi_2}{\partial \mathbf{n}} = 0, & x \in \partial\Omega. \end{cases} \tag{4.9}$$

Recall that when $\mathcal{R}_0 > 1$ and the characteristic value of the operators $s(\mathscr{B})$, the principal eigenvalue of $\mathscr{B}$ of (4.9) is discussed. From the above theorem, we obtain two important results for the existence of the principal eigenvalue of (4.9).

Lemma 4.4. *Suppose that $\mathcal{R}_0 > 1$, $s(\mathscr{B})$ is the principal eigenvalue of system (4.9).*

Proof. Let $\mathcal{L}_\lambda = d_I\Delta + \left(\beta_1(x) + \frac{c(x)\beta_2(x)}{m(x)+\lambda}\right) - \gamma(x)$ be a family of linear operators on domain $C(\Omega)$. We know that $s(\mathcal{L}_\lambda)$ is decreasing continuously. Observe that λ is principal eigenvalue problem of $\mathcal{L}_\lambda u = \lambda u$, then we have

$$s(\mathcal{L}_\lambda) = \sup_{\varphi\in H^1(\Omega),\varphi\neq 0} \left\{ \frac{\int_\Omega (\beta_1(x) + \frac{c(x)\beta_2(x)}{m(x)+\lambda} - d_I|\nabla\varphi|^2 - \gamma(x)\varphi^2)\mathrm{d}x}{\int_\Omega \varphi^2 \mathrm{d}x} \right\}.$$

It is clear that $s(\mathcal{L}_\lambda) < 0$ if λ is large. Indeed, by the condition $\mathcal{R}_0 > 1$ and using Lemma 4.3, $s(\mathcal{L}_0) = \mathscr{K}_0 > 0$ ensures there exists a unique λ such that $s(\mathcal{L}_\lambda) = \lambda$. Let $\psi > 0$ be an eigenvector associated with $s(\mathcal{L}_\lambda)$, then we have $\mathcal{L}_\lambda\psi = \lambda\psi$. By using Theorem 2.3 in [36], we have $\lambda = s(\mathscr{B})$. The proof is completed. □

5. Threshold dynamics of system (2.1)

Theorem 5.1. *If $\mathcal{R}_0 < 1$, then the disease-free steady state (DFSS) Q_0 of system (2.1) is globally asymptotically stable.*

Proof. We prove Theorem 5.1 in two steps.

Step 1. We prove that the DFSS Q_0 of (2.1) is locally stable.

Recall that Theorem 3.1 in [36] and Theorem 4.1 in [26], we can easily know that DFSS Q_0 of system (2.1) is locally stable.

Step 2. We prove that the point Q_0 of system (2.1) is global attractive.

To establish Step 2, we fix the positive solution R_0 of system (2.1). Since (3.1), there are a positive time t_1 and a positive function U such that the inequality $0 < S < U + \epsilon_0$ for all $t > t_1$. If $(I, R) < (z_1, z_2)$ on domain $\Omega\times[t_1, +\infty)$, using Comparison Principle, we know that (z_1, z_2) satisfies the following cooperative systems

$$\begin{cases} \frac{\partial z_1}{\partial t} = d_I\Delta z_1 + \frac{\beta_1(x)(U+\varepsilon_0)z_1}{U+\varepsilon_0+z_1+z_2} + \beta_2(x)\frac{(U+\varepsilon_0)z_2}{U+\varepsilon_0+z_2+z_1} - \gamma(x)z_1, & x \in \Omega, t > t_1, \\ \frac{\partial z_2}{\partial t} = c(x)z_1 - m(x)z_2, & x \in \Omega, t > t_1, \\ \frac{\partial z_1}{\partial \mathbf{n}} = 0, & x \in \partial\Omega. \end{cases} \tag{5.1}$$

Since $\frac{U+\varepsilon_0}{U+\varepsilon_0+z_1+z_2} < 1$, by using Comparison Principle again, a new system is obtained as

$$\begin{cases} \frac{\partial u_1}{\partial t} = d_I\Delta u_1 + \beta_1(x)u_1 + \beta_2(x)u_2 - \gamma(x)u_1 = 0, & x \in \Omega, t > t_1, \\ \frac{\partial u_2}{\partial t} = c(x)u_1 - m(x)u_2, & x \in \Omega, t > t_1, \\ \frac{\partial u_1}{\partial \mathbf{n}} = 0, & x \in \partial\Omega. \end{cases} \tag{5.2}$$

In order to prove Step 2, represent the linear semigroup induced of (5.2) by $\mathcal{T}_{\epsilon_0}(t)$, and the associated generator with (5.2) by $\mathscr{B}_{\epsilon_0}$. Indeed, we transform into prove that

$$||\mathcal{T}_{\varepsilon_0}(t)|| \le \mathscr{C}e^{\omega_{\epsilon_0}t}, \text{ for some } \mathscr{C} > 0,$$

where ω_{ϵ_0} is defined as

$$\omega_{\epsilon_0} := \lim_{t\to+\infty} \ln \frac{||\mathcal{T}_{\varepsilon_0}(t)||}{t}.$$

We set

$$\omega_{\epsilon_0} = \max\{s(B_{\epsilon_0}), \omega_{ess}(\mathcal{T}_{\varepsilon_0}(t))\},$$

where

$$\omega_{ess}(\mathcal{T}_{\varepsilon_0}(t)) := \lim_{t\to+\infty} \frac{\varrho(\mathcal{T}_{\varepsilon_0}(t))}{t},$$

where $\varrho(\cdot)$ is the measure of non-compactness.

Using Lemma 3.4, we know that $\omega_{ess}(\mathcal{T}_{\varepsilon_0}) < -m$. Then, there exists a sufficiently small number ϵ_0 such that $\omega_{\epsilon_0} < 0$. Thus, ω_{ϵ_0} has the same sign as $s(\mathscr{B}_{\epsilon_0})$ if $\omega_{ess}(\mathcal{T}_{\epsilon_0}) < -m$. Then, $s(\mathscr{B}_{\epsilon_0})$ has the same sign as $\mathscr{K}^0_{\epsilon_0}$. To this end, we consider

$$\begin{cases} d_I\Delta\phi - \gamma(x)\phi + \left(\beta_1(x) + \frac{\beta_2(x)c(x)}{m(x)}\right)\phi = \mathscr{K}^0\phi, & x \in \Omega, \\ \frac{\partial\phi}{\partial\mathbf{n}} = 0, & x \in \partial\Omega. \end{cases} \tag{5.3}$$

From Lemma 4.1 and $\mathcal{R}_0 < 1$, we know that $\mathscr{K}^0_\varepsilon$ continuously depends on ϵ_0, which implies that $\mathscr{K}^0_{\epsilon_0} < 0$ when ϵ_0 is small. Thus, we have $\omega_{\epsilon_0} < 0$, that is,

$$(u_1, u_2) \to (0, 0) \text{ uniformly for } x \in \Omega \text{ as } t \to +\infty.$$

Moreover, there exists a point $(0, 0)$ satisfying

$$(z_1, z_2) \to (0, 0) \text{ uniformly for } x \in \Omega \text{ as } t \to +\infty.$$

In other words, there exists a point $(0, 0)$ such that

$$I \to 0 \text{ and } R \to 0 \text{ uniformly for } x \in \Omega \text{ as } t \to +\infty.$$

By Eq.(3.1), we know there is the function $U(x)$ that fulfills

$$S \to U(x) \text{ uniformly for } x \in \Omega \text{ as } t \to +\infty.$$

Then, DFSS Q_0 of system (2.1) is globally attractive.

