Einstein's energy and space isotropy

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Abstract. In this note, we derive an extension of the conventional Einstein variation of mass formula with a specific expression arising from a Lorentz invariant equation for the energy rate de/dp where $e = mc^2$ is the particle energy, $p = mu$ the particle momentum and u the velocity. This is the simplest one-parameter Lorentz-invariant extension of the Einstein mass–energy relation. Implicit in the new expression is space–time anisotropy such that the particle has different rest masses in the positive and negative x directions. While numerous experiments have been undertaken aimed at testing such hypothesis, and all indicate the veracity of the assumption that space is isotropic, nevertheless since it is generally believed that black-holes exist at the centres of galaxies, space must be intrinsically anisotropic in some sense. Finally, we note a very curious connection with both the conventional Einstein energy–mass expression $e = e_0/(1 - (u/c)^2)^{1/2}$ and the new expression derived here with certain singular integral equations usually associated with aero-foil problems, fluid mechanics and punch problems in elasticity, and that this connection is not some vague intangible relationship, but involves an exact correspondence.

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1. Introduction

In this note, we adopt the usual assumptions of special relativity including the Lorentz transformations and the Lorentz invariant energy–momentum relations, but not Einstein's formula for the variation of mass with velocity $u(x, t)$, $m = m_0[1 - (u/c)^2]^{-1/2}$. For a particle moving with velocity u and assuming the relations $e = mc^2$ and $p = mu$ for the particle energy and momentum, respectively, we derive the mass expression

$$m(u) = \frac{m_0}{(1 - (u/c)^2)^{1/2}} \left(\frac{1 + (u/c)}{1 - (u/c)} \right)^{\kappa/2}. \quad (1.1)$$

This expression is based upon the general Lorentz invariant energy rate equation (3.1) for de/dp , namely

$$\frac{de}{dp} = c \left(\frac{\kappa + u/c}{1 + \kappa u/c} \right),$$

involving a new arbitrary constant κ , and providing the simplest one-parameter Lorentz-invariant extension of the Einstein mass–energy relations. Implicit in this new expression is that $\kappa \neq 0$ implies space–time anisotropy with the particle having different rest masses in the positive and negative x -directions. Accordingly, we assume here that symmetry under Lorentz transformations is more fundamental than symmetry under reflections and that the anisotropy is dipolar with no components in the y and z directions. Both the conventional expression $m(u) = m_0[1 - (u/c)^2]^{-1/2}$ and this new expression bear a very curious relationship with certain singular integral equations, generally arising in mechanics, and the connection is exact.

We note here that the particle and wave velocities arise as two special cases corresponding, respectively, to the values $\kappa = 0$ and $\kappa = \pm\infty$, thus

$$\frac{de}{dp} = u, \quad \frac{de}{dp} = \frac{c^2}{u}. \quad (1.2)$$

The case $\kappa = 0$ arises by re-writing the standard relations $m = m_0[1 - (u/c)^2]^{-1/2}$, $e = mc^2$ and $p = mu$ using momentum as the variable, thus

$$\frac{u}{c} = \frac{pc}{(e_0^2 + (pc)^2)^{1/2}}, \quad e = (e_0^2 + (pc)^2)^{1/2}, \quad (1.3)$$

where $e_0 = m_0c^2$ denotes the rest-mass energy. The relationship (1.2)₁ arises immediately on differentiating (1.3)₂ with respect to p , and then using (1.3)₁. The case $\kappa = \pm\infty$ corresponds to the de Broglie wave that is associated with a particle moving with velocity u , and moving with the superluminal wave velocity $w = c^2/u$ (see de Broglie [1]).

The following section deals with the special relativity details that are needed in the derivation, including Lorentz transformations, the various forms of the Einstein addition of velocities law, and the Lorentz invariant energy–momentum relations. The new expression (1.1) is derived in the section thereafter, along with the correspondence with solutions of certain singular integral equations. Some brief conclusions are presented in the final section.

2. Special relativity preliminaries

In this section, we detail the special relativity notation that is used here and some of the preliminary equations that are needed in the derivation.

