Prescribed signal concentration on the boundary: Weak solvability in a chemotaxis-Stokes system with proliferation

Tobias Black and Chunyan Wu

Abstract. We study a chemotaxis-Stokes system with signal consumption and logistic source terms of the form

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c) + \kappa n - \mu n^2, & x \in \Omega, \quad t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nc, & x \in \Omega, \quad t > 0, \\ u_t = \Delta u + \nabla P + n \nabla \phi, & x \in \Omega, \quad t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, \quad t > 0, \\ (\nabla n - n \nabla c) \cdot \nu = 0, \quad c = c_*(x), \quad u = 0, & x \in \partial\Omega, \quad t > 0, \end{cases}$$

where $\kappa \geq 0$, $\mu > 0$ and, in contrast to the commonly investigated variants of chemotaxis-fluid systems, the signal concentration on the boundary of the domain $\Omega \subset \mathbb{R}^N$ with $N \in \{2, 3\}$ is a prescribed time-independent nonnegative function $c_* \in C^2(\overline{\Omega})$. Making use of the boundedness information entailed by the quadratic decay term of the first equation, we will show that the system above has at least one global weak solution for any suitably regular triplet of initial data.

Mathematics Subject Classification. 35D30, 92C17 (primary), 35A01, 35K35, 35K55, 35Q35, 35Q92.

Keywords. Chemotaxis-fluid, Logistic source, Global weak solution, Dirichlet boundary condition.

1. Introduction

Chemotaxis, the oriented movement of bacteria and cells in response to a chemical substance in their surrounding environment, is an important motility scheme in nature. An interesting facet of colonies of such chemotactically active bacteria and cells consists of the possibility to spontaneously generate spatial patterns, as not only witnessed by the experimental findings on the aerobic *Bacillus subtilis* [6, 10, 23] but also in settings where the attracting signal is produced by the cells themselves [14, 45]. This emergence of spatial structures, captivating biologists and mathematicians alike, has led to an intensive study of chemotaxis systems in the past decades and is still garnering attention in the field of mathematical modeling and analysis. (See also the surveys [1, 13, 20].)

In order to study the plume-like aggregation patterns observed to occur when a population of *Bacillus subtilis* is suspended in a drop of water, the authors of [33] proposed a model of the form

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c), & x \in \Omega, \quad t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nc, & x \in \Omega, \quad t > 0, \\ u_t + (u \cdot \nabla)u = \Delta u + \nabla P + n \nabla \phi, & x \in \Omega, \quad t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, \quad t > 0, \end{cases} \quad (1.1)$$

where n, c, u, P denote the density of the bacteria, the oxygen concentration, the velocity field of the incompressible fluid, and the associated pressure, respectively, ϕ is a prescribed gravitational potential and Ω is a bounded domain in $\mathbb{R}^N$. While the authors of [33] suggest to augment the system with a nonzero Dirichlet boundary condition for the chemical at the stress-free fluid–air interface and a no-flux

condition for the bacteria (in fact they even propose mixed boundary conditions distinguishing between the bottom layer of the drop and the fluid–air interface), a large part of the literature on chemotaxis–fluid systems only considers no-flux conditions for both n and c and a no-slip condition for u .

In this setting, the global solvability of (1.1) is well studied and most of the remaining problems remain in the case of $N = 3$. Actually, for $N = 2$ global classical solutions and their stabilization properties have been established in [38] and [39], respectively, whereas, in the higher-dimensional setting, it was shown in [42, 43] that (1.1) possesses at least one global weak solution, which becomes smooth after some possibly large waiting time. A recent study by the same author also reveals that on small time-scales (possible) singularities can only arise in a set of measure zero [44]. Similar results have also been established in models where the bacteria are assumed to obey a logistic population growth (i.e., including the term $+\kappa n - \mu n^2$ on the right-hand side of the first equation). In fact, existence of weak solutions was shown in [34] and [19] considers the eventual smoothness of weak solutions in 3D. Analytical results providing pattern formation as discovered in the experiments, however, are still missing, which raised the question whether the assumed boundary conditions should be adjusted for further advances.

Under consideration of different boundary conditions, the knowledge of (1.1) is quite enigmatic, with most of the current results on existence theory only discussing the two-dimensional setting or relying on the inclusion of small changes to (1.1), like logistic growth terms, an enhanced diffusion rate for the bacteria or the consideration of Stokes fluid (i.e., dropping $(u \cdot \nabla)u$ in the third equation) and even then solutions can often only be obtained with quite mild regularity. In this regard, the work [2] contains the most intricate result in this direction, with the treatment of (1.1) with logistic growth terms under the Robin boundary condition $\frac{\partial c}{\partial \nu} = 1 - c$ on $\partial\Omega$. The author proves the existence of global classical solutions in 2D and global weak solutions in 3D. Additional results featuring a Robin boundary condition in fluid-free (i.e., $u \equiv 0$) variants of (1.1) have been investigated in [3] and [9]. The former considers a stationary (and hence doubly elliptic) system and establishes existence and uniqueness of a classical solution for any prescribed mass $M := \int_{\Omega} n > 0$. The latter studies a parabolic-elliptic variant and attains results on global and bounded classical solutions and their long-term behavior. Concerning nonzero Dirichlet data for c , we are only aware of two works. The first proves global existing generalized solutions in 3D for the Stokes variant of (1.1) [36] and the second provides global generalized solutions for $N \geq 2$ in a Stokes variant of (1.1) with nonlinear diffusion satisfying $m \geq 1$ for $N = 2$ and $m > \frac{3N-2}{2N}$ if $N \geq 3$ [35]. Results on more regular solutions and included logistic population growth appear to be missing for the Dirichlet boundary data case.

(See also [22] and [25, 26] for first analytical results concerning well-posedness of systems closely related to (1.1) with mixed boundary conditions, [15] for a more general fluid-free one-dimensional system with nonzero Dirichlet or Neumann boundary data and [5, 21, 33] for numerical studies related to (1.1).)

Main results. Motivated by the observations above, we are going to consider a chemotaxis–Stokes system with logistic population growth of the form

$$\left\{ \begin{array}{lll} n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \nabla c) + \kappa n - \mu n^2, & x \in \Omega, \quad t > 0, \\ c_t + u \cdot \nabla c &= \Delta c - nc, & x \in \Omega, \quad t > 0, \\ u_t &= \Delta u + \nabla P + n \nabla \phi, & x \in \Omega, \quad t > 0, \\ \nabla \cdot u &= 0, & x \in \Omega, \quad t > 0, \\ (\nabla n - n \nabla c) \cdot \nu &= 0, \quad c = c_*(x), \quad u = 0, & x \in \partial\Omega, \quad t > 0, \\ n(\cdot, 0) &= n_0, \quad c(\cdot, 0) = c_0, \quad u(\cdot, 0) = u_0 & x \in \Omega, \end{array} \right. \quad (1.2)$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^N$ with $N \in \{2, 3\}$ and ν denoting the outward normal vector field on $\partial\Omega$. We prescribe $\kappa \geq 0$, $\mu > 0$, a time constant function c_* satisfying

$$c_* \in C^2(\overline{\Omega}) \quad \text{with} \quad c_* \geq 0, \quad (1.3)$$

a gravitational potential function ϕ fulfilling

$$\phi \in W^{2,\infty}(\Omega) \quad (1.4)$$

and initial data (n_0, c_0, u_0) satisfying

$$\begin{cases} n_0 \in C^0(\overline{\Omega}) & \text{is nonnegative with } n_0 \not\equiv 0, \\ c_0 \in W^{1,q}(\Omega) & \text{is positive in } \Omega \text{ with } c_0 = c_* \text{ on } \partial\Omega, \\ u_0 \in D(A^\varrho) \end{cases} \quad (1.5)$$

with $q > N$, $\varrho \in (\frac{N}{4}, 1)$. Herein, $A := -\mathcal{P}\Delta$ denotes the Stokes operator with its domain $D(A) := W^{2,2}(\Omega; \mathbb{R}^N) \cap W_0^{1,2}(\Omega; \mathbb{R}^N) \cap L_\sigma^2(\Omega)$ with $L_\sigma^2(\Omega) := \{\varphi \in L^2(\Omega; \mathbb{R}^N) \mid \nabla \cdot \varphi = 0\}$ and $\mathcal{P}$ stands for the Helmholtz projection of $L^2(\Omega; \mathbb{R}^N)$ onto $L_\sigma^2(\Omega)$.

Theorem 1.1. *Let $N \in \{2, 3\}$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Suppose that $\kappa \geq 0$ and $\mu > 0$ and that the functions c_* and ϕ satisfy (1.3) and (1.4), respectively. Then, for any n_0, c_0 and u_0 complying with (1.5), system (1.2) admits at least one global weak solution (n, c, u) in the sense of Definition 2.1.*

Outline. In Sect. 2, we will recall the definition of a global weak solution. Section 3 will be devoted to the introduction of families of appropriately regularized systems and their time-global classical solvability. On the path toward time-global classical solvability of the approximating system, we will also establish a first set of basic a priori estimates. The commonly employed testing procedures in chemotaxis systems, however, rely heavily on the Neumann boundary conditions, and hence, adjustments in the treatment of c are necessary here. The substantial regularity information on n , as entailed by the quadratic decay present in the first equation, will be the driving force for the distillation of bounds on the gradient of c (see Lemma 3.5), which are an important cornerstone of our further analysis. In Sect. 4, we will concern ourselves with improving the bounds on n , where, in particular, time-space information on ∇n is the main objective of the section. In Sect. 5, we prepare estimates on the time derivatives, which upon combination with boundedness results of previous sections allow for the construction of a limit object by means of an Aubin–Lions-type argument at the start of Sect. 6. Finally, in the second part of Sect. 6, we will verify that the limit solution indeed satisfies the properties required of a global weak solution.