From Step 1 and Step 2, we can get that the point Q_0 of system (2.1) is globally asymptotically stable if $\mathcal{R}_0 < 1$. The proof is completed. □

Theorem 5.2. *For any $u_0 = (S_0(x), I_0(x), R_0(x)) \in \mathscr{X}_+$ with $I_0(x) \neq 0$ or $R_0(x) \neq 0$. If $\mathcal{R}_0 > 1$, then there is a positive number δ such that the solution (S, I, R) of (2.1) uniformly fulfills*

$$\liminf_{t\to+\infty} S \geq \delta, \ \liminf_{t\to+\infty} I \geq \delta, \ \liminf_{t\to+\infty} R \geq \delta,$$

for variable $x \in \Omega$. Namely, system (2.1) exists a positive endemic steady state (DFSS).

Proof. **To establish Theorem** 5.2, let

$$\mathscr{X}_0 := \{\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathscr{X}_+ : \varphi_2 \neq 0 \text{ and } \varphi_3 \neq 0\}$$

and

$$\partial\mathscr{X}_0 := \mathscr{X}_+ \setminus \mathscr{X}_0 = \{\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathscr{X}_+ : \varphi_2 = 0 \text{ or } \varphi_3 = 0\}.$$

With these settings, $\mathscr{X}_+ = \mathscr{X}_0 \cup \partial\mathscr{X}_0$ with $\mathscr{X}_0$ being relatively open in $\mathscr{X}_+$. Let $M\partial := \{\varphi \in \partial\mathscr{X}_0 : \Phi(t)\varphi \in \partial\mathscr{X}_0, t > 0\}$, where $\Phi(t) : \mathscr{X}_+ \to \mathscr{X}_+$ is the semi-flow generated. To establish Theorem 5.2, we make the following three claims.

Claim 1. We claim that $\mathscr{X}_0$ is positively invariant set with respect to $\Phi(t)$, that is, $\Phi(t)\mathscr{X}_0 \subseteq \mathscr{X}_0$ for all $t > 0$.

Let $u_0 \in \mathscr{X}_0$, observe that $I_0(x) \neq 0$ and $R_0(x) \neq 0$, since the second equation of reaction–diffusion epidemic system (2.1), then $\frac{\partial I}{\partial t} > d_I \Delta I - \gamma(x)I$. Further, we consider the solution I of the system is an upper solution of the problem

$$\begin{cases} z_t = d_I \Delta z - \gamma(x)z, & x \in \Omega, t > 0, \\ \frac{\partial z}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, t > 0, \\ z(x, 0) = z_0, & x \in \Omega. \end{cases} \tag{5.4}$$

By system (5.4), using Maximum principle, and $I_0(x) \neq 0$, we have

$$z > 0 \text{ for all } x \in \Omega \text{ and } t > 0.$$

Furthermore, employing Comparison Principle, we can obtain

$$I > z > 0 \text{ for all } x \in \Omega \text{ and } t > 0.$$

According to the third equation of reaction–diffusion epidemic system (2.1), we have

$$R = \exp\{-m(x)t\}R_0(x) + \int_0^t \exp\{-m(x)(t-s)\}c(x)I(x,s)\mathrm{d}s. \tag{5.5}$$

From (5.5), we have $R > 0$ for all $x \in \Omega$ and $t > 0$. Hence, $\Phi(t)u_0 \in \mathscr{X}_0$.

Claim 2. We claim that the ω limit set $\omega(\varphi)$ is the singleton $\{Q_0\}$ for every $\varphi \in M\partial$.

For convenience, set $S(x,t;\varphi) = S_\varphi, I(x,t;\varphi) = I_\varphi, R(x,t;\varphi) = R_\varphi$. Suppose that $R_\varphi = 0$, from the third equation of reaction–diffusion epidemic system (2.1), we have $I_\varphi = 0$. Furthermore, there exists a function $U(x)$ such that

$$S_\varphi \to U(x) \text{ uniformly for } x \in \Omega.$$

If the solution $I_\varphi = 0$, we need to establish that the solution $R_\varphi = 0$. If not, we can easily know that there exists a time $t_0 > 0$ such that the solution $R(x, t_0; \varphi) \neq 0$. From (5.5), we have the solution $R_\varphi > 0$ for all time $t > t_0$. By the second equation of reaction–diffusion epidemic system(2.1), we have that I is the positive solution, which contradicts with $\varphi \in M\partial$. Then, we know that $I_\varphi = 0$ for all $t > t_0$. Furthermore, from the third equation of reaction–diffusion epidemic system(2.1), we have $R_\varphi \to 0$ for $x \in \Omega$. Thus, $S_\varphi \to U(x)$ uniformly for $x \in \Omega$.

Claim 3. We claim that Q_0 of system (2.1) is a uniform weak repeller, that is, there exists a positive constant δ such that $\limsup\limits_{t\to+\infty} ||\Phi(t)\phi - Q_0|| > \delta$ for all $\varphi \in \mathscr{X}_0$.

To the contrary, suppose that Claim 3 is not satisfied. Thus, for any positive constant δ, $\limsup\limits_{t\to+\infty} ||\Phi(t)\phi - Q_0|| < \delta$. In other words, there exists a positive number t_2 that fulfills $S_\varphi > -\delta$ for all time $t > t_2$. Hence, we consider the upper solution (I_φ, R_φ) of system (2.1) as follows

$$\begin{cases} \frac{\partial y_1}{\partial t} = d_I \Delta y_1 + \frac{\beta_1(x)(U-\delta)y_1}{U-\delta+y_1+y_2} + \beta_2(x)\frac{(U-\delta)y_2}{U-\delta+y_2+y_1} - \gamma(x)y_1, & x \in \Omega, t > t_2, \\ \frac{\partial y_2}{\partial t} = c(x)y_1 - m(x)y_2, & x \in \Omega, t > t_2, \\ \frac{\partial y_1}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, \end{cases} \tag{5.6}$$

and we have

$$\begin{cases} \frac{\partial y_1}{\partial t} > d_I \Delta y_1 + \beta_1(x)y_1 + \beta_2(x)y_2 - \gamma(x)y_1, & x \in \Omega, t > t_2, \\ \frac{\partial y_2}{\partial t} = c(x)y_1 - m(x)y_2, & x \in \Omega, t > t_2, \\ \frac{\partial y_1}{\partial \mathbf{n}} = 0, & x \in \partial\Omega. \end{cases} \tag{5.7}$$

Denote the principal eigenvalue by $\mathscr{K}_\delta^0$, and consider its the eigenvalue problem as follows

$$\begin{cases} d_I\Delta\phi - \gamma(x)\phi + \left(\beta_1(x) + \frac{\beta_2(x)c(x)}{m(x)}\right)\phi = \mathscr{K}^0\phi, & x\in\Omega, t>t_2,\\ \frac{\partial\phi}{\partial\mathbf{n}} = 0, & x\in\partial\Omega. \end{cases} \tag{5.8}$$

Thus, it shows that $\mathscr{K}_\delta^0$ is continuous. Then, choosing a positive number δ such that $\mathscr{K}_\delta^0 > 0$ if $\mathcal{R}_0 > 1$. Furthermore, using Lemma 4.4, for $t > t_2$, we consider the eigenvalue problem as follows

$$\begin{cases} \lambda\psi_2 = d_I\Delta\psi_2 + \beta_1(x)\psi_2 + \beta_2(x)\psi_3 - \gamma(x)\psi_2, & x\in\Omega, t>t_2,\\ \lambda\psi_3 = c(x)\psi_2 - m(x)\psi_3, & x\in\Omega, t>t_2,\\ \frac{\partial\psi_2}{\partial\mathbf{n}} = 0, & x\in\partial\Omega, \end{cases} \tag{5.9}$$

which implies that the principal eigenvalue $\mathscr{K}_\delta^0$ associated with positive eigenvector $(\varphi\delta_2(x), \varphi\delta_3(x))$. To this end, we choose $\alpha > 0$ such that

$$\alpha(\varphi\delta_2(x), \varphi\delta_3(x)) < (I(x,t_2;\varphi), R(x,t_2;\varphi)).$$

Let $(I(x,t_2), R(x,t_2)) = \alpha(\varphi\delta_2(x), \varphi\delta_3(x))$ be the initial data of system (5.9). Then, the linear system (5.8) exists a solution

$$(y_{1,\varphi}, y_{2,\varphi}) = \alpha(\varphi\delta_2(x), \varphi\delta_3(x))e^{\mathscr{K}^0(t-t_2)}.$$

Using Comparison Principle, we can obtain

$$(I_\varphi, R_\varphi) > (y_{1,\varphi}, y_{2,\varphi}) \text{ on } \Omega\times[t_2,+\infty),$$

which implies that when $t\to+\infty$,

$$I_\varphi \to +\infty \text{ and } R_\varphi \to +\infty,$$

which is a contradiction to Lemma 3.2.

