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On existence and uniqueness of weak solutions to nonlocal conservation laws with BV kernels

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Abstract. In this note, we extend the known results on the existence and uniqueness of weak solutions to conservation laws with nonlocal flux. In case the nonlocal term is given by a convolution $\gamma * q$, we weaken the standard assumption on the kernel $\gamma \in L^\infty\big((0,T);W^{1,\infty}(\mathbb{R})\big)$ to the substantially more general condition $\gamma \in L^\infty((0,T);BV(\mathbb{R}))$, which allows for discontinuities in the kernel.

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Keywords. Nonlocal conservation laws, Nonlocal flux, BV kernels, Well-posedness of weak solutions.

1. Introduction

Nonlocal balance laws have been extensively used to describe physical phenomena, including traffic flow [4,15,23,25,28,41], supply chains [24,29,43], crowd dynamics [16], opinion formation [1,40], chemical engineering [39,44], sedimentation [5], or conveyor belt dynamics [42].

Existence and uniqueness of solutions of nonlocal conservation laws has been proved in several papers. In [5,6], existence was established by numerical methods and, in [13], using the vanishing viscosity technique; to ensure uniqueness an entropy condition was prescribed. More recently, existence and uniqueness of weak solutions were established via fixed-point methods and without requiring an entropy condition (see [30,33,34]). For measure-valued solutions, a similar approach had been adopted in [20] requiring specific nonlocal kernels. We also refer to [36,38] for L^p-valued and measure-valued solutions.

Results on the convergence of nonlocal conservation laws to the corresponding local models have been obtained in [7,8,12,14,17,31]. For the study of controllability properties of nonlocal conservation laws, we refer the reader, e.g., to [3,10,11,18,19] and references therein. The aim of this note is to extend the result on the existence and uniqueness of weak solutions to conservation laws with nonlocal flux established in [30]. In the more simplistic case where the nonlocal term is given by a convolution $\gamma * q$, we can weaken the standard assumption $\gamma \in L^\infty\big((0,T);W^{1,\infty}(\mathbb{R})\big)$ to the substantially more general condition $\gamma \in L^\infty((0,T);BV(\mathbb{R}))$, which allows for discontinuities in the kernel.

More precisely, we consider the nonlocal conservation law

$$\begin{aligned} \partial_t q(t,x) + \partial_x\big(V(W[q](t,x))q(t,x)\big) &= 0, \qquad (t,x) \in \Omega_T,\\ q(0,x) &= q_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1.1}$$

with

$$W[q](t,x) := (\gamma(t,\cdot) * q(t,\cdot))(x), \qquad (t,x) \in \Omega_T. \tag{1.2}$$

Thereby, for $T \in \mathbb{R}_{>0}$, $q : \Omega_T \to \mathbb{R}$ with $\Omega_T = (0,T) \times \mathbb{R}$ denotes the space-time dependent density of the conservation law, and $q_0 : \mathbb{R} \to \mathbb{R}$ the initial datum.

We assume that the following conditions are satisfied:

Assumption 1. (*Input datum*) For $T \in \mathbb{R}_{>0}$, let the following hypotheses be satisfied:
(A1) $\gamma \in L^\infty((0,T); BV(\mathbb{R}))$ and $\gamma \geq 0$;
(A2) $V \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$;
(A3) $q_0 \in L^\infty(\mathbb{R})$.
Following [2], we recall that the total variation of $u \in L^1(\mathbb{R})$ is given by

$$|u|_{TV(\mathbb{R})} := \sup \left\{ \int_{\mathbb{R}} u\,\psi' \,\mathrm{d}x : \ \psi \in C^\infty_{\mathrm{c}}(\mathbb{R}), \ \|\psi\|_{C^0(\mathbb{R})} \leq 1 \right\}.$$

The norm $\|u\|_{BV(\mathbb{R})} := \|u\|_{L^1(\mathbb{R})} + |u|_{TV(\mathbb{R})}$ makes the space of functions of bounded variation $\mathrm{BV}(\mathbb{R})$ a Banach space.