2. Definition of global weak solutions

Before we start with our analysis, let us briefly recount the necessary properties for a global weak solution in the following definition, where here and below we set $W_{0,\sigma}^{1,p}(\Omega; \mathbb{R}^N) := W_0^{1,p}(\Omega; \mathbb{R}^N) \cap L_\sigma^2(\Omega)$ for $p \geq 1$.

Definition 2.1. A triple (n, c, u) of functions

$$\begin{aligned} n &\in L_{loc}^2(\overline{\Omega} \times [0, \infty)) \cap L_{loc}^1([0, \infty); W^{1,1}(\Omega)), \\ c &\in L_{loc}^1([0, \infty); W^{1,1}(\Omega)) \quad \text{with} \quad c - c_* \in L_{loc}^1([0, \infty); W_0^{1,1}(\Omega)), \\ u &\in L_{loc}^1([0, \infty); W_{0,\sigma}^{1,1}(\Omega; \mathbb{R}^N)) \end{aligned}$$

with $n \geq 0$ and $c \geq 0$ in $\overline{\Omega} \times [0, \infty)$ will be called a global weak solution of (1.2) if

$$\begin{aligned} nc &\text{ belongs to } L_{loc}^1(\overline{\Omega} \times [0, \infty)), \\ \text{if } n\nabla c, \, nu \text{ and } cu &\text{ belong to } L_{loc}^1(\overline{\Omega} \times [0, \infty); \mathbb{R}^N), \end{aligned}$$

if

$$\begin{aligned} & - \int_0^\infty \int_\Omega n \varphi_t - \int_\Omega n_0 \varphi(\cdot, 0) \\ & = - \int_0^\infty \int_\Omega \nabla n \cdot \nabla \varphi + \int_0^\infty \int_\Omega n (\nabla c \cdot \nabla \varphi) + \kappa \int_0^\infty \int_\Omega n \varphi - \mu \int_0^\infty \int_\Omega n^2 \varphi + \int_0^\infty \int_\Omega n (u \cdot \nabla \varphi) \end{aligned} \quad (2.1)$$

holds for all $\varphi \in C_0^\infty(\bar{\Omega} \times [0, \infty))$, if

$$- \int_0^\infty \int_\Omega c \hat{\varphi}_t - \int_\Omega c_0 \hat{\varphi}(\cdot, 0) = - \int_0^\infty \int_\Omega \nabla c \cdot \nabla \hat{\varphi} - \int_0^\infty \int_\Omega n c \hat{\varphi} + \int_0^\infty \int_\Omega c (u \cdot \nabla \hat{\varphi}) \quad (2.2)$$

is valid for all $\hat{\varphi} \in C_0^\infty(\Omega \times [0, \infty))$, and if

$$- \int_0^\infty \int_\Omega u \cdot \psi_t - \int_\Omega u_0 \cdot \psi(\cdot, 0) = - \int_0^\infty \int_\Omega \nabla u \cdot \nabla \psi + \int_0^\infty \int_\Omega n (\nabla \phi \cdot \psi) \quad (2.3)$$

is fulfilled for all $\psi \in C_0^\infty(\Omega \times [0, \infty); \mathbb{R}^N)$ with $\nabla \cdot \psi \equiv 0$.

3. Global existence of approximate solutions and essential regularity estimates

The global weak solution asserted by Theorem 1.1 will be obtained as a limit object of solutions to certain regularized problems. To this end, for a fixed family $(\rho_\varepsilon)_{\varepsilon \in (0,1)} \subset C_0^\infty(\Omega)$ of smooth cutoff functions satisfying

$$0 \leq \rho_\varepsilon(x) \leq 1 \quad \text{for all } x \in \Omega \quad \text{such that } \rho_\varepsilon \nearrow 1 \text{ as } \varepsilon \searrow 0,$$

we introduce the corresponding family of approximating problems to (1.2) given by

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon &= \Delta n_\varepsilon - \nabla \cdot (\rho_\varepsilon f_\varepsilon(n_\varepsilon) n_\varepsilon \nabla c_\varepsilon) + \kappa n_\varepsilon - \mu n_\varepsilon^2, & x \in \Omega, & t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon &= \Delta c_\varepsilon - g_\varepsilon(n_\varepsilon) c_\varepsilon, & x \in \Omega, & t > 0, \\ u_{\varepsilon t} &= \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon \nabla \phi, \quad \nabla \cdot u_\varepsilon = 0, & x \in \Omega, & t > 0, \\ \frac{\partial n_\varepsilon}{\partial \nu} &= 0, \quad c_\varepsilon = c_*(x), \quad u_\varepsilon = 0, & x \in \partial\Omega, & t > 0, \\ n_\varepsilon(\cdot, 0) &= n_0, \quad c_\varepsilon(\cdot, 0) = c_0, \quad u_\varepsilon(\cdot, 0) = u_0 & x \in \Omega, \end{cases} \quad (3.1)$$

where $f_\varepsilon(s) := \frac{1}{(1+\varepsilon s)^3}$ and $g_\varepsilon(s) := \frac{s}{1+\varepsilon s}$ for $s \geq 0$ and $\varepsilon \in (0, 1)$.

Due to the non-homogeneous boundary condition, this form of the second equation of (3.1), however, is not easily accessible for Dirichlet heat semigroup estimates we will draw on in our following analysis, and hence, we substitute $\hat{c}_\varepsilon := c_* - c_\varepsilon$ and accordingly rewrite the system into the equivalent formulation

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon &= \Delta n_\varepsilon + \nabla \cdot (\rho_\varepsilon f_\varepsilon(n_\varepsilon) n_\varepsilon (\nabla \hat{c}_\varepsilon - \nabla c_*)) + \kappa n_\varepsilon - \mu n_\varepsilon^2, & x \in \Omega, & t > 0, \\ \hat{c}_{\varepsilon t} + u_\varepsilon \cdot \nabla \hat{c}_\varepsilon &= \Delta \hat{c}_\varepsilon - g_\varepsilon(n_\varepsilon) \hat{c}_\varepsilon - \Delta c_* + g_\varepsilon(n_\varepsilon) c_* + u_\varepsilon \cdot \nabla c_*, & x \in \Omega, & t > 0, \\ u_{\varepsilon t} &= \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon \nabla \phi, \quad \nabla \cdot u_\varepsilon = 0, & x \in \Omega, & t > 0, \\ \frac{\partial n_\varepsilon}{\partial \nu} &= 0, \quad \hat{c}_\varepsilon = 0, \quad u_\varepsilon = 0, & x \in \partial\Omega, & t > 0, \\ n_\varepsilon(\cdot, 0) &= n_0, \quad \hat{c}_\varepsilon(\cdot, 0) = c_* - c_0, \quad u_\varepsilon(\cdot, 0) = u_0 & x \in \Omega, \end{cases} \quad (3.2)$$

where, in light of the assumed regularity of c_* , all important properties can be easily transferred back to (3.1). The transformed system will only play a role in the proof of time local existence of solutions (Lemma 3.1) and in the proof that the maximal existence time for fixed $\varepsilon \in (0, 1)$ is actually infinite (Lemma 3.7), as otherwise our analysis in the latter will not necessarily require semigroup arguments for the second component of the systems.

Now, let us begin by establishing time-local existence of solutions to (3.2) (and in turn (3.1)) by means of well-established fixed point arguments.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $q > N$, $\varrho \in (\frac{N}{4}, 1)$, $\kappa \geq 0$, $\mu > 0$. Suppose that c_* and ϕ satisfy (1.3) and (1.4), respectively, and that n_0 , c_0 and u_0 comply with (1.5). Then for any $\varepsilon \in (0, 1)$, there exist $T_{\max, \varepsilon} \in (0, \infty]$ and a uniquely determined triple $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ of functions*

$$\begin{aligned} n_\varepsilon &\in C^0(\overline{\Omega} \times [0, T_{\max, \varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max, \varepsilon})), \\ c_\varepsilon &\in C^0(\overline{\Omega} \times [0, T_{\max, \varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max, \varepsilon})) \cap L_{loc}^\infty((0, T_{\max, \varepsilon}); W^{1,q}(\Omega)), \\ u_\varepsilon &\in C^0(\overline{\Omega} \times [0, T_{\max, \varepsilon}); \mathbb{R}^N) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max, \varepsilon}); \mathbb{R}^N), \end{aligned}$$

which, together with some $P_\varepsilon \in C^{1,0}(\overline{\Omega} \times (0, T_{\max, \varepsilon}))$, solve (3.1) in the classical sense and satisfy $n_\varepsilon \geq 0$ and $c_\varepsilon \geq 0$ in $\overline{\Omega} \times [0, T_{\max, \varepsilon})$. Moreover, either $T_{\max, \varepsilon} = \infty$ or

$$\limsup_{t \nearrow T_{\max, \varepsilon}} (\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot, t)\|_{W^{1,q}(\Omega)} + \|A^\varrho u_\varepsilon(\cdot, t)\|_{L^2(\Omega)}) = \infty. \quad (3.3)$$

