



Modified scattering for the nonlinear nonlocal Schrödinger equation in one-dimensional case

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Abstract. We study the large time asymptotics of solutions to the Cauchy problem for the nonlinear nonlocal Schrödinger equation with critical nonlinearity

$$\begin{cases} i\partial_t (u - \partial_x^2 u) + \partial_x^2 u - a\partial_x^4 u = \lambda |u|^2 u, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $a > \frac{1}{5}$, $\lambda \in \mathbb{R}$. We continue to develop the factorization techniques which was started in papers Hayashi and Naumkin (Z Angew Math Phys 59(6):1002–1028, 2008) for Klein–Gordon, Hayashi and Naumkin (J Math Phys 56(9):093502, 2015) for a fourth-order Schrödinger, Hayashi and Kaikina (Math Methods Appl Sci 40(5):1573–1597, 2017) for a third-order Schrödinger to show the modified scattering of solutions to the equation. The crucial points of our approach presented here are based on the L^2 -boundedness of the pseudodifferential operators.

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1. Introduction

We study the large time asymptotics of solutions to the Cauchy problem for the nonlinear nonlocal Schrödinger equation with a critical nonlinearity in one-dimensional case

$$\begin{cases} i\partial_t (u - \partial_x^2 u) + \partial_x^2 u - a\partial_x^4 u = \lambda |u|^2 u, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \quad (1.1)$$

where $a > \frac{1}{5}$, $\lambda \in \mathbb{R}$. Equation (1.1) can be considered as a particular form of the higher-order nonlinear Schrödinger equation introduced by [30] to describe the nonlinear propagation of pulses through optical fibers. Also it arises in the context of high-speed soliton transmission in long-haul optical communication system [12]. Equation (1.1) represents the propagation of pulses by taking higher dispersion effects into account than those given by the Schrödinger equation (see [14, 22, 27, 31, 34]).

Multiplying Eq. (1.1) by the operator $(1 - \partial_x^2)^{-1}$, we rewrite it in the pseudodifferential form

$$\begin{cases} i\partial_t u - \Lambda u = \lambda (1 - \partial_x^2)^{-1} (|u|^2 u), & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \quad (1.2)$$

where the linear pseudodifferential operator $\Lambda = (1 - \partial_x^2)^{-1} (-\partial_x^2 + a\partial_x^4)$ is characterized by its symbol $\Lambda(\xi) = \frac{\xi^2 + a\xi^4}{1 + \xi^2}$.

As far as we know, there are no results on the large time asymptotics of solutions of the Cauchy problem (1.1). The difficulty of the small data scattering problem lies in the slow time decay rate of the L^∞ -norm of solutions to the linear problem. So the problem on the large time asymptotic behavior becomes more

difficult for low space dimensions and low order of the nonlinearity. Comparing the time decay rates of the main term and the remainder terms of the nonlinearity in (1.1), we find that this equation represents critical behavior for large time. Indeed, below we will prove the modified scattering of solutions to (1.1). On the other hand, the higher-order nonlinear local or homogeneous Schrödinger equations have been widely studied recently. For the local and global well-posedness of the Cauchy problem, we refer to [4, 5, 29] and references cited therein. The dispersive blow-up was obtained in [1], [2]. The Cauchy problem for the higher-order nonlinear Schrödinger equations was intensively studied by many authors. The existence and uniqueness of solutions to (1.1) were proved in [15, 23–26, 28, 33], and the smoothing properties of solutions were studied in [8, 10, 11, 23–26].

In the case of $a = 1$, we find that $\Lambda = -\partial_x^2$, and (1.2) is the well-known cubic nonlinear Schrödinger equation studied by many authors, see [6, 17] and references cited therein. So we exclude the case $a = 1$ from our consideration here. Final value problem for (1.2) with $a = 1$ was studied by Ozawa [32], where the modified wave operator was constructed. Modified scattering of the initial value problem (1.2) with $a = 1$ was obtained in [16] by using *MDFM* decomposition of the free Schrödinger evolution group $U_1(t)$ ($a = 1$) called as the factorization techniques. More precisely, we have the identity

$$\begin{aligned} U_1(t)\phi &= \frac{1}{\sqrt{4\pi it}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4it}} \phi(y) dy = \frac{1}{\sqrt{4\pi it}} e^{-\frac{x^2}{4it}} \int_{\mathbb{R}} e^{-i\frac{xy}{t}} e^{-\frac{y^2}{4it}} \phi(y) dy \\ &= \text{MDFM}\phi, \end{aligned} \tag{1.3}$$

where $M = e^{-\frac{x^2}{4it}}$ is called the multiplication factor, $D\phi = \frac{1}{\sqrt{2it}}\phi\left(\frac{x}{t}\right)$ is called the dilation operator and \mathcal{F} is the Fourier transformation defined by $\mathcal{F}\phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} \phi(y) dy$. The identity $U_1(t) = \text{MDFM}$ was used in [21] to state the relation between the operators $x + 2it\partial_x$, $U_1(t)xU_1(-t)$ and $Mit\partial_x\bar{M}$. In [16], we have used the identity

$$\begin{aligned} \mathcal{F}U_1(-t)|u|^2 u &= \mathcal{F}U_1(-t)|U_1(t)U_1(-t)u|^2 U_1(t)U_1(-t)u \\ &= \mathcal{F}\bar{M}\mathcal{F}^{-1}D^{-1}\bar{M}|MDFMU_1(-t)u|^2 MDFMU_1(-t)u \\ &= \frac{1}{2t}\mathcal{F}\bar{M}\mathcal{F}^{-1}|\mathcal{F}MU_1(-t)u|^2 \mathcal{F}MU_1(-t)u, \end{aligned}$$

to decompose the cubic nonlinearity in the form of the sum of the main and the remainder terms

$$\frac{1}{2t}|\varphi|^2\varphi + R, \text{ with } \varphi = \mathcal{F}U(-t)u$$

and it was shown in [16] that R is the remainder term with respect to time decay. However, this method cannot be applied directly to other dispersive equations since we do not know an explicit representation for the free evolution group. Later we have used the decomposition of free evolution group (also called the factorization techniques) in the case of the cubic nonlinear Klein–Gordon equation [18], the inhomogeneous fourth-order Schrödinger equation [19] and the third-order Schrödinger equation [20]. In the present paper, our purpose is to develop this approach (the factorization techniques) for the case of the evolution group $U_a(t)$ generated by the nonlocal operator with symbol $\Lambda(\xi) = \frac{\xi^2 + a\xi^4}{1 + \xi^2}$, in order to find the large time asymptotic behavior of solutions. We will decompose the cubic nonlinear term into the main part and a remainder term

$$\mathcal{F}U_a(-t)|u|^2 u - \frac{1}{2t}|\varphi_a|^2\varphi_a,$$

where $\varphi_a = \mathcal{F}U_a(-t)u$. In order to prove desired estimates, we use the \mathbf{L}^2 -boundedness of the pseudo-differential operators. This is a crucial point of our approach presented here. Our method can be applied to higher-dimensional cases which will be considered in a separate paper.

