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On a singular limit as $\theta \to 0$ for a model for the evolution of morphogens in a growing tissue

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Abstract. This paper is devoted to the singular limit of a model for the regulation of growth and patterning in developing tissues by diffusing morphogens. The model is governed by a system of nonlinear PDEs. The arguments are based on energy estimates and a change of variable that reduces the system into a nonlinear PDE with singular diffusion.

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1. Introduction

The differentiation and growth of embryonic cells are mainly regulated by morphogens (see [1,8,9,12]). Experimental evidences show that morphogens develop from a localized source spreading in concentration gradients that control the behavior of surrounding cells as a function of their distance from the source, see Wartlick, Mumcu, Kicheva, Bitting, Seum, Jülicher, and González-Gaitán in [10,11].

The experimental observations mentioned in [10,11] have been implemented in the mathematical model proposed by Averbukh, Ben-Zvi, Mishra and Barkai in [2], in which a growth law based on a parameter θ is formulated. It takes into account the fact that a cell divides when it detects that the relative morphogens concentration increases by a factor of $1+\theta$.

The model developed in [2] is the following one

$$\begin{cases} \partial_t M + \partial_x(uM) + \alpha M = D\partial^2_{xx} M, & t>0,\ 0<x<L(t),\\ \partial_t M + u\partial_x M - \dfrac{\theta}{\log 2} M\partial_x u = 0, & t>0,\ 0<x<L(t),\\ \partial_x M(t,0) = -\dfrac{\eta}{D}, \quad \partial_x M(t,L(t)) = 0, & t>0,\\ M(0,x) = M_0(x), & 0<x<L_0,\\ u(t,0)=0, & t>0,\\ L'(t) = u(t,L(t)), & t>0,\\ L(0)=L_0, \end{cases} \tag{1.1}$$

where the unknowns are

$$M = M_\theta(t,x), \qquad u = u_\theta(t,x), \qquad L = L_\theta(t),$$

and

$$L_0 > 0; \quad 0 < c_* \le M_0(x) \le c^*, \quad 0 \le x \le L_0.$$

Here $M_\theta(t,x)$ is the *morphogen concentration* in the *one-dimensional growing tissue* $[0, L_\theta(t)]$, $L_\theta(t)$ is the *length of the tissue*, $u_\theta(t,x)$ is the *(local) flow rate of the growing tissue* with $\partial_x u_\theta(t,x)$ being the *cell proliferation rate*, and α, D, η are positive parameters that correspond to the *morphogen degradation rate*, *diffusion rate* and *incoming morphogen flux rate*, respectively. The evolution of the morphogens concentration in (1.1) is described by the first equation, which is a nonlinear advection–reaction–diffusion PDE. The second equation gives the expression of the cell division rule due to morphogens proliferation and flow rates. Finally, the tissue length $L(t)$ obeys an ODE flow type. The two PDEs are augmented with suitable initial data and non-homogeneous Neumann-type boundary conditions.

Here we are interested in the analysis of $(M_\theta(t,x), u_\theta(t,x), L_\theta(t))$ as

$$\theta \to 0^+,$$

as a consequence in the following we will always assume

$$0 < \theta < \log 2.$$

Under this condition we have established in [4] the well-posedness (existence, uniqueness, and stability) and in [6] the asymptotic behavior as $t \to \infty$ of $(M_\theta(t,x),\ u_\theta(t,x),\ L_\theta(t))$. We will often recall some results of these papers, and therefore, we will assume that the hypotheses assumed therein are satisfied, also here. Before stating them explicitly, we point out that the hypothesis in [6]:

$$DM_0''(x) - \alpha M_0(x),\ 0 \le x \le L_0, \text{has constant sign},$$

in this paper it is assumed by formulating two alternative conditions:

$$DM_0''(x) - \alpha M_0(x) \ge 0 \quad \text{or} \quad DM_0''(x) - \alpha M_0(x) \le 0.$$

The results in the two cases are different while retaining a certain "symmetry". Having said that, in this paper we assume that the following hypotheses are satisfied

$$0 < c_* \le M_0(x) \le c^*, \quad 0 \le x \le L_0; \quad M_0 \in H^2(0, L_0), \tag{1.2}$$

and one within the following

$$M_0'(x) \le 0; \quad DM_0''(x) - \alpha M_0(x) \ge 0, \quad 0 \le x \le L_0, \tag{1.3}$$
$$M_0'(x) \le 0; \quad DM_0''(x) - \alpha M_0(x) \le 0, \quad 0 \le x \le L_0. \tag{1.4}$$

The difference between the two cases is further highlighted by the different initial mean morphogens concentrations, indeed

$$\begin{aligned} DM_0''(x) - \alpha M_0(x) \ge 0,\ 0 \le x \le L_0 \ &\Rightarrow\ \|M_0\|_{L^1(0,L_0)} \le \frac{\eta}{\alpha}, \\ DM_0''(x) - \alpha M_0(x) \le 0,\ 0 \le x \le L_0 \ &\Rightarrow\ \|M_0\|_{L^1(0,L_0)} \ge \frac{\eta}{\alpha}. \end{aligned}$$

Key tool for the analysis of the well-posedness (see [3,4,7]) and of the asymptotic behavior as $t \to \infty$ (see [5,6]) is the definition of a suitable family of "characteristic" curves which start at the points of $[0, L_0]$ and "cover" $\{(t,x) \,|\, t \ge 0, 0 \le x \le L_\theta(t)\}$. Let us briefly recall them because they are also useful in this paper.

Let $(M_\theta(t,x), u_\theta(t,x), L_\theta(t))$ be the solution of (1.1), for every $y \in [0, L_0]$, let $X_\theta(t,y)$ be the solution of

$$\begin{cases} \dfrac{d}{dt} X_\theta(t,y) = u_\theta(t, X_\theta(t,y)), \\ X_\theta(0,y) = y. \end{cases} \tag{1.5}$$

Thanks to (1.1), it is clear that 0 solves (1.5) in correspondence of $y = 0$ and $L_\theta(t)$ solves (1.5) in correspondence of $y = L_0$. The image of $X_\theta(t,\cdot)$ is $[0, L_\theta(t)]$. $X_\theta(t,\cdot)$ is invertible; its inverse $Y_\theta(t,\cdot)$ is defined on $[0, L_\theta(t)]$ and its image is $[0, L_0]$ (see [3,4,7]).

The main results of this paper are the following.

Theorem 1.1. *If* $\|M_0\|_{L^1(0,L_0)} \le \dfrac{\eta}{\alpha}$ *and the assumptions* (1.2), (1.3) *hold, we have that*

$$i)\ \lim_{\theta\to 0}\int_0^{L_0} |M_\theta(t,X_\theta(t,y))-M_0(y)|^r \mathrm{d}y = 0, \quad 1\le r<\infty,$$

uniformly with respect to t *on every compact set* $[0,T]$;

$$ii)\ 0\le u_\theta(t,x); \quad \limsup_{\theta\to 0} u_\theta(t,x) \le e^{-\alpha t}\,\frac{\eta-\alpha\|M_0\|_{L^1(0,L_0)}}{M_0(L_0)};$$

$$iii)\ L_0e^{-\alpha t}+\sqrt{\alpha D}\tanh\Big(L_0\sqrt{\frac{\alpha}{D}}\Big)\frac{1-e^{-\alpha t}}{\alpha} \le \liminf_{\theta\to 0} L_\theta(t)$$
$$\le \limsup_{\theta\to 0} L_\theta(t) \le L_0e^{-\alpha t}+\frac{\eta}{M_0(L_0)}\frac{1-e^{-\alpha t}}{\alpha}.$$

Theorem 1.2. *If* $\|M_0\|_{L^1(0,L_0)} \ge \dfrac{\eta}{\alpha}$ *and the assumptions* (1.2), (1.4) *hold, we have that*

$$i)\ \lim_{\theta\to 0}\int_0^{L(t)} |M_\theta(t,x)-M_0(Y_\theta(t,x)|^r \mathrm{d}y = 0, \quad 1\le r<\infty,$$

uniformly with respect to t *on every compact set* $[0,T]$;

$$ii)\ -e^{-\alpha t}\,\frac{\alpha\|M_0\|_{L^1(0,L_0)}-\eta}{M_0(L_0)}0 \le \liminf_{\theta\to 0} u_\theta(t,x); \quad u_\theta(t,x)\le 0;$$

$$iii)\ L_0e^{-\alpha t}+\frac{\eta}{M_0(0)}\frac{1-e^{-\alpha t}}{\alpha} \le \liminf_{\theta\to 0} L_\theta(t) \le \limsup_{\theta\to 0} L_\theta(t) \le \quad \le L_0e^{-\alpha t}+\frac{\eta}{M_0(0)}\frac{1-e^{-\alpha t}}{\alpha}+ \sqrt{\alpha D}\Big(\log\frac{M_0(0)}{M_0(L_0)}\Big)\frac{1-e^{-\alpha t}}{\alpha}.$$

The paper is organized as follows. In Sect. 2 we recall some preliminary results. Section 3 is devoted to some a priori estimates on the sign of the derivatives of the unknowns. Theorems 1.1 and 1.2 are proved in Sects. 4 and 5, respectively.