Proof. Augmenting well-established fixed point arguments as, e.g., presented in [38, Lemma 2.1] and [1, Lemma 3.1], we will first establish time-local existence for the transformed system (3.2), which afterwards, in view of the substitution $c_\varepsilon = c_* - \hat{c}_\varepsilon$, can be easily transferred back to the corresponding statement for (3.1). For the sake of completeness, let us specify the main steps involved:

First, for some large $R > 0$ and $T \in (0, 1]$, to be specified later, we define the Banach space $X := L^\infty((0, T); C^0(\overline{\Omega}) \times W_0^{1,q}(\Omega) \times D(A^\varrho))$ and its subset

$$S := \left\{ (n_\varepsilon, \hat{c}_\varepsilon, u_\varepsilon) \in X \mid \|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|\hat{c}_\varepsilon(\cdot, t)\|_{W^{1,q}(\Omega)} + \|A^\varrho u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq R \text{ for a.e. } t \in (0, T) \right\}.$$

Next, denoting by $(e^{t\Delta})_{t \geq 0}$, $(e^{t\Delta'})_{t \geq 0}$ and $(e^{-tA})_{t \geq 0}$ the Neumann heat semigroup, the Dirichlet heat semigroup and the Stokes semigroup with Dirichlet boundary data, respectively, we introduce the mapping $\Phi := (\Phi_1, \Phi_2, \Phi_3) : X \rightarrow X$ given by

$$\begin{aligned} \Phi_1(n_\varepsilon, \hat{c}_\varepsilon, u_\varepsilon)(\cdot, t) \\ := e^{t\Delta} n_0 + \int_0^t e^{(t-s)\Delta} \left(\nabla \cdot (-u_\varepsilon n_\varepsilon + \rho_\varepsilon f_\varepsilon(n_\varepsilon) n_\varepsilon (\nabla \hat{c}_\varepsilon - \nabla c_*)) + \kappa n_\varepsilon - \mu n_\varepsilon^2 \right) (\cdot, s) ds, \end{aligned} \quad (3.4)$$

$$\begin{aligned} \Phi_2(n_\varepsilon, \hat{c}_\varepsilon, u_\varepsilon)(\cdot, t) \\ := e^{t\Delta'} (c_* - c_0) + \int_0^t e^{(t-s)\Delta'} (u_\varepsilon \cdot \nabla (c_* - \hat{c}_\varepsilon) + g_\varepsilon(n_\varepsilon)(c_* - \hat{c}_\varepsilon) - \Delta c_*) (\cdot, s) ds, \end{aligned} \quad (3.5)$$

and

$$\Phi_3(n_\varepsilon, \hat{c}_\varepsilon, u_\varepsilon)(\cdot, t) := e^{-tA} u_0 + \int_0^t e^{-(t-s)A} \mathcal{P}(n_\varepsilon \nabla \phi)(\cdot, s) ds \quad \text{for } t \in (0, T). \quad (3.6)$$

We will now show that Φ acts as a contracting self-map on S , provided R and T are suitably fixed beforehand. Dropping the ε -subscript for readability, we pick $(n_1, \hat{c}_1, u_1)$, $(n_2, \hat{c}_2, u_2) \in S$ and observe

that according to (3.4)

$$\begin{aligned}
& \left\| (\Phi_1(n_1, \hat{c}_1, u_1) - \Phi_1(n_2, \hat{c}_2, u_2))(\cdot, t) \right\|_{L^\infty(\Omega)} \\
& \leq \int_0^t \left\| e^{(t-s)\Delta} \nabla \cdot (-n_1(u_1 - u_2) - u_2(n_1 - n_2))(\cdot, s) \right\|_{L^\infty(\Omega)} ds \\
& \quad + \int_0^t \left\| e^{(t-s)\Delta} \nabla \cdot (\rho_\varepsilon f_\varepsilon(n_1)(\nabla \hat{c}_1 - \nabla c_*)(n_1 - n_2))(\cdot, s) \right\|_{L^\infty(\Omega)} ds \\
& \quad + \int_0^t \left\| e^{(t-s)\Delta} \nabla \cdot (\rho_\varepsilon f_\varepsilon(n_1)n_2(\nabla \hat{c}_1 - \nabla \hat{c}_2) + \rho_\varepsilon n_2(\nabla \hat{c}_2 - \nabla c_*)(f_\varepsilon(n_1) - f_\varepsilon(n_2))) (\cdot, s) \right\|_{L^\infty(\Omega)} ds \\
& \quad + \int_0^t \left\| e^{(t-s)\Delta} (\kappa(n_1 - n_2) - \mu(n_1 + n_2)(n_1 - n_2))(\cdot, s) \right\|_{L^\infty(\Omega)} ds \quad \text{for } t \in (0, T).
\end{aligned}$$

Hence, drawing on semigroup estimates as, e.g., provided by [37, Lemma 1.3], [4, Lemma 2.1] and [18, Lemma 3.1], we can find $C_1 = C_1(\Omega) > 0$ such that

$$\begin{aligned}
& \left\| (\Phi_1(n_1, \hat{c}_1, u_1) - \Phi_1(n_2, \hat{c}_2, u_2))(\cdot, t) \right\|_{L^\infty(\Omega)} \\
& \leq C_1 \int_0^t (1 + (t-s)^{-\frac{1}{2}}) (\|n_1\|_{L^\infty(\Omega)} \|u_1 - u_2\|_{L^\infty(\Omega)} + \|u_2\|_{L^\infty(\Omega)} \|n_1 - n_2\|_{L^\infty(\Omega)}) (s) ds \\
& \quad + C_1 \int_0^t (1 + (t-s)^{-\frac{1}{2} - \frac{N}{2q}}) (\|\nabla \hat{c}_1 - \nabla c_*\|_{L^q(\Omega)} \|n_1 - n_2\|_{L^\infty(\Omega)}) (s) ds \\
& \quad + C_1 \int_0^t (1 + (t-s)^{-\frac{1}{2} - \frac{N}{2q}}) (\|n_2\|_{L^\infty(\Omega)} \|\nabla \hat{c}_1 - \nabla \hat{c}_2\|_{L^q(\Omega)}) (s) ds \\
& \quad + C_1 \int_0^t (1 + (t-s)^{-\frac{1}{2} - \frac{N}{2q}}) (\|n_2\|_{L^\infty(\Omega)} \|\nabla \hat{c}_2 - \nabla c_*\|_{L^q(\Omega)} \|f_\varepsilon(n_1) - f_\varepsilon(n_2)\|_{L^\infty(\Omega)}) (s) ds \\
& \quad + \kappa \int_0^t \|n_1(\cdot, s) - n_2(\cdot, s)\|_{L^\infty(\Omega)} ds + \mu \int_0^t (\|n_1 + n_2\|_{L^\infty(\Omega)} \|n_1 - n_2\|_{L^\infty(\Omega)}) (s) ds \quad \text{for } t \in (0, T),
\end{aligned}$$

where we also used the facts that $\rho_\varepsilon \leq 1$ in Ω , $f_\varepsilon \leq 1$ in $[0, \infty)$. Moreover, we have $|f_\varepsilon(a) - f_\varepsilon(b)| \leq |a - b|a^2 + b^2 + ab + 3a + 3b + 3|$ for $a, b \in [0, \infty)$ and all $\varepsilon \in (0, 1)$ and $q > N$ as well as $D(A^\varrho) \hookrightarrow C^\theta(\overline{\Omega})$ for any $\theta \in (0, 2\varrho - \frac{N}{2})$ (e.g., [30, Lemma III.2.4.3] and [8, Thm. 5.6.5]) so that we can find $C_2 = C_2(c_*, \kappa, \mu, \varrho, N, q, R, \Omega)$ such that

$$\begin{aligned}
& \sup_{t \in (0, T)} \left\| (\Phi_1(n_1, \hat{c}_1, u_1) - \Phi_1(n_2, \hat{c}_2, u_2))(\cdot, t) \right\|_{L^\infty(\Omega)} \\
& \leq C_2 (T + T^{\frac{1}{2}} + T^{\frac{1}{2} - \frac{N}{2q}}) \left\| (n_1 - n_2, \hat{c}_1 - \hat{c}_2, u_1 - u_2) \right\|_X. \tag{3.7}
\end{aligned}$$

Similarly, noting that $|g_\varepsilon(a) - g_\varepsilon(b)| \leq |a - b|$ for $a, b \in [0, \infty)$ we can draw on semigroup theory for the Dirichlet heat semigroup (see [28, Proposition 48.4] and [12]) and (3.5) to conclude the existence of

$C_3 = C_3(c_*, \varrho, N, q, R, \Omega) > 0$ satisfying

$$\begin{aligned}
& \sup_{t \in (0, T)} \left\| \left(\Phi_2(n_1, \hat{c}_1, u_1) - \Phi_2(n_2, \hat{c}_2, u_2) \right)(\cdot, t) \right\|_{W^{1, q}(\Omega)} \\
& \leq \sup_{t \in (0, T)} \int_0^t \left\| e^{(t-s)\Delta'} \left((u_1 - u_2) \cdot \nabla(c_* - \hat{c}_1) - u_2 \cdot \nabla(\hat{c}_1 - \hat{c}_2) \right)(\cdot, s) \right\|_{W^{1, q}(\Omega)} ds \\
& \quad + \sup_{t \in (0, T)} \int_0^t \left\| e^{(t-s)\Delta'} \left((g_\varepsilon(n_1) - g_\varepsilon(n_2))(c_* - \hat{c}_1) - g_\varepsilon(n_2)(\hat{c}_1 - \hat{c}_2) \right)(\cdot, s) \right\|_{W^{1, q}(\Omega)} ds \\
& \leq C_3(T + T^{\frac{1}{2}}) \|(n_1 - n_2, \hat{c}_1 - \hat{c}_2, u_1 - u_2)\|_X.
\end{aligned} \tag{3.8}$$

