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(ω, Q)-periodic mild solutions for a class of semilinear abstract differential equations and applications to Hopfield-type neural network model

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Abstract. In this paper, we investigate the existence and uniqueness of (ω, Q)-periodic mild solutions for the following problem

$$x'(t) = Ax(t) + f(t, x(t)), \quad t \in \mathbb{R},$$

on a Banach space X. Here, A is a closed linear operator which generates an exponentially stable C_0-semigroup and the nonlinearity f satisfies suitable properties. The approaches are based on the well-known Banach contraction principle. In addition, a sufficient criterion is established for the existence and uniqueness of (ω, Q)-periodic mild solutions to the Hopfield-type neural network model.

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1. Introduction

Let X be a Banach space and $Q : X \to X$ a linear isomorphism. Of concern in the present paper is the existence and uniqueness of (ω, Q)-periodic mild solutions for a class of semilinear abstract differential equations on X. More precisely, our aim is to investigate regularity of (ω, Q)-periodic mild solutions for

$$x'(t) = Ax(t) + f(t, x(t)), \quad t \in \mathbb{R}. \tag{1.1}$$

In this problem, A is a closed linear operator on X which generates an exponentially stable C_0-semigroup and the nonlinearity $f : \mathbb{R} \times X \to X$ is a given function with suitable properties.

(ω, Q)-periodic functions were introduced by Fečkan et al. in [12] as a generalization of (ω, c)-periodic functions (Alvarez et al. [5]) and Q-affine-periodic functions (Zhang et al [23]). More precisely, a function $h : \mathbb{R}_+ \to X$ is called (ω, Q)-periodic function if there is a pair (ω, Q), such that $h(t + \omega) = Qh(t)$ for all $t \in \mathbb{R}_+$, where $\omega > 0$ and $Q : X \to X$ is a linear isomorphism. In that paper the authors investigated (ω, Q)-periodic solutions of impulsive evolution equations using certain fixed point theorem and Fredholm alternative theorem. More precisely, the regularity of solutions to the studied equation is proved on the space $PC(\mathbb{R}_+, X)$. In order to show their results, they imposed several conditions on the kernel of the operators involved.

In this paper, we shall denote the vector space of (ω, Q)-periodic functions by $A_{ff}P(\mathbb{R}, X, \omega, Q)$. Note that when $Q = cI$, $c \in \mathbb{C}$, we have $A_{ff}P(\mathbb{R}, X, \omega, cI) = P_{\omega c}(\mathbb{R}, X)$ which is the Banach space of (ω, c)-periodic functions (see [5]).

On the other hand, various authors have published works related with different variants and generalizations of periodic functions, namely, (ω, c)-pseudo periodic functions, (ω, c)-asymptotically periodic functions, c-semiperiodic and c-almost periodic functions, pseudo $\mathcal{S}$-asymptotically Bloch type periodic functions, among others (see [1–5, 8, 18–22]).

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Hopfield-type neural network model. A neural network is an assembly of interconnected elementary units which have limited characteristics of real (or biological) neurons. Each unit is capable of receiving many input signals, some of which can activate the receiver while other inputs can inhibit the activities of the receiver. The neuron-like elementary unit computes a weighted sum of the inputs it receives and produces a single response and sends the response down along its axon when the weighted sum exceeds a certain threshold level. Great interest in the dynamical models of neural networks has been steadily increasing during the last 60 years. The Hopfield-type neural network model describes the dynamics of a system formed of n neurons which are massively interconnected. Let u_i be the ith neuron activation state (the membrane potential of the neuron i at time t), g_i be a measure of response or activation to its incoming potentials, T_{ij} be the interconnections (or synaptic strength) weight of the neutron j on the neuron i and $\alpha > 0$ be the rate with which the neuron i self-regulates or resets its potential when isolated from other neurons and inputs. Then the model is given by

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\alpha u_i(t) + \sum_{j=1}^{2} T_{ij}(t) g_j(u_j(t)) + I_i(t), \quad i = 1, 2,\, t \geq 0,$$

see for example [7,9–11,14–17].

Outside the works of [12,13], we remark that not much seems to be known about the regularity of mild solutions of (1.1) in the space $A_{ff}P(\mathbb{R}, X, \omega, Q)$ and its applications to neural networks. In contrast with the papers [12,13], which consider impulses and the study the regularity in the space $PC(\mathbb{R}_+, X)$ with the norm $\sup_{t\in\mathbb{R}_+} \|y\|$ ($y \in PC(\mathbb{R}_+, X)$), the main novelties of the present article are as follows:

- We endow to $A_{ff}P(\mathbb{R}, X, \omega, Q)$ with the norm

$$\|x\|_{\omega Q} := \sup_{\xi\in[0,\omega]} \|x(\xi)\|, \quad x \in A_{ff}P(\mathbb{R}, X, \omega, Q).$$

  Actually, we show that $\|\cdot\|_{\omega Q}$ defines a norm on $A_{ff}P(\mathbb{R}, X, \omega, Q)$. We point out that $\sup_{t\in\mathbb{R}_+} \|x\|$, for $x \in A_{ff}P(\mathbb{R}, X, \omega, Q)$, is not a norm on the space $A_{ff}P(\mathbb{R}, X, \omega, Q)$; this is the reason why we define $\|\cdot\|_{\omega Q}$.
- The result of the existence and uniqueness of (ω, Q)-periodic mild solutions of (1.1) is obtained using the classical Banach fixed point theorem. We do not use degree theory.
- Our abstract results allow to investigate (ω, Q)-periodic mild solutions for the Hopfield's neural network model.