2. Preliminary results

We transform (1.1) into a problem equivalent to it, in the sense that the well-posedness of one of them implies the well-posedness of the other one and from the solution of one of them we obtain at the solution of the other one. Defining

$$\beta := \frac{\log 2}{\theta}, \qquad N_\beta(t,y) := M_\theta(t,X_\theta(t,y)),$$

(1.1) is equivalent to the following problem

$$\begin{cases} \partial_t N_\beta + aN_\beta = d\Big(\dfrac{M_0(y)}{|N_\beta|}\Big)^\beta \partial_y\Big(\Big(\dfrac{M_0(y)}{|N_\beta|}\Big)^\beta \partial_y N_\beta\Big), & t>0,\ 0<y<L_0,\\ \Big(\dfrac{M_0(0)}{|N_\beta(t,0)|}\Big)^\beta \partial_y N_\beta(t,0) = -\dfrac{\eta}{D},\ \Big(\dfrac{M_0(L_0)}{|N_\beta(t,L_0)|}\Big)^\beta \partial_y N_\beta(t,L_0)=0, & t>0,\\ N_\beta(0,y)=M_0(y), & 0<y<L_0, \end{cases} \tag{2.1}$$

where

$$a := \frac{\alpha}{\beta+1}; \quad d := \frac{D}{\beta+1}. \tag{2.2}$$

For every $0 \le T < \infty$ we will use the following notation

$$E_T :=]0, T[\times]0, L_0[; \quad \overline{E}_T := \text{ closure of } E_T.$$

We define in an analogous way E_∞ and $\overline{E}_\infty$.

Let us recall some properties of $N_\beta(t, y)$ useful in the next sections.

Theorem 2.1. *(Existence, uniqueness, and regularity of $N_\beta(t, y)$ [4, Theorem 2.1], [6, Theorem 2.3]) If (1.2) holds, then, for every β and $T > 0$, (2.1) admits a unique solution $N_\beta(t, y)$ such that*

i) $c_* e^{-\frac{\alpha}{\beta+1}t} \le N_\beta(t, y),\ t \ge 0,\ 0 \le y \le L_0$ *and* $N_\beta \in L^\infty(E_\infty)$;

ii) $\partial_t N_\beta,\ \partial_y N_\beta,\ \partial_y \left(\frac{M_0^\beta}{N_\beta^\beta} \partial_y N_\beta\right) \in L^2(E_T),\ \partial_{yy}^2 N \in L^1(E_T)$;

iii) here exists $c(T) > 0$ *such that for every* $(t_1, y_1), (t_2, y_2) \in E_T$ $|N_\beta(t_1, y_1) - N_\beta(t_2, y_2)| \le c(T)\big(\sqrt{|t_1 - t_2|} + |y_1 - y_2|\big)^{\frac{1}{4}}$;

iv) $\partial_y N_\beta \in C(]0, \infty[\times[0, L_0]);\ \ \partial_t N_\beta,\ \partial_{yy}^2 N_\beta \in C(E_\infty)$.

Let us show how to pass from $N_\beta(t, y)$ to $(M_\theta(t, x),\ u_\theta(t, x),\ L_\theta(t))$ and vice versa. Thanks to the properties of $N_\beta(t, y)$, the function $(t, y) \mapsto \left(\frac{M_0(y)}{N_\beta(t,y)}\right)^\beta$ is positive and Hölder continuous in every $\overline{E}_T$ (see [7, Theorem 2.1], [3, Theorem 2.1], and [6, Theorem 2.3]). As a consequence

$$\frac{\mathrm{d}Y_\theta}{\mathrm{d}x} = \Big(\frac{M_0(Y)}{N_\beta(t, Y)}\Big)^\beta,\ Y_\theta(t, 0) = 0$$

([1]) admits a unique (maximal) solution $Y_\theta(t, \cdot)$.

Let $[0, L_\theta(t)]$ be the (maximal) existence interval of $Y_\theta(t, \cdot)$. We have $Y_\theta(t, L_\theta(t)) = L_0$, and defining

$$M_\theta(t, x) = N_\beta(t, Y(t, x));$$

$$u_\theta(t, x) = \beta \int_0^{Y_\theta(t,x)} \frac{N_\beta(t, y)^{\beta-1} \partial_t N_\beta(t, y)}{M_0(y)^\beta} \mathrm{d}y;$$

$$L_\theta(t) = \int_0^{L_0} \big(\frac{N_\beta(t, y)}{M_0(y)}\big)^\beta \mathrm{d}y,$$

$(M_\theta(t, x),\ u_\theta(t, x)\ L_\theta(t))$ is a solution of (1.1). ([2]) As a first step in our analysis, we begin by studying the behavior of $N_\beta(t, y)$ as

$$\beta \to \infty,$$

from now on we assume that

$\beta > 1$ ([3]).

Let us also briefly recall the results on the asymptotic behavior for $t \to \infty$ of $N_\beta(t, y)$ (see [6, Theorem 2.1]) and of $(M_\theta(t, x),\ u_\theta(t, x),\ L_\theta(t))$ (see [6, Theorem1.1]). If the assumptions (1.2) and (1.3) or (1.2) and (1.4) hold, then the function $N_\beta(\cdot, y)$ is monotone and its limit

$$\overline{N}_\beta(y) := \lim_{t\to\infty} N_\beta(t, y)$$

[1] t is a parameter.

[2] $Y_\theta(t, \cdot)$ is the inverse of $X_\theta(t, \cdot)$ for every $t \ge 0$.

[3] The assumption $\theta < \log(2)$ is equivalent to $\beta > 1$.

belongs to $C^2([0,L_0])$, is positive, decreasing and solves the stationary problem

$$\begin{cases} \alpha\overline{N}_\beta = D\Big(\frac{M_0(y)}{\overline{N}_\beta}\Big)^\beta\Big[\Big(\frac{M_0(y)}{\overline{N}_\beta}\Big)^\beta\overline{N}_\beta'\Big]', \quad 0\le y\le L_0,\\ \Big(\frac{M_0(0)}{\overline{N}_\beta(0)}\Big)^\beta\overline{N}_\beta'(0) = -\frac{\eta}{D}; \quad \Big(\frac{M_0(L_0)}{\overline{N}_\beta(L_0)}\Big)^\beta\overline{N}_\beta'(L_0) = 0. \end{cases} \tag{2.3}$$

Moreover, $y\mapsto N_\beta(t,y)$ converges to $\overline{N}_\beta(y)$ uniformly with respect to y as $t\to\infty$.

The triplet $(M_\theta(t,x),\, u_\theta(t,x),\, L_\theta(t))$ satisfies the following statements.

i) $L_\theta(t)$ converges to $\overline{L}_\theta$ as $t\to\infty$, and

$$|L_\theta(t)-\overline{L}_\theta|\le ce^{-\alpha t},$$

for some constant c independent on t.

ii) $\lim\limits_{t\to\infty} u_\theta(t,x) = 0$ uniformly with respect to x.

iii) $M_\theta(t,x)$ converges to $\overline{M}_\theta\in C^2([0,\overline{L}_\theta])$ as $t\to\infty$, $M_\theta(t,\xi L_\theta(t))$ converges to $\overline{M}_\theta(\xi\overline{L}_\theta)$ uniformly with respect to $0\le\xi\le 1$. Moreover, $\overline{M}_\theta(x)$ satisfies

$$\begin{cases} \alpha\overline{M}_\theta = D\overline{M}_\theta'', \ \text{ in } [0,\overline{L}_\theta],\\ \overline{M}_\theta'(0) = -\frac{\eta}{D};\ \overline{M}_\theta'(\overline{L}_\theta) = 0, \end{cases}$$

and its explicit expression is

$$\overline{M}_\theta(x) = \frac{\eta}{\sqrt{\alpha D}}\frac{\cosh\big[(x-\overline{L}_\theta)\sqrt{\frac{\alpha}{D}}\big]}{\sinh\big[\overline{L}_\theta\sqrt{\frac{\alpha}{D}}\big]}, \quad 0\le x\le\overline{L}_\theta.$$

We conclude this section recalling that

$$\overline{N}_\beta(0) = \overline{M}_\theta(0) = \frac{\eta}{\sqrt{\alpha D}}\coth\Big(\overline{L}_\theta\sqrt{\frac{\alpha}{D}}\Big).$$

3. On the signs of $\partial_t N_\beta(t,y)$ and $\partial_y N_\beta(t,y)$

On the sign of $\partial_t N_\beta(t,y)$, we proved the following result.

Theorem 3.1. *(Sign of $\partial_t N_\beta(t,y)$ [6, Theorem 2.5]) For every $t>0$, $0\le y\le L_0$, we have that*

i) $DM_0''(y)-\alpha M_0(y)\ge 0 \ \Rightarrow\ \partial_t N_\beta(t,y)\ge 0$,

ii) $DM_0''(y)-\alpha M_0(y)\le 0 \ \Rightarrow\ \partial_t N_\beta(t,y)\le 0$.

To clarify the link between the hypotheses (1.3), (1.4) and the initial mean morphogens concentration, i.e., $\|M_0\|_{L^1(0,L_0)}$, the following lemma is needed.