For (3.6), we rely on semigroup estimates for the Stokes equation (cf. [4, Lemma 2.3] and [40, Lemma 3.1]) to obtain $C_4 = C_4(\phi, \varrho, N, R, \Omega) > 0$ such that

$$\begin{aligned}
& \sup_{t \in (0, T)} \left\| A^\varrho \left(\Phi_3(n_1, \hat{c}_1, u_1) - \Phi_3(n_2, \hat{c}_2, u_2) \right)(\cdot, t) \right\|_{L^2(\Omega)} \\
& \leq C_4 T^{1-\varrho} \|(n_1 - n_2, \hat{c}_1 - \hat{c}_2, u_1 - u_2)\|_X,
\end{aligned} \tag{3.9}$$

so that collecting (3.7), (3.8) and (3.9) yields

$$\begin{aligned}
& \|\Phi(n_1, \hat{c}_1, u_1) - \Phi(n_2, \hat{c}_2, u_2)\|_X \\
& \leq C_5(T + T^{\frac{1}{2}} + T^{\frac{1}{2} - \frac{N}{2q}} + T^{1-\varrho}) \|(n_1 - n_2, \hat{c}_1 - \hat{c}_2, u_1 - u_2)\|_X,
\end{aligned} \tag{3.10}$$

with some $C_5 = C_5(c_*, \kappa, \mu, \phi, \varrho, N, q, R, \Omega) > 0$. Moreover, since the Dirichlet heat-semigroup estimates provide $C_6 = C_6(\Omega) > 0$ such that

$$\int_0^T \|e^{(t-s)\Delta'} \Delta c_*\|_{W^{1, q}(\Omega)} ds \leq C_6(T + T^{\frac{1}{2}}) \|\Delta c_*\|_{L^q(\Omega)},$$

we find that for some $C_7 = C_7(c_*, \Omega) > 0$

$$\begin{aligned}
& \|\Phi(n, \hat{c}, u)\|_X \leq \|\Phi(n, \hat{c}, u) - \Phi(0, 0, 0)\|_X + \|\Phi(0, 0, 0)\|_X \\
& \leq \|\Phi(n, \hat{c}, u) - \Phi(0, 0, 0)\|_X + \|(n_0, c_* - c_0, u_0)\|_X + C_6(T + T^{\frac{1}{2}}) \|\Delta c_*\|_{L^q(\Omega)} \\
& \leq C_5(T + T^{\frac{1}{2}} + T^{\frac{1}{2} - \frac{N}{2q}} + T^{1-\varrho}) \|(n, \hat{c}, u)\|_X + \|(n_0, c_* - c_0, u_0)\|_X + C_7(T + T^{\frac{1}{2}}).
\end{aligned} \tag{3.11}$$

Hence, by first taking $R > 3 \max \{ \|(n_0, c_* - c_0, u_0)\|_X, 2C_7 \}$ and then $T \in (0, 1]$ sufficiently small such that $C_5(T + T^{\frac{1}{2}} + T^{\frac{1}{2} - \frac{N}{2q}} + T^{1-\varrho}) < \frac{1}{3}$, we firstly see from (3.11) that Φ maps S onto itself and secondly from (3.10) that indeed Φ also acts as contraction on S . Aided by Banach's fixed point theorem, we therefore obtain a unique $(n_\varepsilon, \hat{c}_\varepsilon, u_\varepsilon) \in S$ with $\Phi(n_\varepsilon, \hat{c}_\varepsilon, u_\varepsilon) = (n_\varepsilon, \hat{c}_\varepsilon, u_\varepsilon)$. In light of standard bootstrapping procedures drawing on regularity theories for parabolic equations and the Stokes semigroup [16, 27, 31], one can verify that $(n_\varepsilon, \hat{c}_\varepsilon, u_\varepsilon)$ actually satisfies the claimed regularity properties, which then entails the existence of a corresponding P_ε such that $(n_\varepsilon, \hat{c}_\varepsilon, u_\varepsilon, P_\varepsilon)$ solves (3.2) classically in $\Omega \times (0, T)$. Uniqueness of $(n_\varepsilon, \hat{c}_\varepsilon, u_\varepsilon)$ can be verified by standard L^2 testing procedures for the differences of two assumed solutions. Noticing that the choice of T only depends on fixed system parameters and the initial data, we may iterate the arguments (with different initial data and possibly larger R) to extend the solution on a maximal time interval $(0, T_{max, \varepsilon})$ such that either $T_{max, \varepsilon} = \infty$ or

$$\limsup_{t \nearrow T_{max, \varepsilon}} (\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|\hat{c}_\varepsilon(\cdot, t)\|_{W^{1, q}(\Omega)} + \|A^\varrho u_\varepsilon(\cdot, t)\|_{L^2(\Omega)}) = \infty.$$

Clearly, by substituting $c_\varepsilon = c_* - \hat{c}_\varepsilon$ (and recalling (1.3)), we immediately obtain the desired results for (3.1), where, finally, the nonnegativity of n_ε and c_ε is entailed by two applications of the maximum principle to the first and second equation of (3.1). $\square$

For the remainder of the work, we will now assume that $N \in \{2, 3\}$, $\Omega \subset \mathbb{R}^N$, $\kappa \geq 0$, $\mu > 0$, $q > N$, $\varrho \in (\frac{N}{4}, 1)$, c_*, ϕ satisfying (1.3) and (1.4), respectively, and initial data n_0, c_0, u_0 obeying (1.5) are fixed and, accordingly, for $\varepsilon \in (0, 1)$ denote by $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ the triple of functions provided by Lemma 3.1 and by $T_{\max, \varepsilon}$ the corresponding maximal existence time.

Time-local existence, at hand, we can now proceed with a first set of a priori properties obtained by straightforward integration and an application of the maximum principle.

Lemma 3.2. *There is $C > 0$ such that for any $\varepsilon \in (0, 1)$ the solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ of (3.1) satisfies*

$$\int_{\Omega} n_\varepsilon(\cdot, t) \leq C, \quad \int_{(t-1)_+}^t \int_{\Omega} n_\varepsilon^2 \leq C \quad \text{and} \quad \|c_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max, \varepsilon}).$$

Proof. Making use of the fact that u_ε is divergence free, by integrating the first equation of (3.1) over Ω and utilizing integration by parts as well as Young's inequality, we deduce that for all $\varepsilon \in (0, 1)$

$$\frac{d}{dt} \int_{\Omega} n_\varepsilon + \mu \int_{\Omega} n_\varepsilon^2 = \kappa \int_{\Omega} n_\varepsilon \leq \frac{\mu}{2} \int_{\Omega} n_\varepsilon^2 + \frac{\kappa^2}{2\mu} |\Omega| \quad \text{on } (0, T_{\max, \varepsilon}).$$

Employing Young's inequality once more to estimate the quadratic term on the left from below, we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} n_\varepsilon + \mu \int_{\Omega} n_\varepsilon + \frac{\mu}{4} \int_{\Omega} n_\varepsilon^2 \\ \leq \frac{\kappa^2}{2\mu} |\Omega| + \mu |\Omega| \quad \text{on } (0, T_{\max, \varepsilon}) \text{ for all } \varepsilon \in (0, 1), \end{aligned} \quad (3.12)$$

which, when combined with the nonnegativity of n_ε and an ODE comparison argument, implies

$$\begin{aligned} \int_{\Omega} n_\varepsilon(\cdot, t) &\leq C_1 \\ &:= \max \left\{ \int_{\Omega} n_0, \left(\frac{\kappa^2}{2\mu^2} + 1 \right) |\Omega| \right\} \quad \text{for all } t \in (0, T_{\max, \varepsilon}) \text{ and all } \varepsilon \in (0, 1). \end{aligned}$$

Furthermore, integration of (3.12) over $((t-1)_+, t)$ now provides,

$$\frac{\mu}{4} \int_{(t-1)_+}^t \int_{\Omega} n_\varepsilon^2 \leq \int_{\Omega} n_\varepsilon(\cdot, (t-1)_+) + \frac{\kappa^2}{2\mu} |\Omega| + \mu |\Omega| \leq C_1 + \frac{\kappa^2}{2\mu} |\Omega| + \mu |\Omega|$$

for all $t \in (0, T_{\max, \varepsilon})$ and all $\varepsilon \in (0, 1)$. Finally, by the maximum principle, we instantly obtain that

$$\begin{aligned} \|c_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} &\leq \max \{ \|c_*\|_{L^\infty(\partial\Omega)}, \|c_0\|_{L^\infty(\Omega)} \} \\ &\quad \text{for all } t \in (0, T_{\max, \varepsilon}) \text{ and } \varepsilon \in (0, 1), \end{aligned}$$

which completes the proof. $\square$

In order to distill further uniform bounds from the somewhat sparse (yet sufficiently powerful) space-time information on n_ε^2 provided by Lemma 3.2, we state the following comparison result for ordinary differential equations. This lemma is copied from [17, Lemma 3.4], whereto we refer the reader for details of the proof.