Our first main result (Theorem 2.7) states that $(A_{ff}P(\mathbb{R}, X, \omega, Q), \|\cdot\|_{\omega Q})$ is a Banach space with the norm $\|\cdot\|_{\omega Q}$. The second main result (Theorem 3.4) shows the existence and uniqueness of (ω, Q)-periodic mild solutions of (1.1). In order to do this, we use a fixed point argument defining a suitable map on the Banach space $(A_{ff}P(\mathbb{R}, X, \omega, Q), \|\cdot\|_{\omega Q})$. Finally, our third main result (Theorem 4.1) states the conditions under which the model (4.1) has a unique (ω, Q)-periodic mild solution.

The rest of the paper is structured as follows. In Sect. 2, firstly we introduce some notations and give certain interesting properties of (ω, Q)-periodic functions which will be used throughout the article. In addition, we show that $\|\cdot\|_{\omega Q}$ is a norm on $A_{ff}P(\mathbb{R}, X, \omega, Q)$. Moreover, we prove that $(A_{ff}P(\mathbb{R}, X, \omega, Q), \|\cdot\|_{\omega Q})$ is a Banach space. In Sect. 3, we investigate the existence and uniqueness of (ω, Q)-periodic mild solutions to (1.1). Finally, an application is given in Sect. 4.

2. Basic properties

In this section we show fundamental properties of vector-valued (ω, Q)-periodic functions.

Throughout this paper, we assume that X is a Banach space endowed with the norm $\| \cdot \|$. We will use the following notation:

- $C(\mathbb{R}, X) := \{f : \mathbb{R} \to X : f \text{ is continuous}\}$,

- $C(\mathbb{R} \times X, X) := \{f : \mathbb{R} \times X \to X : f \text{ is continuous}\}$,
- $\mathcal{L}(X) := \{T : X \to X : T \text{ is linear and (bounded) continuous}\}$.

We recall that the space $\mathcal{L}(X)$ is equipped with the usual sup norm.

Definition 2.1. [[12, Definition 2.2]] Let $Q : X \to X$ be a linear isomorphism. We say that $x : \mathbb{R} \to X$ is a (ω, Q)-periodic function if there exists $\omega > 0$ such that

$$x(t+\omega) = Qx(t), \quad \text{for all } t \in \mathbb{R}.$$

Let $A_{ff}P(\mathbb{R}, X, \omega, Q) := \{x \in C(\mathbb{R}, X) : x(t+\omega) = Qx(t), \quad \text{for all } t \in \mathbb{R}\}$.

The next result describes some basic properties of the space $A_{ff}P(\mathbb{R}, X, \omega, Q)$.

Proposition 2.2. *Let $\beta \in \mathbb{C}$, $c \in \mathbb{R}\backslash\{0\}$, $Q : X \to X$ be a linear isomorphism and $\omega > 0$. Then the following conditions hold.*

(i) *If $x, z \in A_{ff}P(\mathbb{R}, X, \omega, Q)$, then $x + \beta z \in A_{ff}P(\mathbb{R}, X, \omega, Q)$.*
(ii) *If $\delta \geqslant 0$ and $x \in A_{ff}P(\mathbb{R}, X, \omega, Q)$, then $x_\delta(\cdot) := x(\cdot + \delta) \in A_{ff}P(\mathbb{R}, X, \omega, Q)$.*
(iii) *If $\alpha \in A_{ff}P(\mathbb{R}, \mathbb{R}, \omega, cI_{\mathbb{R}})$ with $x \in A_{ff}P(\mathbb{R}, X, \omega, Q)$, then $\alpha x \in A_{ff}P(\mathbb{R}, X, \omega, cQ)$*
(iv) *If $x \in A_{ff}P(\mathbb{R}, X, \omega, Q)$ is such that $x'(t)$ exists for all $t \in \mathbb{R}$, then $x' \in A_{ff}P(\mathbb{R}, X, \omega, Q)$.*

Proof. Items (i) and (ii) follow immediately from Definition 2.1.