Lemma 3.1. *([6, Theorem 2.1. ii]) For every $\beta\ge 1$ and $t\ge 0$, we have that*

$$\int_0^{L_0}\frac{N_\beta(t,y)^{\beta+1}}{M_0(y)^\beta}\mathrm{d}y = \frac{\eta}{\alpha} + e^{-\alpha t}\Big(\|M_0\|_{L^1(0,L_0)}-\frac{\eta}{\alpha}\Big).$$

Proof. Let us quickly sketch the proof of [6, Theorem2.1.ii)]. It is not difficult to rewrite the equation of (2.1) as follows

$$\partial_t\Big(e^{\alpha t}\frac{N_\beta(t,y)^{\beta+1}}{M_0(y)^\beta}\Big) = D\partial_y\Big[e^{\alpha t}\Big(\frac{M_0(y)}{N_\beta(t,y)}\Big)^\beta\partial_y N_\beta(t,y)\Big].$$

We integrate both sides in y on $[0, L_0]$, thanks to the boundary and initial data in (2.1),

$$\partial_t\Big(e^{\alpha t}\int_0^{L_0}\frac{N_\beta(t,y)^{\beta+1}}{M_0(y)^\beta}\mathrm{d}y\Big) = De^{\alpha t}\frac{\eta}{D}$$

and then

$$e^{\alpha t}\int_0^{L_0}\frac{N_\beta(t,y)^{\beta+1}}{M_0(y)^\beta}\mathrm{d}y = \int_0^{L_0}M_0(y)\mathrm{d}y + \frac{\eta}{\alpha}(e^{\alpha t}-1),$$

that gives the claim. □

The relation between the assumptions (1.3), (1.4) and $\|M_0\|_{L^1(0,L_0)}$ is clarified in the next statements

$$1.3 \ \Rightarrow\ DM_0^{''}(\cdot) - \alpha M_0(\cdot) \geq 0 \ \Rightarrow\ \|M_0\|_{L^1(0,L_0)} \leq \frac{\eta}{\alpha}, \tag{3.1}$$

$$1.4 \ \Rightarrow\ DM_0^{''}(\cdot) - \alpha M_0(\cdot) \leq 0 \ \Rightarrow\ \|M_0\|_{L^1(0,L_0)} \geq \frac{\eta}{\alpha}. \tag{3.2}$$

We prove only (3.1), because the same argument works also for (3.2)

$$\begin{aligned} DM_0^{''}(y) - \alpha M_0(y) \geq 0 \ \Rightarrow\ \partial_t N_\beta(t,y) \geq 0 \ \Rightarrow\ \partial_t\int_0^{L_0}\frac{N_\beta(t,y)^{\beta+1}}{M_0(y)^\beta}\mathrm{d}y \geq 0 \ \Leftrightarrow \\ \Leftrightarrow\ \partial_t\left\{\frac{\eta}{\alpha} + e^{-\alpha t}\Big(\|M_0\|_{L^1(0,L_0)} - \frac{\eta}{\alpha}\Big)\right\} \geq 0 \ \Leftrightarrow \\ \Leftrightarrow\ \alpha e^{-\alpha t}\Big(\frac{\eta}{\alpha} - \|M_0\|_{L^1(0,L_0)}\Big) \geq 0 \ \Leftrightarrow\ \|M_0\|_{L^1(0,L_0)} \leq \frac{\eta}{\alpha}. \end{aligned}$$

To determine the sign of $\partial_y N_\beta(t,y)$, it is convenient to consider a reformulation of (2.1) useful for partially camouflaging the cumbersome initial datum $M_0(y)$. We will use the following notations, given $0 \leq T < \infty$,

$$Q_T :=]0,T[\times]0,1[; \quad \overline{Q}_T := \text{ closure } Q_T.$$

Similarly, we define Q_∞ and $\overline{Q}_\infty$.

Due to the assumptions on M_0, we can consider the function

$$z = Z_0(y) := \frac{1}{\mu_0}\int_0^y M_0(\xi)^{-\beta}\mathrm{d}\xi, \ \ 0 \leq y \leq L_0,$$

where

$$\mu_0 = \left\|M_0^{-\beta}\right\|_{L^1(0,L_0)}.$$

If $Y_0(z)$ is the inverse of $Z_0(y)$, we define

$$n_\beta(t,z) := N_\beta(t, Y_0(z)).$$

Passing from the unknown $N_\beta(t,y)$ to $n_\beta(t,z)$, we simplify (2.1) in the following way

$$\begin{cases} \partial_t n + an = \dfrac{A}{|n|^\beta}\partial_z\Big(\dfrac{\partial_z n}{|n|^\beta}\Big), & (t,z) \in Q_\infty, \\ \dfrac{\partial_z n(t,0)}{|n(t,0)|^\beta} = -B, \quad \dfrac{\partial_z n(t,1)}{|n(t,1)|^\beta} = 0, & t > 0, \\ n(0,z) = M_0(Y_0(z)), & 0 < z < 1, \end{cases} \tag{3.3}$$

where

$$A := \frac{d}{\mu_0^2} = \frac{D}{(\beta+1)\mu_0^2}; \qquad B := \frac{\eta\mu_0}{D}. \tag{3.4}$$

Theorem 3.2. *(Sign of $\partial_y N_\beta(t,y)$) Let $\beta > 0$ be given. If*

$$M_0'(y) \le 0, \;\; 0 \le y \le L_0,$$

then

$$\partial_y N_\beta(t,y) \le 0, \;\; t > 0, \, 0 \le y \le L_0.$$

Proof. In order to keep the presentation simple and clear, we start considering (3.3) and proving

$$\partial_z n_\beta(t,z) \le 0, \;\; t > 0, \;\; 0 < z < 1. \tag{3.5}$$

([4]) We begin by assuming $\beta \neq 1$. Consider the functions

$$v(t,z) = n_\beta(t,z)^{1-\beta}, \qquad \omega = \partial_z v_\beta,$$

ω satisfies in the weak sense the following identity

$$A\partial_z(v^m \partial_z \omega) + a(\beta-1)\omega = \partial_t \omega \quad \text{in} \;\; Q_\infty, \tag{3.6}$$

with

$$\begin{aligned} m =& \frac{2\beta}{\beta-1} \\ |v^m \partial_z \omega| =& |(n^{1-\beta})^{\frac{2\beta}{\beta-1}} \partial^2_{zz} v| = \big|\frac{1}{n^{2\beta}} \partial_z\big((1-\beta)\frac{\partial_z n}{n^\beta}\big)\big| \le \\ \le& \frac{|\beta-1|}{c_*^{\frac{\alpha\beta t}{\beta+1}}} \big|\frac{1}{n^\beta}\partial_z\big(\frac{\partial_z n}{n^\beta}\big)\big| \in L^2(Q_t), \quad \forall t > 0; \\ |\omega| =& |\partial_z v| = |\partial_z n^{1-\beta}| = |(1-\beta)\frac{\partial_z n}{n^\beta}| \le \frac{|\beta-1|}{c_*^{\frac{\alpha\beta t}{\beta+1}}} |\partial_z n| \in L^2(Q_t); \\ |v^m \partial_z \omega \partial_z \omega| =& (v^{\frac{m}{2}} \partial_z \omega)^2 = \big(\frac{1}{n^\beta}(1-\beta)\partial_z\big(\frac{\partial_z n}{n^\beta}\big)\big)^2 \in L^1(Q_t). \end{aligned}$$

.

([5]) Moreover, the following are satisfied in the sense of traces

$$\begin{aligned} \omega(t,0) = B(\beta-1); \quad \omega(t,1) = 0, \quad t > 0, \\ \omega(0,z) = \mu_0(1-\beta)M_0'(Y_0(z)), \quad 0 < z < 1. \end{aligned}$$

Let us distinguish two cases $0 < \beta < 1$, $1 < \beta$.

$\underline{0 < \beta < 1}$. We multiply (3.6) by

$$\omega(t,z)^+ e^{\lambda t}, \quad \text{whit } \lambda = -2a\beta, \; \omega^+ := \frac{1}{2}(|\omega| + \omega),$$

and integrate over Q_t, $t > 0$. Being

$$\omega(t,0)^+ = \omega(t,1)^+ = 0,$$

we have

$$-A \int\limits_{Q_t} v^m (\partial_z \omega)^2 \frac{\operatorname{sign}(\omega) + 1}{2} e^{\lambda\tau} \mathrm{d}\tau \mathrm{d}z + a(\beta-1) \int\limits_{Q_t} \omega\omega^+ e^{\lambda\tau} \mathrm{d}\tau \mathrm{d}z = \int\limits_{Q_t} \partial_\tau \omega \omega^+ e^{\lambda\tau} \mathrm{d}\tau \mathrm{d}z. \tag{3.7}$$

[4] $n(t,z)$ ed $N_\beta(t,y)$ share the same regularity (see [4, Theorem 2.1] and [6, Theorem2.3]).

[5] The differentiation in the weak sense of the function $v^m \partial_z \omega$ and the role of the test function ω^- can be justified by the following observations

Since

$$\partial_\tau \omega \cdot \omega^+ e^{\lambda\tau} = \partial_\tau(\omega\omega^+ e^{\lambda\tau}) - \omega e^{\lambda\tau}\Big(\partial_\tau\omega \frac{\mathrm{sign}\,(\omega)+1}{2} + \lambda\omega^+\Big)$$
$$= \partial_\tau(\omega\omega^+ e^{\lambda\tau}) - \partial_\tau\omega \cdot \omega^+ e^{\lambda\tau} - \lambda\omega\omega^+ e^{\lambda\tau},$$

we have

$$\partial_\tau \omega \cdot \omega^+ e^{\lambda\tau} = \frac{1}{2}\partial_\tau(\omega\omega^+ e^{\lambda\tau}) - \frac{\lambda}{2}\omega\omega^+ e^{\lambda\tau},$$

and using (3.7)

$$-A\int_{Q_t} v^m(\partial_z\omega)^2\frac{\mathrm{sign}\,(\omega)+1}{2}e^{\lambda\tau}\mathrm{d}\tau\mathrm{d}z + \ a(\beta-1)\int_{Q_t}\omega\omega^+ e^{\lambda\tau}\mathrm{d}\tau\mathrm{d}z$$
$$= \frac{1}{2}\int_0^1 \omega(t,z)\omega(t,z)^+ e^{\lambda t}\mathrm{d}z - \frac{1}{2}\int_0^1 \omega(0,z)\omega(0,z)^+\mathrm{d}z - \frac{\lambda}{2}\int_{Q_t}\omega\omega^+ e^{\lambda\tau}\mathrm{d}\tau\mathrm{d}z.$$

Being $\beta < 1$

$$\omega(0,z)^+ = \Big(\mu_0(1-\beta)M_0'(Y_0(z))\Big)^+ = 0, \qquad \lambda = -2a\beta,$$

and then

$$-A\int_{Q_t} v^m(\partial_z\omega)^2\frac{\mathrm{sign}\,(\omega)+1}{2}e^{\lambda\tau}\mathrm{d}\tau\mathrm{d}z - a\int_{Q_t}\omega\omega^+ e^{\lambda\tau}\mathrm{d}\tau\mathrm{d}z = \frac{1}{2}\int_0^1 \omega(t,z)\omega(t,z)^+ e^{\lambda t}\mathrm{d}z.$$

