Z. Angew. Math. Phys. (2023) 74:59

0044-2275/23/020001-28
published online February 10, 2023
https://doi.org/10.1007/s00033-022-01935-1

Zeitschrift für angewandte Mathematik und Physik ZAMP

Propagation thresholds in a diffusive epidemic model with latency and vaccination

Yahui Wang, Xinjian Wang and Guo Lin

Abstract. This paper studies the propagation thresholds in a diffusive epidemic model with latency and vaccination. When the initial condition satisfies proper exponential decaying behavior, we present the spatial expansion feature of the infected. Different leftward and rightward spreading speeds are obtained with respect to different decaying initial values. Moreover, the convergence in the sense of compact open topology is also studied when the spreading speeds are finite. Finally, we show that the minimal spreading speed is the minimal wave speed of traveling wave solutions, which also presents the precisely asymptotic behavior of traveling wave solutions for the infected branch at the disease-free side. Here, the asymptotic behavior plays an important role that distinguishes the minimal spreading speed from all possible spreading speeds. From the definition of possible spreading speeds, we may find some factors affecting the spatial expansion ability, which includes that the vaccination could decrease the spatial expansion ability of the disease.

Mathematics Subject Classification. 35K57, 92D30.

Keywords. Generalized upper-lower solutions, Asymptotic spreading, Nonmonotone delayed system, Minimal wave speed, Fast propagation.

1. Introduction

The vaccination has been widely utilized to prevent diseases, which greatly reduces risks of getting a disease [36]. To quantitatively understand the role of vaccination, many mathematical models have been established and studied. For example, in very recent works by Papst et al. [26] and Zou et al. [44], the effect of vaccination and other factors were, respectively, investigated to model the spreading of seasonal flu and COVID-19. Moreover, Miyaoka et al. [24] discussed the optimal control of Zika virus by analyzing a reaction-diffusion model and El Alami Laaroussi and Rachik [10] used a reaction-diffusion system to model the spatial spread of Ebola in gorillas in Gabon, in which the nontrivial role of vaccination was discussed.

By introducing an exposed class into the classical SVIR (susceptible, vaccinated, infected and recovered classes) compartmental epidemic model, He et al. [13] established a diffusive epidemic model with latency and vaccination. Since the $R-$branch is linear and does not affect the dynamics for S, V, I in their model, they studied the following model

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} = d_1 \frac{\partial^2 S(x,t)}{\partial x^2} + \Pi - (\mu+\omega)S(x,t) - \frac{\beta_1 S(x,t) I(x,t)}{1+\alpha I(x,t)}, \\ \frac{\partial V(x,t)}{\partial t} = d_2 \frac{\partial^2 V(x,t)}{\partial x^2} + \omega S(x,t) - \mu V(x,t) - \frac{\beta_2 V(x,t) I(x,t)}{1+\alpha I(x,t)}, \\ \frac{\partial I(x,t)}{\partial t} = d_3 \frac{\partial^2 I(x,t)}{\partial x^2} + e^{-\mu\tau} \frac{[\beta_1 S(x,t-\tau)+\beta_2 V(x,t-\tau)] I(x,t-\tau)}{1+\alpha I(x,t-\tau)} - (\mu+\nu) I(x,t), \end{cases} \tag{1.1}$$

Research was partially supported by National Natural Science Foundation of China (No. 11971213, 11731005).

in which $x \in \mathbb{R}, t > 0$, $S(x,t), V(x,t), I(x,t)$ are the population sizes of the susceptible, vaccinated, infected classes at location x and time t, respectively, $d_1 > 0, d_2 > 0, d_3 > 0$ are positive constants describing moving ability of each class. In the above model, $\Pi > 0$ and $\omega > 0$ depend on the entering flux and the vaccination rate of the susceptible, respectively, μ denotes the natural death rate, $\tau \geq 0$ is the latent period, β_1 and β_2 are the transmission coefficients of susceptible and vaccinated individuals, respectively, $\alpha \geq 0$ measures the saturation level, ν is the recovery rate. Here, $e^{-\mu\tau}\frac{[\beta_1 S(x,t-\tau)+\beta_2 V(x,t-\tau)]I(x,t-\tau)}{1+\alpha I(x,t-\tau)}$ reflects the incidence effect.

Evidently, (1.1) always has a disease-free equilibrium

$$E_0 = (S_0, V_0, 0) := \left(\frac{\Pi}{\omega+\mu}, \frac{\Pi\omega}{\mu(\omega+\mu)}, 0\right).$$

When

$$\mathcal{R}_0^1 := \frac{e^{-\mu\tau}\left(\beta_1 S_0 + \beta_2 V_0\right)}{\mu+\nu} > 1,$$

it also has a unique endemic steady state $E_1^* = (S_1^*, V_1^*, I_1^*)$ with

$$S_1^* = \frac{\Pi\left(1+\alpha I_1^*\right)}{\beta_1 I_1^* + (\mu+\omega)\left(1+\alpha I_1^*\right)}, V_1^* = \frac{\omega\Pi\left(1+\alpha I_1^*\right)^2}{\left(\beta_1 I_1^* + (\mu+\omega)\left(1+\alpha I_1^*\right)\right)\left(\beta_2 I_1^* + \mu\left(1+\alpha I_1^*\right)\right)}$$

and I_1^* is the unique positive root of $AI^2 + BI + C = 0$, where

$$A = (\mu\alpha + \beta_2)\left((\mu+\omega)\alpha + \beta_1\right) > 0, \quad C = \mu(\mu+\omega)\left(1 - \mathcal{R}_0^1\right) < 0,$$
$$B = (\mu+\omega)\left(2\mu\alpha + \beta_2\right) + \mu\beta_1 - \frac{\alpha\left(\beta_1\mu + \beta_2\omega\right) + \beta_1\beta_2}{\mu+\nu} e^{-\mu\tau}\Pi.$$

In He et al. [13], the authors formulated the disease spreading process by traveling wave solutions of (1.1). Here, a traveling wave solution of (1.1) is a special entire solution taking the form

$$(S(x,t), V(x,t), I(x,t)) = (\hat{S}(x+ct), \hat{V}(x+ct), \hat{I}(x+ct)), \qquad x, t \in \mathbb{R},$$

and satisfying the asymptotic boundary conditions

$$(\hat{S}, \hat{V}, \hat{I})(-\infty) = (S_0, V_0, 0) \quad \text{ and } \quad (\hat{S}, \hat{V}, \hat{I})(+\infty) = \left(S_1^*, V_1^*, I_1^*\right),$$

where $c > 0$ is the wave speed and $(\hat{S}, \hat{I}, \hat{R})$ is the wave profile. By the definition of traveling wave coordinate $x + ct$, we see that such a traveling wave solution could model the disease spreading process at a fixed speed c. Therefore, the threshold that determines the existence or nonexistence of nontrivial traveling wave solutions becomes very important, which is also called the minimal wave speed. In He et al. [13], the authors gave such a threshold for (1.1). In particular, when the wave speed is the threshold, the existence of traveling wave solutions was proved by passing to a limit function in Ref. [13]. We also refer to some recent works [20,33,38,39,41] for the traveling wave solutions in epidemic models with the vaccinated class.

Based on the above work, the purpose of this article is to study the propagation dynamics for a generalized system of (1.1) by considering general incidence functions. Our model is

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} = d_1 \frac{\partial^2 S(x,t)}{\partial x^2} + \Pi - (\mu+\omega)S(x,t) - S(x,t)f_1(I(x,t)), \\ \frac{\partial V(x,t)}{\partial t} = d_2 \frac{\partial^2 V(x,t)}{\partial x^2} + \omega S(x,t) - \mu V(x,t) - V(x,t)f_2(I(x,t)), \\ \frac{\partial I(x,t)}{\partial t} = d_3 \frac{\partial^2 I(x,t)}{\partial x^2} - (\mu+\nu)I(x,t) \\ \qquad + e^{-\mu\tau}[S\left(x,t-\tau\right)f_1(I(x,t-\tau)) + V(x,t-\tau)f_2(I(x,t-\tau))], \end{cases} \tag{1.2}$$

in which two functions $f_1, f_2 : [0,\infty) \to [0,\infty)$ will be further stated in Sect. 2. Our study is from both the initial value problem and the traveling wave solutions that model the disease expansion process. In what follows, for the initial value problem, we equip (1.2) with the following initial condition

$$(S(x,s),V(x,s),I(x,s)) = (\phi_1(x,s),\phi_2(x,s),\phi_3(x,s)), \quad x\in\mathbb{R}, s\in[-\tau,0]. \tag{1.3}$$

When the initial condition (1.3) satisfies proper limit assumptions, we shall try to show the long time feature of solutions to (1.2) by leftward and rightward spreading speeds. Here, the leftward and rightward spreading speeds of a function are defined as follows (see Aronson and Weinberger [3–5] for related definition).

Definition 1.1. Assume that $u(x,t)\ge 0, x\in\mathbb{R}, t>0$. Then, a finite constant $c_u^l>0$ is called the *leftward spreading speed* of it provided that

(a) $\limsup_{t\to\infty}\sup_{-x>(c_u^l+\epsilon)t} u(x,t)=0$ for any given $\epsilon>0$;
(b) $\liminf_{t\to\infty}\inf_{0<-x<(c_u^l-\epsilon)t} u(x,t)>0$ for any given $\epsilon\in(0,c_u^l)$.

In particular, we define $c_u^l=\infty$ if $\liminf_{t\to\infty}\inf_{0<-x<ct} u(x,t)>0$ for any given $c>0$. Similarly, we can define the rightward spreading speed c_u^r.

For parabolic systems, traveling wave solutions and spreading speeds have been widely studied. For example, when a system could generate monotone semiflows [11,14,19,23,27,35] or a nonmonotone semiflow can be controlled by two monotone semiflows [15,17,31,40], many important conclusions have been established, which often imply that the minimal wave speed equals to the minimal spreading speed. Moreover, when the initial value problem is concerned, the fast propagation or accelerating spreading of cooperative systems or scalar equations was also studied in [2,12,37], which shows the existence of infinite spreading speed. In epidemics models, these thresholds could describe the ability of disease spreading and have been studied in many works, see earlier results in [7,8,29]. In particular, when an epidemic system can not generate monotone semiflows, the spreading speed has attracted much attention in the past decade, see [9,16,22,32,42]. It should be noted that in model (1.2), the third equation of I depends on the two equations of S,V such that the structure is different from that in [1,9,16,22,32,42] and the known results in these works can not be directly applied to it. Moreover, even if only the traveling wave solutions are concerned, the possible deficiency of classical comparison principle in (1.2) makes the analysis more complex than the special monotone model (1.1) in Ref. [13].

In this article, we first show possible different leftward and rightward propagation thresholds when the initial values satisfy different decaying behaviors. The upper bounds of spreading speed (see the item (a) in Definition 1.1) are estimated by the upper bounds of S,V, as well as some auxiliary monotone delayed equations. Furthermore, motivated by Ref. [22,25], we shall construct an auxiliary monotone delayed equation of I according to the estimation on S,V, by which we may obtain the lower bounds of spreading speed (see the item (b) in Definition 1.1). Using these two bounds, we obtain the spreading speeds of I, which shall be discussed in Sect. 3. Moreover, we study the convergence of initial value problem in the sense of compact open topology, which will be presented in Sect. 4. In Sect. 5, we investigate the existence and nonexistence of traveling wave solutions by constructing upper and lower solutions and auxiliary monotone equations. When the wave speed is not less than the minimal wave speed, our conclusion implies the exponentially decaying behavior of traveling wave solutions and completes the known results. Finally, in Sect. 6, we give a brief discussion on the potential investigation and the role of vaccination. In particular, when the vaccination plays a positive role in the sense of $f_1\le f_2, f_1'(0)<f_2'(0)$, the vaccination could decrease the spreading ability of the disease from both the outbreak threshold and the possible spreading speeds.

2. Main results

In this article, we shall use the standard partial ordering in $\mathbb{R}^n, n=1,2,3$. The assumptions on f_1,f_2 are first listed as follows, which will be imposed without further illustration.

(F1) $f_i:[0,\infty)\to[0,\infty)$ is differentiable such that $f_i(0)=0$ and

$$\max\{0, f_i'(0)x - Lx^{1+\sigma}\} < f_i(x) < f'(0)x, \qquad x>0$$

for some $\sigma\in(0,1], L>0$ and $i\in\{1,2\}$;

(F2) $f_i(x)/x$ is strictly decreasing in $x>0, i\in\{1,2\}$, and there exists $I_0>0$ such that

$$e^{-\mu\tau}[S_0\sup_{x\in[0,I]} f_1(x) + V_0\sup_{x\in[0,I]} f_2(x)] < (\mu+\nu)I \text{ for any } I\ge I_0.$$

Evidently, in the model (1.1), the corresponding monotone functions f_1, f_2 satisfy our condition. Besides them, there are many functions satisfying (F1)–(F2) but they are not monotone. For example, $f_1(x)=f_2(x)=pxe^{-x}, 2pe^{-\mu\tau}>\mu+\nu$ also satisfies (F1)–(F2) but the monotone property might be false. With the existence of nonmonotonic f_1, f_2 in (1.2), the comparison principle similar to that in Ref. [13] does not hold always.

To state our main results, we further introduce some constants. Firstly, define

$$\mathcal{R}_0 := \frac{e^{-\mu\tau}\left(S_0 f_1'(0) + V_0 f_2'(0)\right)}{\mu+\nu} S$$

and consider

$$\Lambda(\lambda,c) = d_3\lambda^2 - c\lambda + e^{-(\mu+c\lambda)\tau}\left(S_0 f_1'(0)+V_0 f_2'(0)\right) - (\mu+\nu), \quad \lambda\ge 0, c\ge 0.$$

By the monotonicity in $c>0$ and convex in $\lambda\ge 0$, the following conclusion holds.

Lemma 2.1. *Assume that $\mathcal{R}_0>1$. Then, there exists a constant $c^*>0$ such that $\Lambda(\lambda,c^*)=0$ has a unique positive root λ^* such that*

$$2d_3\lambda - c - c\tau e^{-(\mu+c\lambda)\tau}\left(S_0 f_1'(0)+V_0 f_2'(0)\right)=0, \qquad \lambda=\lambda^*, c=c^*.$$

Moreover, if $c>c^*$, then $\Lambda(\lambda,c)=0$ has two distinct positive roots and we denote the smaller (larger) root by $\lambda^c(\Lambda^c)$.

With these constants, we give the following assumptions and classifications of the initial condition (1.3).

(I1) $(0,0,0)\le(\phi_1(x,s),\phi_2(x,s),\phi_3(x,s))\le(S_0,V_0,I_0)$ for all $x\in\mathbb{R}, s\in[-\tau,0]$;

(I2) $(\phi_1(x,s),\phi_2(x,s),\phi_3(x,s))$ is uniformly continuous for all $x\in\mathbb{R}, s\in[-\tau,0]$ and ϕ_3 has nonempty support in the sense that $\phi_3(x_0,0)>0$ for some $x_0\in\mathbb{R}$;

(I3) $\inf_{x\in\mathbb{R},s\in[-\tau,0]}\phi_3(x,s)>0$;

(IL1) there exists $A_1>0$ such that $\phi_3(x,s)\le A_1(-x+1)e^{\lambda^* x}, x<0, s\in[-\tau,0]$;

(IL2) there exist $A_2>1, c_l>c^*, X_1>0$ such that

$$A_2^{-1}e^{\lambda^{c_l}x}\le\phi_3(x,0)\le A_2 e^{\lambda^{c_l}x}, \quad \phi_3(x,s)\le A_2 e^{\lambda^{c_l}x}, \quad s\in[-\tau,0), \quad x<-X_1;$$

(IL3) for any $\epsilon>0$, we have $\liminf_{x\to-\infty}\phi_3(x,0)e^{-\epsilon x}=\infty$;

(IR1) there exists $A_3>0$ such that $\phi_3(x,s)\le A_3(x+1)e^{-\lambda^* x}, x>0, s\in[-\tau,0]$;

(IR2) there exist $A_4>1, c_r>c^*, X_2>0$ such that if $x>0$ is large, we have

$$A_4^{-1}e^{-\lambda^{c_r}x}\le\phi_3(x,0)\le A_4 e^{-\lambda^{c_r}x}, \quad \phi_3(x,s)\le A_4 e^{-\lambda^{c_r}x}, \quad s\in[-\tau,0), \quad x>X_2;$$

(IR3) for any $\epsilon>0$, we have $\liminf_{x\to+\infty}\phi_3(x,0)e^{\epsilon x}=\infty$.