1.1. Outline

In Sect. 2, we state our main well-posedness result and outline the steps of its proof. In particular, we point out what changes are required to generalize the argument of [30]. In Sect. 3, we obtain the two key lemmata that are needed to extend the proofs of [30] to the more general class of kernels under consideration. Finally, Sect. 4 concludes the paper with some examples and numerical simulations.

2. Main result and outline of the proof

Before stating our main theorem, we recall the notion of weak solution for the nonlocal conservation law in (1.1) stated in [30, Definition 2.13].

Definition 2.1. (*Weak solution of the nonlocal balance law*) We say that $q \in C\big([0,T]; L^1_{\text{loc}}(\mathbb{R})\big)$ is a *weak solution* of the nonlocal conservation law in (1.1) iff for all $\varphi \in C^1_{\mathrm{c}}((-42,T)\times\mathbb{R})$, the following integral equation holds:

$$\iint_{\Omega_T} q(t,x)\Big(\partial_t\varphi(t,x) + \partial_x\varphi(t,x)V\big(W[q](t,x)\big)\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} \varphi(0,x)q_0(x)\,\mathrm{d}x = 0,$$

with $W[q]$ as in (1.2).

Our main theorem establishes the existence and uniqueness of weak solutions to the nonlocal conservation law in (1.1) given Assumption 1.

Theorem 2.1. (Local well-posedness of nonlocal conservation laws with rough kernels) *Let $T \in \mathbb{R}_{>0}$, and let Assumption 1 hold. Then, there exists $T^* \in (0,T]$ such that the nonlocal initial value problem in (1.1) admits a unique weak solution $q \in C\big([0,T^*]; L^1_{\text{loc}}(\mathbb{R})\big) \cap L^\infty((0,T^*); L^\infty(\mathbb{R}))$ in the sense of Definition 2.1. Moreover, the weak solution can be written as*

$$q(t,x) = q_0(\xi_{w^*}(t,x;0))\partial_2\xi_{w^*}(t,x;0), \qquad (t,x) \in [0,T^*]\times\mathbb{R},$$

where w^* is the unique solution on $(0,T^*)\times\mathbb{R}$ of the fixed point problem in (3.1) and ξ_{w^*} the characteristics defined in (2.3).

Under physically reasonable additional monotonicity assumptions on the velocity and the kernel (i.e. the further away the density is from the current space location, the less it contributes in the nonlocal term), we obtain the following existence result for larger time (which is established by means of a comparison principle).

Corollary 2.1. (Global existence and comparison principle) *Under the assumptions of Theorem 2.1, if, in addition, it holds that*

(A4) $V' \leqq 0$,
(A5) $\operatorname{supp}(\gamma(t,\cdot)) \subseteq \mathbb{R}_{\geq 0}$ *and* $\gamma(t,\cdot)$ *monotonically non-increasing on* $\mathbb{R}_{>0}$ $\forall t \in [0,T]$,
(A6) $q_0 \in L^\infty(\mathbb{R};\mathbb{R}_{\geq 0})$,

then the initial-value problem in (1.1) admits, for every $T \in \mathbb{R}_{>0}$, *a unique solution*

$$q \in C\big([0,T]; L^1_{\text{loc}}(\mathbb{R})\big) \cap L^\infty((0,T); L^\infty(\mathbb{R}))$$

satisfying the comparison principle

$$\text{ess-inf}_{x\in\mathbb{R}} q_0(x) \leq q(t,x) \leq \|q_0\|_{L^\infty(\mathbb{R})} \quad \forall (t,x) \in \Omega_T \quad a.e.$$

2.1. Outline of the proof to Theorem 2.1

As we will mimic the argument in [30] for rough kernels, we first shortly introduce the required steps in this proof.