Since $\omega\omega^+ \geq 0$ the two sides of the identity have different sings. As a consequence, they must vanish and

$$\int_0^1 \omega(t,z)\omega(t,z)^+ e^{\lambda t}\mathrm{d}z = 0, \qquad t>0,$$

that gives $\omega(t,z)^+ = 0$, namely $\omega(t,z) \leq 0$. In light of the definition of $\omega(t,z)$ we have

$$\partial_z v(t,z) = \omega(t,z) \leq 0 \ \Leftrightarrow\ (1-\beta)n(t,z)^{-\beta}\partial_z n(t,z) \leq 0 \ \Leftrightarrow\ \partial_z n(t,z) \leq 0.$$

$\underline{\beta > 1}$. We argue as before and multiply (3.6) by

$$\omega(t,z)^- e^{\lambda t}, \qquad \text{with } \lambda = -2a\beta.$$

Being $\beta > 1$ we have

$$\omega(t,0)^- = \omega(t,1)^- = 0, \qquad \omega(0,z)^- = \Big(\mu_0(1-\beta)M_0'(Y_0(z))\Big)^- = 0,$$

and then

$$-A\int_{Q_t} v^m(\partial_z\omega)^2\frac{\mathrm{sign}\,(\omega)-1}{2}e^{\lambda\tau}\mathrm{d}\tau\mathrm{d}z - a\int_{Q_t}\omega\omega^- e^{\lambda\tau}\mathrm{d}\tau\mathrm{d}z = \frac{1}{2}\int_0^1 \omega(t,z)\omega(t,z)^- e^{\lambda t}\mathrm{d}z.$$

Since $\omega\omega^- \leq 0$, the two sides of the identity have different sings. As a consequence, they must vanish and

$$\int_0^1 (\omega\omega^-)(t,z)e^{\lambda t}\mathrm{d}z = 0, \qquad t>0,$$

that gives $\omega(t,z)^- = 0$, namely $\omega(t,z) \geq 0$. As in the previous case

$$\partial_z v(t,z) = \omega(t,z) \geq 0 \ \Leftrightarrow \ (1-\beta) n_\beta(t,z)^{-\beta} \partial_z n_\beta(t,z) \geq 0 \ \Leftrightarrow \ \partial_z n_\beta(t,z) \leq 0.$$

$\underline{\beta = 1}$. Define

$$v = \log n_\beta, \qquad \omega = \partial_z v,$$

ω satisfies in the weak sense the identity

$$A\partial_z \left(\frac{1}{n_\beta^2} \partial_z \omega \right) = \partial_t \omega \quad \text{in} \quad Q_\infty \tag{3.8}$$

and in the sense of traces

$$\omega(t,0) = -B; \quad \omega(t,1) = 0; \quad \omega(0,z) = \mu_0 M_0'(Y_0(z)).$$

We multiply (3.8) by ω^+ and integrate over Q_t. Arguing as in the previous cases we obtain

$$-A \int\limits_{Q_t} \frac{1}{n_\beta^2} (\partial_z \omega)^2 \frac{\operatorname{sign}(\omega) + 1}{2} \mathrm{d}\tau \mathrm{d}z = \frac{1}{2} \int\limits_0^1 \omega(t,z) \omega(t,z)^+ \mathrm{d}z,$$

that implies $\omega(t,z)^+ = 0$, namely $\omega(t,z) \leq 0$. Therefore,

$$\partial_z v(t,z) = \omega(t,z) \leq 0 \ \Leftrightarrow \ \frac{\partial_z n_\beta(t,z)}{n_\beta(t,z)} \leq 0 \ \Leftrightarrow \ \partial_z n_\beta(t,z) \leq 0.$$

In this way we have proved (3.5).

Finally, being $N_\beta(t,y) = n_\beta(t, Z_0(y))$, we have

$$\partial_y N_\beta(t,y) = \partial_z n_\beta(t, Z_0(y)) Z_0'(y) = \frac{\partial_z n_\beta(t, Z_0(y))}{\mu_0 M_0(y)^\beta} \leq 0,$$

that concludes the proof. □

4. Proof of Theorem 1.1

We begin this section by proving some a priori estimates on $N_\beta(t,y)$ and $\partial_y N_\beta(t,y)$ independent on β.

Lemma 4.1. *We have that*

$$M_0(y) \leq N_\beta(t,y) \leq \frac{\eta}{\sqrt{\alpha D}} \coth\left(L_0 \sqrt{\frac{\alpha}{D}} \right), \quad (t,y) \in \overline{E}_\infty; \tag{4.1}$$

$$-\frac{\eta}{D} \leq \left(\frac{M_0(y)}{N_\beta(t,y)} \right)^\beta \partial_y N_\beta(t,y) \leq 0, \quad (t,y) \in]0,\infty[\times [0, L_0]. \tag{4.2}$$

Proof. We prove (4.1). The lower bound on $N_\beta(\cdot,y)$ follows from the monotonicity of $N_\beta(\cdot,y)$ (see (1.3) and Theorem 3.1) and the identity $N_\beta(0,y) = M_0(y)$. We have to prove the upper bound on $N_\beta(\cdot,y)$. Since

$$N_\beta(t,y) \leq \lim_{t \to \infty} N_\beta(t,y) = \overline{N}_\beta(y),$$

the monotonicity of $\overline{N}_\beta(y)$ (see [6, Theorem 2.1]) and (3.8) guarantee

$$N_\beta(t,y) \leq \overline{N}_\beta(0) = \overline{M}_\theta(0) = \frac{\eta}{\sqrt{\alpha D}} \coth\left(\overline{L}_\theta \sqrt{\frac{\alpha}{D}} \right).$$

Moreover, by observing

$$L_\theta(t) = \int_0^{L_0} \left(\frac{N_\beta(t,y)}{M_0(y)}\right)^\beta \mathrm{d}y \geq \int_0^{L_0} \left(\frac{N_\beta(0,y)}{M_0(y)}\right)^\beta \mathrm{d}y = L_0,$$

we must have

$$\overline{L}_\theta \geq L_0, \quad t \geq 0.$$

Since coth is nonincreasing,

$$N_\beta(t,y) \leq \frac{\eta}{\sqrt{\alpha D}} \coth\left(\overline{L}_\theta \sqrt{\frac{\alpha}{D}}\right) \leq \frac{\eta}{\sqrt{\alpha D}} \coth\left(L_0 \sqrt{\frac{\alpha}{D}}\right), \tag{4.3}$$

that proves (4.1).

We have to prove (4.2). Thanks to the assumption (1.3) and Theorem 3.1, we know that $\partial_t N_\beta(t,y) \geq 0$. Using the equation in (2.1) we have that $\left(\frac{M_0(y)}{N_\beta(t,y)}\right)^\beta \partial_y N_\beta(t,y)$ is nondecreasing with respect to y, for every $t > 0$ and $\beta > 1$. Using the boundary conditions, we gain

$$-\frac{\eta}{D} = \left(\frac{M_0(0)}{N_\beta(t,0)}\right)^\beta \partial_y N_\beta(t,0) \leq \left(\frac{M_0(y)}{N_\beta(t,y)}\right)^\beta \partial_y N_\beta(t,y) \leq \left(\frac{M_0(L_0)}{N_\beta(t,L_0)}\right)^\beta \partial_y N_\beta(t,L_0) = 0.$$

Employing (1.2), Theorem 2.1 and the fact

$$\left(\frac{M_0(y)}{N_\beta(t,y)}\right)^\beta > 0, \quad t > 0, \ 0 \leq y \leq L_0,$$

we conclude $\partial_y N_\beta(t,y) \leq 0$, $(t,y) \in]0,\infty[\times[0,L_0]$, that proves (4.2). □

We continue with the following result on the limit of $\partial_t N_\beta(t,y)$ as $\beta \to \infty$.

Theorem 4.1. *We have that*

$$\lim_{\beta\to\infty} \int_{E_T} (\partial_t N_\beta)^2 \mathrm{d}t\mathrm{d}y = 0.$$

The following lemma is needed

Lemma 4.2. *For every $T > 0$*

$$\limsup_{\beta\to\infty} \left\{ \int_{E_T} \left(\frac{N_\beta}{M_0}\right)^\beta (\partial_t N_\beta)^2 \mathrm{d}t\mathrm{d}y + \frac{D\beta}{2(\beta+1)} \int_{E_T} \left(\frac{M_0}{N_\beta}\right)^\beta (\partial_y N_\beta)^2 \frac{\partial_t N_\beta}{N_\beta} \mathrm{d}t\mathrm{d}y \right\} \leq 0.$$