To prove (iii), let $\alpha \in A_{ff}P(\mathbb{R}, \mathbb{R}, \omega, cI_{\mathbb{R}})$ and $x \in A_{ff}P(\mathbb{R}, X, \omega, Q)$. Then

$$(\alpha x)(t+\omega) = \alpha(t+\omega)x(t+\omega) = cI_{\mathbb{R}}\alpha(t)Qx(t) = cQ(\alpha x)(t).$$

Now, we will show (iv). Indeed, for $t \in \mathbb{R}$ fixed, we have that

$$\begin{aligned} x'(t+\omega) &= \lim_{\eta \to 0} \frac{x(t+\omega+\eta) - x(t+\omega)}{\eta} \\ &= \lim_{\eta \to 0} \frac{Qx(t+\eta) - Qx(t)}{\eta} \\ &= Qx'(t), \end{aligned}$$

where we have used the fact that Q is linear and continuous in the last equality. □

From now on $Q : X \to X$ will be a linear isomorphism and $\omega > 0$.

Proposition 2.3. *Assume that $g \in A_{ff}P(\mathbb{R}, X, \omega, Q)$. Let $a \in \mathbb{R}$ and $G(t) := \int_a^t g(s)\ \mathrm{d}s$, for all $t \in \mathbb{R}$. Then $G \in A_{ff}P(\mathbb{R}, X, \omega, Q)$ if and only if $G(a+\omega) = 0$.*

Proof. A simple calculation gives

$$G(t+\omega) - G(a+\omega) = \int_a^t g(s+\omega)\mathrm{d}s = \int_a^t Qg(s)\mathrm{d}s = QG(t).$$

Since $g \in A_{ff}P(\mathbb{R}, X, \omega, Q)$, we have that $G \in A_{ff}P(\mathbb{R}, X, \omega, Q)$ if and only if $G(a+\omega) = 0$. □

Lemma 2.4. *Suppose that $x \in A_{ff}P(\mathbb{R}, X, \omega, Q)$. Then the following conditions hold.*

(i) *$x(s) = Qx(s-\omega)$ for all $s \in \mathbb{R}$.*
(ii) *$x(s) = Q^{-1}x(s+\omega)$ for all $s \in \mathbb{R}$.*

Proof. Item (i) follows from Definition 2.1 and the fact that $s = (s-\omega) + \omega$. On the other hand, (ii) follows immediately from (i) and the fact that Q is an isomorphism. □

Proposition 2.5. *Let k be an integrable and bounded function on $\mathbb{R}$ and $h \in A_{ff}P(\mathbb{R}, X, \omega, Q)$ be an integrable function (in sense of Bochner) on $\mathbb{R}$. Then*

$$(k \star h)(t) := \int_{-\infty}^{\infty} k(t-s)h(s)\,\mathrm{d}s, \quad \textit{for} \quad t \in \mathbb{R},$$

belongs to $A_{ff}P(\mathbb{R}, X, \omega, Q)$.

Proof. By the properties of convolution, we have that $k \star h$ is continuous on $\mathbb{R}$ (see for example [6, Proposition 1.3.2]). Now,

$$\begin{aligned}(k \star h)(t+\omega) &= \int_{-\infty}^{\infty} k(t+\omega-s)h(s)\,\mathrm{d}s \\ &= \int_{-\infty}^{\infty} k(t-\tau)h(\tau+\omega)\,\mathrm{d}\tau \\ &= Q\int_{-\infty}^{\infty} k(t-\tau)h(\tau)\,\mathrm{d}\tau = Q(k \star h)(t), \quad \text{for all} \quad t \in \mathbb{R}.\end{aligned}$$

It follows that $(k \star h) \in A_{ff}P(\mathbb{R}, X, \omega, Q)$. □

In order to obtain our main results, we need the following theorem.

Theorem 2.6. *The function $\|\cdot\|_{\omega Q} : A_{ff}P(\mathbb{R}, X, \omega, Q) \to \mathbb{R}$ given by*

$$\|x\|_{\omega Q} := \sup_{\xi \in [0,\omega]} \|x(\xi)\|, \quad x \in A_{ff}P(\mathbb{R}, X, \omega, Q)$$

defines a norm on $A_{ff}P(\mathbb{R}, X, \omega, Q)$.

Proof. Since $[0, \omega]$ is compact and x is a continuous function, we have $\sup_{\xi\in[0,\omega]} \|x(\xi)\| < \infty$. Thus, $\|x\|_{\omega Q}$ is well-defined. Also, note that $\|x\|_{\omega Q} \geqslant 0$, for all $x \in A_{ff}P(\mathbb{R}, X, \omega, Q)$.

On the other hand, if $x(t) = 0$ for all $t \in \mathbb{R}$, we get $\|x\|_{\omega Q} = 0$.

Now, assume that $\|x\|_{\omega Q} = 0$. Then, $x(\xi) = 0$ for all $\xi \in [0, \omega]$.

Statement $x(t) = 0$ for all $t \in \mathbb{R}$.