A function $\rho : \mathscr{X}_+ \to [0,+\infty)$ is considered as

$$\rho(\varphi) = \min\{\min_{x\in\Omega}\varphi_2(x), \min_{x\in\Omega}\varphi_3(x)\}, \varphi\in\mathscr{X}_+,$$

where $\varphi_2 = I$ and $\varphi_3 = R$. Thus, $\rho^{-1}(0,+\infty) \subseteq \mathscr{X}_0$, and we have two cases: one case is $\rho(\varphi) = 0$ for $\varphi\in\mathscr{X}_0$. The other case is $\rho(\varphi) > 0$, and then, $\rho(\Phi(t)\varphi) > 0$. Thus, using the definition of semi-flow, we know that the generalized distance function ρ is a semi-flow $\Phi(t) : \mathscr{X}_+ \to \mathscr{X}_+$. So far, it shows that any forward orbit of $\Phi(t)$ in $M\partial$ converges to Q_0 of system (2.1), and then, there exists the stable subset $W_s(Q_0)$ of Q_0 such that $W_s(Q_0)\bigcap\mathscr{X}_0 = \emptyset$.

Further, we consider that Q_0 in domain $\mathscr{X}_+$ is an isolated invariant set, i.e., there is no set of Q_0 in $\partial\mathscr{X}_0$. Using Theorem 3 in [31], we know that there has a positive number δ_1 satisfies

$$\min\{\rho(\psi)\} > \delta_1,$$

where $\psi\in\omega(\varphi)$ for any $\psi\in\mathscr{X}_0$. That is, for $\forall\varphi\in\mathscr{X}_0$, we have

$$\liminf_{t\to\infty} I_\varphi > \delta_1 \text{ and } \liminf_{t\to+\infty} R_\varphi > \delta_1.$$

By Lemma 3.2, we can find that there are positive function $\mathscr{M}^{**}$ and time t_3, then the inequality $I_\varphi < M^{**}$ and the inequality $R_\varphi < \mathscr{M}^{**}$ hold. From the first equation of system (2.1), for $t > t_3$ and $x\in\Omega$, we can obtain

$$S'_t > d_S\Delta S + \underline{A} - (\overline{\alpha} + \overline{\beta_1\mathscr{M}^{**}} + \overline{\beta_2\mathscr{M}^{**}})S.$$

Further, by using Comparison Principle, we have

$$\liminf_{t\to+\infty} S_\varphi \ge \delta_2 := \frac{\underline{A}}{\overline{\alpha} + \overline{\beta_1\mathscr{M}^{**}} + \overline{\beta_2\mathscr{M}^{**}}}.$$

Hence,

$$\liminf_{t\to+\infty} S \geq \delta,\ \liminf_{t\to+\infty} I \geq \delta,\ \liminf_{t\to+\infty} R \geq \delta,$$

where $\delta = \min\{\delta_1, \delta_2\}$. Up to now, the uniform persistence is proved. Based on Theorem 4.7 in [29], we know that (2.1) admits at least a DFSS in $\mathscr{X}_0$. The proof is completed. □

6. Asymptotic profiles of system (2.1)

In this section, we only consider that if one of the two positive diffusion rates d_S and d_I of system (2.1) tends to zero. From (4.1), (S, I, R) is a DFSS of (2.1) if only if the point (S, I) satisfies

$$\begin{cases} d_S\Delta S + A - \alpha S - \frac{\beta_1 SI}{S+I+\frac{c}{m}I} - \frac{\beta_2 S\frac{c}{m}I}{S+I+\frac{c}{m}I} = 0, & x\in\Omega, t>0,\\ d_I\Delta I + \frac{\beta_1 SI}{S+I+\frac{c}{m}I} + \frac{\beta_2 S\frac{c}{m}I}{S+\frac{c}{m}I+I} - \gamma I = 0, & x\in\Omega, t>0,\\ \frac{\partial S}{\partial \mathbf{n}} = \frac{\partial I}{\partial \mathbf{n}} = 0, & x\in\partial\Omega \end{cases} \tag{6.1}$$

and the solution $R = \frac{cI}{m}$. We here denote the principal eigenvalue by $\mathscr{K}^0(D,\xi)$ and then, consider the following system

$$\begin{cases} D\Delta\varphi + \xi\varphi = \mathscr{K}\varphi, & x\in\Omega,\\ \frac{\partial\varphi}{\partial \mathbf{n}} = 0, & x\in\partial\Omega, \end{cases} \tag{6.2}$$

where $D > 0$ and $\xi \in C(\Omega)$. $\mathscr{K}^0(D,\xi)$ is continuous function with the two variables D and ξ. For $\varphi \in H^1(\Omega)$ with $\int_\Omega \varphi^2 \mathrm{d}x = 1$, we have

$$\mathscr{K}^0(D,\xi) = -\inf\left\{\int_\Omega (D|\nabla\varphi|^2 - \xi\varphi^2)\mathrm{d}x\right\}. \tag{6.3}$$

Then, from Eq. (6.3), we know that $\mathscr{K}^0(D,\xi)$ is decreasing on the variable D. In addition, for $x \in \Omega$, we denote that

$$\lim_{D\to 0} \mathscr{K}^0(D,\xi) = \max\{\xi(x)\},$$

and it is increasing on the variable ξ.

6.1. Profile as $d_S \to 0$

Lemma 6.1. *Let* $\mathcal{P}_0 = \mathscr{K}^0\left(d_I, ((\beta_1 + \frac{\beta_2 c}{m})\frac{\alpha(1+\frac{c}{m})+\beta_1+\beta_2\frac{c}{m}}{\alpha} - \gamma\right)$ *and* $A - \alpha\hat{S} - \frac{\beta_1\hat{S}\varphi}{\hat{S}+(1+\frac{c}{m})\varphi} - \frac{\beta_2\hat{S}\frac{c}{m}\varphi}{\hat{S}+(1+\frac{c}{m})\varphi} = 0,$ *then the nonlinear problem*

$$\begin{cases} d_I\Delta\varphi + \left(\left(\beta_1 + \frac{\beta_2 c}{m}\right)\frac{A}{\alpha\left(\hat{S}+\left(1+\frac{c}{m}\right)\varphi\right)+\left(\beta_1+\frac{\beta_2 c}{m}\right)\varphi} - \gamma\right)\varphi = 0, & x\in\Omega,\\ \frac{\partial\varphi}{\partial\mathbf{n}} = 0, & x\in\partial\Omega \end{cases} \tag{6.4}$$

has the following results:

1. *If* $\mathcal{P}_0 < 0$, *there is no positive solution of system* (6.4).
2. *If* $\mathcal{P}_0 > 0$, *there is a unique positive solution of system* (6.4).

Theorem 6.1. *Let* $\mathcal{P}_0 = \mathscr{K}^0\left(d_I, ((\beta_1 + \frac{\beta_2 c}{m})\frac{\alpha(1+\frac{c}{m})+\beta_1+\beta_2\frac{c}{m}}{\alpha} - \gamma)\right)$, *we have the following statements:*

1. *If* $\mathcal{P}_0 < 0$, *there is a positive number* d_S^{**} *such that* (6.1) *does not have the DFSS if* $d_S < d_S^{**}$.