Proof. We multiply the equation in (2.1) by $\left(\frac{N_\beta}{M_0}\right)^\beta \partial_t N_\beta(t,y)$:

$$\left(\frac{N_\beta}{M_0}\right)^\beta (\partial_t N_\beta(t,y))^2 + \frac{a}{\beta+2} \partial_t \frac{N_\beta^{\beta+2}}{M_0^\beta} = \mathrm{d}\partial_y \left[\left(\frac{M_0}{N_\beta}\right)^\beta \partial_y N_\beta\right] \partial_t N_\beta$$

and integrate over E_T :

$$\int_{E_T} \left(\frac{N_\beta}{M_0}\right)^\beta (\partial_t N_\beta(t,y))^2 \mathrm{d}t\mathrm{d}y + \frac{a}{\beta+2} \int_0^{L_0} \left(\frac{N_\beta(T,y)^{\beta+2}}{M_0(y)^\beta} - M_0(y)^2\right) \mathrm{d}y$$

$$= \mathrm{d} \int_0^T \left[\left(\frac{M_0}{N_\beta}\right)^\beta \partial_y N_\beta \, \partial_t N_\beta\right]_0^{L_0} \mathrm{d}t - \mathrm{d} \int_{E_T} \left(\frac{M_0}{N}\right)^\beta \partial_y N_\beta \, \partial_{yt}^2 N_\beta \mathrm{d}t\mathrm{d}y$$

$$
\begin{aligned}
&= \frac{\eta}{\beta+1}\int_0^T \partial_t N_\beta(t,0)\mathrm{d}t - \frac{d}{2}\int_{E_T}\left(\frac{M_0}{N_\beta}\right)^\beta \partial_t(\partial_y N_\beta)^2\mathrm{d}t\mathrm{d}y \\
&= \frac{\eta}{\beta+1}(N_\beta(T,0)-M_0(0)) \\
&\quad - \frac{d}{2}\int_0^{L_0}\left[\left(\frac{M_0}{N_\beta}\right)^\beta (\partial_y N_\beta)^2\right]_0^T \mathrm{d}y - \frac{d}{2}\int_{E_T}\beta\frac{M_0^\beta}{N_\beta^{\beta+1}}\,\partial_t N_\beta\,(\partial_y N_\beta)^2\mathrm{d}t\mathrm{d}y \\
&= \frac{\eta}{\beta+1}(N_\beta(T,0)-M_0(0)) - \frac{D}{2(\beta+1)}\int_0^{L_0}\left(\frac{M_0(y)}{N_\beta(T,y)}\right)^\beta (\partial_y N_\beta(y))^2\mathrm{d}y \\
&\quad + \frac{D}{2(\beta+1)}\int_0^{L_0}(M_0'(y))^2\mathrm{d}y - \frac{D\beta}{2(\beta+1)}\int_{E_T}\frac{M_0^\beta}{N_\beta^{\beta+1}}\,\partial_t N_\beta\,(\partial_y N_\beta)^2\mathrm{d}t\mathrm{d}y.
\end{aligned}
$$

Rearranging the terms in the following way

$$
\begin{aligned}
&\int_{E_T}\left(\frac{N_\beta}{M_0}\right)^\beta (\partial_t N_\beta(t,y))^2\mathrm{d}t\mathrm{d}y + \frac{D\beta}{2(\beta+1)}\int_{E_T}\frac{M_0^\beta}{N_\beta^{\beta+1}}\,\partial_t N_\beta\,(\partial_y N_\beta)^2\mathrm{d}t\mathrm{d}y \\
&\qquad + \frac{\alpha}{(\beta+1)(\beta+2)}\int_0^{L_0}\frac{N_\beta(T,y)^{\beta+2}}{M_0(y)^\beta}\mathrm{d}y + \frac{D}{2(\beta+1)}\int_0^{L_0}\left(\frac{M_0(y)}{N_\beta(T,y)}\right)^\beta (\partial_y N_\beta(T,y))^2\mathrm{d}y \\
&= \frac{\alpha}{(\beta+1)(\beta+2)}\left\|M_0\right\|^2_{L^2(0,L_0)} + \frac{\eta}{\beta+1}(N_\beta(T,0)-M_0(0)) + \frac{D}{2(\beta+1)}\left\|M_0'\right\|^2_{L^2(0,L_0)},
\end{aligned}
$$

we get the claim. □

Proof. (Proof of Theorem 4.1) Since $\partial_t N_\beta(t,y)\geq 0$, by Lemma 4.2,

$$
\lim_{\beta\to\infty}\int_{E_T}\left(\frac{N_\beta}{M_0}\right)^\beta (\partial_t N_\beta(t,y))^2\mathrm{d}t\mathrm{d}y = 0.
$$

Being $M_0(y)\leq N_\beta(t,y)$ (see (4.1))

$$
\int_{E_T}(\partial_t N_\beta(t,y))^2\mathrm{d}t\mathrm{d}y \leq \int_{E_T}\left(\frac{N_\beta}{M_0}\right)^\beta (\partial_t N_\beta)^2\mathrm{d}t\mathrm{d}y,
$$

that proves the claim. □

We continue with the behavior of $N_\beta(t,y)$ as $\beta\to\infty$.

Theorem 4.2. *For every $T>0$ and $1\leq r<\infty$*

$$
\lim_{\beta\to\infty}\int_0^{L_0}|N_\beta(t,y)-M_0(y)|^r\mathrm{d}y = 0,
$$

uniformly with respect to $t\in]0,T[$.

Proof. Since

$$\begin{aligned}\int_0^{L_0} |N_\beta(t,y) - M_0(y)|\mathrm{d}y &= \int_0^{L_0} |N_\beta(t,y) - N_\beta(0,y)|\mathrm{d}y\\ &= \int_0^{L_0}\left|\int_0^t \partial_\tau N_\beta(\tau,y)\mathrm{d}\tau\right|\mathrm{d}y \leq \int_{E_t} |\partial_\tau N_\beta(\tau,y)|\,\mathrm{d}\tau\mathrm{d}y \leq \sqrt{TL_0}\cdot \|\partial\tau N_\beta\|_{L^2(E_T)},\end{aligned}$$

thanks to Theorem 4.1

$$\lim_{\beta\to\infty}\int_0^{L_0} |N_\beta(t,y) - M_0(y)|\mathrm{d}y = 0,$$

uniformly with respect to $t \in]0,T[$.

The boundedness of $(N_\beta)_{\beta>1}$ in $L^\infty(E_T)$ (see (4.1)) and the boundedness of $M_0(y)$ (see (1.2)) imply the claim. □

We are finally ready for the proof of Theorem 1.1.

Proof. (Proof of Theorem 1.1) Since $M_\theta(t, X_\theta(t,y)) = N_\beta(t,y)$ and $\beta = \frac{\log 2}{\theta}$, for every $1 \leq r < \infty$, we have

$$\int_0^{L_0} |M_\theta(t, X_\theta(t,y)) - M_0(y)|^r\mathrm{d}y = \int_0^{L_0} |N_\beta(t,y) - M_0(y)|^r\mathrm{d}y.$$

In light of Theorem 4.2, we have $i)$.

Since $\partial_t N_\beta(t,y) \geq 0,\ (t,y)\in E_T$, (see Theorem 3.1.$i$)) and $\beta > 0$,

$$u_\theta(t,x) = \beta\int_0^{Y_\theta(t,x)} \frac{N_\beta(t,y)^{\beta-1}\partial_t N_\beta(t,y)}{M_0(y)^\beta}\mathrm{d}y \geq 0.$$

The monotonicity of $N_\beta(t,y)$ with respect to t, $0 \leq Y_\theta(t,x) \leq L_0$, $N_\beta(0,y) = M_0(y)$ and the definition of $u_\theta(t,x)$ guarantee

$$\begin{aligned}u_\theta(t,x) \leq& \beta\int_0^{L_0} \frac{1}{N_\beta(t,y)}\left(\frac{N_\beta(t,y)}{M_0(y)}\right)^\beta \partial_t N_\beta(t,y)\mathrm{d}y \leq\\ \leq& \beta\int_0^{L_0} \frac{1}{M_0(y)}\left(\frac{N_\beta(t,y)}{M_0(y)}\right)^\beta \partial_t N_\beta(t,y)\mathrm{d}y.\end{aligned}$$

The monotonicity assumption on $M_0(y)$ gives

$$u_\theta(t,x) \leq \frac{\beta}{M_0(L_0)}\int_0^{L_0}\left(\frac{N_\beta(t,y)}{M_0(y)}\right)^\beta \partial_t N_\beta(t,y)\mathrm{d}y,$$

and using the equation in (2.1)

$$u_\theta(t,x) \le \frac{\beta}{M_0(L_0)} \int_0^{L_0} \left\{ \mathrm{d}\partial_y \left(\left(\frac{N_\beta(t,y)}{M_0(y)} \right)^\beta \partial_t N_\beta(t,y) \right) - a \frac{N_\beta(t,y)^{\beta+1}}{M_0(y)^\beta} \right\} \mathrm{d}y$$
$$= \frac{\beta}{\beta+1} \frac{1}{M_0(L_0)} \left\{ \int_0^{L_0} D\partial_y \left(\left(\frac{N_\beta(t,y)}{M_0(y)} \right)^\beta \partial_y N_\beta(t,y) \right) \mathrm{d}y - \alpha \int_0^{L_0} \frac{N_\beta(t,y)^{\beta+1}}{M_0(y)^\beta} \right\} \mathrm{d}y.$$

The boundary conditions in (2.1) and Lemma 3.1:

$$u_\theta(t,x) \le \frac{\beta}{\beta+1} \frac{1}{M_0(L_0)} \left\{ D\frac{\eta}{D} - \alpha \left[\frac{\eta}{\alpha} + e^{-\alpha t} \Big(\|M_0\|_{L^1(0,L_0)} - \frac{\eta}{\alpha} \Big) \right] \right\}$$
$$= \frac{\beta}{\beta+1} \frac{e^{-\alpha t}}{M_0(L_0)} \Big(\eta - \alpha \|M_0\|_{L^1(0,L_0)} \Big).$$

As a consequence,

$$\limsup_{\theta\to 0} u_\theta(t,x) \le e^{-\alpha t} \frac{\eta - \alpha \|M_0\|_{L^1(0,L_0)}}{M_0(L_0)},$$

that proves *ii*)

The equation in (2.1) gives

$$\frac{1}{\beta}\partial_t \left(\frac{N_\beta}{M_0} \right)^\beta + a \left(\frac{N_\beta}{M_0} \right)^\beta = \frac{\mathrm{d}}{N_\beta} \partial_y \left(\left(\frac{M_0}{N_\beta} \right)^\beta \partial_y N_\beta \right) \tag{4.4}$$

and then

$$\partial_t \left(e^{a^* t} \left(\frac{N_\beta}{M_0} \right)^\beta \right) = \frac{\mathrm{d}^* e^{a^* t}}{N_\beta} \partial_y \left(\left(\frac{M_0}{N_\beta} \right)^\beta \partial_y N_\beta \right),$$

where

$$a^* = a\beta = \frac{\alpha\beta}{\beta+1}; \quad \mathrm{d}^* = \mathrm{d}\beta = \frac{D\beta}{\beta+1}.$$