Indeed, let $t \in \mathbb{R}$. Then there exists $n \in \mathbb{Z}$ such that $t \in [n\omega, (n + 1)\omega]$.
If $n \in \mathbb{Z}^+$, by Lemma 2.4 (Item (i)), we have

$$\begin{aligned}x(t) &= Qx(t-\omega) \\ &= Q^2x(t-2\omega) \\ &\ \vdots \\ &= Q^n x(t-n\omega) \\ &= Q^n 0 = 0.\end{aligned}$$

Observe that the last equality follows from the fact that $t - n\omega \in [0, \omega]$.

If $n \in \mathbb{Z}^-$, then $n = -k$ with $k \in \mathbb{Z}^+$. Thus, by Lemma 2.4 (Item (ii)), we obtain

$$\begin{aligned} x(t) &= Q^{-1}x(t+\omega) \\ &= Q^{-2}x(t+2\omega) \\ &\vdots \\ &= Q^{-k}x(t+k\omega) \\ &= Q^{n}x(t-n\omega) \\ &= Q^{n}0 = 0. \end{aligned}$$

Hence, $x(t) = 0$ for all $t \in \mathbb{R}$.

The properties

$$\|cx\|_{\omega Q} = |c|\,\|x\|_{\omega Q} \qquad \text{and} \qquad \|x+y\|_{\omega Q} \leqslant \|x\|_{\omega Q} + \|y\|_{\omega Q}\,,$$

follow immediately. □

The next theorem is the first main result of this paper.

Theorem 2.7. *$A_{ff}P(\mathbb{R}, X, \omega, Q)$ is a Banach space with the norm*

$$\|x\|_{\omega Q} = \sup_{\xi \in [0,\omega]} \|x(\xi)\|\,.$$

Proof. Let $(x_k)_{k\in\mathbb{N}} \subset A_{ff}P(\mathbb{R}, X, \omega, Q)$ be a Cauchy sequence. Let $\epsilon > 0$ be given. Then, there exists $N(\epsilon) \in \mathbb{N}$ such that

$$\|x_n - x_m\|_{\omega Q} = \sup_{\xi \in [0,\omega]} \|x_n(\xi) - x_m(\xi)\| < \epsilon,$$

for all $n, m \geqslant N(\epsilon)$.

For each $n \in \mathbb{N}$, define $\varphi_n := x_n|_{[0,\omega]}$. Hence, for any fixed $\tau \in [0, \omega]$ and $n, m \geqslant N(\epsilon)$, we obtain

$$\|\varphi_n(\tau) - \varphi_m(\tau)\| = \|x_n(\tau) - x_m(\tau)\| < \epsilon.$$

This show that $(\varphi_k(\tau))_{k\in\mathbb{N}} \subset X$ is Cauchy sequence. Since X is a Banach space, the sequence converges, say

$$\varphi_k(\tau) \overset{k\to\infty}{\longrightarrow} \varphi(\tau).$$

In this way we can associate with each $\tau \in [0, \omega]$ a unique element $\varphi(\tau) \in X$. This defines a pointwise function $\varphi : [0, \omega] \to X$ given by

$$\varphi(t) := \lim_{k\to\infty} \varphi_k(t), \quad t \in [0, \omega]. \tag{2.1}$$

Statement 1. $(\varphi_k(t))_{k\in\mathbb{N}}$ converges to $\varphi(t)$ uniformly on $[0, \omega]$.

Let $t \in [0, \omega]$ fixed. According to (2.1), there exists $k(t, \epsilon) \in \mathbb{N}$ such that

$$\|\varphi_n(t) - \varphi(t)\| < \epsilon \quad \text{ for all } \quad n \geqslant k(t, \epsilon).$$

Hence, for all $m \geqslant N(\epsilon)$ and all $n \geqslant \max\{N(\epsilon), k(t, \epsilon)\}$, we have

$$\begin{aligned} \|\varphi_m(t) - \varphi(t)\| &\leqslant \|\varphi_m(t) - \varphi_n(t)\| + \|\varphi_n(t) - \varphi(t)\| \\ &= \|x_n(t) - x_m(t)\| + \|\varphi_n(t) - \varphi(t)\| \\ &< \epsilon + \epsilon = 2\epsilon. \end{aligned}$$

This gives

$$\|\varphi_m(t) - \varphi(t)\| < 2\epsilon, \quad \text{ for all } m \geqslant N(\epsilon) \text{ and all } t \in [0, \omega]. \tag{2.2}$$

This shows that $(\varphi_k(t))_{n\in\mathbb{N}}$ converges to $\varphi(t)$ uniformly on $[0,\omega]$. Also, since each φ_k are continuous on $[0,\omega]$ and the convergence is uniform, the limit function φ is continuous on $[0,\omega]$ i.e., $\varphi \in C([0,\omega],X)$. Moreover, notice that $\varphi(\omega) = Q\varphi(0)$.