2.1. Notation and Lorentz transformations

We consider a rectangular Cartesian frame (X, Y, Z) and another frame (x, y, z) moving with constant velocity v relative to the first frame and the motion is assumed to be in the aligned X and x directions as indicated in Fig. 1. We view the relative velocity v as a parameter measuring the departure of the current frame (x, y, z) from the rest frame (X, Y, Z) , and we adopt a notation employing lower case for variables associated with the moving (x, y, z) frame, and upper case or capitals for those variables associated with the rest (X, Y, Z) frame. Accordingly, time is measured from the (X, Y, Z) frame with the variable T and from the (x, y, z) frame with the variable t . Following normal practice, we assume that $y = Y$ and $z = Z$, so that (X, T) and (x, t) are the variables of principal interest.

For $0 \leq v < c$, the standard Lorentz transformations are

$$X = \frac{x + vt}{[1 - (v/c)^2]^{1/2}}, \quad T = \frac{t + vx/c^2}{[1 - (v/c)^2]^{1/2}},$$

with the inverse transformation characterised by $-v$, thus

$$x = \frac{X - vT}{[1 - (v/c)^2]^{1/2}}, \quad t = \frac{T - vX/c^2}{[1 - (v/c)^2]^{1/2}}. \quad (2.1)$$

The above equations reflect, of course, that the two coordinate frames coincide when the relative velocity v is zero, namely $x = X$, $t = T$, when $v = 0$, and by direct substitution from the above equations, we may readily deduce

$$ct + x = \left(\frac{1 - v/c}{1 + v/c} \right)^{1/2} (cT + X), \quad ct - x = \left(\frac{1 + v/c}{1 - v/c} \right)^{1/2} (cT - X).$$

FIG. 1. Two inertial frames moving along x -axis with relative velocity v

2.2. Einstein addition of velocities law

The relative frame velocity v is assumed to be constant, so that with velocities $U = dX/dT$ and $u = dx/dt$, on taking differentials of equations (2.1), thus

$$dx = \frac{dX - v dT}{[1 - (v/c)^2]^{1/2}}, \quad dt = \frac{dT - v dX/c^2}{[1 - (v/c)^2]^{1/2}},$$

so that with $U = dX/dT$ and $u = dx/dt$ as the respective particle velocities, we obtain the following two forms of the Einstein addition of velocity law

$$u = \frac{U - v}{(1 - Uv/c^2)}, \quad U = \frac{u + v}{(1 + uv/c^2)}. \quad (2.2)$$

An immediate consequence of (2.2) is the identity

$$[1 - (u/c)^2]^{1/2} (1 - Uv/c^2) = [1 - (v/c)^2]^{1/2} [1 - (U/c)^2]^{1/2}, \quad (2.3)$$

which can be easily established by using (2.2)₁ in the left-hand side of (2.3). These velocity equations are fundamental to the development of special relativity, and to establish the Lorentz invariance of various quantities.

A formula arising from (2.2) that is not so well known is

$$\left(\frac{1 + U/c}{1 - U/c} \right) = \left(\frac{1 + u/c}{1 - u/c} \right) \left(\frac{1 + v/c}{1 - v/c} \right), \quad (2.4)$$

so that on introducing velocity variables $(\Theta, \theta, \epsilon)$ defined by

$$\Theta = \tanh^{-1}(U/c), \quad \theta = \tanh^{-1}(u/c), \quad \epsilon = \tanh^{-1}(v/c), \quad (2.5)$$

Equation (2.4) becomes simply the translation $\Theta = \theta + \epsilon$, noting that within the context of special relativity, v and therefore, ϵ are constants. The angle θ assumes an important role and is the angle in which Lorentz invariance appears as a translation, and so for completeness we note the elementary relations

$$\theta = \frac{1}{2} \log \left(\frac{1 + u/c}{1 - u/c} \right) = \tanh^{-1}(u/c), \quad \left(\frac{1 + u/c}{1 - u/c} \right)^{1/2} = e^\theta. \quad (2.6)$$

2.3. Force and energy arising from work done

The basic notions of force, as rate of change of momenta, and physical energy, as the work done (namely force times distance) arise in the two rate-of-working equations (or work done equations) for the physical energies E and e in the (X, T) and (x, t) frames, respectively, and these are as follows:

$$dE = F dX = \frac{dP}{dT} dX, \quad de = f dx = \frac{dp}{dt} dx, \quad (2.7)$$

where $F = dP/dT$ and $f = dp/dt$ denote the physical force in the two frames and $P = MU$ and $p = mu$ the momenta where $U = dX/dT$ and $u = dx/dt$ are the respective particle velocities. Using these relations and the expressions $E = Mc^2$ and $e = mc^2$, Eq. (2.7) on multiplication by their respective masses may be rewritten as:

$$E \frac{dE}{dT} = c^2 P \frac{dP}{dT}, \quad e \frac{de}{dt} = c^2 p \frac{dp}{dt}.$$