Let

$$\left[T_i u(\cdot,s)\right](t)(x) = \frac{1}{\sqrt{4\pi d_i t}}\int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4d_i t}} u(x,s)dy, t\ge 0, x\in\mathbb{R}, i=1,2,3,$$

in which u is a continuous and bounded function. Using the notation, we give the following existence result of bounded solution.

Table 1. *Spreading speeds of* (1.1) *with different initial conditions*

Initial conditions	(IL1) and (IR1)	(IL1) and (IR2)	(IL1) and (IR3)
Spreading speeds	$c_I^l = c_I^r = c^*$	$c_I^l = c^*, c_I^r = c_r$	$c_I^l = c^*, c_I^r = \infty$
Initial conditions	(IL2) and (IR1)	(IL2) and (IR2)	(IL2) and (IR3)
Spreading speeds	$c_I^l = c_l, c_I^r = c^*$	$c_I^l = c_l, c_I^r = c_r$	$c_I^l = c_l, c_I^r = \infty$
Initial conditions	(IL3) and (IR1)	(IL3) and (IR2)	(IL3) and (IR3)
Spreading speeds	$c_I^l = \infty, c_I^r = c^*$	$c_I^l = \infty, c_I^r = c_r$	$c_I^l = c_I^r = \infty$

Theorem 2.2. *Assume that (I1) and (I2) hold. Then,* (1.2) *with* (1.3) *has a mild solution*

$$S(x,t) = [T_1 S(\cdot,0)](t)(x) + \int_0^t [T_1 [\Pi - (\mu+\omega)S(\cdot,s) - S(\cdot,s)f_1(I(\cdot,s))]](t-s)(x)ds,$$

$$V(x,t) = [T_2 V(\cdot,0)](t)(x) + \int_0^t [T_2 [\omega S(\cdot,s) - \mu V(\cdot,s) - V(\cdot,s)f_2(I(\cdot,s))]](t-s)(x)ds,$$

$$I(x,t) = [T_3 I(\cdot,0)](t)(x) - (\mu+\nu)\int_0^t [T_3 I(\cdot,s)](t-s)(x)ds$$
$$+e^{-\mu\tau}\int_0^t [T_3 [S(\cdot,s-\tau)f_1(I(\cdot,s-\tau)) + V(\cdot,s-\tau)f_2(I(\cdot,s-\tau))]](t-s)(x)ds$$

for all $t > 0, x \in \mathbb{R}$*, which is the classical solution when* $t > \tau$ *and also satisfies*

$$(0,0,0) \le (S(x,t),V(x,t),I(x,t)) \le (S_0,V_0,I_0), \quad I(x,t) > 0, \quad x \in \mathbb{R}, t > 0.$$

Moreover, for each $t_0 > 0$*, there exists* $\delta > 0$ *such that*

$$S(x,t) > \delta, V(x,t) > \delta, \qquad x \in \mathbb{R}, t \ge t_0.$$

Finally, the partial derivatives

$$u_t(x,t), u_x(x,t), \qquad u \in \{S,V,I\}$$

are uniformly continuous and bounded for $x \in \mathbb{R}, t \ge \tau + 1$.

For the former part of this conclusion, we may refer to He et al. [13]. The latter part is clear by the boundedness, and we omit the proof here. Using these constants, for the bounded solution of (1.2) with (1.3), we present the following four theorems.

Theorem 2.3. *Assume that* $\mathcal{R}_0 > 1$ *and (I1)–(I2) hold. Then, with different decaying initial values, the leftward and rightward spreading speeds of* (1.2) *with* (1.3) *are given by Table 1.*

It should be noted that $|x|e^{-\lambda^*|x|}$ plays an important role in determining the spreading speeds. In fact, this is related to the asymptotic behavior of traveling wave solutions with minimal wave speed c^*. Here, a traveling wave solution of (1.2) is a special solution having the form

$$(S(x,t),V(x,t),I(x,t)) = (\mathcal{S}(x+ct),\mathcal{V}(x+ct),\mathcal{I}(x+ct)), \qquad x,t \in \mathbb{R}, \tag{2.1}$$

and satisfying the asymptotic boundary conditions

$$(\mathcal{S},\mathcal{V},\mathcal{I})(-\infty) = (S_0,V_0,0) \quad \text{and} \quad \liminf_{z\to\infty}(\mathcal{S}(z)\mathcal{V}(z)\mathcal{I}(z)) > 0. \tag{2.2}$$

Theorem 2.4. *If $\mathcal{R}_0 > 1$, then (2.1)–(2.2) has a positive solution if and only if $c \geq c^*$. When $c = c^*(c > c^*)$, (2.1)–(2.2) has a positive solution such that*

$$\lim_{z\to-\infty}\{\mathcal{I}(z)e^{-\lambda^* z}/z\} < 0 \quad (\lim_{z\to-\infty}\{\mathcal{I}(z)e^{-\lambda^c z}\} > 0).$$

If $\mathcal{R}_0 \leq 1$, then (2.1)-(2.2) does not have a nontrivial positive solution for any $c \in \mathbb{R}$.

Remark 2.5. Combining Theorem 2.4 with [13, Theorem 3.7], for model (1.1) with critical wave speed, we obtain the precise asymptotic behavior of $\widehat{I}(z)$ when $z \to -\infty$.

To state the convergence, we make several assumptions.

(F3) If $\mathcal{R}_0 > 1$, then system (1.2) has a unique positive steady state (S^*, V^*, I^*) that is spatially homogeneous.

(B1) Assume that (F3) holds. If positive constants $\underline{S}, \underline{V}, \underline{I}, \overline{S}, \overline{V}, \overline{I}$ satisfy

$$(0,0,0) \ll (\underline{S}, \underline{V}, \underline{I}) \leq (\overline{S}, \overline{V}, \overline{I}) \ll (S_0, V_0, I_0)$$

and

$$\Pi - (\mu+\omega)\underline{S} - \underline{S}\sup\nolimits_{I\in[\underline{I},\overline{I}]} f_1(I) \leq 0, \tag{2.3}$$
$$\Pi - (\mu+\omega)\overline{S} - \overline{S}\inf\nolimits_{I\in[\underline{I},\overline{I}]} f_1(I) \geq 0, \tag{2.4}$$
$$\omega\underline{S} - \mu\underline{V} - \underline{V}\sup\nolimits_{I\in[\underline{I},\overline{I}]} f_2(I) \leq 0, \tag{2.5}$$
$$\omega\overline{S} - \mu\overline{V} - \overline{V}\inf\nolimits_{I\in[\underline{I},\overline{I}]} f_2(I) \geq 0, \tag{2.6}$$
$$e^{-\mu\tau}\underline{S}\inf\nolimits_{I\in[\underline{I},\overline{I}]} f_1(I) + e^{-\mu\tau}\underline{V}\inf\nolimits_{I\in[\underline{I},\overline{I}]} f_2(I) - (\mu+\nu)\underline{I} \leq 0, \tag{2.7}$$
$$e^{-\mu\tau}\overline{S}\sup\nolimits_{I\in[\underline{I},\overline{I}]} f_1(I) + e^{-\mu\tau}\overline{V}\sup\nolimits_{I\in[\underline{I},\overline{I}]} f_2(I) - (\mu+\nu)\overline{I} \geq 0, \tag{2.8}$$

then $(\underline{S}, \underline{V}, \underline{I}) = (\overline{S}, \overline{V}, \overline{I}) = (S^*, V^*, I^*)$.

(B2) Assume that $\mathcal{R}_0 > 1$, (F3), (I1)–(I3) hold. Then, (1.2) with (1.3) satisfies

$$\lim_{t\to\infty}(|S(x,t)-S^*|, |V(x,t)-V^*|, |I(x,t)-I^*|) = (0,0,0) \text{ uniformly in } x \in \mathbb{R}.$$

Theorem 2.6. *Assume that (I1)–(I2) hold and $\mathcal{R}_0 > 1$ in (1.2). If c_I^l, c_I^r are finite, then*

$$\limsup_{t\to\infty} \sup_{x>(c_I^r+\epsilon)t \text{ or } x<-(c_I^l+\epsilon)t} (|S(x,t)-S_0| + |V(x,t)-V_0| + I(x,t)) = 0 \text{ for any } \epsilon > 0. \tag{2.9}$$

Further suppose that (B1) or (B2) holds, then

$$\limsup_{t\to\infty} \sup_{(c_I^r-\epsilon)t>x>-(c_I^l-\epsilon)t} (|S(x,t)-S^*| + |V(x,t)-V^*| + |I(x,t)-I^*|) = 0 \tag{2.10}$$

for any small $\epsilon > 0$. Therefore, if (B1) or (B2) holds, then a positive solution of (2.1)–(2.2) satisfies

$$\lim_{z\to\infty}(\mathcal{S}(z), \mathcal{V}(z), \mathcal{I}(z)) = (S^*, V^*, I^*). \tag{2.11}$$

Because f_1, f_2 may be nonmonotone, it is difficult to give a sufficient condition on general f_1, f_2 such that (B1) or (B2) holds. But for special functions f_1, f_2, the verification is possible. For the assumption (B1) in (1.1), we also give the following sufficient condition.

Proposition 2.7. *Assume that $\mathcal{R}_0^1 > 1$ in (1.1). Then, (B1) holds provided that*

$$\begin{aligned}\alpha(\mu+\nu)(\overline{I}-\underline{I}) \leq \beta_1\overline{S}\left[\frac{e^{-\mu\tau}\beta_1}{\mu+\omega} + \frac{e^{-\mu\tau}\beta_2}{\mu}\frac{\omega}{(\mu+\omega)}\right]\left[\frac{\overline{I}}{1+\alpha\overline{I}} - \frac{\underline{I}}{1+\alpha\underline{I}}\right] \\ +\beta_2\overline{V}\frac{e^{-\mu\tau}\beta_2}{\mu}\left[\frac{\overline{I}}{1+\alpha\overline{I}} - \frac{\underline{I}}{1+\alpha\underline{I}}\right]\end{aligned} \tag{2.12}$$

leads to $\underline{I} = \overline{I}$. *In particular,* (2.12) *implies* $\underline{I} = \overline{I}$ *if*

$$\alpha(\mu+\nu) \geq \beta_1 S_0 \left[\frac{e^{-\mu\tau}\beta_1}{\mu+\omega} + \frac{e^{-\mu\tau}\beta_2}{\mu}\frac{\omega}{(\mu+\omega)}\right] + \beta_2 V_0 \frac{e^{-\mu\tau}\beta_2}{\mu}, \tag{2.13}$$

which indicates that large α *leads to the convergence.*

Similarly, for (B2), we have the following conclusion.

Proposition 2.8. *Assume that* $\mathcal{R}_0^1 > 1$ *in* (1.1). *Then, by regarding* (1.1) *as a special case of* (1.2), *(B2) holds for large* $\alpha > 0$.

Moreover, if $\mathcal{R}_0 \leq 1$, then the following convergence remains true.

Theorem 2.9. *Assume that (I1) and (I2) hold. If* $\mathcal{R}_0 \leq 1$ *in* (1.2), *then*

$$\lim_{t\to\infty}(S(x,t), V(x,t), I(x,t)) = (S_0, V_0, 0) \text{ uniformly in } x \in \mathbb{R}. \tag{2.14}$$

3. Proof of spreading speeds

In this part, we shall give the proof of Theorems 2.3 and 2.9. We first consider the following initial value problem

$$\begin{cases} \frac{\partial U(x,t)}{\partial t} = D_0 \frac{\partial^2 U(x,t)}{\partial x^2} + \widehat{f}_1(U(x,t-\tau)) + \widehat{f}_2(U(x,t-\tau)) \\ \qquad -\varepsilon U(x,t-\tau) - C_1 U(x,t) - C_2 U^2(x,t), \\ U(x,s) = \Phi(x,s), \end{cases} \tag{3.1}$$

in which $x \in \mathbb{R}, t > 0, s \in [-\tau, 0], D_0, C_1$ are positive constants and $\varepsilon \geq 0, C_2 \geq 0$, $\widehat{f}_1, \widehat{f}_2$ satisfy (F1) and (F2). When $\widehat{f}_1'(0) + \widehat{f}_2'(0) - \varepsilon > C_1$, we define $u^* > 0$ as the unique root of

$$\widehat{f}_1(u^*)/u^* + \widehat{f}_2(u^*)/u^* - \varepsilon - C_1 - C_2 u^* = 0,$$

of which the uniqueness is clear by the monotonicity of $\widehat{f}_i(x)/x$. We also define

$$\Lambda_1(\lambda, c) = D_0\lambda^2 - c\lambda + (\widehat{f}_1'(0) + \widehat{f}_2'(0) - \varepsilon)e^{-\lambda c\tau} - C_1, \quad c \geq 0, \lambda > 0,$$

for which the properties can be studied similar to Lemma 2.1.

Lemma 3.1. *If* $\widehat{f}_1'(0) + \widehat{f}_2'(0) > \varepsilon + C_1$, *then there exists a constant* $c_1' > 0$ *such that* $\Lambda_1(\lambda, c) = 0$ *has a real root if and only if* $c \geq c_1'$. *Therefore, there exists a unique* $\lambda_1' > 0$ *such that* $\Lambda_1(\lambda_1', c_1') = 0$. *In particular,* c_1', λ_1' *are continuous in* ε. *When* $c > c_1'$, *then* $\Lambda_1(\lambda, c) = 0$ *has two positive roots and the smaller one is denoted by* λ_1^c.

Using these constants, we have the following conclusion by the entire solutions in Li et al. [18], spreading speeds in Solar and Trofimchuk [28], Thieme and Zhao [30], stability of traveling waves in Wang et al. [34] and the fast propagation in Zhu [43].

Lemma 3.2. *Assume that there exists* $\overline{u} \geq u^*$ *such that*

$$\widehat{f}_1(u) + \widehat{f}_2(u) - \varepsilon u, \qquad u \in [0, \overline{u}]$$

is nondecreasing. If $\Phi(x,s)$ *is nonnegative, bounded and continuous and* $0 \leq \Phi(x,s) \leq \overline{u}, x \in \mathbb{R}, s \in [-\tau, 0]$, *then* (3.1) *has a bounded mild solution* $0 \leq U(x,t) \leq \overline{u}, x \in \mathbb{R}, t > 0$, *which has the following properties.*

(D1) It is a classical solution for all $t > \tau$.

(D2) If $\widehat{f}_1'(0) + \widehat{f}_2'(0) \leq \varepsilon + C_1$, *then* $\lim_{t\to\infty} U(x,t) = 0$ *uniformly in* $x \in \mathbb{R}$.

Table 2. *Spreading speeds of* (3.1) *with different initial conditions*

Initial conditions	(IL4) and (IR4)	(IL4) and (IR5)	(IL4) and (IR6)
Spreading speeds	$c_U^l = c_U^r = c_1'$	$c_U^l = c_1', c_U^r = c_r$	$c_U^l = c_1', c_U^r = \infty$
Initial conditions	(IL5) and (IR4)	(IL5) and (IR5)	(IL5) and (IR6)
Spreading speeds	$c_U^l = c_l, c_U^r = c_1'$	$c_U^l = c_l, c_U^r = c_r$	$c_U^l = c_l, c_U^r = \infty$
Initial conditions	(IL6) and (IR4)	(IL6) and (IR5)	(IL6) and (IR6)
Spreading speeds	$c_U^l = \infty, c_U^r = c_1'$	$c_U^l = \infty, c_U^r = c_r$	$c_U^l = c_U^r = \infty$

(D3) If the continuous function $\underline{U} : \mathbb{R} \times [-\tau, \infty) \to [0, \overline{u}]$ *satisfies*

$$\underline{U}(x,s) \le (\ge) \Phi(x,s), \qquad x \in \mathbb{R}, s \in [-\tau, 0]$$

and

$$\underline{U}(x,t) \le (\ge) \left[T_3 \underline{U}(\cdot, r)\right](t-r)(x) + \int_r^t \left[T_3 \left[\widehat{f}_1 \left(\underline{U}(\cdot, s-\tau)\right)\right]\right](t-s)(x) ds$$

$$+ \int_r^t \left[T_3 \left[\widehat{f}_2 \left(\underline{U}(\cdot, s-\tau)\right) - \varepsilon \underline{U}(\cdot, s-\tau)\right]\right](t-s)(x) ds$$

$$- \int_r^t \left[T_3 \left[C_1 \underline{U}(\cdot, s) + C_2 \underline{U}^2(\cdot, s)\right]\right](t-s)(x) ds$$

for any $0 \le r \le t$*, then* $\underline{U}(x,t) \le (\ge) U(x,t)$ *for all* $x \in \mathbb{R}, t > 0$.