Lemma 3.3. *For some $T \in (0, \infty]$, let $y \in C^1((0, T)) \cap C^0([0, T))$, $h \in C^0([0, T))$, $h \geq 0$, $C > 0$, $a > 0$ satisfy*

$$y'(t) + ay(t) \leq h(t), \quad \int_{(t-1)_+}^t h(s) \, ds \leq C$$

for all $t \in (0, T)$. Then $y \leq y(0) + \frac{C}{1-e^{-a}}$ throughout $(0, T)$.

With the comparison lemma above, we can make now turn to obtain some uniform bounds for the third solution component.

Lemma 3.4. *There is $C > 0$ such that for any $\varepsilon \in (0, 1)$ the solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ of (3.1) satisfies*

$$\int_{\Omega} |\nabla u_\varepsilon(\cdot, t)|^2 \leq C \quad \text{and} \quad \int_{\Omega} |u_\varepsilon(\cdot, t)|^6 \leq C \quad \text{for all } t \in (0, T_{\max, \varepsilon}).$$

Proof. First, we test the third equation in (3.1) against u_ε , integrate by parts over Ω , and employ the Young and Poincaré inequalities as well as (1.4) to conclude the existence of $C_1 > 0$ such that for all $\varepsilon \in (0, 1)$

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_\varepsilon|^2 + \frac{1}{2} \int_{\Omega} |\nabla u_\varepsilon|^2 \leq C_1 \int_{\Omega} n_\varepsilon^2 \quad (3.13)$$

is valid on $(0, T_{\max, \varepsilon})$. Then, again denoting by $\mathcal{P}$ the Helmholtz projection and by A the Stokes operator, we multiply the projected third equation by Au_ε to obtain $C_2 > 0$ such that for all $\varepsilon \in (0, 1)$ the inequality

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |A^{\frac{1}{2}} u_\varepsilon|^2 + \int_{\Omega} |Au_\varepsilon|^2 \\ &= \int_{\Omega} \mathcal{P}[n_\varepsilon \nabla \phi] \cdot Au_\varepsilon \leq \frac{1}{2} \int_{\Omega} |Au_\varepsilon|^2 + C_2 \int_{\Omega} n_\varepsilon^2 \end{aligned} \quad (3.14)$$

holds on $(0, T_{\max, \varepsilon})$, where we once more made use of the boundedness of $\nabla \phi$ and Young's inequality. Since $A = -\mathcal{P}\Delta$, we have $\|A^{\frac{1}{2}} u_\varepsilon\|_{L^2(\Omega)} = \|\nabla u_\varepsilon\|_{L^2(\Omega)}$, so that in light of the Poincaré inequality, a combination of (3.13) and (3.14) entails the existence of $C_3, C_4 > 0$ such that

$$\begin{aligned} & \frac{d}{dt} \left\{ \int_{\Omega} |u_\varepsilon|^2 + \int_{\Omega} |\nabla u_\varepsilon|^2 \right\} + C_3 \left\{ \int_{\Omega} |u_\varepsilon|^2 + \int_{\Omega} |\nabla u_\varepsilon|^2 \right\} + \int_{\Omega} |Au_\varepsilon|^2 \\ & \leq (2C_1 + 2C_2) \int_{\Omega} n_\varepsilon^2 + C_4 \end{aligned}$$

on $(0, T_{\max, \varepsilon})$ for all $\varepsilon \in (0, 1)$. Drawing on Lemmas 3.2 and 3.3, we conclude that there is $C_5 > 0$ satisfying

$$\int_{\Omega} |\nabla u_\varepsilon(\cdot, t)|^2 \leq C_5 \quad \text{for all } t \in (0, T_{\max, \varepsilon}) \text{ and } \varepsilon \in (0, 1).$$

Finally, relying on the Sobolev embedding theorem, we find $C_6 > 0$ such that

$$\int_{\Omega} |u_\varepsilon(\cdot, t)|^6 \leq C_6 \quad \text{for all } t \in (0, T_{\max, \varepsilon}) \text{ and } \varepsilon \in (0, 1). \quad \square$$

The uniform bounds on u_ε in $L^\infty((0, T_{max, \varepsilon}); L^6(\Omega))$ and n_ε in $L^2(\Omega \times (0, T_{max, \varepsilon}))$ will now be the key ingredient in obtaining information on ∇c_ε . We start by exploiting the fact that c_* is constant in time to establish an ordinary differential inequality for $\int_\Omega |\nabla c_\varepsilon(\cdot, t)|^2$ on $(0, T_{max, \varepsilon})$.

Lemma 3.5. *There exists $C > 0$ such that for all $\varepsilon \in (0, 1)$ the solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ of (3.1) satisfies*

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla c_\varepsilon|^2 + \frac{1}{4} \int_\Omega |\Delta c_\varepsilon|^2 \leq C \int_\Omega n_\varepsilon^2 + C$$

on $(0, T_{max, \varepsilon})$.

Proof. Since the boundary conditions in (3.1) imply that $\frac{\partial}{\partial t} c_\varepsilon|_{\partial\Omega} = 0$ on $(0, T_{max, \varepsilon})$, we can multiply the second equation of (3.1) by $-\Delta c_\varepsilon$ and integrate by parts to find that for all $\varepsilon \in (0, 1)$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla c_\varepsilon|^2 &= - \int_\Omega \Delta c_\varepsilon c_{\varepsilon t} + \int_{\partial\Omega} c_{\varepsilon t} \frac{\partial c_\varepsilon}{\partial \nu} \\ &= - \int_\Omega |\Delta c_\varepsilon|^2 + \int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon) \Delta c_\varepsilon + \int_\Omega g_\varepsilon(n_\varepsilon) c_\varepsilon \Delta c_\varepsilon \end{aligned}$$

on $(0, T_{max, \varepsilon})$. Employing Young's inequality to the last two terms on the right and making use of the fact that $|g_\varepsilon(s)| \leq s$ for all $s \geq 0$ we obtain that for all $\varepsilon \in (0, 1)$

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla c_\varepsilon|^2 + \frac{1}{2} \int_\Omega |\Delta c_\varepsilon|^2 \leq \int_\Omega n_\varepsilon^2 c_\varepsilon^2 + \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^2 \quad \text{on } (0, T_{max, \varepsilon}). \quad (3.15)$$

In view of Lemma 3.2, there is $C_1 > 0$ satisfying

$$\int_\Omega n_\varepsilon^2 c_\varepsilon^2 \leq \|c_\varepsilon\|_{L^\infty(\Omega)}^2 \int_\Omega n_\varepsilon^2 \leq C_1 \int_\Omega n_\varepsilon^2 \quad \text{on } (0, T_{max, \varepsilon}) \quad \text{for all } \varepsilon \in (0, 1). \quad (3.16)$$

Furthermore, relying on the Hölder inequality and Lemma 3.4, we find $C_2 > 0$ such that for all $\varepsilon \in (0, 1)$ we have

$$\int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^2 \leq \|u_\varepsilon\|_{L^6(\Omega)}^2 \|\nabla c_\varepsilon\|_{L^3(\Omega)}^2 \leq C_2 \|\nabla c_\varepsilon\|_{L^3(\Omega)}^2 \quad \text{on } (0, T_{max, \varepsilon}).$$

The Gagliardo–Nirenberg inequality and Lemma 3.2, moreover, imply the existence of $C_3 > 0$ and $C_4 > 0$ satisfying

$$\|\nabla c_\varepsilon\|_{L^3(\Omega)}^2 \leq C_3 \|\Delta c_\varepsilon\|_{L^2(\Omega)} \|c_\varepsilon\|_{L^6(\Omega)} + C_3 \|c_\varepsilon\|_{L^6(\Omega)}^2 \leq C_4 \|\Delta c_\varepsilon\|_{L^2(\Omega)} + C_4$$

on $(0, T_{max, \varepsilon})$ for all $\varepsilon \in (0, 1)$, so that an application of Young's inequality entails the existence of $C_5 > 0$ such that for all $\varepsilon \in (0, 1)$ we have

$$\int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^2 \leq \frac{1}{4} \int_\Omega |\Delta c_\varepsilon|^2 + C_5 \quad \text{on } (0, T_{max, \varepsilon}). \quad (3.17)$$

A combination of (3.15)–(3.17) finally shows that with $C := \max\{C_1, C_5\}$ we obtain

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla c_\varepsilon|^2 + \frac{1}{4} \int_\Omega |\Delta c_\varepsilon|^2 \leq C \int_\Omega n_\varepsilon^2 + C$$

on $(0, T_{max, \varepsilon})$ for all $\varepsilon \in (0, 1)$. □

Next, we combine the recently established differential inequality with the Gagliardo–Nirenberg inequality, the comparison Lemma 3.3 and the space–time bound for n_ε from Lemma 3.2 to obtain the following.

Lemma 3.6. *There is $C > 0$ such that for any $\varepsilon \in (0, 1)$ the solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ of (3.1) fulfills*

$$\int_{\Omega} |\nabla c_\varepsilon(\cdot, t)|^2 \leq C, \quad \int_{(t-1)_+}^t \int_{\Omega} |\Delta c_\varepsilon|^2 \leq C \quad \text{and} \quad \int_{(t-1)_+}^t \int_{\Omega} |\nabla c_\varepsilon|^4 \leq C$$

for all $t \in (0, T_{max, \varepsilon})$.