These equations evidently integrate to yield the respective equations $E^2 = (Pc)^2 + \text{constant}$ and $e^2 = (pc)^2 + \text{constant}$. The arbitrary constants in these equations are generally fixed by taking the particle energy at $E = e = e_0 = m_0 c^2$ at zero velocity, so that we have $E^2 - (Pc)^2 = e^2 - (pc)^2 = e_0^2$. Assuming these arbitrary constants are as generally prescribed, then from the energy statements $E^2 = e_0^2 + (Pc)^2$ and $e^2 = e_0^2 + (pc)^2$, along with $E = Mc^2$ and $e = mc^2$, we might deduce the Einstein formulae

$$M(U) = \frac{m_0}{[1 - (U/c)^2]^{1/2}}, \quad m(u) = \frac{m_0}{[1 - (u/c)^2]^{1/2}}.$$

We comment that the new expression derived below differs from the Einstein formula only by a Lorentz invariant multiplicative factor, and is therefore consistent with the notion of Lorentz invariant energy–momentum relations established in the following subsection.

2.4. Lorentz invariant energy–momentum relations

The Lorentz invariant energy–momentum relations given by Eq. (2.8) can be deduced from (2.2) as follows. From $E = Mc^2$ and $P = MU$, and $e = mc^2$ and $p = mu$, we have on multiplication of (2.2) by $m_0 [1 - (u/c)^2]^{-1/2}$ and by using (2.3), we may readily deduce (2.8)₁, thus

$$\frac{um_0}{[1 - (u/c)^2]^{1/2}} = \frac{m_0 U - m_0 v}{[1 - (v/c)^2]^{1/2} [1 - (U/c)^2]^{1/2}},$$

while (2.8)₂ arises directly from $e(u) = e_0 [1 - (u/c)^2]^{-1/2}$, on using (2.3), thus

$$e = \frac{m_0 c^2}{[1 - (u/c)^2]^{1/2}} = \frac{m_0 c^2 (1 - Uv/c^2)}{[1 - (v/c)^2]^{1/2} [1 - (U/c)^2]^{1/2}},$$

so that altogether we obtain

$$p = \frac{P - Ev/c^2}{[1 - (v/c)^2]^{1/2}}, \quad e = \frac{E - Pv}{[1 - (v/c)^2]^{1/2}}. \quad (2.8)$$

The inverse relations are given by

$$P = \frac{p + ev/c^2}{[1 - (v/c)^2]^{1/2}}, \quad E = \frac{e + pv}{[1 - (v/c)^2]^{1/2}}, \quad (2.9)$$

and together these equations are referred to as the Lorentz invariant energy–momentum relations. In the following section, we derive an extension of the conventional Einstein variation of mass formula from a Lorentz invariant equation for the energy rate de/dp .

3. Alternative energy–mass velocity variation

The Einstein energy–mass expression $e = e_0/(1 - (u/c)^2)^{1/2}$ is the simplest and most logical, and any other hypothesis is necessarily more complicated and might involve additional implied consequences. Here, we derive one such alternative expression arising from the general Lorentz invariant equation

$$\frac{de}{dp} = c \left(\frac{\kappa + u/c}{1 + \kappa u/c} \right), \quad (3.1)$$

which for $\kappa \neq 0$ implies a non-isotropy of space. This equation involves an arbitrary constant κ for which the particle and wave velocities arise as special cases corresponding, respectively, to the values $\kappa = 0$ and $\kappa = \pm\infty$, thus

$$\frac{de}{dp} = u, \quad \frac{de}{dp} = \frac{c^2}{u}.$$

Equation (3.1) is Lorentz invariant in the sense that for fixed relative frame velocities v , by division of the differentials of the inverse energy–momentum relations (2.9), namely