- **Step 1.** *Formulation of the fixed-point equation in the nonlocal term* w. Recalling that for $(t,x) \in \Omega_T$ we have

$$W[q](t,x) = \int_{\mathbb{R}} \gamma(t,x-y) q(t,y)\,\mathrm{d}y =: w(t,x), \tag{2.1}$$

  we assume for now that this nonlocal term is given. Then, the corresponding conservation law is linear with Lipschitz continuous velocity $V(w)$ and we use the method of characteristics to write the solution as

$$q_w(t,x) = q_0(\xi_w(t,x;0))\partial_x \xi_w(t,x;0), \qquad (t,x) \in \Omega_T, \tag{2.2}$$

  where ξ_w solves the characteristic ODE

$$\xi_w(t,x;\tau) = x + \int_t^\tau V\big(w(s,\xi_w(t,x;s)\big)\,\mathrm{d}s, \quad \tau \in [0,T]. \tag{2.3}$$

  This can now be plugged into the nonlocal term in (2.1) once more to obtain, for $(t,x) \in \Omega_T$,

$$\begin{aligned} w(t,x) &= \int_{\mathbb{R}} \gamma(t,x-y) q(t,y)\,\mathrm{d}y = \int_{\mathbb{R}} \gamma(t,x-y) q_0(\xi_w(t,y;0))\partial_y \xi_w(t,y;0)\,\mathrm{d}y \\ &= \int_{\mathbb{R}} \gamma\big(t,x-\xi_w(0;y;t)\big) q_0(y)\,\mathrm{d}y, \end{aligned}$$

  a fixed-point problem in w which is then studied for existence and uniqueness of solutions on a sufficiently small time horizon.
- **Step 2.** *Local existence for the nonlocal conservation law.* Having proven the existence of a $w^* \in L^\infty((0,T); W^{1,\infty}(\mathbb{R}))$ with Banach's fixed-point theorem, we can build a solution of (1.1) in terms of characteristics (analogously to (2.2)):

$$q(t,x) = q_0(\xi_{w^*}(t,x;0))\partial_x \xi_{w^*}(t,x;0),\ (t,x) \in (0,T^*) \times \mathbb{R},$$

  which is presented in [30, Theorem 2.20] and [32, Theorem 3.1] in detail.
- **Step 3.** *Uniqueness for the nonlocal conservation law.* The uniqueness of w^* is shown to imply the uniqueness of the solution q. The main idea is to prove that any weak solution can be written in the same way as instantiated in (2.2) (see [30, Lemma 3.1 and Theorem 3.2]).

- **Step 4.** *Extension of the solution for larger times.* Gluing a sequence of initial value problems with initial data equal to the terminal-time solution of the previous one, we can extend the existence result to a longer (but not necessarily arbitrary) time-horizon (as in [30, Theorem 4.1]).
- **Step 5.** *Extension to arbitrary time-horizons and comparison principle.* Under the stronger assumptions (A4)-(A5), we can extend the solution to an arbitrary time-horizon and show that a comparison principle holds. For the detailed argument, we refer to [34, Lemma 5.8]. It mainly consists of studying the time evolution of the maximum/minimum of the solution and deducing that its time derivative is negative implying that the minimum can only increase and the maximum only decrease over time.

Extension of the proof to rough kernels. The only parts of the proof outlined above that need to be adjusted from [30] to extend the well-posedness result to our more general setting are as follows:

1. proving that for $t \in [0, T]$ the convolution $(x \mapsto \gamma(t, \cdot) * q(t, \cdot))(x)$ is in $W^{1,\infty}(\mathbb{R})$ for $\gamma \in L^\infty((0, T); BV(\mathbb{R}))$;
2. establishing the analogue of [30, Proposition 2.17], where it was shown that the fixed-point mapping induced in **Step 1** in Sect. 2.1 satisfies the required assumptions of Banach's fixed-point theorem by relying on the regularity assumption $\gamma \in L^\infty\big((0, T); W^{1,\infty}(\mathbb{R})\big)$.

To this end, in Sect. 3, we first prove, in Lemma 3.1, that the convolution is Lipschitz in space and then, in Proposition 3.1, we demonstrate the analogue of [30, Proposition 2.17] in our setting.