(D4) If $\widehat{f}_1'(0) + \widehat{f}_2'(0) > \varepsilon + C_1$ *and* $c_1 > 0, c_2 > 0$ *such that* $\liminf_{t\to\infty} \inf_{-c_1 t < x < c_2 t} U(x,t) > 0$*, then* $\limsup_{t\to\infty} \sup_{-c_1 t < x < c_2 t} |U(x,t) - u^*| = 0$.

(D5) With different decaying initial values, the spreading speeds of U *are given by Table* 2*. Here, the initial conditions are defined by the following items:*

(IL4) there exists $A_5 > 0$ *such that* $\Phi(x,s) \le A_5(-x+1)e^{\lambda_1' x}, x < 0, s \in [-\tau, 0]$;

(IL5) there exist $A_6 > 1, c_l > c_1', X_3 > 0$ *such that*

$$A_6^{-1} e^{\lambda_1^{c_l} x} \le \Phi(x,0) \le A_6 e^{\lambda_1^{c_l} x}, \Phi(x,s) \le A_6 e^{\lambda_1^{c_l} x}, s \in [-\tau, 0], x < -X_3;$$

(IL6) for any $\epsilon > 0$*, we have* $\liminf_{x\to-\infty} \Phi(x,0) e^{-\epsilon x} = \infty$;

(IR4) there exists $A_7 > 0$ *such that* $\Phi(x,s) \le A_7(x+1)e^{-\lambda_1' x}, x > 0, s \in [-\tau, 0]$;

(IR5) there exist $A_8 > 1, c_r > c_1', X_4 > 0$ *such that*

$$A_8^{-1} e^{-\lambda_1^{c_r} x} \le \Phi(x,0) \le A_8 e^{-\lambda_1^{c_r} x}, \Phi(x,s) \le A_8 e^{-\lambda_1^{c_r} x}, s \in [-\tau, 0], x > X_4;$$

(IR6) for any $\epsilon > 0$*, we have* $\liminf_{x\to+\infty} \Phi(x,0) e^{\epsilon x} = \infty$.

Lemma 3.3. *Theorem 2.9 holds.*

Proof. By Theorem 2.2, as well as the property of semigroup, we have

$$I(x,t) \le \left[T_3 I(\cdot, r)\right](t-r)(x) + e^{-\mu\tau} S_0 \int_r^t \left[T_3 f_1 \left(I(\cdot, s-\tau)\right)\right](t-s)(x) ds$$

$$+ \int_r^t \left[T_3 \left[e^{-\mu\tau} V_0 f_2 \left(I(\cdot, s-\tau)\right) - (\mu+\nu) I(\cdot, s)\right]\right](t-s)(x) ds$$

$$\leq [T_3 I(\cdot,r)](t-r)(x) + e^{-\mu\tau} S_0 \int_r^t \left[T_3 \overline{f}_1 (I(\cdot, s-\tau))\right](t-s)(x)ds$$

$$+ \int_r^t \left[T_3 \left[e^{-\mu\tau} V_0 \overline{f}_2 (I(\cdot, s-\tau)) - (\mu+\nu) I(\cdot, s)\right]\right](t-s)(x)ds \tag{3.2}$$

for all $0 \leq r \leq t$, where

$$\overline{f}_i(x) = \sup_{y\in[0,x]} f_i(y), \quad x \in [0, I_0], i = 1, 2.$$

Evidently, $\overline{f}_i(x)$ is nondecreasing and $\overline{f}_i'(0) = f_i'(0)$ for $x \in [0, I_0], i = 1, 2$. Using (D2) and (D3) in Lemma 3.2, we complete the proof. □

Lemma 3.4. *Assume that $\mathcal{R}_0 > 1$ is true. Then, the upper bounds of spreading speeds in Table 1 hold.*

Proof. Due to (3.2), the conclusion is clear by Lemma 3.2. □

Lemma 3.5. *Assume that $\mathcal{R}_0 > 1$, (I1) and (I2) hold. Then, Theorem 2.3 is true.*

Proof. We only consider the case $t > 2\tau + 1$ such that the classical solution is concerned. Define

$$\underline{f}(x) - 2(S_0 + V_0)\epsilon x = \inf_{y\in[x,I_0]} \left[S_0 f_1(y) + V_0 f_2(y) - 2(S_0 + V_0)\epsilon y\right], \quad x \in [0, I_0].$$

Evidently, for small $\epsilon > 0$, we have

$$\underline{f}(x) - 2(S_0 + V_0)\epsilon x, \quad x \in (0, I_0],$$

is nondecreasing and positive. In particular, (F1) implies that

$$\underline{f}'(0) = S_0 f_1'(0) + V_0 f_2'(0).$$

Further select $\epsilon > 0$ small enough such that

$$d_3\lambda^2 - c\lambda + e^{-(\mu+c\lambda)\tau} \left(S_0 f_1'(0) + V_0 f_2'(0)\right) - (\mu+\nu) - 2\epsilon(f_1'(0) + f_1'(0))e^{-\lambda c\tau} > 0, \quad \lambda > 0.$$

For any given $\epsilon > 0$ satisfying the above properties, we shall prove that there exist $t_0 \geq 2\tau + 1, M > 0$ such that

$$\frac{\partial I(x,t)}{\partial t} \geq d_3 \frac{\partial^2 I(x,t)}{\partial x^2} + e^{-\mu\tau} \underline{f}\left(I(x, t-\tau)\right)$$
$$-\epsilon(f_1'(0) + f_2'(0))I(x, t-\tau) - (\mu+\nu)I(x,t) - MI^2(x,t) \tag{3.3}$$

for all $t > t_0, x \in \mathbb{R}$. Once (3.3) holds, then $I(x, t_0) > 0, x \in \mathbb{R}$ (see Theorem 2.2), Lemma 3.2 implies the inner spreading speeds are not less than c^* by selecting any small $\epsilon > 0$. From Lemma 3.4, we further find that $c_I^l = c^*$ when (IL1) holds or $c_I^r = c^*$ when (IR1) is true.

For the case (IL2), from

$$\frac{\partial I(x,t)}{\partial t} \geq d_3 \frac{\partial^2 I(x,t)}{\partial x^2} - (\mu+\nu)I(x,t)$$

and

$$I(x,t) \geq e^{-(\mu+\nu)t} \left[T_3 I(\cdot, 0)\right](t)(x),$$

there exists a small constant $\varepsilon_0 > 0$ such that

$$I(x, t_0 + s) \geq \varepsilon_0 e^{\lambda^{c_l} x}, x < 0, s \in [-\tau, 0].$$

Combining this with (3.3), for any fixed $c \in (0, c_l)$, we can find that the leftward spreading speed is not less than c_l by selecting $\epsilon > 0$ small enough. With the help of Lemma 3.4, we have $c_I^l = c_l$. Similarly, the conclusions for $c_I^r = c_r$ hold when (IR2) is true.

When the case (IL3) is concerned, for any given $\widehat{c} > c^*$ and the corresponding $\lambda^{\widehat{c}}$, we can select a small constant $\varepsilon_0 > 0$ such that

$$I(x, t_0 + s) \geq \varepsilon_0 e^{\lambda^{\widehat{c}} x}, x < 0, s \in [-\tau, 0].$$

Using (3.3), the leftward spreading speed of I is not less than $\widehat{c}$. Due to the arbitrary of $\widehat{c}$, $c_I^l = \infty$. Similarly, we can prove that (IR3) implies that $c_I^r = \infty$.

Therefore, it suffices to obtain (3.3), which will be divided into the following two steps.

Step 1 We estimate $S(x,t) - S_0, V(x,t) - V_0$.

Step 2 From the equation of I, we shall find (3.3).

We now discuss the Step 1. Since

$$\begin{aligned}\frac{\partial[S(x,t) - S_0]}{\partial t} &= d_1 \frac{\partial^2[S(x,t) - S_0]}{\partial x^2} + (\mu+\omega)S_0 - (\mu+\omega)S(x,t) - S(x,t)f_1(I(x,t)) \\ &= d_1 \frac{\partial^2[S(x,t) - S_0]}{\partial x^2} - (\mu+\omega)[S(x,t) - S_0] - S(x,t)f_1(I(x,t)) \\ &\geq d_1 \frac{\partial^2[S(x,t) - S_0]}{\partial x^2} - (\mu+\omega)[S(x,t) - S_0] - S_0 f_1'(0) I(x,t),\end{aligned}$$

we have

$$S(x,t) - S_0 \geq -e^{-(\mu+\omega)(t-t_1)} S_0 - S_0 f_1'(0) \int_{t-t_1}^{t} \frac{e^{-(\mu+\omega)(t-s)}}{\sqrt{4\pi d_1 (t-s)}} \int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4d_1(t-s)}} I(y,s) \mathrm{d}y\mathrm{d}s$$

for any $x \in \mathbb{R}, 0 \leq t_1 \leq t < \infty$. By the uniform convergence of $\frac{1}{\sqrt{4\pi d_1 t}} \int_{\mathbb{R}} e^{\frac{-y^2}{4d_1 t}} dy$ when t belongs to a bounded interval, as well as the decay behavior of $e^{-(\mu+\omega)t}$, we can select $T_1 > 2\tau + 1$ large enough such that

$$S(x,t) - S_0 \geq \frac{-\epsilon}{4} - S_0 f_1'(0) \int_{t-T_1}^{t-1/T_1} \frac{e^{-(\mu+\omega)(t-s)}}{\sqrt{4\pi d_1 (t-s)}} \int_{-T_1}^{T_1} e^{\frac{-(x-y)^2}{4d_1(t-s)}} I(y,s) \mathrm{d}y\mathrm{d}s$$

for any $t > 2T_1$.

Furthermore, from

$$\begin{aligned}&\frac{\partial[V(x,t) - V_0]}{\partial t} \\ &= d_2 \frac{\partial^2[V(x,t) - V_0]}{\partial x^2} + \omega[S(x,t) - S_0] - \mu[V(x,t) - V_0] - V(x,t)f_2(I(x,t)) \\ &\geq d_2 \frac{\partial^2[V(x,t) - V_0]}{\partial x^2} + \omega[S(x,t) - S_0] - \mu[V(x,t) - V_0] - V_0 f_2'(0) I(x,t),\end{aligned}$$

we obtain

$$\begin{aligned}V(x,t) - V_0 \geq& -e^{-\mu(t-t_1)} V_0 - V_0 f_2'(0) \int_{t-t_1}^{t} \frac{e^{-\mu(t-s)}}{\sqrt{4\pi d_2 (t-s)}} \int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4d_2(t-s)}} I(y,s) \mathrm{d}y\mathrm{d}s \\ &+\omega \int_{t-t_1}^{t} \frac{e^{-\mu(t-s)}}{\sqrt{4\pi d_2 (t-s)}} \int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4d_2(t-s)}} [S(y,s) - S_0] \mathrm{d}y\mathrm{d}s \\ \geq& -V_0 e^{-\mu(t-t_1)} - S_0 e^{-(\mu+\omega)(t-t_1)} (e^{\omega t_1} - 1)\end{aligned}$$

$$
\begin{aligned}
&-V_0 f_2'(0) \int\limits_{t-t_1}^{t} \frac{e^{-\mu(t-s)}}{\sqrt{4\pi d_2(t-s)}} \int\limits_{\mathbb{R}} e^{\frac{-(x-y)^2}{4d_2(t-s)}} I(y,s)\mathrm{d}y\mathrm{d}s \\
&-\omega S_0 f_1'(0) \int\limits_{t-t_1}^{t} \frac{e^{-\mu(t-s)}}{\sqrt{4\pi d_2(t-s)}} \int\limits_{\mathbb{R}} e^{\frac{-(x-y)^2}{4d_2(t-s)}} \\
&\times \left[\int\limits_{s-t_1}^{s} \frac{e^{-(\mu+\omega)(s-r)}}{\sqrt{4\pi d_1(s-r)}} \int\limits_{\mathbb{R}} e^{\frac{-(z-y)^2}{4d_1(s-r)}} I(z,r)\mathrm{d}z\mathrm{d}r \right] \mathrm{d}y\mathrm{d}s
\end{aligned}
$$

for any $x \in \mathbb{R}, 0 \le t_1 \le t < \infty, s \in [t-t_1, t]$. Similar to the discussion on $S(x,t)$, we can select $T_2 \ge T_1$ such that

$$
\begin{aligned}
V(x,t) - V_0 \ge \frac{-\epsilon}{4} &- V_0 f_2'(0) \int\limits_{t-T_2}^{t-1/T_2} \frac{e^{-\mu(t-s)}}{\sqrt{4\pi d_2(t-s)}} \int\limits_{-T_2}^{T_2} e^{\frac{-(x-y)^2}{4d_2(t-s)}} I(y,s)\mathrm{d}y\mathrm{d}s \\
&-\omega S_0 f_1'(0) \int\limits_{t-T_2}^{t-1/T_2} \frac{e^{-\mu(t-s)}}{\sqrt{4\pi d_2(t-s)}} \int\limits_{-T_2}^{T_2} e^{\frac{-(x-y)^2}{4d_2(t-s)}} \\
&\times \left[\int\limits_{s-T_1}^{s} \frac{e^{-(\mu+\omega)(s-r)}}{\sqrt{4\pi d_1(s-r)}} \int\limits_{-T_1}^{T_1} e^{\frac{-(z-y)^2}{4d_1(s-r)}} I(z,r)dzdr \right] \mathrm{d}y\mathrm{d}s
\end{aligned}
$$

for all $t \ge 2T_1 + 2T_2 + 2\tau + 1, x \in \mathbb{R}$.

Further study the Step 2. From

$$
\begin{aligned}
\frac{\partial I(x,t)}{\partial t} &= d_3 \frac{\partial^2 I(x,t)}{\partial x^2} + e^{-\mu\tau} S(x,t-\tau) f_1(I(x,t-\tau)) \\
&\quad + e^{-\mu\tau} V(x,t-\tau) f_2(I(x,t-\tau)) - (\mu+\nu) I(x,t) \\
&= d_3 \frac{\partial^2 I(x,t)}{\partial x^2} + e^{-\mu\tau} \left[S_0 f_1(I(x,t-\tau)) + V_0 f_2(I(x,t-\tau)) \right] I(x,t-\tau) \\
&\quad + e^{-\mu\tau} [S(x,t-\tau) - S_0] f_1(I(x,t-\tau)) \\
&\quad + e^{-\mu\tau} [V(x,t-\tau) - V_0] f_2(I(x,t-\tau)) - (\mu+\nu) I(x,t) \\
&\ge d_3 \frac{\partial^2 I(x,t)}{\partial x^2} + e^{-\mu\tau} \underline{f}(I(x,t-\tau)) - (\mu+\nu) I(x,t) \\
&\quad + e^{-\mu\tau} \left[[S(x,t-\tau) - S_0] f_1'(0) + [V(x,t-\tau) - V_0] f_2'(0) \right] I(x,t-\tau),
\end{aligned}
$$

we shall look for $M > 0, t_0 > 0$ such that

$$
\begin{aligned}
&e^{-\mu\tau} \left[[S(x,t-\tau) - S_0] f_1'(0) + [V(x,t-\tau) - V_0] f_2'(0) \right] I(x,t-\tau) \\
&\quad \ge -(f_1'(0)\epsilon + f_2'(0)\epsilon) I(x,t-\tau) - M I^2(x,t)
\end{aligned} \tag{3.4}
$$

for all $x \in \mathbb{R}, t \ge t_0$.