$$dP = \frac{dp + vde/c^2}{[1 - (v/c)^2]^{1/2}}, \quad dE = \frac{de + vdp}{[1 - (v/c)^2]^{1/2}},$$

we may deduce the equation

$$\frac{dE}{dP} = \left(\frac{de}{dp} + v \right) / \left(1 + \frac{v}{c^2} \frac{de}{dp} \right),$$

and on substitution of (3.1) in this equation, we obtain

$$\frac{dE}{dP} = \left(\frac{(u+v) + c\kappa(1+uv/c^2)}{(1+uv/c^2) + \kappa(u+v)/c} \right) = c \left(\frac{\kappa + U/c}{1 + \kappa U/c} \right),$$

on using (2.2)₂. Since there is the same velocity dependence in both the moving and reference frames, (3.1) is a Lorentz invariant equation.

Assuming the usual relations $e = mc^2$ and $p = mu$, we have $p = eu/c^2$, and therefore from (3.1) we obtain

$$\frac{dp}{de} = \frac{1}{c^2} \left(u + e \frac{du}{de} \right) = \frac{1}{c} \left(\frac{1 + \kappa u/c}{\kappa + u/c} \right),$$

which simplifies to become

$$\frac{de}{e} = \frac{(\kappa + u/c)du}{c(1 - (u/c)^2)} = \frac{1}{2c} \left(\frac{(\kappa + 1)du}{(1 - (u/c))} + \frac{(\kappa - 1)du}{(1 + (u/c))} \right),$$

and this integrates to give

$$e(u) = \frac{e_0}{(1 - (u/c)^2)^{1/2}} \left(\frac{1 + (u/c)}{1 - (u/c)} \right)^{\kappa/2}, \quad (3.2)$$

where as usual $e_0 = m_0 c^2$ denotes the rest energy, and evidently the Einstein variation arises from the special case $\kappa = 0$.

In terms of the angle θ defined by (2.6) in which the Lorentz invariance appears through a translational invariance, we have the following expressions:

$$e(u) = \frac{e_0}{(1 - (u/c)^2)^{1/2}} e^{\kappa\theta} = e_0 \cosh \theta e^{\kappa\theta} = \frac{e_0}{2} \left(e^{(\kappa+1)\theta} + e^{(\kappa-1)\theta} \right), \quad (3.3)$$

and the following relations also apply

$$e(u) - cp(u) = e_0 e^{(\kappa-1)\theta}, \quad e(u) + cp(u) = e_0 e^{(\kappa+1)\theta},$$

so that in this case we have

$$e(u)^2 - (cp(u))^2 = e_0^2 e^{2\kappa\theta} = e_0^2 \left(\frac{1 + (u/c)}{1 - (u/c)} \right)^\kappa.$$

With the angles $(\Theta, \theta, \epsilon)$ defined by (2.5) and the Lorentz invariance represented by the translation $\Theta = \theta + \epsilon$, it is clear from (3.3)₁ that the energy–momentum relations (2.9) remain properly Lorentz invariant, noting, however, that the rest mass as perceived from the reference frame becomes $E_0 = e_0 e^{-\kappa\epsilon}$.

Specifically, from the inverse energy–momentum relation (2.9)₂, we have

$$E = \frac{e + pv}{[1 - (v/c)^2]^{1/2}} = \frac{e_0 e^{\kappa\theta} (1 + uv/c^2)}{(1 - (u/c)^2)^{1/2} (1 - (v/c)^2)^{1/2}},$$

and on using (2.2) to replace u in the denominator, we might deduce

$$E = \frac{e_0 e^{\kappa\theta} (1 - (v/c)^2)}{(1 - (u/c)^2)^{1/2} (1 - (v/c)^2)^{1/2} (1 - Uv/c^2)} = \frac{e_0 e^{-\kappa\epsilon} e^{\kappa\Theta}}{(1 - (U/c)^2)^{1/2}}, \quad (3.4)$$

demonstrating that E has the same dependence on U as e has on u , except that the rest energy $E_0 = e_0 e^{-\kappa\epsilon}$ which is dependent on the constant relative frame velocity v through $\epsilon = \tanh^{-1}(v/c)$, thus

$$E_0 = e_0 e^{-\kappa\epsilon} = e_0 \left(\frac{1 - (v/c)}{1 + (v/c)} \right)^{\kappa/2}. \quad (3.5)$$

We comment that the final line of (3.4) follows from Eq. (2.3), and that a similar calculation applies to the inverse energy–momentum relation (2.9)₁.