Integrating with respect to y on $[0, L_0]$

$$\partial_t \left(e^{a^* t} \int_0^{L_0} \left(\frac{N_\beta}{M_0} \right)^\beta \mathrm{d}y \right) = \mathrm{d}^* e^{a^* t} \int_0^{L_0} \frac{1}{N_\beta} \partial_y \left(\left(\frac{M_0}{N_\beta} \right)^\beta \partial_y N_\beta \right) \mathrm{d}y$$
$$= \mathrm{d}^* e^{a^* t} \left\{ \left[\frac{1}{N_\beta} \left(\frac{M_0}{N_\beta} \right)^\beta \partial_y N_\beta \right]_0^{L_0} + \int_0^{L_0} \left(\frac{M_0}{N_\beta} \right)^\beta \left(\frac{\partial_y N_\beta}{N_\beta} \right)^2 \mathrm{d}y \right\}$$
$$= \mathrm{d}^* e^{a^* t} \left\{ \frac{1}{N_\beta(t,0)} \frac{\eta}{D} + \int_0^{L_0} \left(\frac{M_0}{N_\beta} \right)^\beta \left(\frac{\partial_y N_\beta}{N_\beta} \right)^2 \mathrm{d}y \right\}.$$

Integrating with respect to t on $[0,T]$

$$e^{a^*T}\int_0^{L_0}\left(\frac{N_\beta(T,y)}{M_0(y)}\right)^\beta \mathrm{d}y - L_0$$
$$= \frac{\eta\beta}{\beta+1}\int_0^T \frac{e^{a^*t}}{N_\beta(t,0)}\mathrm{d}t + \mathrm{d}^*\int_0^T e^{a^*t}\mathrm{d}t\int_0^{L_0}\left(\frac{M_0(y)}{N_\beta(t,y)}\right)^\beta\left(\frac{\partial_y N_\beta(t,y)}{N_\beta(t,y)}\right)^2\mathrm{d}y$$

and then

$$L_\theta(T) = L_0 e^{-a^*T} + \frac{\eta\beta}{\beta+1}\int_0^T\frac{e^{-a^*(T-t)}}{N_\beta(t,0)}\mathrm{d}t +$$
$$+\,\mathrm{d}^*\int_0^T e^{-a^*(T-t)}\mathrm{d}t\int_0^{L_0}\left(\frac{M_0(y)}{N_\beta(t,y)}\right)^\beta\left(\frac{\partial_y N_\beta(t,y)}{N_\beta(t,y)}\right)^2\mathrm{d}y.$$

Thanks to Theorem 3.2

$$\begin{aligned} &L_0 e^{-a^*T} + \frac{\eta\beta}{\beta+1}\int_0^T\frac{e^{-a^*(T-t)}}{N_\beta(t,0)}\mathrm{d}t \le L_\theta(T)\\ &\le L_0 e^{-a^*T} + \frac{\eta\beta}{\beta+1}\int_0^T\frac{e^{-a^*(T-t)}}{N_\beta(t,0)}\mathrm{d}t + \mathrm{d}^*\int_0^T e^{-a^*(T-t)}\mathrm{d}t\int_0^{L_0}\frac{\eta}{D}\frac{|\partial_y N_\beta(t,y)|}{N_\beta(t,y)^2}\mathrm{d}y. \end{aligned} \tag{4.5}$$

Since $\partial_y N_\beta(t,y)\le 0,\ (t,y)\in E_\infty$, (see Theorem 3.2), we observe

$$\begin{aligned} \mathrm{d}^*\int_0^T e^{-a^*(T-t)}\mathrm{d}t\int_0^{L_0}\frac{\eta}{D}\frac{|\partial_y N_\beta(t,y)|}{N_\beta(t,y)^2}\mathrm{d}y &= -\frac{\eta\beta}{\beta+1}\int_0^T e^{-a^*(T-t)}\mathrm{d}t\int_0^{L_0}\frac{\partial_y N_\beta(t,y)}{N_\beta(t,y)^2}\mathrm{d}y\\ &= \frac{\eta\beta}{\beta+1}\int_0^T e^{-a^*(T-t)}\left[\frac{1}{N_\beta(t,y)}\right]_0^{L_0}\mathrm{d}t\\ &= \frac{\eta\beta}{\beta+1}\int_0^T e^{-a^*(T-t)}\left(\frac{1}{N_\beta(t,L_0)}-\frac{1}{N_\beta(t,0)}\right)\mathrm{d}t. \end{aligned}$$

Using (4.5)

$$L_0 e^{-a^*T} + \frac{\eta\beta}{\beta+1}\int_0^T\frac{e^{-a^*(T-t)}}{N_\beta(t,0)}\mathrm{d}t \le L_\theta(T) \le L_0 e^{-a^*T} + \frac{\eta\beta}{\beta+1}\int_0^T\frac{e^{-a^*(T-t)}}{N_\beta(t,L_0)}\mathrm{d}t.$$

Thanks to (4.1)

$$\begin{aligned} \frac{\eta\beta}{\beta+1}\int_0^T\frac{e^{-a^*(T-t)}}{N_\beta(t,0)}\mathrm{d}t &\ge \frac{\eta\beta}{\beta+1}\int_0^T\frac{e^{-a^*(T-t)}\sqrt{\alpha D}}{\eta\coth(L_0\sqrt{\frac{\alpha}{D}})}\mathrm{d}t\\ &= \frac{\eta\beta}{\beta+1}\frac{\sqrt{\alpha D}}{\eta}\tanh\left(L_0\sqrt{\frac{\alpha}{D}}\right)\frac{1-e^{-a^*T}}{a^*}. \end{aligned}$$

Since $a^* = \frac{\alpha\beta}{\beta+1}$

$$\sqrt{\alpha D}\tanh\left(L_0\sqrt{\frac{\alpha}{D}}\right)\frac{1-e^{-\frac{\alpha\beta}{\beta+1}}}{\alpha} \le \frac{\eta\beta}{\beta+1}\int_0^T \frac{e^{-a^*(T-t)}}{N_\beta(t,0)}\mathrm{d}t$$

$$\le L_\theta(T) - L_0 e^{a^* T} \le \frac{\eta\beta}{\beta+1}\int_0^T \frac{e^{-a^*(T-t)}}{N_\beta(t,L_0)}\mathrm{d}t$$

$$\le \frac{\eta\beta}{\beta+1}\int_0^T \frac{e^{-a^*(T-t)}}{M_0(L_0)}\mathrm{d}t = \frac{\eta}{M_0(L_0)}\frac{1-e^{-\frac{\alpha\beta}{\beta+1}}}{\alpha}.$$

Sending $\theta \to 0^+$, namely $\beta \to \infty$, we obtain *iii*). □

5. Proof of Theorem 1.2

We begin by proving some a priori estimates on $N_\beta(t,y)$ and $\partial_y N_\beta(t,y)$ independent on β.

Lemma 5.1. *We have that*

$$c_* e^{-\frac{\alpha}{\beta+1}t} \le N_\beta(t,y) \le M_0(0), \qquad (t,y) \in \overline{E}_\infty; \tag{5.1}$$

$$-M_0(y)\sqrt{\frac{\alpha}{D}} \le \left(\frac{M_0(y)}{N_\beta(t,y)}\right)^\beta \partial_y N_\beta(t,y) \le 0; \quad (t,y) \in]0,\infty[\times[0,L_0]. \tag{5.2}$$

Proof. The lower bound in (5.1) follows from Theorem 2.1.*i*). For the upper bound in (5.1), we observe that $N_\beta(\cdot,y)$ and $M_0(y)$ are nonincreasing (see (1.4) and Theorem 3.1); therefore,

$$N_\beta(t,y) \le M_0(y) \le M_0(0).$$

We multiply the equation in (2.1) by $\partial_y N_\beta(t,y)$

$$\partial_t N_\beta(t,y)\partial_y N_\beta(t,y) = \frac{d}{2}\partial_y\left(\left(\frac{M_0(y)}{N_\beta(t,y)}\right)^{2\beta}(\partial_y N_\beta(t,y))^2\right) - \frac{a}{2}\partial_y N_\beta(t,y)^2. \tag{5.3}$$

Since $\partial_y N_\beta(t,y) \le 0$ (see Theorem 3.2) and $\partial_t N_\beta(t,y) \le 0$ (see Theorem 3.1), thanks to (5.3),

$$\mathrm{d}\partial_y\left(\left(\frac{M_0(y)}{N_\beta(t,y)}\right)^{2\beta}(\partial_y N_\beta(t,y))^2\right) \ge a\partial_y N_\beta(t,y)^2.$$

Integrating with respect to y over $[\xi, L_0]$, $0 \le \xi \le L_0$,

$$\mathrm{d}\int_\xi^{L_0}\partial_y\left(\left(\frac{M_0(y)}{N_\beta(t,y)}\right)^{2\beta}(\partial_y N_\beta(t,y))^2\right)\mathrm{d}y \ge a\int_\xi^{L_0}\partial_y N_\beta(t,y)^2\mathrm{d}y,$$

and using the boundary conditions in (2.1) and (2.2)

$$-\left(\left(\frac{M_0(\xi)}{N_\beta(t,\xi)}\right)^{2\beta}(\partial_\xi N_\beta(t,\xi))^2\right) \ge \frac{\alpha}{D}(N_\beta(t,L_0)^2 - N_\beta(t,\xi)^2),$$

namely

$$\left(\left(\frac{M_0(\xi)}{N_\beta(t,\xi)}\right)^{2\beta}(\partial_\xi N_\beta(t,\xi))^2\right) \le \frac{\alpha}{D}(N_\beta(t,\xi)^2 - N_\beta(t,L_0)^2) \le \frac{\alpha}{D}N_\beta(t,\xi)^2.$$

Since $N_\beta(\cdot,\xi)$ is nonincreasing

$$\Big(\frac{M_0(\xi)}{N_\beta(t,\xi)}\Big)^\beta |\partial\xi N_\beta(t,\xi)| \le \sqrt{\frac{\alpha}{D}} N_\beta(t,\xi) \le \sqrt{\frac{\alpha}{D}} N_\beta(0,\xi) = \sqrt{\frac{\alpha}{D}} M_0(\xi),$$

using $\partial_y N_\beta(t,\xi) \le 0$, we have (5.2). □

We continue with the analysis of the behavior of $\partial_t N_\beta(t,y)$ as $\beta \to \infty$.