To state our results precisely, we introduce *Notation and Function Spaces*. $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$ is the usual Lebesgue space with norm $\|\phi\|_{\mathbf{L}^p} = \left(\int_{\mathbb{R}} |\phi(x)|^p dx\right)^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \sup_{x \in \mathbb{R}} |\phi(x)|$ for $p = \infty$. The weighted Sobolev space is

$$\mathbf{H}_p^{m,s} = \left\{ \varphi \in \mathbf{S}'; \|\varphi\|_{\mathbf{H}_p^{m,s}} = \|\langle x \rangle^s \langle i\partial_x \rangle^m \varphi\|_{\mathbf{L}^p} < \infty \right\},$$

with $m, s \in \mathbb{R}$, $1 \leq p \leq \infty$, $\langle x \rangle = \sqrt{1+x^2}$, $\langle i\partial_x \rangle = \sqrt{1-\partial_x^2}$. Below \mathcal{F} stands for the Fourier transform $\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \phi(x) dx$, and \mathcal{F}^{-1} is the inverse Fourier transformation $\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \phi(\xi) d\xi$. We also use the notations $\mathbf{H}^{m,s} = \mathbf{H}_2^{m,s}$, $\mathbf{H}^m = \mathbf{H}^{m,0}$. Let $\mathbf{C}(\mathbf{I}; \mathbf{B})$ be the space of continuous functions from the time interval \mathbf{I} to a Banach space \mathbf{B} . Define the free evolution group $\mathcal{U}(t) = \mathcal{F}^{-1} e^{-it\Lambda(\xi)} \mathcal{F}$, we redefine the dilation operator $\mathcal{D}_t \phi(x) = t^{-\frac{1}{2}} \phi\left(\frac{x}{t}\right)$, the scaling operator $(\mathcal{B}\phi)(x) = \phi(\mu(x))$, and the multiplication factor $M = e^{-it(\Lambda(\eta) - \eta\Lambda'(\eta))}$, where $\mu(x)$ is defined as a root of equation

$$\Lambda'(\xi) = \frac{2\xi(1 + 2a\xi^2 + a\xi^4)}{(1 + \xi^2)^2} = x.$$

We note here that M is different from the one used previously in (1.3). We use the same notation for simplicity.

We are now in a position to state the main result of this paper.

Theorem 1.1. *Let the initial data $u_0 \in \mathbf{H}^1 \cap \mathbf{H}^{0,1}$ and $a > \frac{1}{5}$. Assume that the norm $0 < \|u_0\|_{\mathbf{H}^1 \cap \mathbf{H}^{0,1}} \leq \varepsilon$. Then, there exists an ε such that (1.1) has a unique global solution $u \in \mathbf{C}([0, \infty); \mathbf{H}^1 \cap \mathbf{H}^{0,1})$. Moreover, there exists a unique modified scattering state $W_+ \in \mathbf{L}^\infty$ such that the asymptotics*

$$u(t) = \mathcal{D}_t \mathcal{B} M \frac{W_+}{\sqrt{i\Lambda''}} e^{-\frac{i\lambda}{\langle \xi \rangle^{2\Lambda''}} |W_+|^2 \log t} + O\left(t^{-\frac{1}{2}-\delta}\right) \tag{1.4}$$

is valid for $t \rightarrow \infty$ uniformly with respect to $x \in \mathbb{R}$, where $\delta > 0$.

Remark 1.1. Large time asymptotics (1.4) can be written more explicitly in the following form

$$u(t, x) = \frac{W_+(\mu(\frac{x}{t})) e^{-it(\Lambda(\mu(\frac{x}{t})) - \mu(\frac{x}{t})\Lambda'(\mu(\frac{x}{t})))} e^{-\frac{i\lambda \log t}{\langle \mu(\frac{x}{t}) \rangle^{2\Lambda''(\mu(\frac{x}{t}))}} |W_+(\mu(\frac{x}{t}))|^2}{\sqrt{it\Lambda''(\mu(\frac{x}{t}))}} + O\left(t^{-\frac{1}{2}-\delta}\right).$$

We note that the main term of the asymptotics differs from the corresponding linear case by the logarithmic oscillation which vanishes in the case of $\lambda = 0$.

We organize the rest of our paper as follows. In Sect. 2, we formulate the factorization techniques. We prove the estimates of the defect operators in the uniform metrics and obtain the estimates for derivatives of the defect operator and its adjoint by applying the \mathbf{L}^2 -boundedness results for pseudodifferential operators. We estimate the nonlinearity in Sect. 3. Section 4 is devoted to the proof of a priori estimates of solutions to the Cauchy problem (1.1) in the norm

$$\|u\|_{\mathbf{X}_T} = \sup_{t \in [0, T]} \left(\|\widehat{\varphi}(t)\|_{\mathbf{L}^\infty} + (1+t)^{-\gamma} \|\widehat{\varphi}(t)\|_{\mathbf{H}^1 \cap \mathbf{H}^{0,1}} \right), \tag{1.5}$$

where $\widehat{\varphi}(t) = \mathcal{F}\mathcal{U}(-t)u(t)$, $\gamma > 0$ is small depending on the size of the data. Finally, we prove Theorem 1.1 in Sect. 5.