Now, define $x:\mathbb{R}\to X$ by

$$x(s) := Q^n\varphi(s-n\omega), \quad s\in[n\omega,(n+1)\omega),\ n\in\mathbb{Z}. \tag{2.3}$$

Note that x is well–defined by part (i) of Lemma 2.4.

Statement 2 $x\in C(\mathbb{R},X)$.

In order to prove that x is continuous on $\mathbb{R}$, we will show the following two assertions.

Assertion 1 x is continuous at $t_0 = n\omega$ with $n\in\mathbb{Z}$.

In fact, for all $t\in(n\omega,(n+1)\omega)$, we have

$$\|x(t)-x(n\omega)\| \leqslant \|Q^n\|_{\mathcal{L}(X)}\,\|\varphi(t-n\omega)-\varphi(0)\|\,.$$

Since $\varphi\in C([0,\omega],X)$ and $t-n\omega \overset{t\to n\omega^+}{\longrightarrow} 0^+$, we get $\varphi(t-n\omega)-\varphi(0) \overset{t\to n\omega^+}{\longrightarrow} 0$ and, therefore,

$$\|x(t)-x(n\omega)\| \overset{t\to n\omega^+}{\longrightarrow} 0.$$

On the other hand, for all $t\in((n-1)\omega,n\omega)$, we obtain

$$\|x(t)-x(n\omega)\| \leqslant \left\|Q^{n-1}\right\|_{\mathcal{L}(X)}\,\|\varphi(t-(n-1)\omega)-\varphi(\omega)\|\,.$$

Notice that the last inequality follows from the fact that $\varphi(0) = Q^{-1}\varphi(\omega)$. Now, since $\varphi\in C([0,\omega],X)$ and $t-(n-1)\omega \overset{t\to n\omega^-}{\longrightarrow} \omega^-$, we get $\varphi(t-(n-1)\omega)-\varphi(\omega) \overset{t\to n\omega^-}{\longrightarrow} 0$ and, therefore,

$$\|x(t)-x(n\omega)\| \overset{t\to n\omega^\cdot}{\longrightarrow} 0.$$

Hence, x is continuous at $t_0 = n\omega$, with $n\in\mathbb{Z}$.

Assertion 2 x is continuous on $(n\omega,(n+1)\omega)$, with $n\in\mathbb{Z}$.

Indeed, let $t_0\in(n\omega,(n+1)\omega)$ be fixed. Then, for all $t\in(n\omega,(n+1)\omega)$, we have

$$\|x(t)-x(t_0)\| \leqslant \|Q^n\|_{\mathcal{L}(X)}\,\|\varphi(t-n\omega)-\varphi(t_0-n\omega)\|\,.$$

Since $\|\varphi(t-n\omega)-\varphi(t_0-n\omega)\| \overset{t\to t_0}{\longrightarrow} 0$, we obtain

$$\|x(t)-x(t_0)\| \overset{t\to t_0}{\longrightarrow} 0.$$

Thus, x is continuous on $(n\omega,(n+1)\omega)$ with $n\in\mathbb{Z}$, proving Statement 2.

Statement 3 $x(t+\omega) = Qx(t)$, for all $t\in\mathbb{R}$.

Let $t\in[n\omega,(n+1)\omega)$. Then $t+\omega\in[(n+1)\omega,(n+2)\omega)$ and

$$\begin{aligned} x(t+\omega) &= Q^{n+1}\varphi(t+\omega-(n+1)\omega)\\ &= Q^{n+1}\varphi(t-n\omega)\\ &= QQ^n\varphi(t-n\omega)\\ &= Qx(t), \end{aligned}$$

obtaining the desired result.

Statement 4 $x_m \overset{\|\cdot\|_{\omega Q}}{\longrightarrow} x$ as $m\to\infty$.

In fact, for all $m\geqslant N(\epsilon)$, by (2.2), we have

$$\begin{aligned} \|x_m - x\|_{\omega Q} &= \sup_{t\in[0,\omega]}\|x_m(t)-x(t)\|\\ &= \sup_{t\in[0,\omega]}\|\varphi_m(t)-\varphi(t)\|\\ &\leqslant 2\epsilon. \end{aligned}$$

Hence $A_{ff}P(\mathbb{R}, X, \omega, Q)$ is a Banach space with the norm $\|x\|_{\omega Q}$. □

3. Existence and uniqueness

In this section our goal is to investigate the existence and uniqueness of (ω, Q)-periodic mild solutions for a class of semilinear abstract differential equations.

Consider the semilinear abstract differential equation given by

$$x'(t) = Ax(t) + f(t, x(t)), \quad t \in \mathbb{R}, \tag{3.1}$$

where $f \in C(\mathbb{R} \times X, X)$ is such that

$$M_0 := \sup_{(t,x)\in\mathbb{R}\times X} \|f(t, x)\| < \infty,$$

and A is a closed linear operator in a Banach space X which generates a C_0-semigroup $\{T(t)\}_{t\geqslant 0}$ such that there exist constants $M > 0$ and $\alpha > 0$ such that

$$\|T(t)x\| \leqslant Me^{-\alpha t} \|x\|, \tag{3.2}$$

for $t \geqslant 0$.