Proof. According to the Gagliardo–Nirenberg inequality, Hölder’s inequality and Lemma 3.2, there are $C_1, C_2 > 0$ such that for all $\varepsilon \in (0, 1)$

$$\|\nabla c_\varepsilon\|_{L^2(\Omega)}^2 \leq C_1 \|\Delta c_\varepsilon\|_{L^2(\Omega)} \|c_\varepsilon\|_{L^2(\Omega)} + C_1 \|c_\varepsilon\|_{L^2(\Omega)}^2 \leq \frac{1}{4} \|\Delta c_\varepsilon\|_{L^2(\Omega)}^2 + C_2$$

on $(0, T_{max, \varepsilon})$, which upon combination with the differential inequality for c_ε in Lemma 3.5, the bounds obtained in Lemma 3.2 and the ODE-comparison of Lemma 3.3 entails the existence of $C_3 > 0$ such that for all $\varepsilon \in (0, 1)$ we have

$$\int_{\Omega} |\nabla c_\varepsilon|^2 \leq C_3 \quad \text{on } (0, T_{max, \varepsilon}). \quad (3.18)$$

Then, returning to the differential inequality for c_ε from Lemma 3.5, we obtain $C_4 > 0$ such that for all $\varepsilon \in (0, 1)$ we have $\int_{(t-1)_+}^t \int_{\Omega} |\Delta c_\varepsilon|^2 \leq C_4$ on $(0, T_{max, \varepsilon})$ from straightforward integration of said inequality in light of (3.18) and Lemma 3.2. Finally, once again in view of the Gagliardo–Nirenberg inequality, we find $C_5 > 0$ such that for all $\varepsilon \in (0, 1)$

$$\int_{(t-1)_+}^t \|\nabla c_\varepsilon\|_{L^4(\Omega)}^4 \leq C_5 \int_{(t-1)_+}^t \|\Delta c_\varepsilon\|_{L^2(\Omega)}^2 \|c_\varepsilon\|_{L^\infty(\Omega)}^2 + C_5 \int_{(t-1)_+}^t \|c_\varepsilon\|_{L^\infty(\Omega)}^4 \quad \text{on } (0, T_{max, \varepsilon}),$$

completing the proof by drawing on the previous parts of this lemma and Lemma 3.2. $\square$

The boundedness property of ∇c_ε in $L^\infty((0, T_{max, \varepsilon}); L^2(\Omega))$ was the last missing piece of information necessary for proving time-global existence of solution to (3.1). Augmenting the bounds we established in this section with additional ε -dependent bounds in the proof below, we will be able to draw on a Moser–Alikakos-type iteration procedure (see [32, Lemma A.1]) to finally conclude that for fixed ε the maximal existence time $T_{max, \varepsilon}$ provided by Lemma 3.1 is indeed not finite.

Lemma 3.7. *For all $\varepsilon \in (0, 1)$, the solution of (3.1) is global in time, i.e., $T_{max, \varepsilon} = \infty$.*

Proof. We fix $\varepsilon \in (0, 1)$ and assume for contradiction that $T := T_{max, \varepsilon} < \infty$. Subsequently, we will consider estimates for the quantities appearing in the extensibility criterion (3.3), and, as our estimation process relies on employing semigroup arguments to the second component, we will once more return to working in the transformed system (3.2). Since an immediate estimation of n_ε in $L^\infty(\Omega \times (0, T))$ is out of our reach, we will first establish the boundedness of n_ε in $L^\infty((0, T); L^6(\Omega))$. To this regard, we multiply the first equation of (3.2) by n_ε^5 and integrate by parts to find that due to u_ε being divergence-free

$$\begin{aligned} & \frac{1}{6} \frac{d}{dt} \int_{\Omega} n_\varepsilon^6 + 5 \int_{\Omega} n_\varepsilon^4 |\nabla n_\varepsilon|^2 \\ &= -5 \int_{\Omega} \rho_\varepsilon f_\varepsilon(n_\varepsilon) n_\varepsilon^5 (\nabla \hat{c}_\varepsilon - \nabla c_*) \cdot \nabla n_\varepsilon + \kappa \int_{\Omega} n_\varepsilon^6 - \mu \int_{\Omega} n_\varepsilon^7 \\ &+ \int_{\partial\Omega} n_\varepsilon^5 (\nabla n_\varepsilon - \rho_\varepsilon f_\varepsilon(n_\varepsilon) (\nabla \hat{c}_\varepsilon - \nabla c_*)) \cdot -\frac{1}{6} \int_{\partial\Omega} n_\varepsilon^6 (u_\varepsilon \cdot \nu) \end{aligned}$$

on $(0, T)$. Here, the last two integrals disappear because of the boundary conditions and the fact that $\rho_\varepsilon = 0$ on $\partial\Omega$. Moreover, noticing that $|f_\varepsilon(s)s^3| \leq \frac{1}{\varepsilon^3}$ for all $s \geq 0$ and that $|\rho_\varepsilon| \leq 1$ on Ω , we make use of two applications of Young's inequality to find $C_1 := C_1(\kappa, \mu) > 0$ such that

$$\begin{aligned} \frac{1}{6} \frac{d}{dt} \int_{\Omega} n_\varepsilon^6 + \frac{5}{2} \int_{\Omega} n_\varepsilon^4 |\nabla n_\varepsilon|^2 + \int_{\Omega} n_\varepsilon^6 \\ \leq \frac{5}{2\varepsilon^6} \int_{\Omega} |\nabla \hat{c}_\varepsilon - \nabla c_*|^2 + C_1 \quad \text{on } (0, T). \end{aligned}$$

Since $|\nabla \hat{c}_\varepsilon - \nabla c_*|^2 = |\nabla c_\varepsilon|^2$, we conclude from Lemma 3.6 and a straightforward comparison argument that there is $C_2 = C_2(\varepsilon) > 0$ satisfying

$$\int_{\Omega} n_\varepsilon^6 \leq C_2 \quad \text{on } (0, T). \quad (3.19)$$

This bound at hand, we pick $\varrho \in (\frac{N}{4}, 1)$ as in (1.5) and then rely on smoothing properties of the Stokes semigroup (e.g., [11, p.201] and [40, Lemma 3.1]), (1.5), (1.4) and (3.19) to obtain $C_3 = C_3(\varepsilon) > 0$ satisfying

$$\begin{aligned} \|A^\varrho u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} &\leq \|A^\varrho u_0\|_{L^2(\Omega)} + \int_0^t \|A^\varrho e^{-(t-s)A} \mathcal{P}(n(\cdot, s) \nabla \phi)\|_{L^2(\Omega)} ds \\ &\leq C_3 + \frac{C_3 T^{1-\varrho}}{1-\varrho} \end{aligned}$$

for all $t \in (0, T)$, which, due to the embedding $D(A^\varrho) \hookrightarrow C^\theta(\overline{\Omega})$ for any $\theta \in (0, 2\varrho - \frac{N}{2})$ (cf. [30, Lemma III.2.4.3] and [8, Thm. 5.6.5]), also entails that there is some $C_4 = C_4(T, \varepsilon) > 0$ such that

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C_4 \quad \text{for all } t \in (0, T). \quad (3.20)$$

Next, drawing on the Dirichlet heat-semigroup representation of $\hat{c}_\varepsilon$ we find that

$$\begin{aligned} \nabla \hat{c}_\varepsilon(\cdot, t) &= \nabla e^{t\Delta'}(c_* - c_0) \\ &\quad + \int_0^t e^{(t-s)\Delta'} \nabla (u_\varepsilon \cdot \nabla (c_* - \hat{c}_\varepsilon) + g_\varepsilon(n_\varepsilon)(c_* - \hat{c}_\varepsilon) - \Delta c_*) ds \end{aligned}$$

for all $t \in (0, T)$. Picking $q \in (N+2, 6)$ and letting $t_0 := \min\{1, \frac{T}{2}\}$, in light of well-known semigroup estimates ([12]), we can hence obtain $C_5 > 0$ satisfying

$$\begin{aligned} &\|\nabla \hat{c}_\varepsilon(\cdot, t)\|_{L^q(\Omega)} \\ &\leq C_5 \left(1 + t_0^{-\frac{1}{2} - \frac{N}{2}(\frac{1}{N} - \frac{1}{q})}\right) \|c_* - c_0\|_{L^N(\Omega)} + C_5 \int_0^t \left(1 + (t-s)^{-\frac{1}{2} - \frac{N}{2}(\frac{1}{2} - \frac{1}{q})}\right) \|(u_\varepsilon \nabla (c_* - \hat{c}_\varepsilon))(\cdot, s)\|_{L^2(\Omega)} ds \\ &\quad + C_5 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) \|(n_\varepsilon |c_* - \hat{c}_\varepsilon| + |\Delta c_*|)(\cdot, s)\|_{L^q(\Omega)} ds \quad \text{for all } t \in (t_0, T), \end{aligned}$$

where we also relied on the estimate $g_\varepsilon(s) \leq s$ for $s \geq 0$. Here, due to $c_* - \hat{c}_\varepsilon = c_\varepsilon$, we can make use of (3.20) and Lemma 3.6 for the first integral and (3.19) combined with $q < 6$, Lemma 3.2 and (1.3) for the