2. Preliminaries

2.1. Factorization techniques

Denote the symbol $\Lambda(\xi) = \frac{\xi^2 + a\xi^4}{1 + \xi^2}$, then the free evolution group has the form $\mathcal{U}(t) = \mathcal{F}^{-1}e^{-it\Lambda(\xi)}\mathcal{F}$. We have

$$\begin{aligned} \mathcal{U}(t)\mathcal{F}^{-1}\phi &= \mathcal{F}^{-1}e^{-it\Lambda(\xi)}\phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi - it\Lambda(\xi)}\phi(\xi) \, d\xi \\ &= \frac{1}{\sqrt{2\pi}} M \int_{\mathbb{R}} e^{ix\xi - it(\Lambda(\xi) - \Lambda(\eta) + \eta\Lambda'(\eta))}\phi(\xi) \, d\xi \end{aligned}$$

where

$$M = e^{-it(\Lambda(\eta) - \eta\Lambda'(\eta))}.$$

By a direct calculation

$$\Lambda''(\xi) = \frac{2(1 + (6a - 3)\xi^2 + 3a\xi^4 + a\xi^6)}{(1 + \xi^2)^3}.$$

It is clear that $\Lambda''(\xi) > 0$ if $a \geq \frac{1}{2}$. We consider the case $0 \leq a < \frac{1}{2}$. We put

$$F(x) = 1 + (6a - 3)x + 3ax^2 + ax^3, \quad x = \xi^2 \geq 0,$$

then

$$F'(x) = (6a - 3) + 6ax + 3ax^2$$

and

$$F'(x) < 0 \text{ for } 0 < x < x_a, F'(x) > 0 \text{ for } x > x_a, \text{ and } F'(x_a) = 0$$

where $x_a = \sqrt{\frac{1}{a} - 1} - 1 \geq 0$. Therefore, $F(x) \geq F(x_a)$ for any $x \geq 0$. By a simple computation, we have

$$F(x_a) = 3a - 1 + \sqrt{a - a^2} > 0$$

if $a > \frac{1}{5}$. Therefore under the condition $a > \frac{1}{5}$, we have $\Lambda''(\xi) > 0$ for all $\xi \in \mathbb{R}$. This guaranties $\Lambda'(\xi)$ is monotone increasing function and the unique stationary point $\mu(x)$, defined as a root of equation

$$\Lambda'(\xi) = \frac{2\xi(1 + 2a\xi^2 + a\xi^4)}{(1 + \xi^2)^2} = x$$

for all $x \in \mathbb{R}$. Hence, we have

$$\begin{aligned} \mathcal{U}(t)\mathcal{F}^{-1}\phi &= \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} \mathcal{D}_t e^{-it(\Lambda(\mu) - \mu\Lambda'(\mu))} \int_{\mathbb{R}} e^{itx\xi - it(\Lambda(\xi) - \Lambda(\mu) + \mu\Lambda'(\mu))}\phi(\xi) \, d\xi \\ &= \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} \mathcal{D}_t e^{-it(\Lambda(\mu) - \mu\Lambda'(\mu))} \int_{\mathbb{R}} e^{it\xi\Lambda'(\mu) - it(\Lambda(\xi) - \Lambda(\mu) + \mu\Lambda'(\mu))}\phi(\xi) \, d\xi \\ &= \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} \mathcal{D}_t e^{-it(\Lambda(\mu) - \mu\Lambda'(\mu))} \int_{\mathbb{R}} e^{-it(\Lambda(\xi) - \Lambda(\mu) - (\xi - \mu)\Lambda'(\mu))}\phi(\xi) \, d\xi \end{aligned}$$

$$= \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} \mathcal{B} \mathcal{D}_t M \int_{\mathbb{R}} e^{-it(\Lambda(\xi) - \Lambda(\eta) - (\xi - \eta)\Lambda'(\eta))} \phi(\xi) d\xi.$$

Now we write the factorization formula $\mathcal{U}(t) \mathcal{F}^{-1} \phi = \mathcal{D}_t \mathcal{B} M \mathcal{Q} \phi$, where $\mathcal{D}_t \phi = t^{-\frac{1}{2}} \phi\left(\frac{x}{t}\right)$, $(\mathcal{B}\phi)(x) = \phi(\mu)$, and the defect operator

$$\mathcal{Q}(t) \phi = \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi, \eta)} \phi(\xi) d\xi$$

with phase function $S(\xi, \eta) = \Lambda(\xi) - \Lambda(\eta) - \Lambda'(\eta)(\xi - \eta)$. Also we need the representation for the inverse evolution group $\mathcal{F}\mathcal{U}(-t) \phi = \mathcal{Q}^* \bar{M} \mathcal{B}^{-1} \mathcal{D}_t^{-1}$, where $\mathcal{D}_t^{-1} \phi = t^{\frac{1}{2}} \phi(xt)$, $\mathcal{B}^{-1} \phi = \phi(\Lambda'(\eta))$ and the adjoint defect operator

$$\mathcal{Q}^*(t) \phi = \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{itS(\xi, \eta)} \phi(\eta) \Lambda''(\eta) d\eta.$$

Indeed we have

$$\begin{aligned} \mathcal{Q}^* \mathcal{Q} \phi &= \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{itS(\xi, \eta)} (\mathcal{Q}\phi)(\eta) \Lambda''(\eta) d\eta \\ &= \frac{t}{2\pi} \int_{\mathbb{R}} e^{itS(\xi, \eta)} \int_{\mathbb{R}} e^{-itS(\tilde{\xi}, \eta)} \phi(\tilde{\xi}) d\tilde{\xi} d\Lambda'(\eta) \\ &= \frac{t}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{it(\Lambda(\xi) - \Lambda(\tilde{\xi})) + itx(\xi - \tilde{\xi})} \phi(\tilde{\xi}) d\tilde{\xi} dx \\ &= \frac{1}{2\pi} e^{it\Lambda(\xi)} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix(\xi - \tilde{\xi})} e^{-it\Lambda(\tilde{\xi})} \phi(\tilde{\xi}) d\tilde{\xi} dx \\ &= \phi(\xi). \end{aligned}$$

Define the new dependent variable $\hat{\varphi} = \mathcal{F}\mathcal{U}(-t) u(t)$. Since $\mathcal{F}\mathcal{U}(-t) \mathcal{L} = i\partial_t \mathcal{F}\mathcal{U}(-t)$, where $\mathcal{L} = i\partial_t - \Lambda$, applying the operator $\mathcal{F}\mathcal{U}(-t)$ to Eq. (1.2), we get

$$i\partial_t \hat{\varphi} = \lambda \langle \xi \rangle^{-2} t^{-1} \mathcal{Q}^* \left(|v|^2 v \right), \quad (2.1)$$

where $v = \mathcal{Q}\hat{\varphi}$. This is our target equation. Our function space is based on the norm defined by (1.5).