If

$$
S_0 f_1'(0) \int\limits_{t-t_1}^{t} \frac{e^{-(\mu+\omega)(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int\limits_{\mathbb{R}} e^{\frac{-(x-y)^2}{4d_1(t-s)}} I(y,s)\mathrm{d}y\mathrm{d}s < \frac{\epsilon}{8}, \tag{3.5}
$$

$$V_0 f_2'(0) \int\limits_{t-t_1}^{t} \frac{e^{-\mu(t-s)}}{\sqrt{4\pi d_2(t-s)}} \int\limits_{\mathbb{R}} e^{\frac{-(x-y)^2}{4d_2(t-s)}} I(y,s)\mathrm{d}y\mathrm{d}s < \frac{\epsilon}{8} \tag{3.6}$$

and

$$\omega f_1'(0) S_0 \int\limits_{t-T_2}^{t-1/T_2} \frac{e^{-\mu(t-s)}}{\sqrt{4\pi d_2(t-s)}} \int\limits_{-T_2}^{T_2} e^{\frac{-(x-y)^2}{4d_2(t-s)}} \times \left[\int\limits_{s-T_1}^{s} \frac{e^{-(\mu+\omega)(s-r)}}{\sqrt{4\pi d_1(s-r)}} \int\limits_{-T_1}^{T_1} e^{\frac{-(z-y)^2}{4d_1(s-r)}} I(z,r)dzdr \right] \mathrm{d}y\mathrm{d}s < \frac{\epsilon}{8}, \tag{3.7}$$

then we obtain

$$\begin{aligned} &e^{-\mu\tau} \left[[S(x,t-\tau) - S_0] f_1'(0) + [V(x,t-\tau) - V_0] f_2'(0) \right] I(x,t-\tau) \\ &\geq -e^{-\mu\tau} \left[f_1'(0)\epsilon + f_2'(0)\epsilon \right] I(x,t-\tau)/2. \end{aligned} \tag{3.8}$$

If Eq. (3.5) is false, then

$$f_1'(0) S_0 \int\limits_{t-T_1}^{t-1/T_1} \frac{e^{-(\mu+\omega)(t-s)}}{\sqrt{4\pi d_1(t-s)}} \int\limits_{-T_1}^{T_1} e^{\frac{-(x-y)^2}{4d_1(t-s)}} I(y,s)\mathrm{d}y\mathrm{d}s \geq \frac{\epsilon}{8} \tag{3.9}$$

for some $t \geq 2T_1 + 2T_2 + 2\tau + 1, x \in \mathbb{R}$. From Theorem 2.2, I is uniformly continuous, then we can select two constants $\eta > 0, \delta > 0$ such that

$$\begin{aligned} &I(y,s) > \eta \text{ for some } s \in [t - T_1, t - 1/T_1], y_0 \in (x - T_1, x + T_1) \\ &\qquad \text{and all } y \in [y_0 - \delta, y_0 + \delta] \subseteq [x - T_1, x + T_1]. \end{aligned}$$

In particular, $\eta > 0, \delta > 0$ are independent of x, t once (3.9) holds. That is, we may fix $\eta > 0, \delta > 0$ for all $x, t \geq 2T_1 + 2T_2 + 2\tau + 1$ such that (3.9) holds.

Consider the initial value problem

$$\frac{\partial i(x,t)}{\partial t} = d_3 \frac{\partial^2 i(x,t)}{\partial x^2} - (\gamma + \mu)\, i(x,t), \quad x \in \mathbb{R}, t > 0$$

with a continuous initial condition

$$i(x,0) \begin{cases} = \eta, |x| \leq \delta/2, \\ = 0, |x| \geq \delta, \\ \in [0,\eta], \delta/2 \leq |x| \leq \delta. \end{cases}$$

Let

$$\kappa = \inf_{t \in [1/T_1, T_1+\tau], x \in [-T_1, T_1]} i(x,t).$$

Then, κ is independent on x, t when (3.9) holds. That is, we can fix $\kappa > 0$ for all $x, t \geq 2T_1 + 2T_2 + 2\tau + 1$ such that (3.4) holds. Using the comparison principle, we have $I(x,t) > \kappa$ when (3.9) is true, which implies the existence of $M > 0$ by the boundedness of the left of (3.4). When (3.6) or (3.7) does not hold, we can get a similar conclusion by selecting M, t_0 large enough but finite. The proof is complete. □

4. Proof of convergence

Lemma 4.1. *Assume that (I1)–(I2) hold and $\mathcal{R}_0 > 1$. If c_I^l, c_I^r are finite, then (2.9) is true.*

Proof. We prove the result for given $\epsilon > 0$. Let $\{\epsilon_n\}$ be a strictly increasing sequence such that $\epsilon_1 = \epsilon/2, \lim_{n\to\infty}\epsilon_n = \epsilon$. Define

$$\liminf_{t\to\infty} \inf_{-(c_I^l+\epsilon_n)t>x \text{ or } x>(c_I^r+\epsilon_n)t} S(x,t) = s_n, \limsup_{t\to\infty} \sup_{-(c_I^l+\epsilon_n)t>x \text{ or } x>(c_I^r+\epsilon_n)t} I(x,t) = I_n.$$

Then, $\{s_n\}$ is nondecreasing and positive in n and $I_n = 0$ for all $n \in \mathbb{R}$ Therefore, the limit of $\{s_n\}$ exists, we denote it by s'. Let

$$\liminf_{t\to\infty} \inf_{-(c_I^l+\epsilon)t>x \text{ or } x>(c_I^r+\epsilon)t} S(x,t) = s_0.$$

Then, we have $s_0 \geq s_n$ and $s_0 \geq s'$. Select $\theta > 0$ such that

$$\theta s + \Pi - (\mu+\omega)s - sf_1(i), \quad s \in [0, S_0], i \in [0, I_0]$$

is nondecreasing in s. For given s_n, s_{n+1}, the dominated convergence in the integral equation of S implies that

$$s_{n+1} \geq (\theta s_n + \Pi - (\mu+\omega)s_n)/\theta.$$

Letting $n \to \infty$, we have $s' \geq S_0$. Note that, $S(x,t) \leq S_0$, we obtain $s' = s_0 = S_0$.

In a similar way, we can prove the result on V. The proof is complete. □

Lemma 4.2. *Assume that $\mathcal{R}_0 > 1$ and (B1) hold. Then, (2.10) holds.*

Proof. Let

$$\liminf_{t\to\infty} \inf_{-(c_u^l-\epsilon)t<x<(c_u^r-\epsilon)t} u(x,t) = \underline{u}, \quad \limsup_{t\to\infty} \sup_{-(c_u^l-\epsilon)t<x<(c_u^r-\epsilon)t} u(x,t) = \overline{u}, \quad u \in \{S, V, I\}.$$

Then, these are positive constants such that

$$\overline{S} < S_0, \overline{V} < V_0, \overline{I} < I_0.$$

Similar to the proof of Lemma 4.1, we obtain (2.3)–(2.8). Then, (B1) implies what we wanted. The proof is complete. □

Lemma 4.3. *Assume that $\mathcal{R}_0 > 1$ and (B2) hold. Then, (2.10) holds.*

Proof. We can prove this similar to Bo et al. [6, Section 4] and omit the detail here. □

Lemma 4.4. *Proposition 2.7 is true.*

Proof. From the monotonicity, the assumption (B1) becomes if

$$\Pi - (\mu+\omega)\underline{S} - \frac{\beta_1 \underline{S}\overline{I}}{1+\alpha\overline{I}} \leq 0, \tag{4.1}$$

$$\Pi - (\mu+\omega)\overline{S} - \frac{\beta_1 \overline{S}\underline{I}}{1+\alpha\underline{I}} \geq 0, \tag{4.2}$$

$$\omega\underline{S} - \mu\underline{V} - \frac{\beta_2 \underline{V}\overline{I}}{1+\alpha\overline{I}} \leq 0, \tag{4.3}$$

$$\omega\overline{S} - \mu\overline{V} - \frac{\beta_2 \overline{V}\underline{I}}{1+\alpha\underline{I}} \geq 0, \tag{4.4}$$

$$\frac{e^{-\mu\tau}\beta_1\underline{S}}{1+\alpha\underline{I}} + \frac{e^{-\mu\tau}\beta_2\underline{V}}{1+\alpha\underline{I}} - (\mu+\nu) \leq 0, \tag{4.5}$$

$$\frac{e^{-\mu\tau}\beta_1\overline{S}}{1+\alpha\overline{I}}+\frac{e^{-\mu\tau}\beta_2\overline{V}}{1+\alpha\overline{I}}-(\mu+\nu)\geq 0, \tag{4.6}$$

then $(\underline{S},\underline{V},\underline{I})=(\overline{S},\overline{V},\overline{I})=(S^*,V^*,I^*)$. From (4.1) and (4.2), we have

$$-\left(\overline{S}-\underline{S}\right)\geq\frac{1}{(\mu+\omega)}\left[\frac{\beta_1\overline{S}\underline{I}}{1+\alpha\underline{I}}-\frac{\beta_1\underline{S}\overline{I}}{1+\alpha\overline{I}}\right]. \tag{4.7}$$

Applying (4.3) and (4.4), we obtain

$$\begin{aligned}-\left(\overline{V}-\underline{V}\right)&\geq\frac{1}{\mu}\left[\omega\left(\underline{S}-\overline{S}\right)+\frac{\beta_2\overline{V}\underline{I}}{1+\alpha\underline{I}}-\frac{\beta_2\underline{V}\overline{I}}{1+\alpha\overline{I}}\right]\\&\geq\frac{1}{\mu}\left[\frac{\omega}{(\mu+\omega)}\left[\frac{\beta_1\overline{S}\underline{I}}{1+\alpha\underline{I}}-\frac{\beta_1\underline{S}\overline{I}}{1+\alpha\overline{I}}\right]+\frac{\beta_2\overline{V}\underline{I}}{1+\alpha\underline{I}}-\frac{\beta_2\underline{V}\overline{I}}{1+\alpha\overline{I}}\right].\end{aligned} \tag{4.8}$$

Using (4.5) and (4.6), we find

$$e^{-\mu\tau}\beta_1(\overline{S}-\underline{S})+e^{-\mu\tau}\beta_2(\overline{V}-\underline{V})\geq\alpha(\mu+\nu)(\overline{I}-\underline{I}). \tag{4.9}$$

From (4.7) to (4.9), we further have

$$\begin{aligned}0&\geq\frac{e^{-\mu\tau}\beta_1}{(\mu+\omega)}\left[\frac{\beta_1\overline{S}\underline{I}}{1+\alpha\underline{I}}-\frac{\beta_1\underline{S}\overline{I}}{1+\alpha\overline{I}}\right]+\alpha(\mu+\nu)(\overline{I}-\underline{I})\\&\quad+\frac{e^{-\mu\tau}\beta_2}{\mu}\left[\frac{\omega}{(\mu+\omega)}\left[\frac{\beta_1\overline{S}\underline{I}}{1+\alpha\underline{I}}-\frac{\beta_1\underline{S}\overline{I}}{1+\alpha\overline{I}}\right]+\frac{\beta_2\overline{V}\underline{I}}{1+\alpha\underline{I}}-\frac{\beta_2\underline{V}\overline{I}}{1+\alpha\overline{I}}\right]\\&=\left[\frac{e^{-\mu\tau}\beta_1}{\mu+\omega}+\frac{e^{-\mu\tau}\beta_2}{\mu}\frac{\omega}{(\mu+\omega)}\right]\left[\frac{\beta_1\overline{S}\underline{I}}{1+\alpha\underline{I}}-\frac{\beta_1\underline{S}\overline{I}}{1+\alpha\overline{I}}\right]+\alpha(\mu+\nu)(\overline{I}-\underline{I})\\&\quad+\frac{e^{-\mu\tau}\beta_2}{\mu}\left[\frac{\beta_2\overline{V}\underline{I}}{1+\alpha\underline{I}}-\frac{\beta_2\underline{V}\overline{I}}{1+\alpha\overline{I}}\right]\end{aligned}$$

or

$$\begin{aligned}\alpha(\mu+\nu)(\overline{I}-\underline{I})&\leq\left[\frac{e^{-\mu\tau}\beta_1}{\mu+\omega}+\frac{e^{-\mu\tau}\beta_2}{\mu}\frac{\omega}{(\mu+\omega)}\right]\left[\frac{\beta_1\underline{S}\overline{I}}{1+\alpha\overline{I}}-\frac{\beta_1\overline{S}\underline{I}}{1+\alpha\underline{I}}\right]\\&\quad+\frac{e^{-\mu\tau}\beta_2}{\mu}\left[\frac{\beta_2\underline{V}\overline{I}}{1+\alpha\overline{I}}-\frac{\beta_2\overline{V}\underline{I}}{1+\alpha\underline{I}}\right]\\&\leq\left[\frac{e^{-\mu\tau}\beta_1}{\mu+\omega}+\frac{e^{-\mu\tau}\beta_2}{\mu}\frac{\omega}{(\mu+\omega)}\right]\left[\frac{\beta_1\overline{S}\overline{I}}{1+\alpha\overline{I}}-\frac{\beta_1\overline{S}\underline{I}}{1+\alpha\underline{I}}\right]\\&\quad+\frac{e^{-\mu\tau}\beta_2}{\mu}\left[\frac{\beta_2\overline{V}\overline{I}}{1+\alpha\overline{I}}-\frac{\beta_2\overline{V}\underline{I}}{1+\alpha\underline{I}}\right]\\&\leq\beta_1\overline{S}\left[\frac{e^{-\mu\tau}\beta_1}{\mu+\omega}+\frac{e^{-\mu\tau}\beta_2}{\mu}\frac{\omega}{(\mu+\omega)}\right]\left[\frac{\overline{I}}{1+\alpha\overline{I}}-\frac{\underline{I}}{1+\alpha\underline{I}}\right]\\&\quad+\beta_2\overline{V}\frac{e^{-\mu\tau}\beta_2}{\mu}\left[\frac{\overline{I}}{1+\alpha\overline{I}}-\frac{\underline{I}}{1+\alpha\underline{I}}\right]\\&\leq\beta_1\overline{S}\left[\frac{e^{-\mu\tau}\beta_1}{\mu+\omega}+\frac{e^{-\mu\tau}\beta_2}{\mu}\frac{\omega}{(\mu+\omega)}\right](\overline{I}-\underline{I})+\beta_2\overline{V}\frac{e^{-\mu\tau}\beta_2}{\mu}(\overline{I}-\underline{I}).\end{aligned}$$

Thus, we find that if (2.12) leads to $\overline{I}=\underline{I}$, then further from $\overline{I}=\underline{I}$, (4.7) leads to $\overline{S}=\underline{S}$ and (4.8) leads to $\overline{V}=\underline{V}$. Due to the uniqueness of constant steady states, we obtain $(\overline{S},\overline{V},\overline{I})=(\underline{S},\underline{V},\underline{I})=(S^*,V^*,I^*)$. Thus, (B1) holds true. Moreover, (2.13) also makes $\overline{I}=\underline{I}$ by the fact $(\frac{x}{1+\alpha x})'\leq 1, x\geq 0$, which also implies (B1). We complete the proof. □

On the assumption (B1), we may give further bounds of $\underline{S}, \underline{V}, \underline{I}, \overline{S}, \overline{V}, \overline{I}$, which possibly weakens the verification of (B1). We give a special case here. Let $\underline{s} > 0, \underline{v} > 0$ be defined by

$$\Pi - (\mu + \omega)\underline{s} - \underline{s} \sup_{i \in [0, I_0]} f_1(i) = 0, \qquad \omega \underline{s} - \mu \underline{v} - \underline{v} \sup_{i \in [0, I_0]} f_1(i) = 0, \tag{4.10}$$

which are well defined once $\mathcal{R}_0 > 1$. When

$$e^{-\mu\tau} f_1'(0)\underline{s} + e^{-\mu\tau} f_2'(0)\underline{v} > \mu + \nu, \tag{4.11}$$

we further fix $\underline{i} > 0$ such that

$$e^{-\mu\tau}\underline{s} \inf_{i \in [\underline{i}, I_0]} f_1(i) + e^{-\mu\tau}\underline{v} \inf_{i \in [\underline{i}, I_0]} f_2(i) = (\mu + \nu)\underline{i}.$$

Using the notations in the proof of Lemma 4.4, we can verify that

$$(\underline{S}, \underline{V}, \underline{I}) \geq (\underline{s}, \underline{v}, \underline{i}).$$

From $(\underline{s}, \underline{v}, \underline{i})$, we may obtain a new bound $(\overline{S}_1, \overline{V}_1, \overline{I}_1)$ of $(\overline{S}, \overline{V}, \overline{I})$, where $\overline{S}_1$ is defined by

$$\Pi - (\mu + \omega)\overline{S}_1 - \overline{S}_1 \inf_{i \in [\underline{i}, I_0]} f_1(i) = 0,$$

and $\overline{V}_1, \overline{I}_1$ are defined in the similar way. Repeating the process, we obtain a strictly decreasing sequence $\{(\overline{S}_n, \overline{V}_n, \overline{I}_n)\}$ of upper bounds and a strictly increasing sequence $\{(\underline{S}_n, \underline{V}_n, \underline{I}_n)\}$ of lower bounds. For each n, we consider (B1) in the rectangle $[(\underline{S}_n, \underline{V}_n, \underline{I}_n), (\overline{S}_n, \overline{V}_n, \overline{I}_n)]$. With the larger n, it is possible to obtain a weaker condition such that (B1) holds.