Theorem 5.1. *For every $T > 0$*

$$\lim_{\beta\to\infty} \int\limits_{E_T} \Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta |\partial_t N_\beta(t,y)| \mathrm{d}t\mathrm{d}y = 0.$$

The following lemma is needed.

Lemma 5.2. *We have that*

$$\lim_{\beta\to\infty} \int\limits_0^{L_0} \partial_t N_\beta(t,y)\partial_y N_\beta(t,y)\mathrm{d}y = 0, \ \ \textit{uniformly with respecttot} \in [0,\infty[; \tag{5.4}$$

$$\lim_{\beta\to\infty} \int\limits_{E_T} \Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta (\partial_t N_\beta(t,y))^2 \mathrm{d}t\mathrm{d}y = 0, \ \ T > 0. \tag{5.5}$$

Proof. We multiply the equation in (2.1) by $\partial_y N_\beta(t,y)$

$$\partial_t N_\beta(t,y)\partial_y N_\beta(t,y) + \frac{a}{2}\partial_y N_\beta(t,y)^2 = \frac{d}{2}\partial_y\left(\Big(\frac{M_0(y)}{N_\beta(t,y)}\Big)^{2\beta}(\partial_y N_\beta(t,y))^2\right),$$

Integrating with respect to y on $[0, L_0]$ and using Theorem 2.1 and the boundary conditions in (2.1)

$$\int\limits_0^{L_0} \partial_t N_\beta(t,y)\partial_y N_\beta(t,y)\mathrm{d}y + \frac{\alpha}{2(\beta+1)}(N_\beta(t,L_0)^2 - N_\beta(t,0)^2) = \frac{D}{2(\beta+1)}\big(-\frac{\eta^2}{D^2}\big).$$

Thanks to (5.1)

$$\left|\int\limits_0^{L_0} \partial_t N_\beta(t,y)\partial_y N_\beta(t,y)\mathrm{d}y\right| \le \frac{1}{2(\beta+1)}\big(\alpha M_0(0)^2 + \frac{\eta^2}{D}\big),$$

and the (5.4).

Lemma 4.2 holds independently on the sign of $DM_0''(y) - \alpha M_(y)$, (1.3) and (1.4). Indeed, its proof uses only (1.2) and Theorem 2.1. Therefore

$$\limsup_{\beta\to\infty}\Big\{ \int\limits_{E_T} \big(\frac{N_\beta(t,y)}{M_0(y)}\big)^\beta (\partial_t N_\beta(t,y))^2 \mathrm{d}t\mathrm{d}y + \\ + \frac{D\beta}{2(\beta+1)} \int\limits_{E_T} \big(\frac{M_0(y)}{N_\beta(t,y)}\big)^\beta (\partial_y N_\beta(t,y))^2 \frac{\partial_t N_\beta(t,y)}{N_\beta(t,y)} \mathrm{d}t\mathrm{d}y\Big\} \le 0. \tag{5.6}$$

Since $N_\beta(\cdot,y)$ is nonincreasing, the second term is negative; as a consequence in order to prove (5.5), it is enough to prove that the second term vanishes as $\beta \to \infty$.

Thanks to Lemma 5.1

$$
\begin{aligned}
I_\beta := & \left| \int_{E_T} \left(\frac{M_0(y)}{N_\beta(t,y)}\right)^\beta (\partial_y N_\beta(t,y))^2 \frac{\partial_t N_\beta(t,y)}{N_\beta(t,y)} \mathrm{d}t\mathrm{d}y \right| \\
= & \int_{E_T} \left(\frac{M_0(y)}{N_\beta(t,y)}\right)^\beta |\partial_y N_\beta(t,y)| \frac{|\partial_t N_\beta(t,y)\partial_y N_\beta(t,y)|}{N_\beta(t,y)} \mathrm{d}t\mathrm{d}y \\
\leq & \sqrt{\frac{\alpha}{D}} \int_{E_T} \frac{M_0(y)}{N_\beta(t,y)} |\partial_t N_\beta(t,y)\partial_y N_\beta(t,y)| \mathrm{d}t\mathrm{d}y.
\end{aligned}
$$

Using (1.2) and Lemma 5.1

$$
I_\beta \leq \sqrt{\frac{\alpha}{D}} \frac{c^*}{c_* e^{-\frac{\alpha}{\beta+1}T}} \int_{E_T} |\partial_t N_\beta(t,y)\partial_y N_\beta(t,y)| \mathrm{d}t\mathrm{d}y,
$$

since $\partial_y N_\beta(t,y) \leq 0$ (see Theorem 3.2) and $\partial_t N_\beta(t,y) \leq 0$ (see Theorem 3.1)

$$
I_\beta \leq \sqrt{\frac{\alpha}{D}} \frac{c^*}{c_* e^{-\frac{\alpha}{\beta+1}T}} \int_{E_T} \partial_t N_\beta(t,y)\partial_y N_\beta(t,y) \mathrm{d}t\mathrm{d}y.
$$

By (5.4)

$$
\lim_{\beta\to\infty} I_\beta = 0,
$$

that gives (5.5). □

Proof of Theorem 5.1. Since $N_\beta(\cdot,y)$ in nonincreasing

$$
\frac{N_\beta(t,y)}{M_0(y)} \leq 1, \quad (t,y) \in E_\infty,
$$

and then

$$
\begin{aligned}
\int_{E_T} \left(\frac{N_\beta(t,y)}{M_0(y)}\right)^\beta |\partial_t N_\beta(t,y)| \mathrm{d}t\mathrm{d}y \leq & \int_{E_T} \left(\frac{N_\beta(t,y)}{M_0(y)}\right)^{\frac{\beta}{2}} |\partial_t N_\beta(t,y)| \mathrm{d}t\mathrm{d}y \leq \\
& \leq \sqrt{TL_0} \sqrt{\int_{E_T} \left(\frac{N_\beta(t,y)}{M_0(y)}\right)^\beta (\partial_t N_\beta(t,y))^2 \mathrm{d}t\mathrm{d}y}.
\end{aligned}
$$

The claim follows from (5.5). □

We study the behavior of $N_\beta(t,y)$ as $\beta \to \infty$.

Theorem 5.2. *For every* $0 \leq T < \infty$

$$
\lim_{\beta\to\infty} \int_0^{L_0} \left(\frac{N_\beta(t,y)}{M_0(y)}\right)^\beta |N_\beta(t,y) - M_0(y)|^r \mathrm{d}y = 0, \quad 1 \leq r < \infty,
$$

uniformly with respect to $t \in [0,T]$.

Proof. Consider

$$I_\beta(t) := \int_0^{L_0} \Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta |N_\beta(t,y) - M_0(y)|\mathrm{d}y$$
$$= \int_0^{L_0} \Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta \Big|\int_0^t \partial_\tau N_\beta(\tau,y)\mathrm{d}\tau\Big|\mathrm{d}y.$$

Since $\Big(\frac{N_\beta(\cdot,y)}{M_0(y)}\Big)^\beta$ is nonincreasing, for every $0 \le \tau \le t$

$$\Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta \le \Big(\frac{N_\beta(\tau,y)}{M_0(y)}\Big)^\beta$$

and then

$$I_\beta(t) \le \int_0^{L_0} \mathrm{d}y \int_0^t \Big(\frac{N_\beta(\tau,y)}{M_0(y)}\Big)^\beta \Big|\partial_\tau N_\beta(\tau,y)\Big|\mathrm{d}\tau.$$

Given $0 \le T < \infty$

$$I_\beta(t) \le \int_{E_T} \Big(\frac{N_\beta(\tau,y)}{M_0(y)}\Big)^\beta \Big|\partial_\tau N_\beta(\tau,y)\Big|\mathrm{d}\tau\mathrm{d}y, \qquad 0 \le t \le T.$$

Using Theorem 5.1

$$\lim_{\beta\to\infty} I_\beta(t) = 0$$

uniformly with respect to $t \in [0,T]$. Finally, since $(N_\beta)_{\beta>1}$ is bounded in $L^\infty(E_T)$ (see Lemma 5.1.*i*)) and $M_0 \in L^\infty(0,L_0)$ (see (1.2)),

$$\int_0^{L_0} \Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta |N_\beta(t,y) - M_0(y)|^r\mathrm{d}y \le \big(M_0(0) + c^*\big)^{r-1} I_\beta(t), \qquad 1 < r < \infty,$$

that gives the claim. □

We are finally ready for the proof of Theorem 1.2.

Proof of Theorem 1.2.

$$\Delta_\beta(t) = \int_0^{L_\theta(t)} |M_\theta(t,x) - M_0(Y_\theta(t,x))|^r\mathrm{d}x$$

and consider the change of variable $y = Y_\theta(t,x)$. For every t, $x = X_\theta(t,y)$ is the inverse of $y = Y_\theta(t,x)$, therefore

$$\mathrm{d}x = \partial_y X_\theta(t,y)\mathrm{d}y = \frac{\mathrm{d}y}{\partial_x Y_\theta(t, X_\theta(t,y))} = \Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta \mathrm{d}y.$$

Thanks to the definition of $N_\beta(t,y)$,

$$\Delta_\beta(t) = \int_0^{L_0} |M_\theta(t, X_\theta(t,y)) - M_0(y)|^r \Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta \mathrm{d}y = \int_0^{L_0} \Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta |N_\beta(t,y) - M_0(y)|^r \mathrm{d}y,$$

and, using Theorem 5.2, we get i).