For the convenience of the reader, we now state our strategy of the proof of the theorem briefly. By using the stationary phase method and integration by parts, we estimate $v = \mathcal{Q}\hat{\varphi}$ as follows

$$v(\eta) = (\mathcal{Q}\hat{\varphi}(\xi))(\eta) = \frac{1}{\sqrt{i\Lambda''(\eta)}} \hat{\varphi}(\eta) + Ct^{-\frac{1}{4}} \|\partial_\xi \hat{\varphi}\|_{\mathbf{L}^2} \quad (2.2)$$

in Lemma 2.1 (estimate of the defect operator). This estimate will be justified if we could be able to obtain the estimate of $\|\partial_\xi \hat{\varphi}\|_{\mathbf{L}^2}$, which requires us to estimate the norm $\left\| \partial_\xi \mathcal{Q}^* \left(|v|^2 v \right) \right\|_{\mathbf{L}^2}$ via Eq. (2.1).

In Lemma 2.6 we show that

$$\left\| \partial_\xi \mathcal{Q}^* \left(|v|^2 v \right) \right\|_{\mathbf{L}^2} \leq C \left\| |v|^2 v \right\|_{\mathbf{H}^1} \leq C \|v\|_{\mathbf{L}^\infty}^2 \|v\|_{\mathbf{H}^1}.$$

In order to get the estimate of $\|\hat{\varphi}\|_{\mathbf{L}^\infty}$, in Lemma 2.2 we prove the estimate of $\|\mathcal{Q}^* \phi\|_{\mathbf{L}^\infty}$ (estimate of the adjoint defect operator) with $\phi = |v|^2 v$, such as

$$\begin{aligned} (\mathcal{Q}^* \phi(\eta))(\xi) &= \sqrt{i\Lambda''(\xi)} \phi(\xi) + Ct^{-\frac{1}{4}} \|\partial_\eta \phi\|_{\mathbf{L}^2} \\ &= \sqrt{i\Lambda''(\xi)} \phi(\xi) + Ct^{-\frac{1}{4}} \|v\|_{\mathbf{L}^\infty}^2 \|\partial_\eta v\|_{\mathbf{L}^2}. \end{aligned}$$

This can be considered as the adjoint estimate to (2.2). Furthermore we show the estimate for $\|\partial_\eta v\|_{\mathbf{L}^2} = \|\partial_\eta \mathcal{Q}\widehat{\phi}\|_{\mathbf{L}^2}$ in Lemma 2.5, given by

$$\|\partial_\eta \mathcal{Q}\widehat{\phi}\|_{\mathbf{L}^2} \leq C \|\widehat{\phi}\|_{\mathbf{H}^1}.$$

Therefore, the crucial estimates are the following

$$\|\partial_\xi \mathcal{Q}^* \phi\|_{\mathbf{L}^2} \leq C \|\phi\|_{\mathbf{H}^1}, \|\partial_\eta \mathcal{Q}\widehat{\phi}\|_{\mathbf{L}^2} \leq C \|\widehat{\phi}\|_{\mathbf{H}^1}.$$

In order to get these estimates, we use the \mathbf{L}^2 - boundedness of pseudodifferential operators given by Lemma 2.3 and Lemma 2.4.

2.2. Estimate for the defect operator in the uniform norm

Consider the kernel

$$A(t, \eta) = \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi, \eta)} d\xi.$$

To compute the asymptotics of A for large time, we apply the stationary phase method (see [13], p. 110) to find

$$\int_{\mathbb{R}} e^{itg(y)} f(y) dy = e^{itg(y_0)} f(y_0) \sqrt{\frac{2\pi}{t|g''(y_0)|}} e^{i\frac{\pi}{4} \text{sgn} g''(y_0)} + O\left(t^{-\frac{3}{2}}\right) \tag{2.3}$$

for $t \rightarrow \infty$, where the stationary point y_0 is defined by the equation $g'(y_0) = 0$. By virtue of formula (2.3), we get

$$A(t, \eta) = \frac{1}{\sqrt{i\Lambda''(\eta)}} (1 + O(t^{-1}))$$

as $t \rightarrow \infty$.

In the next lemma, we estimate the defect operator \mathcal{Q} in the uniform norm.

Lemma 2.1. *The estimate $|\mathcal{Q}\phi - A\phi| \leq Ct^{-\frac{1}{4}} \|\partial_\xi \phi\|_{\mathbf{L}^2}$ is valid for all $t \geq 1$.*

Proof. By a simple calculation, we have

$$\mathcal{Q}\phi - A\phi = \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi, \eta)} (\phi(\xi) - \phi(\eta)) d\xi.$$

Integration by parts via identity

$$e^{-itS(\xi, \eta)} = H_1 \partial_\xi \left((\xi - \eta) e^{-itS(\xi, \eta)} \right),$$

where

$$H_1 = (1 - it(\xi - \eta) \partial_\xi S(\xi, \eta))^{-1},$$

yields

$$\begin{aligned} \mathcal{Q}\phi - A\phi &= Ct^{\frac{1}{2}} \int_{\mathbb{R}} e^{-itS(\xi, \eta)} (\phi(\xi) - \phi(\eta)) (\xi - \eta) \partial_\xi H_1 d\xi \\ &+ Ct^{\frac{1}{2}} \int_{\mathbb{R}} e^{-itS(\xi, \eta)} (\xi - \eta) H_1 \partial_\xi \phi(\xi) d\xi. \end{aligned}$$

Note that $\Lambda''(\xi) = O(1)$. Also we have

$$\partial_\xi S(\xi, \eta) = \Lambda'(\xi) - \Lambda'(\eta) = (\xi - \eta) \int_0^1 \Lambda''(\eta + (\xi - \eta)z) dz = O(\xi - \eta).$$

Hence the estimate follows

$$|H_1| + |(\xi - \eta) \partial_\xi H_1| \leq \frac{C}{1 + t(\xi - \eta)^2}.$$

Therefore by Hardy and Cauchy–Schwarz inequalities, we obtain

$$\begin{aligned} |\mathcal{Q}\phi - A\phi| &\leq Ct^{\frac{1}{2}} \int_{\mathbb{R}} \left(\frac{|\phi(\xi) - \phi(\eta)|}{|\xi - \eta|} + |\partial_\xi \phi(\xi)| \right) \frac{|\xi - \eta| d\xi}{1 + t(\xi - \eta)^2} \\ &\leq Ct^{\frac{1}{2}} \|\partial_\xi \phi\|_{\mathbf{L}^2} \left(\int_{\mathbb{R}} \frac{(\xi - \eta)^2 d\xi}{(1 + t(\xi - \eta)^2)^2} \right)^{\frac{1}{2}} \leq Ct^{-\frac{1}{4}} \|\partial_\xi \phi\|_{\mathbf{L}^2}. \end{aligned}$$

Lemma 2.1 is proved. □

2.3. Estimate for the adjoint defect operator in the uniform norm

We consider the kernel

$$A^*(t, \xi) = \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{itS(\xi, \eta)} \Lambda''(\eta) d\eta.$$

By virtue of formula (2.3), we obtain the large time asymptotics

$$A^*(t, \xi) = \sqrt{i\Lambda''(\xi)} (1 + O(t^{-1}))$$

for $t \rightarrow \infty$. In the next lemma, we estimate the adjoint defect operator \mathcal{Q}^* in the uniform norm.