3. Proof of the properties of the fixed-point mapping

We start by proving the Lipschitz-continuity (in space) of the convolution.

Lemma 3.1. (Smoothing via convolution with BV functions) *Let* $\gamma \in BV(\mathbb{R})$ *and* $f \in L^\infty(\mathbb{R})$*. Then* $\gamma * f \in W^{1,\infty}(\mathbb{R})$.

Proof. For $h \in \mathbb{R}$ by [2, Remark 3.5] or [37, Corollary 2.17], the right h-translation of γ, i.e., $\tau_h\gamma(x) := \gamma(x + h)\ \forall x \in \mathbb{R}$ a.e., satisfies

$$\|\tau_h\gamma - \gamma\|_{L^1(\mathbb{R})} \le |\gamma|_{TV(\mathbb{R})}|h|.$$

As a consequence, we can estimate, by using Young's convolution inequality (see [9, Theorem 4.33]),

$$\|\tau_h(\gamma * f) - \gamma * f\|_{L^\infty(\mathbb{R})} = \|(\tau_h\gamma - \gamma) * f\|_{L^\infty(\mathbb{R})} \le \|f\|_{L^\infty(\mathbb{R})}\|\tau_h\gamma - \gamma\|_{L^1(\mathbb{R})} \le |\gamma|_{TV(\mathbb{R})}\|f\|_{L^\infty(\mathbb{R})}|h|.$$

We thus conclude that $\gamma * f \in W^{1,\infty}(\mathbb{R})$. □

We now review the proof of the fixed-point argument contained in [30, Proposition 2.17]. As mentioned in Sect. 2, this is the main step that needs to be taken to adapt the arguments of [30] to the case of a nonlocal term given by the convolution of the density q with a rough kernel $\gamma \in L^\infty((0, T); BV(\mathbb{R}))$.

Proposition 3.1. (Properties of the fixed-point mapping) *Let*

$$F : \begin{cases} \tilde{\Omega} \to L^\infty((0, T); W^{1,\infty}(\mathbb{R})), \\ w \mapsto \Big((t, x) \mapsto \int_{\mathbb{R}} \gamma\big(t, x - \xi_w(0, z; t)\big) q_0(z)\,\mathrm{d}z\Big), \end{cases} \tag{3.1}$$

*be the fixed-point mapping as introduced in **Step 1** of Sect. 2.1 and let* $\tilde{\Omega}$ *be defined by*

$$M := 42\|\gamma\|_{L^\infty((0,T);L^1(\mathbb{R}))}\|q_0\|_{L^\infty(\mathbb{R})}, \qquad M' := 42|\gamma|_{L^\infty((0,T);TV(\mathbb{R}))}\|q_0\|_{L^\infty(\mathbb{R})},$$

$$\Omega_M^{M'}(T) := \Big\{ w \in L^\infty((0, T); W^{1,\infty}(\mathbb{R})) : \ \|w\|_{L^\infty((0,T);L^\infty(\mathbb{R}))} \le M \ \wedge \ \|\partial_2 w\|_{L^\infty((0,T);L^\infty(\mathbb{R}))} \le M' \Big\}. \tag{3.2}$$

Then, the fixed-point mapping defined in (3.1) satisfies the following properties:

1. $\exists T^* \in (0, T] : \|F[w]\|_{L^\infty((0,T^*);L^\infty(\mathbb{R}))} \le M$ *for all* $w \in \Omega_M^{M'}(T^*)$;

2. *$\exists T' \in (0,T] : \|\partial_2 F[w]\|_{L^\infty((0,T');L^\infty(\mathbb{R}))} \leq M'$ for all $w \in \Omega_M^{M'}(T')$;*
3. *F is Lipschitz continuous with respect to the uniform topology, i.e., for $w, \tilde{w} \in \Omega_M^{M'}(\bar{T})$, $\bar{T} := \min\{T^*, T'\}$,*
$$\|F[w] - F[\tilde{w}]\|_{L^\infty((0,\bar{T});L^\infty(\mathbb{R}))} \leq |\gamma|_{L^\infty((0,T);TV(\mathbb{R}))}\bar{T}\|w - \tilde{w}\|_{L^\infty((0,\bar{T});L^\infty(\mathbb{R}))} \cdot \|V'\|_{L^\infty((-M,M))}\mathrm{e}^{2\bar{T}\|V'\|_{L^\infty((-M,M))}M'}$$
and thus, for small time $\hat{T} \in (0, \bar{T}]$, F is a contraction on $\Omega_M^{M'}(\hat{T})$.