Lemma 4.5. *Proposition 2.8 is true.*

Proof. We prove this by the idea of comparison principle. Consider

$$\begin{cases} x'(t) = \Pi - (\mu + \omega)x(t) - \frac{\beta_1 x(t) z(t)}{1 + \alpha z(t)}, \\ y'(t) = \omega x(t) - \frac{\beta_2 y(t) z(t)}{1 + \alpha z(t)} - \mu y(t), \\ z'(t) = e^{-\mu\tau} \frac{\beta_1 x(t-\tau) z(t-\tau)}{1 + \alpha z(t-\tau)} + e^{-\mu\tau} \frac{\beta_2 y(t-\tau) z(t-\tau)}{1 + \alpha z(t-\tau)} - (\mu + \nu) z(t). \end{cases} \tag{4.12}$$

For (1.1), I_0 is defined by

$$\frac{e^{-\mu\tau}\beta_1 S_0}{1 + \alpha I_0} + \frac{e^{-\mu\tau}\beta_2 V_0}{1 + \alpha I_0} = \mu + \nu.$$

Let $\underline{s} > 0, \underline{v} > 0$ such that

$$\Pi - (\mu + \omega)\underline{s} - \frac{\beta_1 \underline{s} I_0}{1 + \alpha I_0} = 0, \quad \omega \underline{s} - \frac{\beta_2 \underline{v} I_0}{1 + \alpha I_0} - \mu \underline{v} = 0.$$

When

$$e^{-\mu\tau}\beta_1 \underline{s} + e^{-\mu\tau}\beta_2 \underline{v} > \mu + \nu, \tag{4.13}$$

fix $\underline{i} > 0$ such that

$$\frac{e^{-\mu\tau}\beta_1 \underline{s}}{1 + \alpha \underline{i}} + \frac{e^{-\mu\tau}\beta_2 \underline{v}}{1 + \alpha \underline{i}} = \mu + \nu.$$

It should be noted that (4.11) is true if $\alpha > 0$ is large. Evidently, there exist $m > 1, \alpha_0 > 0$ such that $\alpha \geq \alpha_0$ implies that

$$\frac{1}{m\alpha} \leq S_0 - \underline{s}, V_0 - \underline{v}, I_0, \underline{i} \leq \frac{m}{\alpha}.$$

We now consider the question if $\alpha \geq \alpha_0$. Define continuous functions

$$(\overline{x}(t), \overline{y}(t), \overline{z}(t)) = (S_1^* + (S_0 - S_1^* + \epsilon_1)e^{-rt}, V_1^* + (V_0 - V_1^* + \epsilon_2)e^{-rt}, I_1^* + (I_0 - I_1^* + \epsilon_3)e^{-rt}),$$

$$(\underline{x}(t), \underline{y}(t), \underline{z}(t)) = (S_1^* - (S_1^* - \underline{s} + \epsilon_4)e^{-rt}, V_1^* - (V_1^* - \underline{v} + \epsilon_5)e^{-rt}, I_1^* - (I_1^* - \underline{i} + \epsilon_6)e^{-rt}),$$

in which

$$r > 0, \epsilon_i > 0, i \in \{1, 2, 3, 4, 5, 6\},$$

are small constants clarified later. In what follows, we finish the proof by two steps.

Step 1 We verify the necessary differentiable inequalities (See Appendix).

Step 2 We study the initial condition and apply the comparison principle.

For the Step 2, we verify that if (I3) holds, there exists $T > 0$ such that

$$(\overline{x}(0), \overline{y}(0), \overline{z}(0)) \geq (S(x, T+s), V(x, T+s), I(x, T+s)) \geq (\underline{x}(0), \underline{y}(0), \underline{z}(0))$$

for all $x \in \mathbb{R}, s \in [-\tau, 0]$. Once this is proved, then the comparison principle implies what we wanted. Note that,

$$\frac{\partial S(x,t)}{\partial t} \leq d_1 \frac{\partial^2 S(x,t)}{\partial x^2} + \Pi - (\mu + \omega) S(x,t),$$

then

$$\limsup_{t\to\infty} \sup_{x\in\mathbb{R}} S(x,t) \leq \frac{\Pi}{\mu + \omega}$$

and the conclusion holds. Similarly, we can finish the proof. □

5. Traveling waves

In this part, we investigate the existence and nonexistence of Eqs. (2.1)–(2.2). By Eq. (2.1), we study the positive solution of

$$\begin{cases} c\mathcal{S}'(z) = d_1 \mathcal{S}''(z) + \Pi - (\mu + \omega)\mathcal{S}(z) - \mathcal{S}(z) f_1(\mathcal{I}(z)), \\ c\mathcal{V}'(z) = d_2 \mathcal{V}''(z) + \omega \mathcal{S}(z) - \mu \mathcal{V}(z) - \mathcal{V}(z) f_2(\mathcal{I}(z)), \\ c\mathcal{I}'(z) = d_3 \mathcal{I}''(z) - (\mu + \nu)\mathcal{I}(z) \\ \qquad\qquad + e^{-\mu\tau} [\mathcal{S}(z - c\tau) f_1(\mathcal{I}(z - c\tau)) + \mathcal{V}(z - c\tau) f_2(\mathcal{I}(z - c\tau))] \end{cases} \tag{5.1}$$

for all $z \in \mathbb{R}$. Note that, a traveling wave solution is a special entire solution, then the results on asymptotic spreading imply the following conclusion.

Lemma 5.1. *Assume that $\mathcal{R}_0 > 1$. If $(\mathcal{S}, \mathcal{V}, \mathcal{I})$ is positive and satisfies (5.1) and*

$$\lim_{z\to-\infty} (\mathcal{S}(z), \mathcal{V}(z), \mathcal{I}(z)) = (S_0, V_0, 0), \quad \mathcal{I}(z_0) > 0 \text{ for some } z_0 \in \mathbb{R}, \tag{5.2}$$

then (2.2) holds. Further suppose that Theorem 2.9 holds. Then, (2.11) is true.

Therefore, when the existence of Eqs. (2.1)–(2.2) is concerned, we only study the existence of nontrivial solutions to Eq. (5.1). For this, we introduce the following definition and result.

Definition 5.2. Assume that continuous functions

$$(0,0,0) \leq (\underline{S}(z), \underline{V}(z), \underline{I}(z)) \leq (\overline{S}(z), \overline{V}(z), \overline{I}(z)) \leq (S_0, V_0, I_0), \ z \in \mathbb{R}$$

satisfy the following inequalities

$$c\overline{S}'(z) \geq d_1 \overline{S}''(z) + \Pi - (\mu + \omega)\overline{S}(z) - \overline{S}(z) f_1(I(z)), \tag{5.3}$$

$$c\overline{V}'(z) \geq d_2 \overline{V}''(z) + \omega \overline{S}(z) - \mu \overline{V}(z) - \overline{V}(z) f_2(I(z)), \tag{5.4}$$

$$\begin{aligned} c\overline{I}'(z) \geq\ & d_3 \overline{I}''(z) - (\mu + \nu)\overline{I}(z) \\ & + e^{-\mu\tau} \left[\overline{S}(z - c\tau) f_1(I(z - c\tau)) + \overline{V}(z - c\tau) f_2(I(z - c\tau))\right], \end{aligned} \tag{5.5}$$

$$c\underline{S}'(z) \leq d_1 \underline{S}''(z) + \Pi - (\mu + \omega)\underline{S}(z) - \underline{S}(z) f_1(I(z)), \tag{5.6}$$

$$c\underline{V}'(z) \le d_2\underline{V}''(z) + \omega\underline{S}(z) - \mu\underline{V}(z) - \underline{V}(z)f_2(I(z)), \tag{5.7}$$

$$\begin{aligned} c\underline{I}'(z) \le {} & d_3\underline{I}''(z) - (\mu+\nu)\underline{I}(z) \\ & + e^{-\mu\tau}\left[\underline{S}(z-c\tau)f_1(I(z-c\tau)) + \underline{V}(z-c\tau)f_2(I(z-c\tau))\right] \end{aligned} \tag{5.8}$$

except finite points for $z \in \mathbb{R}$, in which I is any continuous function with $\underline{I}(z) \le I(z) \le \overline{I}(z), z \in \mathbb{R}$. Then, they are a pair of generalized upper and lower solutions of (5.1).

Lemma 5.3. *Assume that* $(\overline{S}(z), \overline{V}(z), \overline{I}(z)), (\underline{S}(z), \underline{V}(z), \underline{I}(z))$ *are a pair of generalized upper and lower solutions of* (5.1) *such that*

$$u'(z+) \le u'(z-), \ \ u \in \{\overline{S}(z), \overline{V}(z), \overline{I}(z)\}, \quad z \in \mathbb{R},$$
$$u'(z+) \ge u'(z-), \ \ u \in \{\underline{S}(z), \underline{V}(z), \underline{I}(z)\}, \quad z \in \mathbb{R}.$$

Then, (5.1) *has a solution* $(\mathcal{S}, \mathcal{V}, \mathcal{I})$ *such that*

$$(\underline{S}(z), \underline{V}(z), \underline{I}(z)) \le (\mathcal{S}(z), \mathcal{V}(z), \mathcal{I}(z)) \le (\overline{S}(z), \overline{V}(z), \overline{I}(z)), z \in \mathbb{R}.$$

This lemma can be obtained by Lin and Ruan [21], and we omit the proof here.

5.1. Critical wave speed

For simplicity, in this subsection, we use c, λ instead of c^*, λ^*, respectively. Define continuous functions as follows

$$\begin{aligned} &\overline{S}(z) = S_0, \quad \overline{V}(z) = V_0, \\ &\overline{I}(z) = \begin{cases} \min\{(-z+K_1)e^{\lambda z}, I_0\}, z < 0, \\ I_0, z \ge 0, \end{cases} \\ &\underline{S}(z) = \max\{S_0 - K_2 e^{\gamma z}, \underline{s}\}, \quad \underline{V}(z) = \max\{V_0 - K_3 e^{\gamma z}, \underline{v}\}, \\ &\underline{I}(z) = \begin{cases} (-z - K_4\sqrt{-z})e^{\lambda z}, z < -K_4^2, \\ 0, z \ge -K_4^2, \end{cases} \end{aligned}$$

in which $\gamma = \min\{c/(d_1+d_2+1), \lambda/4\}$, K_1, K_2, K_3, K_4 are positive constants clarified later, and $\underline{s}, \underline{v}$ are defined by (4.10). From Lemma 5.3, we can obtain Theorem 2.4 for $c = c^*$ by the following conclusion.

Lemma 5.4. *There exist* K_1, K_2, K_3, K_4 *large enough such that Eqs.* (5.3)–(5.8) *hold.*

Proof. We prove them one by one. From the definitions of S_0, V_0, (5.3), (5.4) are clear. Let $K_1 > 0$ be large enough such that $\sup_{z<0}\{(-z+K_1)e^{\lambda z}\} > 2I_0$ and $z_1 < 0$ be the smaller root of $(-z+K_1)e^{\lambda z} = I_0$. Then, (5.5) holds when $z > z_1$. When $z < z_1$, (5.5) holds if

$$c\overline{I}'(z) \ge d_3\overline{I}''(z) - (\mu+\nu)\overline{I}(z) + e^{-\mu\tau}\left[S_0 f_1'(0) + V_0 f_2'(0)\right]\overline{I}(z-c\tau)$$

by (F1). In fact,

$$[(-z)e^{\lambda z}]' = -e^{\lambda z} - \lambda z e^{\lambda z}, \quad [(-z)e^{\lambda z}]'' = -\lambda^2 z e^{\lambda z} - 2\lambda e^{\lambda z}$$

implies that

$$\begin{aligned} & d_3\overline{I}''(z) - c\overline{I}'(z) - (\mu+\nu)\overline{I}(z) + e^{-\mu\tau}S_0 f_1'(0)\overline{I}(z-c\tau) + e^{-\mu\tau}V_0 f_2'(0)\overline{I}(z-c\tau). \\ & = K_1\Lambda(\lambda, c)e^{\lambda z} + d_3[-\lambda^2 z e^{\lambda z} - 2\lambda e^{\lambda z}] - c[-e^{\lambda z} - \lambda z e^{\lambda z}] \\ & \quad -(\mu+\nu)(-z)e^{\lambda z} + e^{-\mu\tau}\left[S_0 f_1'(0) + V_0 f_2'(0)\right](-z+c\tau)e^{\lambda(z-c\tau)} \\ & = 0, \end{aligned}$$

and we complete the proof of Eq. (5.5).

For fixed $K_1 > 0$, we now select K_2, K_3 such that (5.6)–(5.7) hold. When $\underline{S}'(z) = 0$ or $\underline{V}'(z) = 0$, the conclusions are clear. Select $K_2 > S_0$ large enough such that $S_0 - K_2 e^{\gamma z} > \underline{s}$ implies that $\overline{I}(z) \le e^{\lambda z/2}$. When $S_0 - K_2 e^{\gamma z} > \underline{s}$, we have $z < 0$ and it suffices to prove that

$$c\underline{S}'(z) \le d_1 \underline{S}''(z) + \Pi - (\mu+\omega)\underline{S}(z) - f_1'(0) S_0 e^{\lambda z/2}. \tag{5.9}$$

Since

$$\begin{aligned}
&d_1\underline{S}''(z) - c\underline{S}'(z) + \Pi - (\mu+\omega)\underline{S}(z) - f_1'(0)S_0 e^{\lambda z/2}\\
&= -d_1 K_2 \gamma^2 e^{\gamma z} + cK_2\gamma e^{\gamma z} + (\mu+\omega)K_2 e^{\gamma z} - f_1'(0)S_0 e^{\lambda z/2}\\
&\ge (\mu+\omega)K_2 e^{\gamma z} - f_1'(0)S_0 e^{\lambda z/2}\\
&\ge (\mu+\omega)K_2 e^{\gamma z} - f_1'(0)S_0 e^{\gamma z}\\
&= e^{\gamma z}\left[(\mu+\omega)K_2 - f_1'(0)S_0\right],
\end{aligned}$$

(5.9) is true by selecting large $K_2 > S_0$. Similarly, we can fix K_3 such that (5.7) holds.