2. *If $\mathcal{P}_0 > 0$, there exists a positive number d_S^{**} such that system (6.1) exists a DFSS (S, I) if $d_S < d_S^{**}$. Furthermore, $S \to S_{**}$ and $I \to I_{**}$ as $d_S \to 0$ uniformly on Ω. System (6.4) has a unique positive solution (S_{**}, I_{**}).*

6.2. Profile as $d_I \to 0$

We denote $\widetilde{l}$ by the spatial average of l, that is, $\widetilde{l} = \frac{\int_\Omega l dx}{|\Omega|}$, where $\widetilde{l} \in C(\Omega)$.

Lemma 6.2. *Let $\mathcal{P}_1 = \mathscr{K}^0\left(d_I, ((\beta_1 + \frac{\beta_2 c}{m})\frac{\widetilde{\alpha}(1+\widetilde{\frac{c}{m}})+\widetilde{\beta_1}+\widetilde{\beta_2 \frac{c}{m}}}{\widetilde{\alpha}} - \gamma)\right)$, then the nonlinear nonlocal problem*

$$\begin{cases} d_I \Delta\varphi + \left(\left(\beta_1 + \frac{\beta_2 c}{m}\right)\frac{\widetilde{A}}{\widetilde{\alpha \hat{S}} + \widetilde{\alpha}\left(1+\widetilde{\frac{c}{m}}\right)\varphi) + \left(\widetilde{\beta_1 + \frac{\beta_2 c}{m}}\right)\varphi} - \gamma\right)\varphi = 0, & x \in \Omega, \\ \frac{\partial \varphi}{\partial \mathbf{n}} = 0, & x \in \partial\Omega \end{cases} \tag{6.5}$$

has the following properties:

1. *If $\mathcal{P}_1 < 0$, there is no positive solution of system (6.5).*
2. *If $\mathcal{P}_1 > 0$, there exists a unique positive solution of system (6.5).*

Theorem 6.2. *Let $\mathcal{P}_1 = \mathscr{K}^0\left(d_I, (\beta_1 + \frac{\beta_2 c}{m})\frac{\widetilde{\alpha}(1+\widetilde{\frac{c}{m}})+\widetilde{\beta_1}+\widetilde{\beta_2 \frac{c}{m}}}{\widetilde{\alpha}} - \gamma)\right)$, we have the following statements:*

1. *If $\mathcal{P}_1 < 0$, there is a positive number d_S^* such that system (6.1) has not solution if the diffusion rate $d_S > d_S^*$.*
2. *If $\mathcal{P}_1 > 0$, there exists a positive number d_S^* such that system (6.1) exists $(S, I) > 0$ if $d_S > d_S^*$. Moreover, $S \to S_{**}$ and $I \to I_{**}$ uniformly on region Ω as $d_S \to \infty$, then system (6.4) exists a unique $(S_{**}, I_{**}) > 0$.*

Theorem 6.3. *Assume that $\left(\beta_1 + \frac{\beta_2 c}{m}\right)(\frac{\widetilde{\alpha}(1+\widetilde{\frac{c}{m}})+\widetilde{\beta_1}+\widetilde{\beta_2 \frac{c}{m}}}{\widetilde{\alpha}}) - \gamma > 0$, let H be defined by (6.9), then there exists a positive numbers d_I^* for some $x \in \Omega$ such that (6.4) has $I > 0$ if $d_I < d_I^*$ and $\widetilde{\alpha f_1(I)}I + \widetilde{\alpha}(1 + \widetilde{\frac{c}{m}})\widetilde{I} + (\widetilde{\beta_1 + \frac{\beta_2 c}{m}})\widetilde{I} \to H$ as $d_I \to 0$.*

6.3. The proof of main results

Proof. **To establish Lemma** 6.1, the first case of Lemma 6.1 is studied. To prove this by using the reduction to absurdity, suppose that (6.4) has a solution $\varphi > 0$ if $\mathcal{P}_0 < 0$. Multiplying both sides of first equation of system (6.4) by φ, and then integrating it over Ω, we can obtain

$$-d_I \int_\Omega |\nabla\varphi|^2 \mathrm{d}x + \int_\Omega \left(\left(\beta_1 + \frac{\beta_2 c}{m}\right)\frac{A}{\alpha\left(\hat{S} + \left(1 + \frac{c}{m}\right)\varphi\right) + \left(\beta_1 + \frac{\beta_2 c}{m}\right)\varphi} - \gamma\right)\varphi^2 \mathrm{d}x = 0.$$

From

$$0 = A - \alpha\hat{S} - \frac{\beta_1 \hat{S}\varphi}{\hat{S} + \left(1 + \frac{c}{m}\right)\varphi} - \frac{\beta_2 \hat{S}\frac{c}{m}\varphi}{\hat{S} + \left(1 + \frac{c}{m}\right)\varphi} > A - \alpha\hat{S} - \frac{\beta_1 \hat{S}\varphi}{\left(1 + \frac{c}{m}\right)\varphi} - \frac{\beta_2 \hat{S}\frac{c}{m}\varphi}{\left(1 + \frac{c}{m}\right)\varphi},$$

we have

$$\frac{A}{\alpha\left(1 + \frac{c}{m}\right) + \beta_1 + \beta_2 \frac{c}{m}} < \hat{S} \to \frac{A}{\alpha}.$$

Then, we can conclude

$$-d_I\int\limits_\Omega |\nabla\varphi|^2\mathrm{d}x+\int\limits_\Omega\left(\left(\beta_1+\frac{\beta_2 c}{m}\right)\frac{A}{\alpha\hat{S}}-\gamma\right)\varphi^2\mathrm{d}x>0. \tag{6.6}$$

From (6.6), we get

$$\mathcal{P}_0>\frac{-d_I\int\limits_\Omega|\nabla\varphi|^2\mathrm{d}x+\int\limits_\Omega\left(\left(\beta_1+\frac{\beta_2 c}{m}\right)\frac{\alpha(1+\frac{c}{m})+\beta_1+\beta_2\frac{c}{m}}{\alpha}-\gamma\right)\varphi^2\mathrm{d}x}{\int\limits_\Omega\phi^2\mathrm{d}x}>0,$$

which is a contradiction with $\mathcal{P}_0<0$. The proof of first case of Lemma 6.1 is completed.

In this second case of Lemma 6.1, we will prove it in two steps as follows:

Step 1. We prove that system (6.2) has a positive solution. To establish this by using the method of upper/lower solution, we prove the positive eigenfunction of (6.2) by ϕ tending to $\mathcal{P}_0$. Next, we claim that $\epsilon\phi$ and M are a pair of lower and upper solutions for $\epsilon>0$ and $M>0$. $f(\varphi)$ is given by

$$f(\varphi)=d_I\Delta\varphi+\left(\left(\beta_1+\frac{\beta_2 c}{m}\right)\frac{\alpha\left(1+\frac{c}{m}\right)+\beta_1+\beta_2\frac{c}{m}}{\alpha}-\gamma\right)\varphi.$$

Then,

$$\begin{aligned}
f(\epsilon\phi)&=\epsilon\left[(d_I\Delta\phi+\left(\left(\beta_1+\frac{\beta_2 c}{m}\right)\frac{A}{\alpha\left(\hat{S}+\epsilon\left(1+\frac{c}{m}\right)\phi\right)+\epsilon\left(\beta_1+\frac{\beta_2 c}{m}\right)\phi}-\gamma\right)\phi\right]\\
&=\epsilon\left(\beta_1+\frac{\beta_2 c}{m}\right)\left[\frac{A}{\alpha\left(\hat{S}+\epsilon(1+\frac{c}{m})\phi\right)+\epsilon\left(\beta_1+\frac{\beta_2 c}{m}\right)\phi}-\frac{A}{\left(\alpha\hat{S}+\left(1+\frac{c}{m}\right)\phi\right)+\left(\beta_1+\frac{\beta_2 c}{m}\right)\phi}\right]\phi\\
&>0.
\end{aligned}$$

If ϵ is a small positive number, $\mathcal{P}_0>0$, and M is a large positive number, we have $f(M)<0$. Thus, there is a positive solution of system (6.4) if $\mathcal{P}_0>0$.