Proof. 1. For $w \in \Omega_M$ and $t \in [0,T]$, we estimate—recalling the definition of F in (3.1)—
$$\|F[w](t,\cdot)\|_{L^\infty(\mathbb{R})} = \left\|\int_{\mathbb{R}} \gamma(t, \cdot - \xi_w(0,z;t))q_0(z)\,\mathrm{d}z\right\|_{L^\infty(\mathbb{R})} \leq \|\gamma(t,\cdot)\|_{L^1(\mathbb{R})}\|\partial_2\xi_w(t,\cdot;0)\|_{L^\infty}\|q_0\|_{L^\infty(\mathbb{R})}$$
$$\leq \|\gamma(t,\cdot)\|_{L^1(\mathbb{R})}\mathrm{e}^{t\|V'\|_{L^\infty((-M,M))}M'}\|q_0\|_{L^\infty(\mathbb{R})},$$
where we have used the substitution rule and the properties of the characteristics ([30, Lemma 2.6] and in particular [30, Lemma 2.6(3)], see also [32, Corollary 2.1]): namely,
$$\|\partial_2\xi_w(t,\cdot;0)\| \leq \mathrm{e}^{t\|V'\|_{L^\infty((-M,M))}M'} \quad \forall t \in [0,T], \tag{3.3}$$
which is an immediate consequence of differentiating (2.3) with regard to $x \in \mathbb{R}$ to obtain a linear IVP in $\partial_2\xi_w$. However, as M, M' are fixed, we can find a time horizon $T^* \in (0,T]$ such that
$$\|\gamma\|_{L^\infty((0,T);L^1(\mathbb{R}))}\mathrm{e}^{T^*\|V'\|_{L^\infty((-M,M))}M'}\|q_0\|_{L^\infty(\mathbb{R})} \leq M \iff \mathrm{e}^{T^*\|V'\|_{L^\infty((-M,M))}M'} \leq 42,$$
which indeed proves the existence of such a T^*.

2. Next, we estimate the spatial derivative of the fixed-point mapping in (3.1) (which is well-defined according to Lemma 3.1), for $w \in \Omega_M^{M'}(T)$ and $(t,x) \in \Omega_T$. Some technical details are left out and can be found in [30, Lemma 2.6(2)], however, the argument is as follows (applying once more the substitution rule and the estimate in (3.3)):
$$|\partial_x F[w](t,x)| \leq \int_{\mathbb{R}} \left|\partial_x\gamma(t, x - \xi_w(0,z;t))q_0(z)\right|\mathrm{d}z$$
$$\leq \|\partial_2\xi(t,\cdot;0)\|_{L^\infty(\Omega_T)}\|q_0\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|\partial_z\gamma(t,x-y)|\,\mathrm{d}y$$
$$\leq |\gamma(t,\cdot)|_{TV(\mathbb{R})}\mathrm{e}^{t\|V'\|_{L^\infty((-M,M))}M'}\|q_0\|_{L^\infty(\mathbb{R})}.$$
Making this uniform in $(t,x) \in \Omega_T$, and since M, M' are fixed, we can find a time horizon $T' \in (0,T]$ so that
$$|\gamma|_{L^\infty((0,T);TV(\mathbb{R}))}\mathrm{e}^{T'\|V'\|_{L^\infty((-M,M))}M'}\|q_0\|_{L^\infty(\mathbb{R})} \leq M' \iff \mathrm{e}^{T'\|V'\|_{L^\infty((-M,M))}M'} \leq 42.$$
This proves the existence of such a T' and we can indeed chose $T' = T^*$. Thus, we can conclude with the two previous results
$$F\Big(\Omega_M^{M'}(T')\Big) \subseteq \Omega_M^{M'}(T'),$$
i.e., F is a self-mapping on $\Omega_M^{M'}(T')$.