Finally, we verify (5.8). By what we have done, K_1 has been fixed. Let $K_5 > 1$ be large such that $z < -K_5$ implies that

$$\overline{I}(z) < e^{\lambda' z} \text{ for some } \lambda' \in (\lambda/2, \lambda) \text{ with } (1+\sigma)\lambda' > \lambda,\ \lambda' + \gamma > \lambda.$$

By direct calculation, we have

$$\begin{aligned}
\underline{I}'(z) &= \lambda \underline{I}(z) - e^{\lambda z} + \frac{K_4}{2\sqrt{-z}} e^{\lambda z},\\
\underline{I}''(z) &= \lambda^2 \underline{I}(z) - 2\lambda e^{\lambda z} + \frac{K_4\lambda}{\sqrt{-z}} e^{\lambda z} + \frac{K_4}{4\sqrt{(-z)^3}} e^{\lambda z}
\end{aligned}$$

and

$$\begin{aligned}
&d_3\underline{I}''(z) - c\underline{I}'(z) - (\mu+\nu)\underline{I}(z) + e^{-\mu\tau}\left[f_1'(0)S_0 + f_2'(0)V_0\right]\underline{I}(z - c\tau)\\
&= d_3\underline{I}''(z) - c\underline{I}'(z) - (\mu+\nu)\underline{I}(z) + e^{-\mu\tau}\left[f_1'(0)S_0 + f_2'(0)V_0\right]\left[(-z - K_4\sqrt{-z})e^{\lambda(z-c\tau)}\right]\\
&\quad + e^{-\mu\tau}\left[f_1'(0)S_0 + f_2'(0)V_0\right]\underline{I}(z - c\tau)\\
&\quad - e^{-\mu\tau}\left[f_1'(0)S_0 + f_2'(0)V_0\right]\left[(-z - K_4\sqrt{-z})e^{\lambda(z-c\tau)}\right]\\
&= d_3\left[-2\lambda e^{\lambda z} + \frac{K_4\lambda}{\sqrt{-z}}e^{\lambda z} + \frac{K_4}{4\sqrt{(-z)^3}}e^{\lambda z}\right] - c\left[-e^{\lambda z} + \frac{K_4}{2\sqrt{-z}}e^{\lambda z}\right]\\
&\quad + e^{-\mu\tau}\left[f_1'(0)S_0 + f_2'(0)V_0\right]\left[\underline{I}(z - c\tau) - (-z - K_4\sqrt{-z})e^{\lambda(z-c\tau)}\right]\\
&= c\tau\left[f_1'(0)S_0 + f_2'(0)V_0\right]\left[-1 + \frac{K_4}{2\sqrt{-z}}\right]e^{-(\mu+c\lambda)\tau + \lambda z} + \frac{d_3 K_4 e^{\lambda z}}{4\sqrt{(-z)^3}}\\
&\quad + c\tau\left[f_1'(0)S_0 + f_2'(0)V_0\right]\left[1 - \frac{K_4}{\sqrt{-(z-c\tau)} + \sqrt{-z}}\right]e^{-(\mu+c\lambda)\tau + \lambda z}\\
&= \frac{d_3 K_4 e^{\lambda z}}{4\sqrt{(-z)^3}} + c\tau K_4\left[f_1'(0)S_0 + f_2'(0)V_0\right]\left[\frac{1}{2\sqrt{-z}} - \frac{1}{\sqrt{-(z-c\tau)} + \sqrt{-z}}\right]e^{-(\mu+c\lambda)\tau + \lambda z}\\
&\ge \frac{d_3 K_4 e^{\lambda z}}{4\sqrt{(-z)^3}}
\end{aligned}$$

for $z < -K_5$. Therefore,

$$d_3\underline{I}''(z) - c\underline{I}'(z) - (\mu+\nu)\underline{I}(z)$$

$$
\begin{aligned}
&+e^{-\mu\tau}\underline{S}(z-c\tau)f_1(I(z-c\tau))+e^{-\mu\tau}\underline{V}(z-c\tau)f_2(I(z-c\tau))\\
&= d_3\underline{I}''(z)-c\underline{I}'(z)-(\mu+\nu)\underline{I}(z)+\\
&+e^{-\mu\tau}S_0f_1(I(z-c\tau))+e^{-\mu\tau}V_0f_2(I(z-c\tau))\\
&+e^{-\mu\tau}\left[\underline{S}(z-c\tau)-S_0\right]f_1(I(z-c\tau))+e^{-\mu\tau}\left[\underline{V}(z-c\tau)-V_0\right]f_2(I(z-c\tau))\\
&\geq d_3\underline{I}''(z)-c\underline{I}'(z)-(\mu+\nu)\underline{I}(z)+\\
&+e^{-\mu\tau}S_0f_1'(0)I(z-c\tau)+e^{-\mu\tau}V_0f_2'(0)I(z-c\tau)-Le^{-\mu\tau}[S_0+V_0]I^{1+\sigma}(z-c\tau)\\
&+e^{-\mu\tau}\left[\underline{S}(z-c\tau)-S_0\right]f_1'(0)I(z-c\tau)+e^{-\mu\tau}\left[\underline{V}(z-c\tau)-V_0\right]f_2'(0)I(z-c\tau)\\
&\geq d_3\underline{I}''(z)-c\underline{I}'(z)-(\mu+\nu)\underline{I}(z)\\
&+e^{-\mu\tau}S_0f_1'(0)\underline{I}(z-c\tau)+e^{-\mu\tau}V_0f_2'(0)\underline{I}(z-c\tau)-Le^{-\mu\tau}[S_0+V_0]I^{1+\sigma}(z)\\
&+e^{-\mu\tau}\left[\underline{S}(z-c\tau)-S_0\right]f_1'(0)I(z)+e^{-\mu\tau}\left[\underline{V}(z-c\tau)-V_0\right]f_2'(0)I(z)\\
&\geq d_3\underline{I}''(z)-c\underline{I}'(z)-(\mu+\nu)\underline{I}(z)\\
&+e^{-\mu\tau}S_0f_1'(0)\underline{I}(z-c\tau)+e^{-\mu\tau}V_0f_2'(0)\underline{I}(z-c\tau)-Le^{-\mu\tau}[S_0+V_0]\overline{I}^{1+\sigma}(z)\\
&+e^{-\mu\tau}\left[\underline{S}(z-c\tau)-S_0\right]f_1'(0)\overline{I}(z)+e^{-\mu\tau}\left[\underline{V}(z-c\tau)-V_0\right]f_2'(0)\overline{I}(z)\\
&\geq \frac{d_3K_4e^{\lambda z}}{4\sqrt{(-z)^3}}-Le^{-\mu\tau}[S_0+V_0]e^{(1+\sigma)\lambda' z}-e^{-\mu\tau}\left[f_1'(0)K_2+f_2'(0)K_3\right]e^{\gamma z}e^{\lambda' z}\\
&= e^{\lambda z}\left[\frac{d_3K_4}{4\sqrt{(-z)^3}}-Le^{-\mu\tau}[S_0+V_0]e^{\left[(1+\sigma)\lambda'-\lambda\right]z}-e^{-\mu\tau}\left[f_1'(0)K_2+f_2'(0)K_3\right]e^{(\gamma+\lambda'-\lambda)z}\right].
\end{aligned}
$$

Let $K_6>1$ such that

$$\frac{d_3K_4}{4\sqrt{(-z)^3}}>Le^{-\mu\tau}[S_0+V_0]e^{\left[(1+\sigma)\lambda'-\lambda\right]z}+e^{-\mu\tau}\left[f_1'(0)K_2+f_2'(0)K_3\right]e^{(\gamma+\lambda'-\lambda)z},z<-K_6.$$

Then, (5.8) holds by selecting $K_4=K_5+K_6$. The proof is complete. □

5.2. Other cases

In this subsection, we first present the existence of nontrivial traveling wave solutions.

Lemma 5.5. *Assume that $\mathcal{R}_0>1, c>c^*$. Then, (5.1) has a positive solution satisfying (5.2).*

Proof. For $z\in\mathbb{R}$, define

$$
\begin{aligned}
&\overline{S}(z)=S_0,\quad \underline{S}(z)=\max\{S_0-K_7e^{\gamma z},\underline{s}\},\\
&\overline{V}(z)=V_0,\quad \underline{V}(z)=\max\{V_0-K_8e^{\gamma z},\underline{v}\},\\
&\overline{I}(z)=\min\{e^{\lambda^c z},I_0\},\quad \underline{I}(z)=\max\{e^{\lambda^c z}-K_9e^{\eta\lambda^c z},0\},
\end{aligned}
$$

where $K_7>0,K_8>0,K_9>1$ are large enough and $\gamma>0,\eta-1>0$ are small such that

$$\eta\lambda^c<\min\{(1+\sigma)\lambda^c,\lambda^c+\gamma,\Lambda^c\}.$$

Then, similar to that in Sect. 5.1, we can verify these are generalized upper and lower solutions of Eq. (5.1). By what we have done, we can finish the proof. □

Before considering the nonexistence, we give the following strict positivity.

Lemma 5.6. *For given $c\in\mathbb{R}$, if Eq. (5.1) has a positive solution $(\mathcal{S},\mathcal{V},\mathcal{I})$ satisfying Eq. (5.2), then*

$$\underline{s}<\mathcal{S}(z)<S_0,\underline{v}<\mathcal{V}(z)<V_0,0<\mathcal{I}(z)<I_0,\ z\in\mathbb{R}.$$

Proof. Let $\gamma_1 < 0 < \gamma_2$ be the roots of

$$d_3x^2 - cx - (\mu + \nu) = 0.$$

Then, $\mathcal{I}(z)$ also satisfies

$$\mathcal{I}(z) = \frac{e^{-\mu\tau}}{d_3(\gamma_2 - \gamma_1)} \int_{\mathbb{R}} \min\{e^{\gamma_1(z-s)}, e^{\gamma_2(z-s)}\} \times [\mathcal{S}(s - c\tau)f_1(\mathcal{I}(s - c\tau)) + \mathcal{V}(s - c\tau)f_2(\mathcal{I}(s - c\tau))]ds$$

for all $z \in \mathbb{R}$. From (F1) and (5.2), we see that $I(z) > 0, z \in \mathbb{R}$. The remainder can be proved in a similar way, and we omit them here. □

From the special invariant form of traveling wave solutions, we obtain the following nonexistence result, see Theorem 2.9.

Lemma 5.7. *If $\mathcal{R}_0 \leq 1$, then Eqs.* (5.1)–(5.2) *has not a solution.*

The second result of nonexistence is given as follows.

Lemma 5.8. *Assume that $\mathcal{R}_0 > 1, c < c^*$. Then,* (2.1) *has not a solution satisfying* (2.2).

Proof. We prove this by contradiction. Were the statement false for some $c_1 < c^*$ and $(\overline{S}, \overline{I}, \overline{V})$. From (5.2) and Theorem 2.3, we see that (2.2) holds. Let $\epsilon > 0$ such that

$$d_3\lambda^2 - c_1\lambda + e^{-(\mu+c_1\lambda)\tau}\left((S_0 - 2\epsilon)f_1'(0) + (V_0 - 2\epsilon)f_2'(0)\right) - (\mu + \nu) > 0, \quad \lambda \geq 0.$$

Also define

$$c_2 = \inf\{c > 0 : d_3\lambda^2 - c\lambda + e^{-(\mu+c\lambda)\tau}\left((S_0 - 2\epsilon)f_1'(0) + (V_0 - 2\epsilon)f_2'(0)\right) - (\mu + \nu) = 0 \text{ has a real root}\}.$$

Then, $c_2 \in (c_1, c^*)$.

Select z' such that

$$\overline{S}(z) > S_0 - \epsilon, \quad \overline{V}(z) > V_0 - \epsilon, \quad z \leq z'.$$

Then, we have

$$c_1\overline{I}'(z) \geq d_3\overline{I}''(z) - (\mu + \nu)\overline{I}(z) + e^{-\mu\tau}[(S_0 - \epsilon)f_1(\overline{I}(z - c_1\tau)) + (V_0 - \epsilon)f_2(\overline{I}(z - c_1\tau))]$$

for $z \leq z'$. When $z > z'$, the positivity in Lemma 5.6 and limit behavior (2.2) imply that $\inf_{z>z'} \overline{I}(z) > 0$ such that

$$c_1\overline{I}'(z) \geq d_3\overline{I}''(z) - (\mu + \nu)\overline{I}(z) + e^{-\mu\tau}[S_0f_1(\overline{I}(z - c_1\tau)) + V_0f_2(\overline{I}(z - c_1\tau))] - M\overline{I}^2(z)$$

for $z > z'$ and large $M > 0$. Then, $I(x, t) = \overline{I}(x + c_1t)$ satisfies

$$\begin{aligned}\frac{\partial I(x,t)}{\partial t} &\geq d_3\frac{\partial^2 I(x,t)}{\partial x^2} - (\mu + \nu)I(x,t) - MI^2(x,t) \\ &\quad + e^{-\mu\tau}[(S_0 - \epsilon)f_1(I(x, t - \tau)) + (V_0 - \epsilon)f_2(I(x, t - \tau))] \\ &\geq d_3\frac{\partial^2 I(x,t)}{\partial x^2} - (\mu + \nu)I(x,t) - MI^2(x,t) \\ &\quad + e^{-\mu\tau}[(S_0 - \epsilon)\underline{f}_1(I(x, t - \tau)) + (V_0 - \epsilon)\underline{f}_2(I(x, t - \tau))]\end{aligned}$$

with $I(x, 0) = \overline{I}(x) > 0, x \in, t > 0$, where

$$\underline{f}_i(x) = \inf_{y\in[x,I_0]} f_i(y), \qquad x \in [0, I_0], i = 1, 2.$$

Let $i_* > 0$ be the unique real rot of

$$-(\mu+\nu)i_* - Mi_*^2 + e^{-\mu\tau}[(S_0-\epsilon)\underline{f}_1(i_*) + (V_0-\epsilon)\underline{f}_2(i_*)] = 0.$$

From Lemma 3.2, we have $\liminf_{t\to\infty} I(-c_2t,t) \geq i_*$. Note that, $-c_2t + c_1t \to \infty$ when $t \to \infty$. By the invariant form of traveling waves, we obtain $\liminf_{z\to-\infty} \overline{I}(z) \geq i_*$. A contradiction occurs. The proof is complete. □

6. Discussion

We have studied the initial value problem and traveling wave solutions for noncooperative system (1.2). With different special decaying initial values, we could classify the corresponding leftward and rightward spreading speeds. When the disease spreads successfully, there is a minimal spreading speed that is the minimal wave speed of traveling wave solutions. Moreover, our results on traveling wave solutions have completed He et al. [13, Theorem 3.8] by presenting the precise asymptotic behavior of critical traveling wave solutions. The behavior is important to classify the spreading speeds with different initial conditions.

Without the vaccination branch, then (1.1) becomes

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} = d_1 \frac{\partial^2 S(x,t)}{\partial x^2} + \Pi - \mu S(x,t) - \frac{\beta_1 S(x,t) I(x,t)}{1+\alpha I(x,t)}, \\ \frac{\partial I(x,t)}{\partial t} = d_3 \frac{\partial^2 I(x,t)}{\partial x^2} + e^{-\mu\tau} \frac{\beta_1 S(x,t-\tau) I(x,t-\tau)}{1+\alpha I(x,t-\tau)} - (\mu+\nu) I(x,t). \end{cases} \tag{6.1}$$

Clearly, the basic reproduction ratio of the corresponding kinetic system is $e^{-\mu\tau}\beta_1\Pi/(\mu(\mu+\nu))$. By Lin et al. [22], the minimal spreading speed of (6.1) depends on the eigenvalue problem

$$d_3\lambda^2 - c\lambda + \beta_1\Pi e^{-(\mu+c\lambda)\tau}/\mu - (\mu+\nu) = 0, \quad \lambda \geq 0, c \geq 0. \tag{6.2}$$

For this Eq. (6.2), we may analyze it similar to that in Lemma 2.1. Note that, $S_0 + V_0 = \Pi/\mu$, if $\beta_1 > \beta_2$ (see [13, pp. 1973]), then we find that the minimal spreading speed of (1.1) is smaller than that of (6.1), which shows that the vaccination could decrease the spreading ability of the disease from both the outbreak threshold (the basic reproduction ratio), as well as the minimal spreading speed. For the general system (1.2), when $f_1 \leq f_2, f_1'(0) < f_2'(0)$ (the vaccination plays a positive role controlling the disease), we may obtain a similar conclusion. At the same time, the role of the latent period τ can be discussed similar to that in many works, e.g., [13, pp. 1993]. Moreover, besides for the minimal spreading speed, we may show the effect of vaccination and latency for other possible spreading speeds and obtain similar conclusions.

Although we have investigated different decaying initial values, it is still a long way to further describe the propagation dynamics of these values. For example, when the slowly decaying initial values (IL3) and (IR3) are concerned, the corresponding fast propagation needs further investigation, which at least incudes the estimation of moving speed for level sets. For cooperative systems, the estimation has been done in several works [2,12,37]. To the best of our knowledge, this topic remains open for many noncooperative systems including (1.2). Moreover, when the asymptotic behavior of traveling waves for (1.1) is concerned, He et al. [13, Theorem 3.8] confirmed the limit behavior by a Lyapunov type discussion, of which the condition is weaker than that in (B1) and (B2). For the convergence of solutions to initial value problem, is it possible to obtain the convergence by such an idea?

The assumption (F2) is important to obtain our conclusion since this leads to a bounded result of I. In fact, the boundedness is not clear in some models. We only illustrate this from special model (1.1). When $\alpha > 0$ as that in this paper and [13], the boundedness of I-branch is clear such that we can follow our discussion. If $\alpha = 0$, the boundedness is not clear enough although this should be true. When the boundedness of I was proven, we can make a similar discussion. In fact, the boundedness of some noncooperative systems is not trivial but may be obtained. For example, Ducrot [9] studied a noncooperative system by using local L^p estimation and other techniques. We hope the idea in [9] can be further developed to such a delayed system (1.1) with $\alpha = 0$.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

Verification of step 1 in the proof of lemma 4.5 To avoid many superscripts and subscripts in our calculations, we denote

$$(S, V, I) := (S_0, V_0, I_0), \qquad (s, v, i) := (S_1^*, V_1^*, I_1^*)$$

in this part.