second integral, to find $C_6 = C_6(T, \varepsilon) > 0$ and, since $\frac{2N}{N-2} \geq 6 > q$ entails $-\frac{1}{2} - \frac{N}{2}(\frac{1}{2} - \frac{1}{q}) > -1$, also $C_7 = C_7(T, \varepsilon)$ satisfying

$$\|\nabla \hat{c}_\varepsilon(\cdot, t)\|_{L^q(\Omega)} \leq C_6 + C_6 \int_0^t \left(2 + (t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{1}{2} - \frac{N}{2}(\frac{1}{2} - \frac{1}{q})} \right) ds \leq C_7$$

for all $t \in (t_0, T)$. With these bounds, the facts $|f_\varepsilon(s)s| \leq \frac{1}{3\varepsilon}$ for all $s \geq 0$ and that $|\rho_\varepsilon| \leq 1$ on Ω , we can easily check that a Moser–Alikakos-type iteration procedure (e.g., [32, Lemma A.1] employed with $(p_0, q_1, q_2) = (6, q, 3)$) becomes applicable to (3.2) and that hence there is $C_8 = C_8(T, \varepsilon) > 0$ such that $\|n_\varepsilon\|_{L^\infty(\Omega)} \leq C_8$ on $(0, T)$. Hence, we find

$$\begin{aligned} & \limsup_{t \nearrow T} \left(\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot, t)\|_{W^{1,q}(\Omega)} + \|A^q u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \right) \\ & \leq \limsup_{t \nearrow T} \left(\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|\hat{c}_\varepsilon(\cdot, t)\|_{W^{1,q}(\Omega)} \right. \\ & \quad \left. + \|c_*\|_{W^{1,q}(\Omega)} + \|A^q u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \right) < \infty, \end{aligned}$$

contradicting (3.3) and therefore proving $T_{max, \varepsilon} = \infty$. $\square$

4. Refined a priori information on n_ε

While the uniform bounds for c_ε and u_ε provided by Sect. 3 would already be strong enough for our limit procedure, we still lack sufficiently good uniform bounds for n_ε . As it turns out, the space–time bound for ∇c_ε of Lemma 3.6, however, can be exploited when considering the functional $y_\varepsilon(t) := \int_\Omega (n_\varepsilon \ln n_\varepsilon)(\cdot, t)$, which has often been a good resource for information in chemotaxis settings ([7, 19, 38, 42]). We start by formulating a corresponding functional inequality.

Lemma 4.1. *There exists $C > 0$ such that for all $\varepsilon \in (0, 1)$ the solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ of (3.1) satisfies*

$$\begin{aligned} \frac{d}{dt} \int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2} \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{\mu}{2} \int_\Omega n_\varepsilon^2 \ln n_\varepsilon \\ \leq C \int_\Omega n_\varepsilon^2 + C \int_\Omega |\nabla c_\varepsilon|^4 + C \end{aligned}$$

on $(0, \infty)$.

Proof. In light of the first equation of (3.1), the fact that u_ε is divergence-free and two integrations by parts we see that

$$\begin{aligned} \frac{d}{dt} \int_\Omega n_\varepsilon \ln n_\varepsilon &= \int_\Omega \left(\nabla \cdot (\nabla n_\varepsilon - \rho_\varepsilon f_\varepsilon(n_\varepsilon) n_\varepsilon \nabla c_\varepsilon) \right. \\ & \quad \left. - u_\varepsilon \cdot \nabla n_\varepsilon + \kappa n_\varepsilon - \mu n_\varepsilon^2 \right) (\ln n_\varepsilon + 1) \\ &= - \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \int_\Omega \rho_\varepsilon f_\varepsilon(n_\varepsilon) (\nabla n_\varepsilon \cdot \nabla c_\varepsilon) \\ & \quad + \int_{\partial\Omega} (\ln n_\varepsilon + 1) (\nabla n_\varepsilon - \rho_\varepsilon f_\varepsilon(n_\varepsilon) n_\varepsilon \nabla c_\varepsilon) \cdot \nu \end{aligned}$$

$$\begin{aligned}
& - \int_{\partial\Omega} n_\varepsilon \ln n_\varepsilon (u_\varepsilon \cdot \nu) + \kappa \int_{\Omega} n_\varepsilon \ln n_\varepsilon \\
& + \kappa \int_{\Omega} n_\varepsilon - \mu \int_{\Omega} n_\varepsilon^2 \ln n_\varepsilon - \mu \int_{\Omega} n_\varepsilon^2
\end{aligned}$$

on $(0, \infty)$ for all $\varepsilon \in (0, 1)$. Observing that again both boundary integrals disappear due to the prescribed boundary conditions and the fact that $\rho_\varepsilon(x) = 0$ for $x \in \partial\Omega$ and noting that there is some $C_1 > 0$ satisfying $\kappa s - \mu s^2 \leq C_1$ for all $s \geq 0$ and such that $(\kappa s - \frac{\mu}{2}s^2) \ln(s) \leq C_1$ for all $s > 0$, we may hence estimate

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} n_\varepsilon \ln n_\varepsilon + \int_{\Omega} \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{\mu}{2} \int_{\Omega} n_\varepsilon^2 \ln n_\varepsilon \\
& \leq \int_{\Omega} \rho_\varepsilon f_\varepsilon(n_\varepsilon) (\nabla n_\varepsilon \cdot \nabla c_\varepsilon) + 2C_1
\end{aligned}$$

on $(0, \infty)$ for all $\varepsilon \in (0, 1)$. To further estimate the integral on the right, we make use of the fact that $|\rho_\varepsilon(x) f_\varepsilon(s)| = \frac{\rho_\varepsilon(x)}{(1+\varepsilon s)^3} \leq 1$ for all $s \geq 0$, $x \in \Omega$ and $\varepsilon \in (0, 1)$ and two applications of Young's inequality to obtain

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} n_\varepsilon \ln n_\varepsilon + \int_{\Omega} \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{\mu}{2} \int_{\Omega} n_\varepsilon^2 \ln n_\varepsilon \\
& \leq \frac{1}{2} \int_{\Omega} \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{1}{2} \int_{\Omega} n_\varepsilon |\nabla c_\varepsilon|^2 + 2C_1 \\
& \leq \frac{1}{2} \int_{\Omega} \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{1}{4} \int_{\Omega} n_\varepsilon^2 + \frac{1}{4} \int_{\Omega} |\nabla c_\varepsilon|^4 + 2C_1
\end{aligned}$$

on $(0, \infty)$ for all $\varepsilon \in (0, 1)$, which concludes the proof upon the choice of $C := \max\{\frac{1}{4}, 2C_1\}$. $\square$

Clearly, we can draw on previously established space–time bounds to extract additional space–time information on $\nabla \sqrt{n_\varepsilon}$ from the previous Lemma, which in a second interpolation step can also be refined to a bound on ∇n_ε in $L^{\frac{4}{3}}(\Omega \times (0, \infty))$.

Lemma 4.2. *For any $T > 0$ there is $C(T) > 0$ such that for all $\varepsilon \in (0, 1)$ the solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ of (3.1) satisfies*

$$\int_0^T \int_{\Omega} \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} \leq C(T) \quad \text{and} \quad \int_0^T \int_{\Omega} n_\varepsilon^2 \ln n_\varepsilon \leq C(T).$$

Proof. Integration of the differential inequality featured in Lemma 4.1 over $(0, T)$ provides $C_1 > 0$ such that for all $\varepsilon \in (0, 1)$

$$\begin{aligned}
& \frac{1}{2} \int_0^T \int_{\Omega} \frac{|\nabla n_{\varepsilon}|^2}{n_{\varepsilon}} + \frac{\mu}{2} \int_0^T \int_{\Omega} n_{\varepsilon}^2 \ln n_{\varepsilon} \\
& \leq C_1 \int_0^T \int_{\Omega} n_{\varepsilon}^2 + C_1 \int_0^T \int_{\Omega} |\nabla c_{\varepsilon}|^4 - \int_{\Omega} n_{\varepsilon}(\cdot, T) \ln n_{\varepsilon}(\cdot, T) \\
& \quad + \int_{\Omega} n_{\varepsilon}(\cdot, 0) \ln n_{\varepsilon}(\cdot, 0) + C_1 T.
\end{aligned}$$

Recalling the bounds provided by Lemma 3.2, Lemma 3.6 and (1.5) as well as the obvious estimate $-\frac{1}{e} \leq s \ln s$ for all $s \geq 0$, the conclusion is immediate. $\square$

Lemma 4.3. *For all $T > 0$, there exists $C(T) > 0$ such that for all $\varepsilon \in (0, 1)$ the solution $(n_{\varepsilon}, c_{\varepsilon}, u_{\varepsilon})$ of (3.1) fulfills*

$$\int_0^T \int_{\Omega} |\nabla n_{\varepsilon}|^{\frac{4}{3}} \leq C(T).$$

Proof. Rewriting the integral under consideration and employing Young's inequality twice, we find that

$$\begin{aligned}
\int_0^T \int_{\Omega} |\nabla n_{\varepsilon}|^{\frac{4}{3}} &= \int_0^T \int_{\Omega} \frac{|\nabla n_{\varepsilon}| |\nabla n_{\varepsilon}|^{\frac{1}{3}} \sqrt{n_{\varepsilon}}}{\sqrt{n_{\varepsilon}}} \\
&\leq \frac{1}{2} \int_0^T \int_{\Omega} \frac{|\nabla n_{\varepsilon}|^2}{n_{\varepsilon}} + \frac{1}{2} \int_0^T \int_{\Omega} |\nabla n_{\varepsilon}|^{\frac{2}{3}} n_{\varepsilon} \\
&\leq \frac{1}{2} \int_0^T \int_{\Omega} \frac{|\nabla n_{\varepsilon}|^2}{n_{\varepsilon}} + \frac{1}{4} \int_0^T \int_{\Omega} |\nabla n_{\varepsilon}|^{\frac{4}{3}} + \frac{1}{4} \int_0^T \int_{\Omega} n_{\varepsilon}^2
\end{aligned}$$

for all $\varepsilon \in (0, 1)$. Reordering and making use of the bounds provided by Lemmas 3.2 and 4.2, we obtain the asserted bound.