The following definition is analogous to [3, Definition 3.1].

Definition 3.1. A function $u : \mathbb{R} \to X$ is said to be a mild solution of (3.1) if

$$u(t) = \int_{-\infty}^{t} T(t - s)f(s, u(s))\, \mathrm{d}s, \quad t \in \mathbb{R}.$$

Remark 3.2. Since $\int_{-\infty}^{t} e^{\alpha s}\, \mathrm{d}s < \infty$ and

$$\begin{aligned}\|T(t - s)f(s, u(s))\| &\leq Me^{-\alpha(t-s)}\|f(s, u(s))\| \\ &\leq MM_0e^{-\alpha t}e^{\alpha s},\end{aligned}$$

we can conclude that the integral in the Definition 3.1 is well defined.

Let us assume the following conditions:

(C1) A generates a C_0-semigroup $\{T(t)\}_{t\geqslant 0}$ that satisfies (3.2).
(C2) $Q : X \to X$ is a linear isomorphism, $T(\tau)Q = QT(\tau)$, for $\tau \geqslant 0$.
(C3) $f(t + \omega, x) = Qf(t, Q^{-1}x)$ for all $x \in X$ and for all $t \in \mathbb{R}$.
(C4) There exists a nonnegative function $L : \mathbb{R} \to \mathbb{R}$ such that

$$\|f(t, x(t)) - f(t, y(t))\| \leqslant L(t)\, \|x - y\|_{\omega Q} \quad \text{for all} \quad x, y \in A_{ff}P(\mathbb{R}, X, \omega, Q).$$

(C5) $\sup_{t\in[0,\omega]} (S * L)(t) < 1$, where

$$S(t) = \|T(t)\| \quad \text{and} \quad (S * L)(t) := \int_0^{\infty} S(t - s)L(s)\, \mathrm{d}s.$$

Remark 3.3. Condition (C1) implies that A generates a uniformly integrable C_0-semigroup $\{T(t)\}_{t\geqslant 0}$, that is,

$$\int_0^{\infty} \|T(t)\| < \infty.$$

The next theorem is our second main result.

Theorem 3.4. *Under conditions* $(C1)$-$(C5)$, *the Eq.* (3.1) *has a unique mild solution in* $A_{ff}P(\mathbb{R},X,\omega,Q)$.

Proof. Define $\mathcal{J}: A_{ff}P(\mathbb{R},X,\omega,Q) \to A_{ff}P(\mathbb{R},X,\omega,Q)$ given by

$$\mathcal{J}(x)(t) = \int_{-\infty}^{t} T(t-s)f(s,x(s))\,\mathrm{d}s, \quad t \in \mathbb{R}.$$

Step 1 $\mathcal{J}(A_{ff}P(\mathbb{R},X,\omega,Q)) \subset A_{ff}P(\mathbb{R},X,\omega,Q)$.
Indeed, let $x \in A_{ff}P(\mathbb{R},X,\omega,Q)$. By (C2) and (C3), we have

$$\begin{aligned}\mathcal{J}(x)(t+\omega) &= \int_{-\infty}^{t+\omega} T(t+\omega-s)f(s,x(s))\,\mathrm{d}s\\ &= \int_{-\infty}^{t} T(t+\omega-(s+\omega))f(s+\omega,x(s+\omega))\,\mathrm{d}s\\ &= \int_{-\infty}^{t} T(t-s)Qf(s,Q^{-1}x(s+\omega))\,\mathrm{d}s\\ &= \int_{-\infty}^{t} QT(t-s)f(s,Q^{-1}Qx(s))\,\mathrm{d}s\\ &= Q\int_{-\infty}^{t} T(t-s)f(s,x(s))\,\mathrm{d}s\\ &= Q\mathcal{J}(x)(t), \quad \text{for all} \quad t \in \mathbb{R}.\end{aligned}$$

On the other hand, $\mathcal{J}(x)$ is continuous. In fact, let $t_0 \in \mathbb{R}$ fixed. Then, for $t > t_0$ one has

$$\begin{aligned}\|\mathcal{J}(x)(t) - \mathcal{J}(x)(t_0)\| &= \left\| \int_{t_0}^{t} T(t-s)f(s,x(s))\mathrm{d}s + \int_{-\infty}^{t_0} [T(t-s)-T(t_0-s)]f(s,x(s))\mathrm{d}s \right\|\\ &\le \int_{t_0}^{t} \|T(t-s)f(s,x(s))\|\,\mathrm{d}s + \int_{-\infty}^{t_0} \|[T(t-s)-T(t_0-s)]f(s,x(s))\|\,\mathrm{d}s\\ &\le Me^{-\alpha t}\int_{t_0}^{t} e^{\alpha s}\,\|f(s,x(s))\|\,\mathrm{d}s + \int_{-\infty}^{t_0} \|[T(t-s)-T(t_0-s)]f(s,x(s))\|\,\mathrm{d}s\\ &=: I_1 + I_2.\end{aligned}$$

We claim that $I_1 \to 0$ as $t \to t_0^+$. Indeed, this is a direct consequence of the continuity of the function $t \mapsto Me^{-\alpha t}\int_{t_0}^{t} e^{\alpha s}\,\|f(s,x(s))\|\,\mathrm{d}s$.