(1). For $\overline{x}(t)$, we shall prove that

$$\overline{x}'(t) = -r(S - s + \epsilon_1)e^{-rt} \geq \Pi - (\mu + \omega)\overline{x}(t) - \frac{\beta_1 \overline{x}(t)\underline{z}(t)}{1 + \alpha \underline{z}(t)}, \quad t > 0. \tag{6.3}$$

By direct calculation, we have

$$\begin{aligned}
&\Pi - (\mu + \omega)\overline{x}(t) - \frac{\beta_1 \overline{x}(t)\underline{z}(t)}{1 + \alpha \underline{z}(t)} \\
&\quad = \Pi - (\mu + \omega)[s + (S - s + \epsilon_1)e^{-rt}] - \frac{\beta_1 [s + (S - s + \epsilon_1)e^{-rt}][i - (i - \underline{i} + \epsilon_6)e^{-rt}]}{1 + \alpha[i - (i - \underline{i} + \epsilon_6)e^{-rt}]} \\
&\quad = (\mu + \omega)[S - s] - (\mu + \omega)(S - s + \epsilon_1)e^{-rt} - \frac{\beta_1 [s + (S - s + \epsilon_1)e^{-rt}][i - (i - \underline{i} + \epsilon_6)e^{-rt}]}{1 + \alpha[i - (i - \underline{i} + \epsilon_6)e^{-rt}]} \\
&\quad = -\epsilon_1(\mu + \omega)e^{-rt} + (\mu + \omega)[S - s][1 - e^{-rt}] - \frac{\beta_1 [s + (S - s + \epsilon_1)e^{-rt}][i - (i - \underline{i} + \epsilon_6)e^{-rt}]}{1 + \alpha[i - (i - \underline{i} + \epsilon_6)e^{-rt}]} \\
&\quad = -\epsilon_1(\mu + \omega)e^{-rt} + \frac{\beta_1 is[1 - e^{-rt}]}{1 + \alpha i} - \frac{\beta_1 [s + (S - s + \epsilon_1)e^{-rt}][i - (i - \underline{i} + \epsilon_6)e^{-rt}]}{1 + \alpha[i - (i - \underline{i} + \epsilon_6)e^{-rt}]} \\
&\quad \leq -\epsilon_1(\mu + \omega)e^{-rt} + \frac{\beta_1 is[1 - e^{-rt}]}{1 + \alpha i} - \frac{\beta_1 s[i - (i - \underline{i} + \epsilon_6)e^{-rt}]}{1 + \alpha i} \\
&\quad = -\epsilon_1(\mu + \omega)e^{-rt} - \frac{\beta_1 s(\underline{i} - \epsilon_6)e^{-rt}}{1 + \alpha i} \\
&\quad \leq -\epsilon_1(\mu + \omega)e^{-rt} \\
&\quad \leq -r(S - s + \epsilon_1)e^{-rt}
\end{aligned}$$

if $r > 0$ is small enough. We complete the verification of (6.3).

(2). For $\overline{y}(t)$, we shall prove

$$\overline{y}'(t) \geq \omega \overline{x}(t) - \frac{\beta_2 \overline{y}(t)\underline{z}(t)}{1 + \alpha \underline{z}(t)} - \mu \overline{y}(t), \quad t > 0. \tag{6.4}$$

Similar to the calculation in (1), we have

$$\begin{aligned}
&\omega\overline{x}(t) - \frac{\beta_2\overline{y}(t)\underline{z}(t)}{1+\alpha\underline{z}(t)} - \mu\overline{y}(t) \\
&\quad = \omega\left[s+(S-s+\epsilon_1)e^{-rt}\right] - \mu\left[v+(V-v+\epsilon_2)e^{-rt}\right] \\
&\qquad - \frac{\beta_2\left[v+(V-v+\epsilon_2)e^{-rt}\right]\left[i-(i-\underline{i}+\epsilon_6)e^{-rt}\right]}{1+\alpha[i-(i-\underline{i}+\epsilon_6)e^{-rt}]} \\
&\quad = [\omega\epsilon_1-\mu\epsilon_2]e^{-rt} + \frac{\beta_2 vi[1-e^{-rt}]}{1+\alpha i} - \frac{\beta_2\left[v+(V-v+\epsilon_2)e^{-rt}\right]\left[i-(i-\underline{i}+\epsilon_6)e^{-rt}\right]}{1+\alpha[i-(i-\underline{i}+\epsilon_6)e^{-rt}]} \\
&\quad \le [\omega\epsilon_1-\mu\epsilon_2]e^{-rt} + \frac{\beta_2 vi[1-e^{-rt}]}{1+\alpha i} - \frac{\beta_2 v[i-(i-\underline{i}+\epsilon_6)e^{-rt}]}{1+\alpha i} \\
&\quad \le [\omega\epsilon_1-\mu\epsilon_2]e^{-rt} \\
&\quad < -r(V-v+\epsilon_2)e^{-rt}
\end{aligned}$$

if $r>0$ is small and

$$\omega\epsilon_1-\mu\epsilon_2<0. \tag{6.5}$$

We complete the verification of (6.4).

(3). We shall verify that

$$\begin{aligned}
\overline{z}'(t) &= -r(I-i+\epsilon_3)e^{-rt} \\
&\ge e^{-\mu\tau}\frac{\beta_1\overline{x}\,(t-\tau)\,\overline{z}(t-\tau)}{1+\alpha\overline{z}(t-\tau)} + e^{-\mu\tau}\frac{\beta_2\overline{y}(t-\tau)\overline{z}(t-\tau)}{1+\alpha\overline{z}(t-\tau)} - (\mu+\nu)\overline{z}(t), \quad t>0.
\end{aligned} \tag{6.6}$$

By direct calculation, we have

$$\begin{aligned}
&e^{-\mu\tau}\frac{\beta_1\overline{x}\,(t-\tau)\,\overline{z}(t-\tau)}{1+\alpha\overline{z}(t-\tau)} + e^{-\mu\tau}\frac{\beta_2\overline{y}(t-\tau)\overline{z}(t-\tau)}{1+\alpha\overline{z}(t-\tau)} - (\mu+\nu)\overline{z}(t) \\
&\quad = e^{-\mu\tau}\frac{\beta_1[s+(S-s+\epsilon_1)e^{-r(t-\tau)}][i+(I-i+\epsilon_3)e^{-r(t-\tau)}]}{1+\alpha[i+(I-i+\epsilon_3)e^{-r(t-\tau)}]} \\
&\qquad + e^{-\mu\tau}\frac{\beta_2\left[v+(V-v+\epsilon_2)e^{-r(t-\tau)}\right]\left[i+(I-i+\epsilon_3)e^{-r(t-\tau)}\right]}{1+\alpha[i+(I-i+\epsilon_3)e^{-r(t-\tau)}]} \\
&\qquad - (\mu+\nu)[i+(I-i+\epsilon_3)e^{-rt}] \\
&\quad = e^{-\mu\tau}\frac{\beta_1[s+(S-s+\epsilon_1)e^{-r(t-\tau)}][i+(I-i+\epsilon_3)e^{-r(t-\tau)}]}{1+\alpha[i+(I-i+\epsilon_3)e^{-r(t-\tau)}]} \\
&\qquad + e^{-\mu\tau}\frac{\beta_2\left[v+(V-v+\epsilon_2)e^{-r(t-\tau)}\right]\left[i+(I-i+\epsilon_3)e^{-r(t-\tau)}\right]}{1+\alpha[i+(I-i+\epsilon_3)e^{-r(t-\tau)}]} \\
&\qquad - (\mu+\nu)[i+(I-i+\epsilon_3)e^{-r(t-\tau)}] \\
&\qquad + (\mu+\nu)[i+(I-i+\epsilon_3)e^{-r(t-\tau)}] - (\mu+\nu)[i+(I-i+\epsilon_3)e^{-rt}] \\
&\quad = e^{-\mu\tau}\frac{\beta_1[s+(S-s+\epsilon_1)e^{-r(t-\tau)}][i+(I-i+\epsilon_3)e^{-r(t-\tau)}]}{1+\alpha[i+(I-i+\epsilon_3)e^{-r(t-\tau)}]} \\
&\qquad + e^{-\mu\tau}\frac{\beta_2\left[v+(V-v+\epsilon_2)e^{-r(t-\tau)}\right]\left[i+(I-i+\epsilon_3)e^{-r(t-\tau)}\right]}{1+\alpha[i+(I-i+\epsilon_3)e^{-r(t-\tau)}]} \\
&\qquad - e^{-\mu\tau}\frac{\beta_1 s+\beta_2 v}{1+\alpha i}[i+(I-i+\epsilon_3)e^{-r(t-\tau)}] \\
&\qquad + (\mu+\nu)[i+(I-i+\epsilon_3)e^{-r(t-\tau)}] - (\mu+\nu)[i+(I-i+\epsilon_3)e^{-rt}].
\end{aligned}$$

If $\epsilon_1, \epsilon_2, \epsilon_3$ are small enough such that

$$(1+\alpha i)\left[\beta_1\epsilon_1+\beta_2\epsilon_2\right] < \alpha\left(\beta_1 s+\beta_2 v\right)\epsilon_3, \tag{6.7}$$

then

$$\frac{\beta_1 S+\beta_2 V}{1+\alpha I}=\frac{\beta_1 s+\beta_2 v}{1+\alpha i}=e^{\mu\tau}(\mu+\nu)$$

implies that

$$\begin{aligned}
&\frac{\beta_1[s+(S-s+\epsilon_1)e^{-r(t-\tau)}]+\beta_2\left[v+(V-v+\epsilon_2)e^{-r(t-\tau)}\right]}{1+\alpha[i+(I-i+\epsilon_3)e^{-r(t-\tau)}]}-\frac{\beta_1 s+\beta_2 v}{1+\alpha i}\\
&=\frac{e^{-r(t-\tau)}\left\{(1+\alpha i)\left[\beta_1\left(S+\epsilon_1\right)+\beta_2\left(V+\epsilon_2\right)\right]-\left(\beta_1 s+\beta_2 v\right)\left[1+\alpha\left(I+\epsilon_3\right)\right]\right\}}{[1+\alpha i]\left[1+\alpha[i+(I-i+\epsilon_3)e^{-r(t-\tau)}]\right]}\\
&=\frac{e^{-r(t-\tau)}\left[(1+\alpha i)\left[\beta_1\epsilon_1+\beta_2\epsilon_2\right]-\alpha\left(\beta_1 s+\beta_2 v\right)\epsilon_3\right]}{[1+\alpha i]\left[1+\alpha[i+(I-i+\epsilon_3)e^{-r(t-\tau)}]\right]}\\
&\le\frac{e^{-r(t-\tau)}\left[(1+\alpha i)\left[\beta_1\epsilon_1+\beta_2\epsilon_2\right]-\alpha\left(\beta_1 s+\beta_2 v\right)\epsilon_3\right]}{\left[1+\alpha[I+\epsilon_3]\right]^2}\\
&<0.
\end{aligned}$$

Since

$$\begin{aligned}
&(\mu+\nu)[i+(I-i+\epsilon_3)e^{-r(t-\tau)}]-(\mu+\nu)[i+(I-i+\epsilon_3)e^{-rt}]\\
&=(\mu+\nu)(I-i+\epsilon_3)e^{-rt}[e^{r\tau}-1]
\end{aligned}$$

and

$$\lim_{r\to 0}\frac{[e^{r\tau}-1]}{r\tau}=1,$$

then (6.6) holds if $r>0$ is small enough and (6.7) is true.

(4). For $\underline{x}(t)$, we shall prove that

$$\underline{x}'(t)=r(s-\underline{s}+\epsilon_4)e^{-rt}\le\Pi-(\mu+\omega)\underline{x}(t)-\frac{\beta_1\underline{x}(t)\overline{z}(t)}{1+\alpha\overline{z}(t)},\quad t>0. \tag{6.8}$$

The right side is

$$\begin{aligned}
&\Pi-(\mu+\omega)\underline{x}(t)-\frac{\beta_1\underline{x}(t)\overline{z}(t)}{1+\alpha\overline{z}(t)}\\
&=\Pi-(\mu+\omega)[(s-(s-\underline{s}+\epsilon_4)e^{-rt}]-\frac{\beta_1[(s-(s-\underline{s}+\epsilon_4)e^{-rt}][i+(i-\underline{i}+\epsilon_3)e^{-rt})]}{1+\alpha[i+(i-\underline{i}+\epsilon_3)e^{-rt})]}\\
&=(\mu+\omega)(s-\underline{s}+\epsilon_4)e^{-rt}+\frac{\beta_1 si}{1+\alpha i}-\frac{\beta_1[(s-(s-\underline{s}+\epsilon_4)e^{-rt}][i+(i-\underline{i}+\epsilon_3)e^{-rt})]}{1+\alpha[i+(i-\underline{i}+\epsilon_3)e^{-rt})]}\\
&\ge(\mu+\omega)(s-\underline{s}+\epsilon_4)e^{-rt}+\frac{\beta_1 si}{1+\alpha i}-\frac{\beta_1[(s-(s-\underline{s}+\epsilon_4)e^{-rt}][i+(i-\underline{i}+\epsilon_3)e^{-rt})]}{1+\alpha i}\\
&\ge(\mu+\omega)(s-\underline{s}+\epsilon_4)e^{-rt}+\frac{-\beta_1 s(i-\underline{i}+\epsilon_3)e^{-rt}+\beta_1(s-\underline{s}+\epsilon_4)ie^{-rt}}{1+\alpha i}\\
&=(\mu+\omega)(s-\underline{s}+\epsilon_4)e^{-rt}-\frac{\beta_1 sie^{-rt}}{1+\alpha i}+\frac{\beta_1(s-\underline{s}+\epsilon_4)ie^{-rt}}{1+\alpha i}\\
&=\left[\Pi\frac{(s-\underline{s}+\epsilon_4)}{s}-\frac{\beta_1 si}{1+\alpha i}\right]e^{-rt}
\end{aligned}$$

$$= \left[\Pi - \Pi\frac{(\underline{s} - \epsilon_4)}{s} - \frac{\beta_1 si}{1+\alpha i}\right] e^{-rt}$$
$$= \left[(\mu+\omega)s - \Pi\frac{(\underline{s} - \epsilon_4)}{s}\right] e^{-rt}.$$

Evidently, if $\alpha \to \infty$, then $\underline{s} \to s$ such that

$$(\mu+\omega)s - \Pi\frac{(\underline{s}-\epsilon_4)}{s} > \frac{\Pi\epsilon_4}{2s}$$

for large α. Selecting small $r > 0$, we obtain (6.8).