Lemma 2.2. *The estimate $|\mathcal{Q}^*\phi - A^*\phi| \leq Ct^{-\frac{1}{4}} \|\partial_\eta \phi\|_{\mathbf{L}^2}$ is valid for all $t \geq 1$.*

Proof. As above, we integrate by parts via the identity

$$e^{itS(\xi, \eta)} = H_2 \partial_\eta \left((\eta - \xi) e^{itS(\xi, \eta)} \right),$$

with

$$H_2 = (1 + it(\eta - \xi) \partial_\eta S(\xi, \eta))^{-1}$$

and $\partial_\eta S(\xi, \eta) = \Lambda''(\eta)(\eta - \xi) = O(\eta - \xi)$. Then, we find

$$\begin{aligned} \mathcal{Q}^*\phi - A^*\phi &= \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{itS(\xi, \eta)} (\phi(\eta) - \phi(\xi)) \Lambda''(\eta) d\eta \\ &= Ct^{\frac{1}{2}} \int_{\mathbb{R}} e^{itS(\xi, \eta)} \frac{\phi(\eta) - \phi(\xi)}{\eta - \xi} (\eta - \xi)^2 \partial_\eta (H_2 \Lambda''(\eta)) d\eta \\ &\quad + Ct^{\frac{1}{2}} \int_{\mathbb{R}} e^{itS(\xi, \eta)} (\eta - \xi) H_2 \Lambda''(\eta) \partial_\eta \phi(\eta) d\eta. \end{aligned}$$

From which via the estimate

$$|H_2\Lambda''(\eta)| + |(\eta - \xi) \partial_\eta (H_2\Lambda''(\eta))| \leq \frac{C}{1 + t(\eta - \xi)^2},$$

and Hardy and Cauchy–Schwarz inequalities, we obtain

$$|\mathcal{Q}^*\phi - A^*\phi| \leq Ct^{\frac{1}{2}} \|\partial_\eta\phi\|_{\mathbf{L}^2} \left(\int_{\mathbb{R}} \frac{(\eta - \xi)^2 d\eta}{(1 + t(\eta - \xi)^2)^2} \right)^{\frac{1}{2}} \leq Ct^{-\frac{1}{4}} \|\partial_\eta\phi\|_{\mathbf{L}^2}.$$

Lemma 2.2 is proved. □

2.4. Boundedness of pseudodifferential operators

There are many papers devoted to the \mathbf{L}^2 estimates of pseudodifferential operators (see, e.g. [3, 7, 9]). Below we will use the following result on the \mathbf{L}^2 boundedness of pseudodifferential operator

$$\mathbf{a}(x, \mathbf{D})\phi \equiv \int_{\mathbb{R}} e^{ix\xi} \mathbf{a}(x, \xi) \widehat{\phi}(\xi) d\xi$$

(see [9]).

Lemma 2.3. *Let the symbol $\mathbf{a}(x, \xi)$ be such that*

$$\sup_{x, \xi \in \mathbb{R}} |\partial_x^k \partial_\xi^l \mathbf{a}(x, \xi)| \leq C$$

for $k, l = 0, 1$. Then

$$\|\mathbf{a}(x, \mathbf{D})\phi\|_{\mathbf{L}_x^2} \leq C \|\phi\|_{\mathbf{L}^2}.$$

Analogously for the conjugate pseudodifferential operator

$$\mathbf{a}^*(\xi, \mathbf{D})\phi \equiv \int_{\mathbb{R}} e^{-ix\xi} \mathbf{a}^*(x, \xi) \widehat{\phi}(x) dx$$

we get the following result.

Lemma 2.4. *Let the symbol*

$$\sup_{x, \xi \in \mathbb{R}} |\partial_x^k \partial_\xi^l \mathbf{a}^*(x, \xi)| \leq C$$

for $k, l = 0, 1$. Then

$$\|\mathbf{a}^*(\xi, \mathbf{D})\phi\|_{\mathbf{L}_\xi^2} \leq C \|\phi\|_{\mathbf{L}^2}.$$

2.5. Estimate for derivative of the defect operator

In the next lemma, we estimate a derivative of the defect operator \mathcal{Q} .

Lemma 2.5. *The estimate $\|\partial_\eta \mathcal{Q}\phi\|_{\mathbf{L}^2} \leq C \|\phi\|_{\mathbf{H}^1}$ is true for all $t \geq 1$.*

Proof. We integrate by parts to get

$$\partial_\eta \mathcal{Q}\phi = Ct^{\frac{1}{2}} \int_{\mathbb{R}} e^{-itS(\xi, \eta)} (q_1(\xi, \eta) \partial_\xi \phi(\xi) + q_2(\xi, \eta) \phi(\xi)) d\xi,$$

where

$$q_1(\xi, \eta) = \frac{\partial_\eta S(\xi, \eta)}{\partial_\xi S(\xi, \eta)}$$

and

$$q_2(\xi, \eta) = \partial_\xi \left(\frac{\partial_\eta S(\xi, \eta)}{\partial_\xi S(\xi, \eta)} \right).$$