Step 2. We prove that the positive solution of system (6.4) is unique.

On the contrary, suppose that (6.4) has two positive solutions φ_1 and φ_2 with $\varphi_i\in[\epsilon\phi,M], i=1,2$. For a sufficiently small number ϵ, a sufficiently large number M, and $m,M>0$, φ_m is the lower solution and φ_M is the upper solution of (6.4), namely,

$$\varphi_m<\varphi_1,\varphi_2<\varphi_M,$$

where φ_m is a minimal solution of system (6.4) in interval $[\varphi_m,\varphi_M]$, φ_M is a maximal solution of system (6.4) in interval $[\varphi_m,\varphi_M]$.

We first multiply both sides of system (6.4) with $\varphi=\varphi_m$ by φ_M and with $\varphi=\varphi_M$ by φ_m, and we can obtain

$$\int\limits_\Omega\left(\beta_1+\frac{\beta_2 c}{m}\right)\varphi_M\varphi_m\left(G(m)-G(M)\right)\mathrm{d}x=0,$$

where $G(j):=\frac{A}{\alpha(\hat{S}+(1+\frac{c}{m})\varphi_j)+\beta_1+\frac{\beta_2 c}{m})\varphi_j}, j=m,M$. By implicit differentiation, we know that the function $\hat{S}(\varphi_j)$ is monotonous, which in turn implies that $\varphi_M=\varphi_m$. This leads to a contradiction with $\varphi_M>\varphi_m$. Thus, the positive solution of system (6.4) is unique. The proof of the second case of Lemma 6.1 is completed. □

Proof. **To establish Theorem** 6.1, notice that $\mathscr{K}^0\left(d_I, (\beta_1+\frac{\beta_2 c}{m})\frac{\alpha(1+\frac{c}{m})+\beta_1+\beta_2\frac{c}{m}}{\alpha}-\gamma\right)$ is the principal eigenvalue of system (4.6). From (3.1) that $U\to\frac{A}{\alpha}$ as $d_S\to 0$, we have

$$\mathscr{K}^0\left(d_I, \left(\left(\beta_1+\frac{\beta_2 c}{m}\right)\frac{A}{\alpha\left(\hat{S}+\left(1+\frac{c}{m}\right)\varphi\right)+\left(\beta_1+\frac{\beta_2 c}{m}\right)\varphi}-\gamma\right)\right)\to\mathscr{K}^0\left(d_I, \left(\left(\beta_1+\frac{\beta_2 c}{m}\right)-\gamma\right)\right).$$

Then,

$$\mathscr{K}^0\left(d_I, \left(\left(\beta_1+\frac{\beta_2 c}{m}\right)-\gamma\right)\right)<\mathcal{P}_0,\ \text{as } d_S\to 0, \tag{6.7}$$

where $A-\alpha\hat{S}-\frac{\beta_1\hat{S}\varphi}{\hat{S}+(1+\frac{c}{m})\varphi}-\frac{\beta_2\hat{S}\frac{c}{m}\varphi}{\hat{S}+(1+\frac{c}{m})\varphi}=0$. Based on Lemma 4.1 and Lemma 4.3, if $\mathcal{P}_0<0$, there is a positive number d_S^{**} such that

$$\mathcal{R}_0<1 \text{ if } d_S<d_S^{**}.$$

Moreover, Q_0 is globally stable for $\mathcal{R}_0<1$. Based on Theorem 5.1, (6.1) has no DFSS if $d_S<d_S^{**}$. The proof of the first case of Theorem 6.1 is completed.

Now, the second case of Theorem 6.1 is considered. From (6.7), we prove that when $\mathcal{P}_0>0$, there is a positive number d_S^{**} such that $\mathcal{R}_0>1$ if $d_S<d_S^{**}$. From Theorem 5.2, we know that (2.1) has a PESS, which implies $(S,I)>0$ if $d_S<d_S^{**}$. Next, we prove that

$$S\to S_{**} \text{ and } I\to I_{**} \text{ as } d_S\to 0.$$

To establish this result, firstly, we prove that (S,I) of system (6.1) has a priori estimate.

For the first equation of (6.1), we know that $d_S\Delta S<A-\alpha S$. Thus, we have $||S||<\mathcal{C}_1$, where $\mathcal{C}_1=\frac{\max\{A(x):x\in\Omega\}}{\min\{\alpha(x):x\in\Omega\}}$. From the first and second equations of (6.1), we have

$$\int_\Omega \gamma I \mathrm{d}x=\int_\Omega (A-\alpha S)\mathrm{d}x<||A|||\Omega|.$$

Hence,

$$||I||_1<\frac{||A|||\Omega|}{\min\{\gamma(x):x\in\Omega\}}.$$

By the second equation of reaction–diffusion system (6.1), we know the variable I is uniform boundedness and then, obtain the elliptic estimate of the variable I. Furthermore, similar to the above discussion, we easy to know that for all positive number d_S, there exists $C_2>0$ fulfills $||I||_{2,p}<C_2$, where p is a positive number. Setting $p>n$, and based on $||S||<C_1$, there is a sequence d_{S_k} with $d_{S_k}\to 0$ such that the corresponding solution (S_k,I_k) of (6.1) that fulfills the estimate as follows

$$S_k\to S_{**} \text{ weakly in } L^p(\Omega).$$

When $k\to\infty$, we know that $I\to I_{**}$ weakly in domain $W^{2,p}(\Omega)$ and $I\to I_{**}$ strongly in domain $C(\Omega)$ such that $I_k\to I_{**}$. Due to $I_k\to I_{**}$ in domain $C(\Omega)$ and the first equation of reaction–diffusion system (6.1), we obtain $S_k\to S_{**}$ in $C(\Omega)$. Hence, we have $I>0$. The proof of the second case of Theorem 6.1 is completed. □

Proof. **To establish Lemma** 6.2, the first case of Lemma 6.2 is now studied. To the contrary, if $\mathcal{P}_1<0$, system (6.5) exists a positive solution φ, and then, we consider a principal eigenvector of (6.5) problem as follows

$$\mathscr{K}^0\left(d_I, \left(\left(\beta_1+\frac{\beta_2 c}{m}\right)\frac{\widetilde{A}}{\widetilde{\alpha}\hat{S}+\widetilde{\alpha}\left(1+\frac{\widetilde{c}}{m}\right)\varphi+\left(\widetilde{\beta_1}+\frac{\widetilde{\beta_2 c}}{m}\right)\varphi}-\gamma\right)\right)=0,$$

where $\hat{S}$ satisfies $A-\alpha\hat{S}-\frac{\beta_1\hat{S}\varphi}{\hat{S}+(1+\frac{c}{m})\varphi}-\frac{\beta_2\hat{S}\frac{c}{m}\varphi}{\hat{S}+(1+\frac{c}{m})\varphi}=0$. It is easy to know that

$$\mathscr{K}^0\left(d_I,\left(\left(\beta_1+\frac{\beta_2c}{m}\right)\frac{\widetilde{A}}{\widetilde{\alpha\hat{S}}+\widetilde{\alpha}\left(1+\frac{\widetilde{c}}{m}\right)\varphi+\left(\widetilde{\beta_1}+\widetilde{\frac{\beta_2c}{m}}\right)\varphi}-\gamma\right)\right)<\mathscr{K}^0\left(d_I,(\beta_1+\frac{\beta_2c}{m}-\gamma)\right).$$

Moreover,

$$\mathscr{K}^0\left(d_I,\left(\left(\beta_1+\frac{\beta_2c}{m}\right)-\gamma\right)\right)<\mathscr{K}^0\left(d_I,\left(\left(\beta_1+\frac{\beta_2c}{m}\right)\frac{\widetilde{\alpha}\left(1+\frac{\widetilde{c}}{m}\right)+\widetilde{\beta_1}+\widetilde{\beta_2\frac{c}{m}}}{\widetilde{\alpha}}-\gamma\right)\right)=\mathcal{P}_1.$$

Then, we have $\mathcal{P}_1>0$, which is a contradiction with $\mathcal{P}_1<0$. The proof of the first case of Lemma 6.2 is completed.