Thanks to (1.4) and Theorem 3.1.ii), we have $\partial_t N_\beta(t,y) \le 0$, $(t,y) \in E_\infty$. Moreover, since $0 \le Y_\theta(t,x) \le L_0$,

$$u_\theta(t,x) = \beta \int_0^{Y_\theta(t,x)} \frac{N_\beta(t,y)^{\beta-1}\partial_t N_\beta(t,y)}{M_0(y)^\beta} \mathrm{d}y \le 0 \tag{5.7}$$

and

$$|u_\theta(t,x)| \le \beta \int_0^{L_0} \frac{1}{N_\beta(t,y)} \Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta |\partial_t N_\beta(t,y)| \mathrm{d}y.$$

Due to (5.1)

$$|u_\theta(t,x)| \le \frac{\beta}{c_* e^{-\frac{\alpha}{\beta+1}t}} \int_0^{L_0} \Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta |\partial_t N_\beta(t,y)| \mathrm{d}y,$$

where c_* is defined in (1.2). Since $M_0(y)$ is nonincreasing (see (1.4))

$$|u_\theta(t,x)| \le \frac{\beta e^{\frac{\alpha}{\beta+1}t}}{\inf\limits_{0\le y\le L_0} M_0(y)} \int_0^{L_0} \Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta |\partial_t N_\beta(t,y)| \mathrm{d}y = \frac{\beta e^{\frac{\alpha}{\beta+1}t}}{M_0(L_0)} \int_0^{L_0} \Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta |\partial_t N_\beta(t,y)| \mathrm{d}y.$$

Being $|\partial_t N_\beta(t,y)| = -\partial_t N_\beta(t,y)$, thanks to the equation in (2.1) and (2.2)

$$|u_\theta(t,x)| \le \frac{\beta e^{\frac{\alpha}{\beta+1}t}}{(\beta+1)M_0(L_0)} \left\{ \alpha \int_0^{L_0} \frac{N_\beta(t,y)^{\beta+1}}{M_0(y)^\beta} \mathrm{d}y - D \int_0^{L_0} \partial_y \left(\Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta \partial_y N_\beta(t,y) \right) \mathrm{d}y \right\}.$$

Lemma 3.1 and the boundary conditions in (2.1) imply

$$|u_\theta(t,x)| \le \frac{\beta}{\beta+1} \frac{e^{\frac{\alpha}{\beta+1}t}}{M_0(L_0)} \left\{ \alpha \Big(\frac{\eta}{\alpha} + e^{-\alpha t} (\|M_0\|_{L^1(0,L_0)} - \frac{\eta}{\alpha}) \Big) - D\frac{\eta}{D} \right\} = \frac{\beta}{\beta+1} \frac{e^{\frac{\alpha}{\beta+1}t}}{M_0(L_0)} \left\{ \alpha \|M_0\|_{L^1(0,L_0)} - \eta \right\}.$$

In light of (5.7) we get

$$-\frac{\beta}{\beta+1} \frac{e^{\frac{\alpha}{\beta+1}t}}{M_0(L_0)} \left\{ \alpha \|M_0\|_{L^1(0,L_0)} - \eta \right\} \le u_\theta(t,x) \le 0,$$

that gives ii).

We observe that

$$
\begin{aligned}
L_\theta(t) &= \int_0^{L_0} \Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta \mathrm{d}y \\
&= \underbrace{\int_0^{L_0} \Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta \frac{M_0(y)-N_\beta(t,y)}{M_0(y)}\mathrm{d}y}_{I_\beta(t)} + \underbrace{\int_0^{L_0} \Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta \frac{\mathrm{d}y}{M_0(y)}}_{II_\beta(t)}.
\end{aligned}
$$

Being $N_\beta(\cdot,y)$ and $M_0(y)$ nonincreasing, we have

$$
\begin{aligned}
N_\beta(t,y) \leq & N_\beta(0,y) = M_0(y), \qquad M_0(L_0) \leq M_0(y), \\
0 \leq I_\beta(t) \leq & \frac{1}{M_0(L_0)} \int_0^{L_0} \Big(\frac{N_\beta(t,y)}{M_0(y)}\Big)^\beta (M_0(y)-N_\beta(t,y))\mathrm{d}y,
\end{aligned}
$$

and using Theorem 5.2

$$
\lim_{\beta\to\infty} I_\beta(t) = 0, \quad \text{uniformly with respect to} t \in [0,T],\ 0 \leq T < \infty.
$$

From the equation in (2.1), we get

$$
\partial_t \left(e^{\alpha t} \frac{N_\beta(t,y)^{\beta+1}}{M_0(y)^\beta} \right) = D\partial_y \left(e^{\alpha t} \Big(\frac{M_0(y)}{N_\beta(t,y)}\Big)^\beta \partial_y N_\beta(t,y) \right).
$$

Multiplying by $1/M_0(y)$ and integrating over $[0,L_0]$

$$
\int_0^{L_0} \partial_t \left(e^{\alpha t} \frac{N_\beta(t,y)^{\beta+1}}{M_0(y)^\beta} \frac{1}{M_0(y)} \right) \mathrm{d}y = D \int_0^{L_0} \partial_y \left(e^{\alpha t} \Big(\frac{M_0(y)}{N_\beta(t,y)}\Big)^\beta \partial_y N_\beta(t,y) \right) \frac{\mathrm{d}y}{M_0(y)},
$$

and then

$$
\begin{aligned}
&\partial_t \left(e^{\alpha t} \int_0^{L_0} \frac{N_\beta(t,y)^{\beta+1}}{M_0(y)^\beta} \frac{\mathrm{d}y}{M_0(y)} \right) \\
&\quad = De^{\alpha t} \left[\Big(\frac{M_0(y)}{N_\beta(t,y)}\Big)^\beta \frac{\partial_y N_\beta(t,y)}{M_0(y)} \right]_0^{L_0} - De^{\alpha t} \int_0^{L_0} \Big(\frac{M_0(y)}{N_\beta(t,y)}\Big)^\beta \partial_y N_\beta(t,y) \Big(-\frac{M_0'(y)}{M_0(y)^2} \Big) \mathrm{d}y \\
&\quad = De^{\alpha t} \frac{\eta}{D} \frac{1}{M_0(0)} + De^{\alpha t} \int_0^{L_0} \Big(\frac{M_0(y)}{N_\beta(t,y)}\Big)^\beta \frac{\partial_y N_\beta(t,y) M_0'(y)}{M_0(y)^2} \mathrm{d}y.
\end{aligned}
$$

Using (1.4) and Theorem 3.2, we have $\partial_y N_\beta(t,y) M_0'(y) \geq 0$, that gives

$$
\begin{aligned}
e^{\alpha t}\frac{\eta}{M_0(0)} \leq & \partial_t \left(e^{\alpha t} \int_0^{L_0} \frac{N_\beta(t,y)^{\beta+1}}{M_0(y)^\beta} \frac{\mathrm{d}y}{M_0(y)} \right) \leq \\
& \leq e^{\alpha t}\frac{\eta}{M_0(0)} + De^{\alpha t} \int_0^{L_0} \Big(\frac{M_0(y)}{N_\beta(t,y)} \Big)^\beta \frac{\partial_y N_\beta(t,y) M_0'(y)}{M_0(y)^2} \mathrm{d}y.
\end{aligned}
$$

Thanks to (5.2) and (1.4)

$$
\begin{aligned}
e^{\alpha t}\frac{\eta}{M_0(0)} &\leq \partial_t \left(e^{\alpha t} \int_0^{L_0} \frac{N_\beta(t,y)^{\beta+1}}{M_0(y)^\beta} \frac{\mathrm{d}y}{M_0(y)} \right) \\
&\leq e^{\alpha t}\frac{\eta}{M_0(0)} + De^{\alpha t}\sqrt{\frac{\alpha}{D}} \int_0^{L_0} \frac{|M_0'(y)|}{M_0(y)} \mathrm{d}y \\
&= e^{\alpha t}\frac{\eta}{M_0(0)} - e^{\alpha t}\sqrt{\alpha D} \int_0^{L_0} \frac{M_0'(y)}{M_0(y)} \mathrm{d}y \\
&= e^{\alpha t}\frac{\eta}{M_0(0)} + e^{\alpha t}\sqrt{\alpha D} \log \frac{M_0(0)}{M_0(L_0)}.
\end{aligned}
$$

Integrating with respect to t on $[0, T]$

$$
\begin{aligned}
\frac{e^{\alpha T}-1}{\alpha}\frac{\eta}{M_0(0)} &\leq e^{\alpha T} \int_0^{L_0} \frac{N_\beta(T,y)^{\beta+1}}{M_0(y)^\beta} \frac{\mathrm{d}y}{M_0(y)} - L_0 \\
&\leq \frac{e^{\alpha T}-1}{\alpha} \Big(\frac{\eta}{M_0(0)} + \sqrt{\alpha D} \log \frac{M_0(0)}{M_0(L_0)} \Big).
\end{aligned}
$$

Since

$$
L_\theta(T) - I_\beta(T) = II_\beta(T) = \int_0^{L_0} \frac{N_\beta(T,y)^{\beta+1}}{M_0(y)^\beta} \frac{\mathrm{d}y}{M_0(y)},
$$

we have

$$
\begin{aligned}
L_0 e^{-\alpha T} + \frac{1-e^{-\alpha T}}{\alpha}\frac{\eta}{M_0(0)} &\leq L_\theta(T) - I_\beta(T) \leq \\
\leq L_0 e^{-\alpha T} + \frac{1-e^{-\alpha T}}{\alpha}\frac{\eta}{M_0(0)} &+ \frac{1-e^{-\alpha T}}{\alpha}\sqrt{\alpha D} \log \frac{M_0(0)}{M_0(L_0)}.
\end{aligned}
$$

Sending $\beta \to \infty$ we get the claim. $\square$

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