Next, let us show that $I_2 \to 0$ as $t \to t_0^+$. Indeed, since $t \mapsto T(t)x$ is continuous for all $t \in \mathbb{R}$ and all $x \in X$,

$$\|[T(t-s)-T(t_0-s)]f(s,x(s))\| \le 2MM_0e^{-\alpha t_0}e^{\alpha s},$$

and $s \mapsto e^{\alpha s}$ is integrable in $(-\infty, t_0]$, the claim follows from the Dominated Convergence Theorem. Hence,

$$\lim_{t\to t_0^+} \|\mathcal{J}(x)(t) - \mathcal{J}(x)(t_0)\| = 0.$$

Analogously, we get that $\lim_{t\to t_0^-} \|\mathcal{J}(x)(t) - \mathcal{J}(x)(t_0)\| = 0$. **Step 2.** Now, for $x, y \in A_{ff}P(\mathbb{R}, X, \omega, Q)$, by (C4) and (C5), we obtain

$$\begin{aligned}
\|\mathcal{J}(x) - \mathcal{J}(y)\|_{\omega Q} &= \sup_{t\in[0,\omega]} \|\mathcal{J}(x)(t) - \mathcal{J}(y)(t)\| \\
&= \sup_{t\in[0,\omega]} \left\| \int_{-\infty}^{t} T(t-s)[f(s, x(s)) - f(s, y(s))]\, \mathrm{d}s \right\| \\
&\leqslant \|x - y\|_{\omega Q} \sup_{t\in[0,\omega]} \int_0^{\infty} \|T(s)\|\, L(t-s)\, \mathrm{d}s \\
&= \|x - y\|_{\omega Q} \sup_{t\in[0,\omega]} (S * L)(t) \\
&< \|x - y\|_{\omega Q}\,.
\end{aligned}$$

Thus, the Banach Fixed Point Theorem guarantees that there exists a unique $u \in A_{ff}P(\mathbb{R}, X, \omega, Q)$ such that $u(t) = \int_{-\infty}^{t} T(t-s)f(s, u(s))\, \mathrm{d}s$, for all $t \in \mathbb{R}$. □

4. Applications

The Hopfield's neural network model is a dynamical system of ordinary differential equations of the form:

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\alpha u_i(t) + \sum_{j=1}^{2} T_{ij}(t) g_j(u_j(t)) + I_i(t), \quad i = 1, 2,\ t \geq 0. \tag{4.1}$$

Here,

u_i : corresponds to the ith neuron activation state.

g_i : denotes a measure of response or activation to its incoming potentials.

T_{ij} : define the interconnections weight of the neutron j on the neuron i.

$\alpha > 0$: is the rate with which the neuron i self-regulates or resets its potential when isolated from other neurons and inputs.

The system (4.1) can be written as a semilinear differential equation of the form (3.1) by setting

$$A = \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \quad u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \quad f(t, u(t)) = \begin{pmatrix} \sum_{i=1}^{2} T_{1j}(t) g_j(u_j(t)) + I_1(t) \\ \sum_{i=1}^{2} T_{2j}(t) g_j(u_j(t)) + I_2(t) \end{pmatrix}.$$

In this example, we will consider $\mathbb{R}^2$ with the norm given by $\|u\| = |u_1| + |u_2|$.

Theorem 4.1. *Let $Q = I_2$ be the 2×2 identity matrix and $\omega > 0$. We assume the following conditions.*

(a) *g_j are continuous and bounded on $\mathbb{R}$, $T_{ij}(t)$ and $I_i(t)$ belong to $P_\omega(\mathbb{R}, X)$ (i.e. are continuous and periodic), for $i, j = 1, 2$.*

(b) *There are nonnegative functions L_{g_j} such that $|g_j(u_j(t)) - g_j(v_j(t))| \leqslant L_{g_j}(t) \|u_j - v_j\|_{\omega Q}$ for all $u_j, v_j \in A_{ff}P(\mathbb{R}, \mathbb{R}^2, \omega, I_2)$ and for $j = 1, 2$.*

Then, Eq. (4.1) has a unique mild solution in $A_{ff}P(\mathbb{R}, \mathbb{R}^2, \omega, I_2)$.