3. Finally, we approach the contraction property of F in $L^\infty((0,T');L^\infty(\mathbb{R}))$ and estimate for $w, \tilde{w} \in \Omega_M^{M'}(T')$ (which is due to its uniform bounds on the involved functions and its derivatives closed in $L^\infty((0,T');L^\infty(\mathbb{R})))$ and $(t,x) \in (0,T') \times \mathbb{R}$:
$$|F[w](t,x) - F[\tilde{w}](t,x)|$$

$$= \left| \int_{\mathbb{R}} \gamma(t, \xi_w(0,z;t)) q_0(z)\,\mathrm{d}z - \int_{\mathbb{R}} \gamma(t, \xi_{\tilde{w}}(0,z;t)) q_0(z)\,\mathrm{d}z \right|$$

$$\leq \int_{\mathbb{R}} |\gamma(t, \xi_w(0,z;t)) - \gamma(t, \xi_{\tilde{w}}(0,z;t))|\, q_0(z)\,\mathrm{d}z$$

$$\leq |\gamma|_{L^\infty((0,T);TV(\mathbb{R}))} \|\xi_w - \xi_{\tilde{w}}\|_{L^\infty((0,t)\times\mathbb{R}\times(0,t))} \|q_0\|_{L^\infty(\mathbb{R})} \mathrm{e}^{t\|V'\|_{L^\infty((-M,M))}M'} \tag{3.4}$$

$$\leq |\gamma|_{L^\infty((0,T);TV(\mathbb{R}))} t \|V'\|_{L^\infty((-M,M))} \|w - \tilde{w}\|_{L^\infty((0,t);L^\infty(\mathbb{R}))} \|q_0\|_{L^\infty(\mathbb{R})} \mathrm{e}^{2t\|V'\|_{L^\infty((-M,M))}M'}, \tag{3.5}$$

where we have used, in (3.4), the substitution rule and the uniform bound on $\partial_2\xi_w, \partial_2\xi_{\tilde{w}}$ as in (2.2) thanks to the bounds on $w, \tilde{w}$ in $L^\infty\big(W^{1,\infty}\big)$ (compare (3.2)); and, in (3.5), the stability of the characteristics with regard to the nonlocal term (see [30, Lemma 2.6(3)] and [32, Theorem 2.4]). This last stability result can be obtained when comparing the solution with the "perturbed" solution of the IVP in (2.3) in w and a typical Gronwall estimate (see [21, Appendix B k) ii]) can be used to derive it. Making the previous estimate uniform in (t,x) and recalling that M, M' are fixed there exists $\hat{T} \in (0, T']$ so that

$$|\gamma|_{L^\infty((0,T);TV(\mathbb{R}))} \hat{T} \|V'\|_{L^\infty((-M,M))} \|w - \tilde{w}\|_{L^\infty((0,\hat{T});L^\infty(\mathbb{R}))} \|q_0\|_{L^\infty(\mathbb{R})} \mathrm{e}^{2\hat{T}\|V'\|_{L^\infty((-M,M))}M'} \leq \tfrac{1}{2}.$$

From this, it follows that F is also a contraction in $\Omega_M^{M'}\big(\hat{T}\big)$ for a sufficiently small $\hat{T} \in \big(0, \bar{T}\big)$. □