We observe that if we require $e(u) = e(-u)$, then necessarily $\kappa = 0$, but note that conventionally this requirement does not hold for light for which the de Broglie relations become $e = \pm pc$, dependent upon the direction. For $\kappa \neq 0$ Eqs. (3.2) and (3.5) impinge on one of the most basic postulates in special relativity relating to the assumed isotropy of space. These equations predict that for $\kappa \neq 0$, the rest mass values will vary with the direction of motion, namely two different values are obtained for positive and negative velocities v . While numerous experiments have been undertaken aimed at testing such hypothesis, and all indicate the veracity of the assumed isotropy of space, nevertheless the validity or otherwise of (3.5) might only be properly tested in those situations for which both rest masses m_0 and M_0 are non-zero and the fraction $(1 + v/c)/(1 - v/c)$ significantly differs from unity. Accordingly, any test must involve speeds close to the speed of light but involving finite (non-zero) rest masses which therefore excludes those tests dealing with light such as the Michelson–Morley experiments [6].

It might also be worth noting that since it is generally believed that black-holes exist at the centres of galaxies, then space–time must be intrinsically anisotropic in some sense. It is conceivable that space–time is anisotropic at galactic scales and possibly a massive black hole might cause particle rest mass to depend on the direction of particle velocity. This might be the case when the black hole is not at rest in the rest frame of the cosmic microwave background. The cosmic microwave background does have a sizeable dipole component, and its rest frame is measured to be travelling at 627 km/s relative to the centre of mass of our galaxy group (see the mini-review of cosmic microwave background by Scott and Smoot [7]).

Further, while the space–time of special relativity is assumed to be isotropic, this is not taken as an assumption in general relativity. The use of the isotropy assumption in cosmology to select the basic models tends to reflect known experimental outcomes rather than being a necessary part of the theory. We further comment that [8] provides a very general approach to mechanical anisotropy in relativistic mechanics which includes the simple model described here, although derived differently.

We comment that the known relations for light arise from $\kappa = 1$ and $\kappa = -1$, and from (3.2) we have, respectively, the following relations:

$$e(u) = \frac{e_0}{1 - (u/c)}, \quad p(u) = \frac{e_0(u/c)}{c(1 - (u/c))}, \quad e(u) - cp(u) = e_0,$$

$$e(u) = \frac{e_0}{1 + (u/c)}, \quad p(u) = \frac{e_0(u/c)}{c(1 + (u/c))}, \quad e(u) + cp(u) = e_0,$$

allowing the possibility of $e = \pm pc + e_0$ with non-zero rest energy e_0 .

Finally, we comment that it is a very curious fact that both the conventional Einstein energy–mass expression $e = e_0/(1 - (u/c)^2)^{1/2}$ and the generalisation derived here (3.2) bear a relationship with certain singular integral equations associated with aero-foil problems, fluid mechanics and punch problems in elasticity, and that this relationship is not some vague intangible connection, but involves an exact correspondence. Linear singular equations arise in many areas of applied mathematics but particularly within fluid and solid mechanics. Specifically, in non-dimensional variables, the formal solution of the singular integral equation of the second kind

$$\phi(x) + \frac{\lambda}{\pi} \int_{-1}^1 \frac{\phi(y)dy}{(x-y)} = g(x), \quad (3.6)$$

is given by

$$\begin{aligned} \phi(x) = & \frac{C\lambda}{\pi(1+\lambda^2)^{1/2}(1-x^2)^{1/2}} \left(\frac{1+x}{1-x} \right)^\gamma + \frac{g(x)}{(1+\lambda^2)} \\ & + \frac{\lambda}{\pi(1+\lambda^2)(1-x^2)^{1/2}} \left(\frac{1+x}{1-x} \right)^\gamma \int_{-1}^1 (1-y^2)^{1/2} \left(\frac{1-y}{1+y} \right)^\gamma \frac{g(y)dy}{(y-x)}, \end{aligned} \quad (3.7)$$

where λ and γ are related by $\lambda = \cot(\pi\gamma)$ and the constant C is defined by