Next, the second case of Lemma 6.2 is considered. By $\widetilde{A}-\widetilde{\alpha}\hat{S}-\frac{\beta_1\hat{S}\varphi}{\hat{S}+(1+\frac{c}{m})\varphi}-\frac{\beta_2\hat{S}\frac{c}{m}\varphi}{\hat{S}+(1+\frac{c}{m})\varphi}=0$, we know that $\hat{S}$ is monotonous. By using implicit function theorem, there is a compact map $f_1:\mathscr{X}_+\to\mathscr{X}_+$ such that $\hat{S}=\frac{\widetilde{A}}{\widetilde{\alpha}}+\varphi f_1(\varphi)$. Define $\mathcal{P}_1(\omega)$ as follows

$$\mathcal{P}_1(\omega)=\mathscr{K}^0\left(d_I,\left(\left(\beta_1+\frac{\beta_2c}{m}\right)\frac{\widetilde{A}}{\widetilde{A}+\omega}-\gamma\right)\right).$$

Again using $\mathscr{K}^0(D,\xi)$ is monotonous, $\mathcal{P}_1(\omega)$ is continuous and strictly decreasing, and then we have $\mathcal{P}_1(0)=\mathcal{P}_1>0$ and $\mathcal{P}_1(\infty)=\mathcal{P}_1(d_I,-\gamma)<0$. To this end, there is a unique positive number ω^{**} such that $\mathcal{P}_1(\omega^{**})=0$. Denote a eigenvector by $\varphi>0$ corresponding to $\mathcal{P}_1(\omega^{**})=0$, then we have

$$\begin{cases} d_I\Delta\varphi+\left(\left(\beta_1+\frac{\beta_2c}{m}\right)\frac{\widetilde{A}}{\widetilde{A}+\omega^{**}}-\gamma\right)\varphi=0, & x\in\Omega,\\ \frac{\partial\varphi}{\partial\mathbf{n}}=0, & x\in\partial\Omega.\end{cases}\tag{6.8}$$

So for some positive number τ, we know that the solution $\tau\varphi>0$ of (6.5) and satisfies

$$\omega^{**}=\tau\int\limits_{\Omega}\left(\alpha f_1(\varphi)+\alpha\left(1+\frac{c}{m}\right)+\left(\beta_1+\frac{\beta_2c}{m}\right)\right)\varphi\mathrm{d}x.$$

Since ω^{**} is uniqueness, it ensures system (6.5) exists at least a positive solution. The proof of the second case of Lemma 6.2 is completed. □

Proof. **To establish Theorem** 6.2, now the first case of Theorem 6.2 is studied. Noticed that

$$\mathscr{K}^0\left(d_I,\left(\left(\beta_1+\frac{\beta_2c}{m}\right)\frac{\widetilde{A}}{\widetilde{\alpha\hat{S}}+\widetilde{\alpha}\left(1+\frac{\widetilde{c}}{m}\right)\varphi+\left(\widetilde{\beta_1}+\widetilde{\frac{\beta_2c}{m}}\right)\varphi}-\gamma\right)\right)=0,$$

we have

$$\mathscr{K}^0\left(d_I,\left(\left(\beta_1+\frac{\beta_2c}{m}\right)\frac{\widetilde{A}}{\widetilde{\alpha\hat{S}}+\widetilde{\alpha}\left(1+\frac{\widetilde{c}}{m}\right)\varphi+\left(\widetilde{\beta_1}+\overline{\frac{\beta_2c}{m}}\right)\varphi}-\gamma\right)\right)<\mathscr{K}^0\left(d_I,\left(\beta_1+\frac{\beta_2c}{m}-\gamma\right)\right),$$

then

$$\mathscr{K}^0\left(d_I,\left(\beta_1+\frac{\beta_2c}{m}-\gamma\right)\right)<\mathcal{P}_1,\ \text{as}\ d_S\to\infty,$$

where $\hat{S}$ satisfies $A - \alpha\hat{S} - \frac{\beta_1 \hat{S}\varphi}{\hat{S}+(1+\frac{c}{m})\varphi} - \frac{\beta_2 \hat{S}\frac{c}{m}\varphi}{\hat{S}+(1+\frac{c}{m})\varphi} = 0$. If $\mathcal{P}_1 < 0$, then there exists a positive number d_S^*, for $d_S > d_S^*$ such that $\mathscr{K}^0 < 0$. Observe that $\mathscr{K}^0$ and $\mathcal{R}_0 - 1$ have the same sign, it implies that for $d_S > d_S^*$, we have $\mathcal{R}_0 < 1$. Using Theorem 5.1, if $d_S > d_S^*$, then system (6.1) does not have a positive solution. The proof is completed.

Now, the second case of Theorem 6.2 is considered. If $\mathcal{P}_1 > 0$, from Theorem 5.2 that (2.1) has a DFSS. In other words, there exists a positive numbers d_S^*, system (6.1) has $(S, I) > 0$ when $d_S > d_S^*$.

Next, we will show that the convergence of the solution (S, I) when $d_S \to \infty$. Due to Theorem 6.1, we get

$$\{S_{d>d^{**}}\} \text{ is uniformly bounded } in\ C(\Omega)$$

and for a positive number d_S, we have

$$\{I_{d>d^{**}}\} \text{ is uniformly bounded } in\ W^{2,p}(\Omega).$$

Thus, we can obtain

$$S \text{ and } I \text{ are uniformly bounded in } W^{2,p}(\Omega).$$

Therefore, there exists a sequence d_{S_k} such that the corresponding $(S_k, I_k) > 0$ of (6.1) when $d_{S_k} \to \infty$, namely,

$$(S_k, I_k) \to (S_{**}, I_{**}) \text{ weakly in } W^{2,p}(\Omega) \times W^{2,p}(\Omega).$$

Since S_{**} is constant, then $\Delta S_{**} = 0$. Then, (6.5) has a unique positive solution (S_{**}, I_{**}). The proof is completed.