4. Conclusions and numerical illustrations

In what follows, we present some numerical simulations (based on a non-dissipative discretization scheme using the method of characteristics, see [35]) to illustrate the effect of a discontinuity in the kernel. We consider the Cauchy problem in (1.1) with initial datum

$$q_0(x) = \chi_{(0,\frac{1}{2})}(x) + \chi_{\mathbb{R}_{>1}}(x), \qquad x \in \mathbb{R} \tag{4.1}$$

and focus on **LWR-type velocity** (see [27, Formula (1.26), p. 11]), i.e., $V(\xi) := 1 - \xi^2, \xi \in \mathbb{R}$, and **Burgers-type velocity** (see [27, Formula (1.8), p. 3]), i.e., $V(\xi) := \xi, \xi \in \mathbb{R}$, (see also [22, Section 3.1.2] for the fundamental diagrams and generalized Greenshields [26]). As examples of convolution kernels, we consider

$$\gamma_1(x) = 2\chi_{(0,1)}(2x), \qquad \gamma_2(x) = \tfrac{8}{3}\chi_{(0,1)}(4x) + \tfrac{4}{3}\chi_{(1,2)}(4x), \qquad x \in \mathbb{R}, \tag{4.2}$$

and remark that $\gamma_1, \gamma_2 \in BV(\mathbb{R})$.

For the LWR-type velocity, a comparison principle (see Corollary 2.1) is satisfied. Since the initial datum is chosen in a way that it has maximum density and zero velocity in $\mathbb{R}_{>1}$, the initial density for $x < 1$ slows down as it gets closer to $x = 1$. We remark that the second illustration in Fig. 1 indicates a disturbance evolving from points where the discontinuities of γ and q "intersect", i.e., 0.25 left of the spatial discontinuities of q.

For the example involving the Burgers-type velocity (due to the chosen initial datum and the right-looking nonlocal term), a comparison principle does not hold (see [30, Example 6.1]) and the entire mass concentrates at the point $x = 0.5$ as time evolves. Thus, the solution ceases to exist for large time. Again, the impact of the discontinuous kernel (the fourth (right) illustration in Fig. 1) is destroying the rather "smooth" structure of the solution which we would obtain when using a smooth kernel.

Possible generalizations of this work may consist of **(1)** weakening the assumptions on V to be discontinuous in space; **(2)** determining the precise regularity assumptions on initial datum and weight to have the nonlocal conservation law be well-posed (including measure valued solutions and kernels); and **(3)** generalizing the results to multi-dimensional nonlocal balance laws.

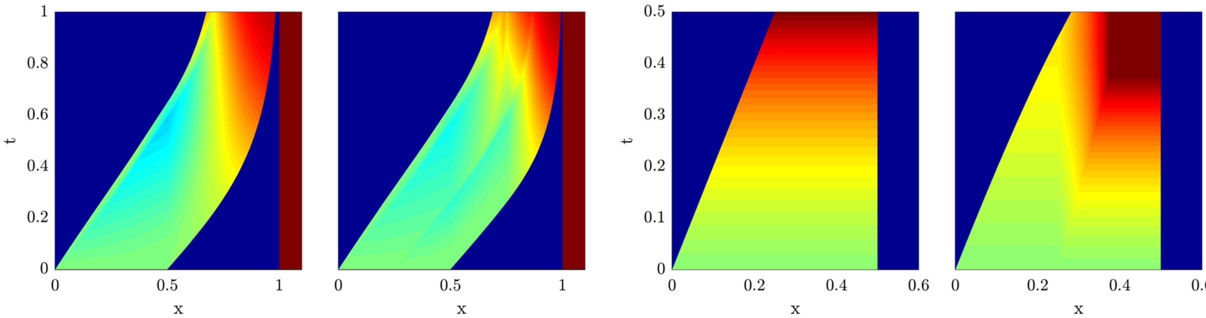

Fig. 1. First and second: plot of the solution for the LWR-type velocity function, i.e., $V \equiv 1 - \cdot$ with γ_1, γ_2 as in (4.2) in the first and second plot respectively. Third and fourth: Plot of the solution for the Burgers-type velocity, i.e. $V \equiv \cdot$ with γ_1 in the third and γ_2 in the fourth plot. Colorbar: 0 1. Note that for the rightmost figure the maximal density exceeds 1 but still visualized by the dark red color (Color figure online)

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