Then, we obtain

$$\begin{aligned} \partial_\eta \mathcal{Q}\phi &= C\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1} \int_{\mathbb{R}} e^{ix\xi} q_1\left(\xi, \mu\left(\frac{x}{t}\right)\right) e^{-it\Lambda(\xi)} \partial_\xi \phi(\xi) d\xi \\ &+ C\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1} \int_{\mathbb{R}} e^{ix\xi} q_2\left(\xi, \mu\left(\frac{x}{t}\right)\right) e^{-it\Lambda(\xi)} \phi(\xi) d\xi \\ &= C\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1} \mathbf{a}_1(x, \mathbf{D}) \mathcal{F}^{-1} e^{-it\Lambda} \partial_\xi \phi + C\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1} \mathbf{a}_2(x, \mathbf{D}) \mathcal{F}^{-1} e^{-it\Lambda} \phi, \end{aligned}$$

where the symbols $\mathbf{a}_1(x, \xi) = q_1\left(\xi, \mu\left(\frac{x}{t}\right)\right)$ and $\mathbf{a}_2(x, \xi) = q_2\left(\xi, \mu\left(\frac{x}{t}\right)\right)$. Since

$$\mu(x) = O(x), \mu'(x) = \frac{1}{\Lambda''(\mu(x))} = O(1)$$

,

$$q_1(\xi, \eta) = \frac{\Lambda''(\eta)}{\int_0^1 \Lambda''(\eta + (\xi - \eta)z) dz} = O(1)$$

and

$$q_2(\xi, \eta) = \partial_\xi \left(\frac{\Lambda''(\eta)}{\int_0^1 \Lambda''(\eta + (\xi - \eta)z) dz} \right) = O(1),$$

we find

$$\sup_{x, \xi \in \mathbb{R}} |\partial_x^k \partial_\xi^l \mathbf{a}_j(x, \xi)| \leq C \text{ for } k, l = 0, 1, j = 1, 2.$$

Hence by Lemma 2.3, we get

$$\|\mathbf{a}_j(x, \mathbf{D}) \phi\|_{\mathbf{L}_x^2} \leq C \|\phi\|_{\mathbf{L}^2}.$$

Thus in view of equalities

$$\|\mathcal{B}^{-1}\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2}, \|\mathcal{D}_t^{-1}\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2}$$

and $\|\mathcal{F}^{-1}\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2}$, we obtain

$$\begin{aligned} \|\partial_\eta \mathcal{Q}\phi\|_{\mathbf{L}^2_\xi} &\leq C \|\mathcal{B}^{-1} \mathcal{D}_t^{-1} \mathbf{a}_1(x, \mathbf{D}) \mathcal{F}^{-1} e^{-it\Lambda} \partial_\xi \phi\|_{\mathbf{L}^2} \\ &+ C \|\mathcal{B}^{-1} \mathcal{D}_t^{-1} \mathbf{a}_2(x, \mathbf{D}) \mathcal{F}^{-1} e^{-it\Lambda} \phi\|_{\mathbf{L}^2} \leq C \|\partial_\xi \phi\|_{\mathbf{L}^2} + C \|\phi\|_{\mathbf{L}^2}. \end{aligned}$$

Lemma 2.5 is proved. □

2.6. Estimate for derivative of the adjoint defect operator

Next we prove the estimates for derivatives of adjoint defect operator \mathcal{Q}^* .

Lemma 2.6. *The estimate $\|\partial_\xi \mathcal{Q}^* \phi\|_{\mathbf{L}^2} \leq C \|\phi\|_{\mathbf{H}^1}$ is true for all $t \geq 1$.*

Proof. Integrating by parts, we get

$$\partial_\xi \mathcal{Q}^* \phi = Ct^{\frac{1}{2}} \int_{\mathbb{R}} e^{itS(\xi, \eta)} (q_3(\xi, \eta) \partial_\eta \phi(\eta) + q_4(\xi, \eta) \phi(\eta)) \Lambda''(\eta) d\eta,$$

where

$$q_3(\xi, \eta) = \frac{\partial_\xi S(\xi, \eta)}{\partial_\eta S(\xi, \eta)}$$

and

$$q_4(\xi, \eta) = \frac{1}{\Lambda''(\eta)} \partial_\eta \left(\frac{\partial_\xi S(\xi, \eta)}{\partial_\eta S(\xi, \eta)} \Lambda''(\eta) \right).$$

Then changing the variable of integration $\eta = \mu(x)$ and after that $x = \frac{x'}{t}$, we find

$$\begin{aligned} \partial_\xi \mathcal{Q}^* \phi &= Ce^{it\Lambda(\xi)} \int_{\mathbb{R}} e^{-ix\xi} q_3\left(\xi, \mu\left(\frac{x}{t}\right)\right) (\mathcal{D}_t \mathcal{B} M \partial_\eta \phi) dx \\ &+ Ce^{it\Lambda(\xi)} \int_{\mathbb{R}} e^{-ix\xi} q_4\left(\xi, \mu\left(\frac{x}{t}\right)\right) (\mathcal{D}_t \mathcal{B} M \phi) dx. \end{aligned}$$

Denote $\mathbf{a}_3^*(\xi, x) = q_3\left(\xi, \mu\left(\frac{x}{t}\right)\right)$ and $\mathbf{a}_4^*(\xi, x) = q_4\left(\xi, \mu\left(\frac{x}{t}\right)\right)$, then

$$\partial_\xi \mathcal{Q}^* \phi = Ce^{it\Lambda(\xi)} \left(\mathbf{a}_3^*(\xi, \mathbf{D}) \overline{\mathcal{F}^{-1} \mathcal{D}_t \mathcal{B} M \partial_\eta \phi} + \mathbf{a}_4^*(\xi, \mathbf{D}) \overline{\mathcal{F}^{-1} \mathcal{D}_t \mathcal{B} M \phi} \right).$$

We have $\mu(x) = O(x)$, $\mu'(x) = \frac{1}{\Lambda''(\mu(x))} = O(1)$,

$$q_3(\xi, \eta) = \frac{1}{\Lambda''(\eta)} \int_0^1 \Lambda''(\eta + (\xi - \eta)z) dz = O(1)$$

and

$$q_4(\xi, \eta) = \frac{1}{\Lambda''(\eta)} \partial_\eta \int_0^1 \Lambda''(\eta + (\xi - \eta)z) dz = O(1),$$

hence

$$\sup_{x, \xi \in \mathbb{R}} |\partial_x^k \partial_\xi^l \mathbf{a}_j^*(\xi, x)| \leq C$$

for $k, l = 0, 1, j = 3, 4$. Application of Lemma 2.4 yields

$$\|\mathbf{a}_j^*(\xi, \mathbf{D}) \phi\|_{\mathbf{L}^2_\xi} \leq C \|\phi\|_{\mathbf{L}^2}.$$

Then in view of equalities $\|\mathcal{B}\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2}$, $\|\mathcal{D}_t\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2}$ and $\|\mathcal{F}^{-1}\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2}$, we find

$$\begin{aligned} \|\partial_\xi \mathcal{Q}^* \phi\|_{\mathbf{L}_x^2} &\leq C \left\| \mathbf{a}_3^*(\xi, \mathbf{D}) \overline{\mathcal{F}^{-1} \mathcal{D}_t \mathcal{B} M \partial_\eta \phi} \right\|_{\mathbf{L}^2} \\ &+ C \left\| \mathbf{a}_4^*(\xi, \mathbf{D}) \overline{\mathcal{F}^{-1} \mathcal{D}_t \mathcal{B} M \phi} \right\|_{\mathbf{L}^2} \leq C \|\partial_\eta \phi\|_{\mathbf{L}^2} + C \|\phi\|_{\mathbf{L}^2}. \end{aligned}$$