Now, we study the solution of (6.4) if d_I is small. In what follows, we consider that there exists $x \in \Omega$ such that

$$\left(\beta_1 + \frac{\beta_2 c}{m}\right) \frac{\widetilde{\alpha}\left(1 + \frac{\widetilde{c}}{m}\right) + \widetilde{\beta_1} + \widetilde{\beta_2 \frac{c}{m}}}{\widetilde{\alpha}} - \gamma > 0.$$

We need to find H such that

$$\max\left\{\left(\beta_1 + \frac{\beta_2 c}{m}\right) \frac{\widetilde{A}}{\widetilde{A} + H} - \gamma\right\} = 0. \tag{6.9}$$

Clearly, for many points in region Ω, the above maximum can be fulfilled. Hence, let a nonempty set be

$$\mathcal{N} = \left\{x \in \Omega : \left(\beta_1 + \frac{\beta_2 c}{m}\right) \frac{\widetilde{A}}{\widetilde{A} + H} - \gamma = 0\right\},$$

then, we know that the set $\mathcal{N}$ includes all points if $d_I \to 0$. The proof of the second case of Theorem 6.2 is completed. □

Proof. **To establish Theorem** 6.3, as the rate $d_I \to 0$, we have

$$\begin{aligned}
&\mathscr{K}^0\left(d_I, \left(\left(\beta_1 + \frac{\beta_2 c}{m}\right) \frac{\widetilde{A}}{\widetilde{\alpha\hat{S}} + \widetilde{\alpha}(1 + \frac{\widetilde{c}}{m})\varphi + \left(\widetilde{\beta_1} + \widetilde{\frac{\beta_2 c}{m}}\right)\varphi} - \gamma\right)\right) < \mathscr{K}^0\left(d_I, \left(\left(\beta_1 + \frac{\beta_2 c}{m}\right) - \gamma\right)\right) \\
&< \mathscr{K}^0\left(d_I, \left(\left(\beta_1 + \frac{\beta_2 c}{m}\right) \frac{\widetilde{\alpha}\left(1 + \frac{\widetilde{c}}{m}\right) + \widetilde{\beta_1} + \widetilde{\beta_2 \frac{c}{m}}}{\widetilde{\alpha}} - \gamma\right)\right) \\
&\to \max\left\{x \in \Omega : \mathscr{K}^0\left(d_I, \left(\left(\beta_1 + \frac{\beta_2 c}{m}\right) \frac{\widetilde{\alpha}\left(1 + \frac{\widetilde{c}}{m}\right) + \widetilde{\beta_1} + \widetilde{\beta_2 \frac{c}{m}}}{\widetilde{\alpha}} - \gamma\right)\right)\right\} > 0,
\end{aligned}$$

where $\hat{S}$ satisfies $A-\alpha\hat{S}-\frac{\beta_1\hat{S}\varphi}{\hat{S}+(1+\frac{c}{m})\varphi}-\frac{\beta_2\hat{S}\frac{c}{m}\varphi}{\hat{S}+(1+\frac{c}{m})\varphi}=0$. Hence, there is a positive number d_I^* for all $d_I<d_I^*$ such that $\mathcal{P}_1>0$. Using Lemma 6.2, system (6.5) exists a unique $I>0$ if $d_I<d_I^*$. From (6.2), we have

$$\mathscr{K}^0\left(d_I,\left(\left(\beta_1+\frac{\beta_2 c}{m}\right)\frac{\widetilde{A}}{\widetilde{A}+\widetilde{\alpha f_1(I)}I+\widetilde{\alpha}\left(1+\frac{\widetilde{c}}{m}\right)\widetilde{I}+\left(\widetilde{\beta_1}+\frac{\widetilde{\beta_2 c}}{m}\right)I}-\gamma\right)\right)=0.$$

By the monotonicity of $\mathscr{K}^0(D,\xi)$, we can show that $\widetilde{\alpha f_1(I)}I+\widetilde{\alpha}(1+\frac{\widetilde{c}}{m})\widetilde{I}+(\widetilde{\beta_1}+\frac{\widetilde{\beta_2 c}}{m})\widetilde{I}$ is decreasing on d_I. Thus,

$$\widetilde{\alpha f_1(I)}I+\left(1+\frac{\widetilde{c}}{m}\right)\widetilde{I}+\widetilde{\alpha}\left(\widetilde{\beta_1}+\frac{\widetilde{\beta_2 c}}{m}\right)\widetilde{I}\to H_0 \text{ for some } H_0>0 \text{ as } d_I\to 0.$$

Further,

$$\begin{aligned}0&=\lim_{d_I\to 0}\mathscr{K}^0\left(d_I,\left(\left(\beta_1+\frac{\beta_2 c}{m}\right)\frac{\widetilde{A}}{\widetilde{A}+\widetilde{\alpha f_1(I)}I+\widetilde{\alpha}\left(1+\frac{\widetilde{c}}{m}\right)\widetilde{I}+\left(\widetilde{\beta_1}+\frac{\widetilde{\beta_2 c}}{m}\right)\widetilde{I}}-\gamma\right)\right)\\&=\max\left\{\left(\beta_1+\frac{\beta_2 c}{m}\right)\frac{\widetilde{A}}{\widetilde{A}+H_0}-\gamma\right\}.\end{aligned}$$

From (6.9), we have $H_0=H$. It follows that $\widetilde{\alpha f_1(I)}I+(1+\frac{\widetilde{c}}{m})\widetilde{I}+(\widetilde{\beta_1}+\frac{\widetilde{\beta_2 c}}{m})\widetilde{I}\to H_0$ as $d_I\to 0$. The proof is completed. □

7. Conclusion and discussion

This paper studied the dynamics of the reaction–diffusion epidemic system with distinct dispersal rates. The overall solution for the diffusion epidemic system was given, and the uniformly boundedness of the solution was obtained by using L^p theory and the Young's inequality. Then, the asymptotic smoothness of the semi-flow and the existence of the global attractor were discussed using linear differential operator. In addition, the basic reproduction number, $\mathcal{R}_0$, was defined to spread the disease model in a spatially uniform environment, and the asymptotic profiles of the system were studied when the spread rates of the susceptible and the infected individuals were close to zero.

In this paper, by using the spectral radius of the next generation operator, the $\mathcal{R}_0$ was given. The information on how system depends the parameters was investigated by employing variational formula. In addition, Theorem 4.1 indicates that how the dispersal rate of the infected individuals affects $\mathcal{R}_0$. It was shown when $\mathcal{R}_0>1$, the system is uniformly persistent and exists a positive steady state.

Furthermore, the asymptotic profiles of the DFSS were studied when the dispersal rate of susceptible or infected hosts tends to zero. From Theorem 6.1, when $\mathcal{P}_0<0$, there exists a positive number d_S^* such that $\mathcal{R}_0<1$ when $d_S<d_S^*$. It was found that the recovered individuals could be eliminated by limiting the movement of the susceptible individuals. We gave the local basic reproduction number as

$\mathcal{R}_{local}(x) = (\beta_1 + \frac{\beta_2 c}{m})\frac{A}{\alpha} - \gamma$. It follows that

$$\begin{aligned}\mathcal{P}_0 &= \mathscr{K}^0\left(d_I, \left(\left(\beta_1 + \frac{\beta_2 c}{m}\right)\frac{\alpha\left(1+\frac{c}{m}\right)+\beta_1+\beta_2\frac{c}{m}}{\alpha} - \gamma\right)\right) \\ &= \sup\left\{\int_\Omega \left(\gamma(\mathcal{R}_{local}(x)-1)\varphi^2 - d_I|\nabla\varphi|^2\right)\mathrm{d}x : \varphi \in H^1(\Omega) \text{ with } \int_\Omega \varphi^2 \mathrm{d}x = 1\right\}.\end{aligned} \tag{7.1}$$

Furthermore, the asymptotic profiles of the DFSS were studied when the dispersal rate of susceptible or infected hosts tends to zero. From Theorem 6.1, when $\mathcal{P}_0 < 0$, there exists a positive number d_S^* such that $\mathcal{R}_0 < 1$ when $d_S < d_S^*$. It was found that the recovered individuals could be eliminated by limiting the movement of the susceptible individuals. We gave the local basic reproduction number as $\mathcal{R}_{local}(x) = (\beta_1 + \frac{\beta_2 c}{m})\frac{A}{\alpha} - \gamma$. It follows that

$$\begin{aligned}\mathcal{P}_0 &= \mathscr{K}^0\left(d_I, \left(\left(\beta_1 + \frac{\beta_2 c}{m}\right)\frac{\alpha\left(1+\frac{c}{m}\right)+\beta_1+\beta_2\frac{c}{m}}{\alpha} - \gamma\right)\right) \\ &= \sup\left\{\int_\Omega (\gamma(\mathcal{R}_{local}(x)-1)\varphi^2 - d_I|\nabla\varphi|^2)\mathrm{d}x : \varphi \in H^1(\Omega) \text{ with } \int_\Omega \varphi^2 \mathrm{d}x = 1\right\}.\end{aligned} \tag{7.2}$$

And hence, if $\mathcal{R}_{local}(x) < 1$ for $x \in \Omega$, then $\mathcal{P}_0 < 0$ regardless of the value of d_I. Limiting d_S can eradicate the recovered individuals directly from Theorem 6.1.