$$C = \int_{-1}^1 \phi(x)dx. \quad (3.8)$$

The singular integral appearing in (3.6) is sometimes referred to as the finite Hilbert transform. There are numerous standard results available such as (see, for example, [2–5] or [9])

$$\begin{aligned} \frac{1}{\pi} \int_{-1}^1 \frac{dy}{(1-y^2)^{1/2}(x-y)} &= 0, \\ \frac{1}{\pi} \int_{-1}^1 \frac{(1-y^2)^{1/2}dy}{(x-y)} &= x, \\ \frac{1}{\pi} \int_{-1}^1 \frac{y(1-y^2)^{1/2}dy}{(x-y)} &= x^2 - \frac{1}{2}, \\ \frac{1}{\pi} \int_{-1}^1 \frac{y^3(1-y^2)^{1/2}dy}{(x-y)} &= x^4 - \frac{x^2}{2} - \frac{1}{8}, \end{aligned}$$

and there are usually other constraints such as $\phi(\pm 1) = 0$ and $g(x)$ is assumed to be an odd function, but since the issue here is the connection with the energy expressions, we do not concern ourselves with such details.

Strictly speaking, the Einstein expression arising from the case $\gamma = 0$ corresponds to $\lambda \rightarrow \infty$ and accordingly arises from the integral equation

$$\frac{1}{\pi} \int_{-1}^1 \frac{\phi(y) dy}{(x-y)} = f(x),$$

which has solution given by [9] (p. 173)

$$\phi(x) = \frac{C}{\pi(1-x^2)^{1/2}} + \frac{1}{\pi(1-x^2)^{1/2}} \int_{-1}^1 (1-y^2)^{1/2} \frac{f(y) dy}{(y-x)}, \quad (3.9)$$

with the constant C as previously given by (3.8) and there are several equivalent expressions available for this solution. We comment that the expression (3.9) formally emerges from (3.7) with $g(x) \approx \lambda f(x)$ and in the limit $\lambda \rightarrow \infty$ and $\gamma \rightarrow 0$. With $x = u/c$, there is an exact correspondence with $e = e_0/(1 - (u/c)^2)^{1/2}$ and Eq. (3.9), and with the new expression (3.2) and (3.7), and both arise as the solution of the homogeneous problem ($f(x) = g(x) = 0$).

4. Conclusions

Assuming $e = mc^2$, $p = mu$, and a Lorentz invariant equation for the energy rate de/dp , one possible simple extension of the conventional Einstein variation of the energy–mass formula is derived. This generalisation involves a new arbitrary constant κ , and provides the simplest one-parameter Lorentz-invariant extension of the Einstein mass–energy relations. For $\kappa \neq 0$, there is space–time anisotropy with the particle having different rest masses in the positive and negative x directions, so that symmetry under Lorentz transformations is taken to be more fundamental than symmetry under reflections and the space–time anisotropy is dipolar with no components in the y and z directions.

The resulting energy–mass formula (3.2), involving an arbitrary constant κ , predicts in particular that the rest masses in the moving and reference frames, respectively, m_0 and M_0 , are related by the equation

$$m_0 = M_0 \left(\frac{1 + (v/c)}{1 - (v/c)} \right)^{\kappa/2}, \quad (4.1)$$

which for $\kappa \neq 0$ therefore impinges on the basic assumption relating to the isotropy of space. If $m_0 = M_0$, then necessarily κ is zero, and for $\kappa \neq 0$ the rest mass values will vary with the direction of motion, namely two different values are obtained for positive and negative velocities v . While numerous experiments have been undertaken aimed at testing such hypothesis, and all indicate the veracity of the assumed isotropy of space–time, nevertheless the validity or otherwise of (4.1) might only be properly tested in those situations for which both rest masses m_0 and M_0 are nonzero and the fraction $(1 + v/c)/(1 - v/c)$ significantly differs from unity.

The correspondence between Einstein’s fundamental energy expression and areas of classical fluid and solid mechanics is very curious to say the least, and perhaps it is just one of those coincidences. However, perhaps also if these inter-connections were properly understood, they might spark the onset of some fundamental revelations in particle physics. In particular, it is natural to pose the question as to what might be the physical meaning for the corresponding energy–mass expressions arising from the above singular integral equations with $f(x), g(x) \neq 0$?

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Conflict of interest The authors declare that they have no conflict of interest.

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