Lemma 2.6 is proved. \square

3. Estimates for the nonlinearity

Define the norm

$$\|u\|_{\mathbf{X}_T} = \sup_{t \in [0, T]} \left(\|\widehat{\varphi}(t)\|_{\mathbf{L}^\infty} + (1+t)^{-\gamma} \|\widehat{\varphi}(t)\|_{\mathbf{H}^1 \cap \mathbf{H}^{0,1}} \right),$$

where $\widehat{\varphi}(t) = \mathcal{F}\mathcal{U}(-t)u(t)$, $\gamma = C\varepsilon > 0$ is small.

3.1. Asymptotics of the nonlinearity

In the next lemma, we calculate the asymptotic representation for the nonlinearity.

Lemma 3.1. *Suppose that $\|u\|_{\mathbf{X}_T} \leq C\varepsilon$. Then the asymptotics*

$$\mathcal{Q}^* |v|^2 v = \frac{1}{\Lambda''} |\widehat{\varphi}|^2 \widehat{\varphi} + O\left(\varepsilon^3 t^{\gamma - \frac{1}{4}}\right)$$

is true for all $t \in [1, T]$, where $v = \mathcal{Q}\widehat{\varphi}$, $\widehat{\varphi}(t) = \mathcal{F}\mathcal{U}(-t)u(t)$.

Proof. Applying Lemma 2.2 with $\phi = |v|^2 v$, we find

$$\mathcal{Q}^* |v|^2 v = \sqrt{i\Lambda''} |v|^2 v + O\left(t^{-1} \|v\|_{\mathbf{L}^\infty}^3\right) + O\left(t^{-\frac{1}{4}} \|v\|_{\mathbf{L}^\infty}^2 \|\partial_\eta v\|_{\mathbf{L}^2}\right).$$

In view of Lemma 2.1, we have the asymptotics $v = \frac{1}{\sqrt{i\Lambda''}} \widehat{\varphi} + O\left(\varepsilon t^{\gamma - \frac{1}{4}}\right)$, and estimate $\|v\|_{\mathbf{L}^\infty} \leq C\varepsilon$. Then by Lemma 2.5 and condition of the lemma we find $\|\partial_\eta v\|_{\mathbf{L}^2} \leq C\varepsilon t^\gamma$. Hence, the result of the lemma follows. Lemma 3.1 is proved. \square

3.2. Estimate for derivative of the nonlinearity in Eq. (2.1).

Lemma 3.2. *Suppose that $\|u\|_{\mathbf{X}_T} \leq C\varepsilon$. Then, the estimate $\left\| \partial_\xi \mathcal{Q}^* |v|^2 v \right\|_{\mathbf{L}^2} \leq C\varepsilon^3 t^\gamma$ is true for all $t \geq 1$, where $v = \mathcal{Q}\widehat{\varphi}$, $\widehat{\varphi}(t) = \mathcal{F}\mathcal{U}(-t)u(t)$, $\gamma = C\varepsilon$.*

Proof. By virtue of Lemma 2.6 with $\phi = |v|^2 v$, we get $\left\| \partial_\xi \mathcal{Q}^* |v|^2 v \right\|_{\mathbf{L}^2} \leq C \|v\|_{\mathbf{L}^\infty}^2 \|v\|_{\mathbf{H}^1}$. Using Lemma 2.1, we find $\|v\|_{\mathbf{L}^\infty} \leq C\varepsilon$. Then by Lemma 2.5, we obtain the result of the lemma. Lemma 3.2 is proved. \square

4. A priori estimate of solutions

We state the local existence of solutions to the Cauchy problem (1.1) in the functional space $\mathbf{H}^1 \cap \mathbf{H}^{0,1}$ which is shown by the well-known contraction mapping principle.

Theorem 4.1. *Assume that the initial data $u_0 \in \mathbf{H}^1 \cap \mathbf{H}^{0,1}$. Then, there exists a time $T > 0$, which depends on the norm $\|u_0\|_{\mathbf{H}^1 \cap \mathbf{H}^{0,1}}$, such that (1.1) has a unique solution $\mathcal{U}(-t)u \in \mathbf{C}([0, T]; \mathbf{H}^1 \cap \mathbf{H}^{0,1})$ such that $\|u\|_{\mathbf{X}_T} < C$. If the norm $\|u_0\|_{\mathbf{H}^1 \cap \mathbf{H}^{0,1}}$ is small, then the existence time $T \geq 1$.*

To prove the global result, we need a priori estimate of the norm $\|u\|_{\mathbf{X}_T}$ uniformly with respect to $T \geq 1$.

Lemma 4.1. *Let the initial data $u_0 \in \mathbf{H}^1 \cap \mathbf{H}^{0,1}$ have a small norm $\|u_0\|_{\mathbf{H}^1 \cap \mathbf{H}^{0,1}}$. Then, the estimate $\|u\|_{\mathbf{X}_T} < C\varepsilon$ is true for all $T \geq 1$.*

Proof. Arguing by the contradiction, we can find the first moment of time $T > 0$, such that $\|u\|_{\mathbf{X}_T} = C\varepsilon$. By Lemma 3.1 and (2.1) we get

$$i\partial_t \widehat{\varphi} = \frac{\lambda}{t \langle \xi \rangle^2 \Lambda''} |\widehat{\varphi}|^2 \widehat{\varphi} + O\left(\langle \xi \rangle^{-2} \varepsilon^3 t^{\gamma - \frac{5}{4}}\right). \tag{4.1}$$

Multiplying (4.1) by $\overline{\widehat{\varphi}}$ and taking the real part of the result, we get

$$\partial_t |\widehat{\varphi}|^2 = O\left(\langle \xi \rangle^{-2} \varepsilon^3 t^{\gamma - \frac{5}{4}}\right) |\widehat{\varphi}|$$

from which it follows that

$$\partial_t |\widehat{\varphi}| = O\left(\langle \xi \rangle^{-2} \varepsilon^3 t^{\gamma - \frac{5}{4}}\right).$$