(5). We need to prove that

$$\underline{y}'(t) = r(v - \underline{v} + \epsilon_5)e^{-rt} \le \omega\underline{x}(t) - \frac{\beta_2 \underline{y}(t)\overline{z}(t)}{1+\alpha\overline{z}(t)} - \mu\underline{y}(t), \quad t > 0. \tag{6.9}$$

Since

$$\omega\underline{s} - \mu\underline{v} - \frac{\beta_2 \underline{v} I}{1+\alpha I} = \omega s - \mu v - \frac{\beta_2 vi}{1+\alpha i},$$

and

$$\begin{aligned}
&vi[1+\alpha[i+(I-i+\epsilon_3)e^{-rt}]] - [v-(v-\underline{v}+\epsilon_5)e^{-rt}][i+(I-i+\epsilon_3)e^{-rt}]\,[1+\alpha i]\\
&\quad = vi(I-i+\epsilon_3)e^{-rt} - v(I-i+\epsilon_3)\,[1+\alpha i]\,e^{-rt}\\
&\qquad +(v-\underline{v}+\epsilon_5)[i+(I-i+\epsilon_3)e^{-rt}]\,[1+\alpha i]\,e^{-rt}\\
&\quad \ge vi(I-i+\epsilon_3)e^{-rt} - v(I-i+\epsilon_3)\,[1+\alpha i]\,e^{-rt}\\
&\qquad +(v-\underline{v}+\epsilon_5)i\,[1+\alpha i]\,e^{-rt},
\end{aligned}$$

we have

$$\begin{aligned}
&\omega\underline{x}(t) - \frac{\beta_2\underline{y}(t)\overline{z}(t)}{1+\alpha\overline{z}(t)} - \mu\underline{y}(t)\\
&\quad = \omega[s-(s-\underline{s}+\epsilon_4)e^{-rt}] - \mu[v-(v-\underline{v}+\epsilon_5)e^{-rt}]\\
&\qquad - \frac{\beta_2[v-(v-\underline{v}+\epsilon_5)e^{-rt}][i+(I-i+\epsilon_3)e^{-rt}]}{1+\alpha[i+(I-i+\epsilon_3)e^{-rt}]}\\
&\quad = -\omega(s-\underline{s}+\epsilon_4)e^{-rt} + \mu(v-\underline{v}+\epsilon_5)e^{-rt}\\
&\qquad + \frac{\beta_2 vi}{1+\alpha i} - \frac{\beta_2[v-(v-\underline{v}+\epsilon_5)e^{-rt}][i+(I-i+\epsilon_3)e^{-rt}]}{1+\alpha[i+(I-i+\epsilon_3)e^{-rt}]}\\
&\quad \ge -\omega(s-\underline{s}+\epsilon_4)e^{-rt} + \mu(v-\underline{v}+\epsilon_5)e^{-rt}\\
&\qquad + \frac{\beta_2 vi}{1+\alpha i} - \frac{\beta_2[v-(v-\underline{v}+\epsilon_5)e^{-rt}][i+(I-i+\epsilon_3)e^{-rt}]}{1+\alpha i}\\
&\quad \ge -\omega(s-\underline{s}+\epsilon_4)e^{-rt} + \mu(v-\underline{v}+\epsilon_5)e^{-rt}\\
&\qquad + \frac{\beta_2 i(v-\underline{v}+\epsilon_5)e^{-rt}}{1+\alpha i} - \frac{\beta_2 v(I-i+\epsilon_3)e^{-rt}}{1+\alpha i}.
\end{aligned}$$

Note that, $\alpha \to \infty$ implies that

$$v - \underline{v} \to 0, s - \underline{s} \to 0, I \to 0,$$

then (6.9) holds if we select $r > 0$ small enough and

$$\mu\epsilon_5 > 2\omega\epsilon_4. \tag{6.10}$$

(6). We need to prove that

$$\underline{z}'(t) = r(i - \underline{i} + \epsilon_6)e^{-rt}$$

$$\leq e^{-\mu\tau}\frac{\beta_1\underline{x}(t-\tau)\underline{z}(t-\tau)}{1+\alpha\underline{z}(t-\tau)}+e^{-\mu\tau}\frac{\beta_2\underline{y}(t-\tau)\underline{z}(t-\tau)}{1+\alpha\underline{z}(t-\tau)}-(\mu+\nu)\underline{z}(t),\quad t>0. \tag{6.11}$$

By the definition, we have

$$\begin{aligned}
&e^{-\mu\tau}\frac{\beta_1\underline{x}(t-\tau)\underline{z}(t-\tau)}{1+\alpha\underline{z}(t-\tau)}+e^{-\mu\tau}\frac{\beta_2\underline{y}(t-\tau)\underline{z}(t-\tau)}{1+\alpha\underline{z}(t-\tau)}-(\mu+\nu)\underline{z}(t)\\
&=e^{-\mu\tau}\frac{\beta_1[s-(s-\underline{s}+\epsilon_4)e^{-r(t-\tau)}][i-(i-\underline{i}+\epsilon_6)e^{-r(t-\tau)}]}{1+\alpha[i-(i-\underline{i}+\epsilon_6)e^{-r(t-\tau)}]}\\
&\quad+e^{-\mu\tau}\frac{\beta_2[v-(v-\underline{v}+\epsilon_5)e^{-r(t-\tau)}][i-(i-\underline{i}+\epsilon_6)e^{-r(t-\tau)}]}{1+\alpha[i-(i-\underline{i}+\epsilon_6)e^{-r(t-\tau)}]}-(\mu+\nu)[i-(i-\underline{i}+\epsilon_6)e^{-rt}]\\
&=e^{-\mu\tau}\frac{\beta_1[s-(s-\underline{s}+\epsilon_4)e^{-r(t-\tau)}][i-(i-\underline{i}+\epsilon_6)e^{-r(t-\tau)}]}{1+\alpha[i-(i-\underline{i}+\epsilon_6)e^{-r(t-\tau)}]}\\
&\quad+e^{-\mu\tau}\frac{\beta_2[v-(v-\underline{v}+\epsilon_5)e^{-r(t-\tau)}][i-(i-\underline{i}+\epsilon_6)e^{-r(t-\tau)}]}{1+\alpha[i-(i-\underline{i}+\epsilon_6)e^{-r(t-\tau)}]}-(\mu+\nu)[i-(i-\underline{i}+\epsilon_6)e^{-r(t-\tau)}]\\
&\quad+(\mu+\nu)[i-(i-\underline{i}+\epsilon_6)e^{-r(t-\tau)}]-(\mu+\nu)[i-(i-\underline{i}+\epsilon_6)e^{-rt}].
\end{aligned}$$

Similar to that in Step 3, since

$$e^{-\mu\tau}\frac{\beta_1 s+\beta_2 v}{1+\alpha i}=\mu+\nu,\quad e^{-\mu\tau}\frac{\beta_1\underline{s}+\beta_2\underline{v}}{1+\alpha\underline{i}}=\mu+\nu$$

and

$$\begin{aligned}
&\frac{\beta_1[s-(s-\underline{s}+\epsilon_4)e^{-r(t-\tau)}]+\beta_2[v-(v-\underline{v}+\epsilon_5)e^{-r(t-\tau)}]}{1+\alpha[i-(i-\underline{i}+\epsilon_6)e^{-r(t-\tau)}]}-\frac{\beta_1 s+\beta_2 v}{1+\alpha i}\\
&=\frac{e^{-r(t-\tau)}(1+\alpha i)\{e^{\mu\tau}(\mu+\nu)[\alpha(\epsilon_6-\underline{i})-1]+[\beta_1(\underline{s}-\epsilon_4)+\beta_2(\underline{v}-\epsilon_5)]\}}{[1+\alpha[i-(i-\underline{i}+\epsilon_6)e^{-r(t-\tau)}]][1+\alpha i]}\\
&=\frac{e^{-r(t-\tau)}(1+\alpha i)\{\beta_1\underline{s}+\beta_2\underline{v}-e^{\mu\tau}(\mu+\nu)(1+\alpha\underline{i})+\alpha\epsilon_6 e^{\mu\tau}(\mu+\nu)-\beta_1\epsilon_4-\beta_2\epsilon_5\}}{[1+\alpha[i-(i-\underline{i}+\epsilon_6)e^{-r(t-\tau)}]][1+\alpha i]}\\
&>\frac{e^{-r(t-\tau)}(1+\alpha i)[\alpha\epsilon_6 e^{\mu\tau}(\mu+\nu)-\beta_1\epsilon_4-\beta_2\epsilon_5]}{[1+\alpha i]^2}\\
&>0
\end{aligned}$$

when $\epsilon_4,\epsilon_5,\epsilon_6$ are small such that

$$\alpha\epsilon_6 e^{\mu\tau}(\mu+\nu)>\beta_1\epsilon_4+\beta_2\epsilon_5, \tag{6.12}$$

then (6.11) is true if $r>0$ is small enough.

Before ending this part, we show the selection of parameters. Firstly, we see that (6.5), (6.7), (6.10) and (6.12) are admissible by selecting ϵ_i small enough. In particular, small ϵ_i leads to

$$(\underline{x}(0),\underline{y}(0),\underline{z}(0))\gg(0,0,0),$$

and they are independent of r. After fixing ϵ_i, we can give a small $r>0$ satisfying our discussion. The verification is complete.

References

[1] Abi Rizk, L., Burie, J.-B., Ducrot, A.: Asymptotic speed of spread for a nonlocal evolutionary-epidemic system. Discret. Contin. Dyn. Syst. **41**, 4959–4985 (2021)
[2] Alfaro, M., Coville, J.: Propagation phenomena in monostable integro-differential equations: acceleration or not? J. Differ. Equ. **263**, 5727–5758 (2017)

[3] Aronson, D.G., Weinberger, H.F.: Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. In: Goldstein, J.A. (ed.) Partial differential equations and related topics, pp. 5–49. Springer, Berlin (1975)

[4] Aronson, D.G., Weinberger, H.F.: The asymptotic speed of propagation of a simple epidemic. In: Fitzgibbon, W.E., Walker, H.F. (eds.) Nonlinear diffusion, pp. 1–23. Pitman, London (1977)

[5] Aronson, D.G., Weinberger, H.F.: Multidimensional nonlinear diffusion arising in population dynamics. Adv. Math. **30**, 33–76 (1978)

[6] Bo, W.-J., Lin, G., Ruan, S.: Traveling wave solutions for time periodic reaction-diffusion systems. Discret. Contin. Dyn. Syst. **38**, 4329–4351 (2018)

[7] Diekmann, O.: Thresholds and travelling waves for the geographical spread of infection. J. Math. Biol. **69**, 109–130 (1978)

[8] Diekmann, O.: Run for your life. A note on the asymptotic speed of propagation of an epidemic. J. Differ. Equ. **3**, 58–73 (1979)

[9] Ducrot, A.: Spatial propagation for a two component reaction-diffusion system arising in population dynamics. J. Differ. Equ. **260**, 8316–8357 (2016)

[10] El Alami Laaroussi, A., Rachik, M.: On the regional control of a reaction-diffusion system SIR. Bull. Math. Biol. **82**(5), 25 (2020)

[11] Fang, J., Yu, X., Zhao, X.-Q.: Traveling waves and spreading speeds for time-space periodic monotone systems. J. Funct. Anal. **272**, 4222–4262 (2017)

[12] Hamel, F., Roques, L.: Fast propagation for KPP equations with slowly decaying initial conditions. J. Differ. Equ. **249**, 1726–1745 (2010)

[13] He, G., Wang, J.-B., Huang, G.: Wave propagation of a diffusive epidemic model with latency and vaccination. Appl. Anal. **100**, 1972–1995 (2021)

[14] Huang, M., Wu, S.-L., Zhao, X.-Q.: Propagation dynamics for time-periodic and partially degenerate reaction-diffusion systems. SIAM J. Math. Anal. **54**, 1860–1897 (2022)

[15] Hsu, S.-B., Zhao, X.-Q.: Spreading speeds and traveling waves for nonmonotone integrodifference equations. SIAM J. Math. Anal. **40**, 776–789 (2008)

[16] Lai, X., Zou, X.: Modeling HIV-1 virus dynamics with both virus-to-cell infection and cell-to-cell transmission. SIAM J. Appl. Math. **74**, 898–917 (2014)

[17] Li, B., Lewis, M.A., Weinberger, H.F.: Existence of traveling waves for integral recursions with nonmonotone growth functions. J. Math. Biol. **58**, 323–338 (2009)

[18] Li, W.-T., Wang, Z.-C., Wu, J.: Entire solutions in monostable reaction-diffusion equations with delayed nonlinearity. J. Differ. Equ. **245**, 102–129 (2008)

[19] Liang, X., Zhao, X.-Q.: Asymptotic speeds of spread and traveling waves for monotone semiflows with applications. Comm. Pure Appl. Math. **60**, 1–40 (2007)

[20] Liao, S., Yang, W., Fang, F.: Traveling waves for a cholera vaccination model with nonlocal dispersal. Math. Methods Appl. Sci. **44**, 5150–5171 (2021)

[21] Lin, G., Ruan, S.: Traveling wave solutions for delayed reaction-diffusion systems and applications to Lotka-Volterra competition-diffusion models with distributed delays. J. Dyn. Differ. Equ. **26**, 583–605 (2014)

[22] Lin, G., Pan, S., Yan, X.-P.: Spreading speeds of epidemic models with nonlocal delays. Math. Biosci. Eng. **16**, 7562–7588 (2019)

[23] Lui, R.: Biological growth and spread modeled by systems of recursions. I. mathematical theory. Math. Biosci. **93**, 269–295 (1989)

[24] Miyaoka, T.Y., Lenhart, S., Meyer, J.F.C.A.: Optimal control of vaccination in a vector-borne reaction-diffusion model applied to Zika virus. J. Math. Biol. **79**, 1077–1104 (2019)

[25] Pan, S.: Invasion speed of a predator-prey system. Appl. Math. Lett. **74**, 46–51 (2017)

[26] Papst, I., O'Keeffe, K.P., Strogatz, S.H.: Modeling the interplay between seasonal flu outcomes and individual vaccination decisions. Bull. Math. Biol. **84**(3), 36 (2022)

[27] Sheng, W.J., Wang, M., Wang, Z.C.: Propagation phenomena in a diffusion system with the Belousov-Zhabotinskii chemical reaction. Commun. Contemp. Math. (2022). https://doi.org/10.1142/S0219199722500018

[28] Solar, A., Trofimchuk, S.: Speed selection and stability of wavefronts for delayed monostable reaction-diffusion equations. J. Dyn. Differ. Equ. **28**, 1265–1292 (2016)

[29] Thieme, H.R.: Density-dependent regulation of spatially distributed populations and their asymptotic speed of spread. J. Math. Biol. **8**, 173–187 (1979)

[30] Thieme, H.R., Zhao, X.Q.: Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction diffusion models. J. Differ. Equ. **195**, 430–470 (2003)

[31] Wang, H., Castillo-Chavez, C.: Spreading speeds and traveling waves for non-cooperative integro-difference systems. Discret. Contin. Dyn. Syst. Ser. B **17**, 2243–2266 (2012)

[32] Wang, X., Lin, G., Ruan, S.: Spreading speeds and traveling wave solutions of diffusive vector-borne disease models without monotonicity. Proc. R. Soc. Edinb. Sect. A (2021). https://doi.org/10.1017/prm.2021.76

[33] Wang, Z., Xu, R.: Traveling waves of an epidemic model with vaccination. Int. J. Biomath. **6**, 1350033 (2013)
[34] Wang, Z.-C., Li, W.-T., Ruan, S.: Traveling fronts in monostable equations with nonlocal delayed effects. J. Dyn. Differ. Equ. **20**, 573–603 (2008)
[35] Weinberger, H.F., Lewis, M.A., Li, B.: Analysis of linear determinacy for spread in cooperative models. J. Math. Biol. **45**, 183–218 (2002)
[36] WHO. https://www.who.int/health-topics/vaccines-and-immunization
[37] Xu, W.-B., Li, W.-T., Ruan, S.: Fast propagation for reaction-diffusion cooperative systems. J. Differ. Equ. **265**, 645–670 (2018)
[38] Xu, Z., Xu, Y., Huang, Y.: Stability and traveling waves of a vaccination model with nonlinear incidence. Comput. Math. Appl. **75**, 561–581 (2018)
[39] Yang, Y., Zhou, J., Hsu, C.-H.: Critical traveling wave solutions for a vaccination model with general incidence. Discret. Contin. Dyn. Syst. Ser. B **27**, 1209–1225 (2022)
[40] Yi, T., Chen, Y., Wu, J.: Unimodal dynamical systems: comparison principles, spreading speeds and travelling waves. J. Differ. Equ. **254**, 3538–3572 (2013)
[41] Zhang, R., Liu, S.: Traveling waves for SVIR epidemic model with nonlocal dispersal. Math. Biosci. Eng. **16**, 1654–1682 (2019)
[42] Zhao, L., Wang, Z.C., Zhang, L.: Propagation dynamics for a time-periodic reaction-diffusion SI epidemic model with periodic recruitment. Z. Angew. Math. Phys. **72**, 142 (2021)
[43] Zhu, F.: Fast propagation for nonlocal delay equations with slowly decaying initial values. Electron. J. Differ. Equ. **140**, 1–10 (2012)
[44] Zou, Y., Yang, W., Lai, J., Hou, J., Lin, W.: Vaccination and quarantine effect on COVID-19 transmission dynamics incorporating Chinese-Spring-Festival travel rush: modeling and simulations. Bull. Math. Biol. **84**(2), 30 (2022)

Yahui Wang, Xinjian Wang and Guo Lin
School of Mathematics and Statistics
Lanzhou University
Lanzhou 730000 Gansu
People's Republic of China
e-mail: ling@lzu.edu.cn

Yahui Wang
e-mail: yhwang15@lzu.edu.cn

Xinjian Wang
e-mail: wangxinjian14@lzu.edu.cn

(Received: August 22, 2022; revised: August 24, 2022; accepted: December 28, 2022)