Suppose that $\mathcal{R}_{local}(x) > 1$ for $x \in \Omega$, which implies the limiting case

$$\beta_1 + \frac{\beta_2 c}{m} > \gamma \tag{7.3}$$

for $U \to A/\alpha$ as $d_S \to 0$, then (7.2) may be positive number or negative number. From (7.2), it is shown that $\mathcal{P}_0$ is decreasing on d_I. Then, we obtain

$$\mathcal{P}_0 \to \gamma(\widetilde{R_{local}(x)} - 1) = \frac{1}{|\Omega|}\int_\Omega \gamma(\mathcal{R}_{local}(x)-1)\mathrm{d}x, d_I \to +\infty.$$

Notice that $\gamma(\widetilde{\mathcal{R}_{local}(x)} - 1) < 0$ is equivalent to

$$\int_\Omega \left(\beta_1 + \frac{\beta_2 c}{m}\right)\mathrm{d}x < \int_\Omega \gamma \mathrm{d}x, \tag{7.4}$$

it shows that it is the limiting case when $U \to \frac{A}{\alpha}$ and $d_S \to 0$. Then, below we give biological explanation: Suppose that the Ω itself is not favorable sites for the recovered individuals in the sense that (7.4), then we know that the recovered individuals can be eliminated by limiting the movement of the susceptible individuals, although there are pathogen favored sites in domain Ω exist in the sense that (7.3).

In addition, Theorem 6.2 and Theorem 6.3 indicate that limiting the mobility of the infected hosts, the infected individuals concentrate on certain points, which are the recovered individuals's most favored sites. It is indeed the set of locations where infected individuals will stay in the sense that $(1+\overline{\frac{c}{m}})\overline{I} + \overline{(\beta_1 + \frac{\beta_2 c}{m})}\overline{I} \to H$ as the diffusion rate $d_I \to 0$.

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Declarations

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Appendix A

Let $\mathscr{X} := C(\bar{\Omega}, R^3)$ be a functional space and satisfy the following norm form:

$$||\varphi||_{\mathscr{X}} := \max\{\sup_{x\in\Omega}|\varphi_1|, \sup_{x\in\Omega}|\varphi_2|, \sup_{x\in\Omega}|\varphi_3|\}, \varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathscr{X}.$$

For $i = 1, 2$, let $A_i : \mathscr{D}(A_i) \to C(\Omega, R)$ be the linear operator, which is described by

$$A_1\varphi := d_S\Delta\varphi, A_2\varphi := d_I\Delta\varphi,$$

where

$$\mathscr{D}(A_i) := \{\varphi \in \cap_{p>1}W^{2,p}(\Omega) : \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega \text{ and } A_i\varphi \in C(\bar{\Omega}, \mathbb{R})\},$$

which implies that A_i is the infinitesimal generator and it is a strong and continuous semigroup. Let $e^{A_i t} > 0, i = 1, 2$, in $C(\Omega, R)$, then the infinitesimal generator $\mathcal{A} : \mathscr{X} \to \mathscr{X}$ is defined by

$$\mathcal{A}\phi(x) := \begin{pmatrix} A_1\phi_1 \\ A_2\phi_2 \\ 0 \end{pmatrix}, \phi = (\phi_1, \phi_2, \phi_3) \in \mathscr{D}(\mathcal{A}), \tag{7.5}$$

then, the infinitesimal generator $\mathcal{A}$ is a strong and continuous semigroup. Let $(e^{t\mathcal{A}})_{t>0}$ in $\mathscr{X}$, $\mathscr{D}(\mathcal{A}) := \mathscr{D}(A_1) \times \mathscr{D}(A_2) \times C(\bar{\Omega}, R) \subset \mathscr{X}$, the nonlinear operator $\mathscr{F} : \mathscr{X} \to \mathscr{X}$ is defined by

$$\mathscr{F}(\phi)(x) := \begin{pmatrix} A(x) - \alpha(x)\phi_1 - \dfrac{\beta_1(x)\phi_1\phi_2}{\phi_1 + \phi_2 + \phi_3} - \dfrac{\beta_2(x)\phi_1\phi_3}{\phi_1 + \phi_2 + \phi_3} \\ \dfrac{\beta_1(x)\phi_1 I}{\phi_1 + \phi_2 + \phi_3} + \dfrac{\beta_2(x)\phi_1\phi_3}{\phi_1 + \phi_2 + \phi_3} - \gamma(x)\phi_2 \\ c(x)\phi_2 - m(x)\phi_3 \end{pmatrix}, \tag{7.6}$$

where $\phi = (\phi_1, \phi_2, \phi_3) \in \mathscr{X}_+$. Thus, the Cauchy problem of system (2.1) in $\mathscr{X}_+$ can be described by:

$$\frac{\partial}{\partial t}u(x, t; u_0) = \mathcal{A}u(x, t; u_0) + \mathscr{F}(u(x, t; u_0)), u(x, 0; u_0) = u_0. \tag{7.7}$$

Some properties of system (2.1) on $\mathscr{X}_+$ are given below:

Lemma 7.1. *For $u_0 \in \mathscr{D}(\mathcal{A}) \subset \mathscr{X}_+$, there exists $T_{max} > 0$ such that the problem (3.20) has*

$$u(x,t;u_0) = e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}}\mathscr{F}(u(x,s;u_0))\mathrm{d}s, t \in [0, T_{max})$$

with $T_{max} < +\infty$, where the variable $\mathcal{A}$ and the variable $\mathscr{F}$ are defined by Eqs. (7.5) and (7.6). Moreover, when $T_{max} = +\infty$, $\lim_{t\to T_{max}} ||u(x,t,u_0)|| = \infty$, which implies that system has nonnegative solution.

For detailed explanation, please refer to Lemma 4.16 in Ref. [26].

Appendix B

Based on Refs. [26,38], for any initial condition $u_0 \in \mathscr{X}_+$, we denote the solution $u(x,t)$ of (2.1) as follows

$$u(x,t;u_0) = (S(x,t;u_0), I(x,t;u_0), R(x,t;u_0)).$$

In addition, based on Ref. [26], we denote the semiflow generated of (2.1) as $\phi(t) : \mathscr{X}_+ \to \mathscr{X}_+, t > 0$, that is,

$$\phi(t)u_0 := u(x,t;u_0) = (S(x,t;u_0), I(x,t;u_0), R(x,t;u_0)),$$

and denote $\kappa(\cdot)$ as a Kuratowski measure of non-compactness. Then, we have $\kappa(\mathscr{B}) := \inf\{r\}$, the Kuratowski measure $\mathscr{B}$ exists a finite cover of diameter less than r, where set $\mathscr{B}$ is bounded. Then, if the Kuratowski measure $\kappa(\mathscr{B}) = 0$, we know that $\mathscr{B}$ is pre-compact.

In other words, we need to claim that $\phi(t)$ is a κ-contraction. Its equivalent condition is as follows. For any time $t > 0$, there exists the Kuratowski measure $\kappa(t) : R_+ \to R_+$ and satisfies the inequality $0 < \kappa(t) < 1$ such that set $\mathscr{B}$ is bounded, $\{\phi(s)\mathscr{B}, 0 < s < t\}$ is bounded and the inequality $\kappa((t)\mathscr{B}) < \kappa(t)\kappa(\mathscr{B})$.

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