Proof. Since $A = -\alpha I_2$, we have $T(t) = e^{-\alpha t} I_2$, $t \geq 0$, and $S(t) = \|T(t)\| = e^{-\alpha t}$. On the other hand, note that $Q = I_2$ is a linear isomorphism and $T(\tau)Q = QT(\tau)$, for $\tau \geqslant 0$, getting condition $(C2)$.

To show (C3), note that

$$\begin{aligned} f(t+\omega, u) &= \begin{pmatrix} \sum_{i=1}^{2} T_{1j}(t+\omega) g_j(u_j) + I_1(t+\omega) \\ \sum_{i=1}^{2} T_{2j}(t+\omega) g_j(u_j) + I_2(t+\omega) \end{pmatrix} \\ &= \begin{pmatrix} \sum_{i=1}^{2} T_{1j}(t) g_j(u_j) + I_1(t) \\ \sum_{i=1}^{2} T_{2j}(t) g_j(u_j) + I_2(t) \end{pmatrix} \\ &= I_2 f(t, I_2^{-1} u) \\ &= Q f(t, Q^{-1} u). \end{aligned}$$

Finally, item (b) implies that the condition (C4) is satisfied with $L_f(t) := \sum_{i=1}^{2} \sum_{j=1}^{2} |T_{ij}(t)| L_{g_j}(t)$.

Hence, by Theorem 3.4 the system (4.1) has a unique mild solution that satisfies

$$u(t) = \int_{-\infty}^{t} e^{-\alpha(t-s)} f(s, u(s))\, \mathrm{d}s$$

and which belongs to the space $A_{ff}P(\mathbb{R}, \mathbb{R}^2, \omega, I_2)$ whenever $\sup_{t \in [0,\omega]} (S * L_f)(t) < 1$. □

Now, we present an example in the infinite dimensional case.

Example 4.2. Let X be a Banach space, $Q := I$, where $I : X \to X$ is the identity operator on X, and $A := \alpha I$ with $\alpha > 0$. Moreover, suppose that $f \in C(\mathbb{R} \times X, X)$ is bounded and satisfies conditions (C3) and (C4) of Theorem 3.4.

Since $A = \alpha I$, we have $T(t) = e^{-\alpha t} I$, for all $t \geqslant 0$. It is easy to see that condition (C2) from Theorem 3.4 is satisfied. Hence, the equation

$$x'(t) = Ax(t) + f(t, x(t)), \quad t \in \mathbb{R},$$

has a unique (ω, Q)-periodic mild solution whenever $\sup_{t \in [0,\omega]} (S * L)(t) < 1$.

Finally, let us consider a case where $Q \neq I$.

Let us consider

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\alpha u_i(t) + \sum_{j=1}^{2} T_{ij}(t) g_j(u_j(t)) + I_i(t), \quad i = 1, 2, t \geq 0. \tag{4.2}$$

If we take $\alpha = 2$, $T_{11}(t) = 3$, $T_{12}(t) = 2$, $T_{21}(t) = 3$,$T_{22}(t) = 4$, $g_1 = g_2 = I_{\mathbb{R}}$, $I_1(t) = 4e^{3t}$ and $I_2(t) = 2e^{3t}$, the system (4.2) is

$$\begin{cases} \dfrac{\mathrm{d}u_1}{\mathrm{d}t} = u_1(t) + 2u_2(t) + 4e^{3t}, \\ \dfrac{\mathrm{d}u_2}{\mathrm{d}t} = 3u_1(t) + 2u_2(t) + 2e^{3t}, \end{cases} \tag{4.3}$$

and its solution is given by

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} -2e^{3t} - c_1 e^{-t} + 2c_2 e^{4t} \\ -4e^{3t} + c_1 e^{-t} + 3c_2 e^{4t} \end{pmatrix}, \tag{4.4}$$

where $c_1, c_2 \in \mathbb{R}$. In particular, $x(t) = \begin{pmatrix} -2e^{3t} \\ -4e^{3t} \end{pmatrix}$ is a solution of (4.3). On the other hand, taking $Q = \begin{pmatrix} e^{3\omega} & 0 \\ 0 & e^{3\omega} \end{pmatrix}$ with $\omega > 0$, we have that

$$\begin{aligned} x(t+\omega) &= \begin{pmatrix} -2e^{3(t+\omega)} \\ -4e^{3(t+\omega)} \end{pmatrix} \\ &= \begin{pmatrix} e^{3\omega} & 0 \\ 0 & e^{3\omega} \end{pmatrix} \begin{pmatrix} -2e^{3t} \\ -4e^{3t} \end{pmatrix} \\ &= Qx(t). \end{aligned}$$

Hence x is a (Q, ω)-affine-periodic solution of (4.3). It is worthwhile to emphasize that according to (4.4) the system (4.3) has no periodic solutions, but it has (ω, Q)-periodic solutions.

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