Hence integration in time yields $|\widehat{\varphi}(t, \xi)| \leq |\widehat{\varphi}(0, \xi)| + C\varepsilon^3 < C\varepsilon$ for all $t \in [1, T]$. Next we estimate the norm $\|\widehat{\varphi}\|_{\mathbf{H}^1}$. We have from (2.1)

$$\frac{d}{dt} \|\widehat{\varphi}(t)\|_{\mathbf{H}^1}^2 \leq Ct^{-1} \left\| \mathcal{Q}^* |v|^2 v \right\|_{\mathbf{H}^1} \|\widehat{\varphi}(t)\|_{\mathbf{H}^1}$$

and by Lemma 3.2

$$\frac{d}{dt} \|\widehat{\varphi}(t)\|_{\mathbf{H}^1} \leq Ct^{-1} \left\| \mathcal{Q}^* |v|^2 v \right\|_{\mathbf{H}^1} \leq C\varepsilon^3 t^{\gamma - 1}.$$

Then integrating in time, we get $\|\widehat{\varphi}(t)\|_{\mathbf{H}^1} \leq \|\widehat{\varphi}(0)\|_{\mathbf{H}^1} + C\varepsilon^3 t^\gamma < C\varepsilon t^\gamma$ for all $t \in [1, T]$. Also multiplying (1.1) by \bar{u} and taking the imaginary part of the result we have

$$\frac{d}{dt} (\|u(t)\|_{\mathbf{L}^2} + \|\partial_x u(t)\|_{\mathbf{L}^2}) = 0.$$

Therefore, integration in time gives us $\|\widehat{\varphi}(t)\|_{\mathbf{H}^{0,1}} = \|u(t)\|_{\mathbf{H}^1} \leq \|u_0\|_{\mathbf{H}^1} < C\varepsilon$. Thus, we get $\|u\|_{\mathbf{X}_T} < C\varepsilon$ for all $T > 1$. We obtain the desired contradiction. Lemma 4.1 is proved. \square

5. Proof of Theorem 1.1

By Lemma 4.1, we see that a priori estimate of the norm $\|u\|_{\mathbf{X}_T} \leq C\varepsilon$ is true for all $T > 0$. Therefore, the global existence of solutions of the Cauchy problem (1.1) satisfying estimate $\|u\|_{\mathbf{X}_\infty} \leq C\varepsilon$ follows by a standard continuation argument and the local existence Theorem 4.1. Now we turn to the proof of the asymptotic formula (1.4) for the solutions u of the Cauchy problem (1.1). As in the proof of Lemma 4.1, we obtain Eq. (4.1)

$$\partial_t \widehat{\varphi} = -\frac{i\lambda}{t \langle \xi \rangle^2 \Lambda''(\xi)} |\widehat{\varphi}|^2 \widehat{\varphi} + O\left(\varepsilon^3 t^{\gamma - \frac{5}{4}}\right).$$

Changing

$$g(t, \xi) = \widehat{\varphi}(t, \xi) \exp \left(\frac{i\lambda}{\langle \xi \rangle^2 \Lambda''(\xi)} \int_1^t |\widehat{\varphi}(\tau, \xi)|^2 \frac{d\tau}{\tau} \right),$$

we get $\partial_t g = O(\varepsilon^3 t^{-1-\delta})$, where $\delta = \frac{1}{4} - \gamma$. Integrating in time, we find

$$\|g(t) - g(s)\|_{\mathbf{L}^\infty} \leq C\varepsilon \int_s^t \tau^{-1-\delta} d\tau \leq C\varepsilon s^{-\delta}$$

for all $t > s > 0$. Therefore, there exists a unique final state $g_+ \in \mathbf{L}^\infty$, such that

$$\|g(t) - g_+\|_{\mathbf{L}^\infty} \leq C\varepsilon t^{-\delta}$$

for all $t > 0$. Denote

$$\Phi(t) = \int_1^t |g(\tau)|^2 \frac{d\tau}{\tau} - |g_+|^2 \log t.$$

We have

$$\Phi(t) - \Phi(s) = \int_s^t (|g(\tau)|^2 - |g(t)|^2) \frac{d\tau}{\tau} + (|g(t)|^2 - |g_+|^2) \log \frac{t}{s}.$$

Hence,

$$\|\Phi(t) - \Phi(s)\|_{\mathbf{L}^\infty} \leq C\varepsilon^2 s^{-\delta}$$

for all $t > s > 0$. Thus, there exists a unique real-valued function Φ_+ , such that $\Phi_+ \in \mathbf{L}^\infty$ and

$$\|\Phi(t) - \Phi_+\|_{\mathbf{L}^\infty} \leq C\varepsilon^2 t^{-\delta}.$$

Therefore, we obtain

$$\int_1^t |\widehat{\varphi}(\tau)|^2 \frac{d\tau}{\tau} = \Phi(t) + |g_+|^2 \log t = \Phi_+ + |g_+|^2 \log t + O(\varepsilon t^{-\delta}).$$

Then, we find the asymptotics

$$\begin{aligned} \widehat{\varphi}(t) &= g_+ \exp \left(-\frac{i\lambda}{\langle \xi \rangle^2 \Lambda''(\xi)} \int_1^t |\widehat{\varphi}(\tau)|^2 \frac{d\tau}{\tau} \right) + O(t^{-\delta}) \\ &= g_+ \exp \left(-\frac{i\lambda}{\langle \xi \rangle^2 \Lambda''} |g_+|^2 \log t - \frac{i\lambda}{\langle \xi \rangle^2 \Lambda''} \Phi_+ \right) + O(t^{-\delta}) \\ &= W_+ \exp \left(-\frac{i\lambda}{\langle \xi \rangle^2 \Lambda''} |W_+|^2 \log t \right) + O(t^{-\delta}), \end{aligned}$$

where

$$W_+ = g_+ \exp \left(-\frac{i\lambda}{\langle \xi \rangle^2 \Lambda''} \Phi_+ \right) \in \mathbf{L}^\infty.$$

Finally using the factorization formulas for $\mathcal{U}(t)$ and the result of Lemma 2.1, we have

$$u(t) = \mathcal{D}_t \mathcal{B} M Q \hat{\varphi} = \mathcal{D}_t \mathcal{B} M \frac{W_+}{\sqrt{i\Lambda''}} \exp\left(-\frac{i\lambda}{\langle \xi \rangle^2 \Lambda''} |W_+|^2 \log t\right) + O\left(t^{-\frac{1}{2}-\delta}\right).$$

This completes the proof of asymptotics (1.4). Theorem 1.1 is proved.

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