5. Regularity estimates for the time derivatives

As final element for an Aubin–Lions-type argument we are going to undertake in Sect. 6, we now prepare uniform bounds for the time derivatives in suitable spaces.

Lemma 5.1. *For every $T > 0$ there exists $C(T) > 0$ such that for any $\varepsilon \in (0, 1)$ the solution $(n_{\varepsilon}, c_{\varepsilon}, u_{\varepsilon})$ of (3.1) fulfills*

$$\int_0^T \|\partial_t n_{\varepsilon}\|_{(W_0^{1,4}(\Omega))^*} \leq C(T), \quad (5.1)$$

$$\int_0^T \|\partial_t c_{\varepsilon}\|_{(W_0^{1,4}(\Omega))^*} \leq C(T), \quad (5.2)$$

and

$$\int_0^T \|\partial_t u_\varepsilon\|_{(W_{0,\sigma}^{1,2}(\Omega))^*} \leq C(T). \quad (5.3)$$

Proof. Given $T > 0$, we fix $\varphi \in L^\infty((0, T); W_0^{1,4}(\Omega))$ with $\|\varphi\|_{L^\infty((0, T); W_0^{1,4}(\Omega))} \leq 1$ and test the first equation of (3.1) against φ to obtain

$$\begin{aligned} \left| \int_\Omega \partial_t n_\varepsilon \varphi \right| &= \left| \int_\Omega \left(-u_\varepsilon \cdot \nabla n_\varepsilon + \Delta n_\varepsilon - \nabla \cdot (\rho_\varepsilon f_\varepsilon(n_\varepsilon) n_\varepsilon \nabla c_\varepsilon) + \kappa n_\varepsilon - \mu n_\varepsilon^2 \right) \varphi \right| \\ &= \left| \int_\Omega n_\varepsilon (u_\varepsilon \cdot \nabla \varphi) - \int_\Omega \nabla n_\varepsilon \cdot \nabla \varphi + \int_\Omega \rho_\varepsilon f_\varepsilon(n_\varepsilon) n_\varepsilon (\nabla c_\varepsilon \cdot \nabla \varphi) + \kappa \int_\Omega n_\varepsilon \varphi - \mu \int_\Omega n_\varepsilon^2 \varphi \right| \end{aligned}$$

for all $t > 0$ and $\varepsilon \in (0, 1)$, where the boundary integrals again disappear due to the boundary conditions for n_ε and u_ε and the fact that $\rho_\varepsilon = 0$ on $\partial\Omega$. Here, we deduce from multiple applications of Hölder's inequality and the fact that for all $\varepsilon \in (0, 1)$ we have $\rho_\varepsilon f(n_\varepsilon) \leq 1$ on $\Omega \times (0, \infty)$ that

$$\begin{aligned} \left| \int_\Omega \partial_t n_\varepsilon \varphi \right| &\leq (\|n_\varepsilon\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^4(\Omega)} + \|\nabla n_\varepsilon\|_{L^{\frac{4}{3}}(\Omega)} + \|n_\varepsilon\|_{L^2(\Omega)} \|\nabla c_\varepsilon\|_{L^4(\Omega)}) \|\nabla \varphi\|_{L^4(\Omega)} \\ &\quad + (\kappa \|n_\varepsilon\|_{L^1(\Omega)} + \mu \|n_\varepsilon\|_{L^2(\Omega)}^2) \|\varphi\|_{L^\infty(\Omega)} \quad \text{for all } t > 0. \end{aligned}$$

Here, we make use of Lemma 3.4 and multiple uses of Young's inequality to conclude that there is $C_1 > 0$ such that for all $\varepsilon \in (0, 1)$

$$\begin{aligned} \left| \int_\Omega \partial_t n_\varepsilon \varphi \right| &\leq C_1 (\|n_\varepsilon\|_{L^2(\Omega)}^2 + \|\nabla n_\varepsilon\|_{L^{\frac{4}{3}}(\Omega)}^{\frac{4}{3}} + \|\nabla c_\varepsilon\|_{L^4(\Omega)}^4 + 1) \|\nabla \varphi\|_{L^4(\Omega)} \\ &\quad + (\kappa \|n_\varepsilon\|_{L^1(\Omega)} + \mu \|n_\varepsilon\|_{L^2(\Omega)}^2) \|\varphi\|_{L^\infty(\Omega)} \end{aligned}$$

holds for all $t > 0$. Then, since $\|\varphi\|_{L^\infty((0, T); W_0^{1,4}(\Omega))} \leq 1$ and $W_0^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$, an integration over $(0, T)$ immediately entails (5.1) thanks to Lemmas 3.2, 3.6 and 4.3.

Similarly, fixing $\hat{\varphi} \in L^\infty((0, T); W_0^{1,4}(\Omega))$ with $\|\hat{\varphi}\|_{L^\infty((0, T); W_0^{1,4}(\Omega))} \leq 1$ and testing the second equation of (3.1) against $\hat{\varphi}$, we find $C_2 > 0$ satisfying

$$\begin{aligned} &\int_0^T \left| \int_\Omega \partial_t c_\varepsilon \hat{\varphi} \right| \\ &= \int_0^T \left| \int_\Omega \left(-u_\varepsilon \cdot \nabla c_\varepsilon + \Delta c_\varepsilon - g_\varepsilon(n_\varepsilon) c_\varepsilon \right) \hat{\varphi} \right| \\ &\leq C_2 \int_0^T \left(\|\nabla c_\varepsilon\|_{L^4(\Omega)}^4 + \|\Delta c_\varepsilon\|_{L^2(\Omega)}^2 + \|n_\varepsilon\|_{L^2(\Omega)}^2 + 1 \right) \|\hat{\varphi}\|_{L^2(\Omega)} \end{aligned}$$

for all $\varepsilon \in (0, 1)$, where we again made use of Lemmas 3.4 and 3.2, Young's inequality and the fact that $g_\varepsilon(s) \leq s$ for all $s \geq 0$, so that (5.2) is an evident consequence of the spatiotemporal bounds provided by Lemmas 3.6 and 3.2.

Finally, for any fixed $\psi \in L^\infty((0, T); W_{0,\sigma}^{1,2}(\Omega))$ with $\|\psi\|_{L^\infty((0,T);W_{0,\sigma}^{1,2}(\Omega))} \leq 1$, we multiply the third equation in 3.1 by ψ and integrate the resulting equation to derive that

$$\begin{aligned}
& \int_0^T \left| \int_\Omega \partial_t u_\varepsilon \cdot \psi \right| \\
&= \int_0^T \left| - \int_\Omega \nabla u_\varepsilon \cdot \nabla \psi + \int_\Omega n_\varepsilon (\nabla \phi \cdot \psi) \right| \\
&\leq C_3 \int_0^T \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|\nabla \psi\|_{L^2(\Omega)} \\
&\quad + C_3 \int_0^T \|n_\varepsilon\|_{L^2(\Omega)} \|\nabla \phi\|_{L^\infty(\Omega)} \|\psi\|_{L^2(\Omega)} \\
&\leq C_3 \int_0^T \left(\|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 + 1 \right) \|\nabla \psi\|_{L^2(\Omega)} \\
&\quad + C_3 \int_0^T \left(\|n_\varepsilon\|_{L^2(\Omega)}^2 + \|\nabla \phi\|_{L^\infty(\Omega)}^2 \right) \|\psi\|_{L^2(\Omega)} \\
&\leq C_4(T)
\end{aligned}$$

for all $\varepsilon \in (0, 1)$, in light of Lemma 3.4, (1.4) and Lemma 3.2, and from which we conclude (5.3). $\square$

6. Existence of a limit solution. The proof of Theorem 1.1

Collecting the uniform bounds presented in Sects. 2–5, we can now construct a limit object, which satisfies all the regularity requirements presented in Definition 2.1.

Proposition 6.1. *There exists a sequence $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)$ with $\varepsilon_j \searrow 0$ as $j \rightarrow \infty$ and functions*

$$\begin{aligned}
n &\in L_{loc}^2(\overline{\Omega} \times [0, \infty)) \quad \text{with} \quad \nabla n \in L_{loc}^{\frac{4}{3}}(\overline{\Omega} \times [0, \infty); \mathbb{R}^N), \\
c &\in L^\infty(\Omega \times (0, \infty)) \quad \text{with} \quad c - c_* \in L_{loc}^4([0, \infty); W_0^{1,4}(\Omega)), \\
u &\in L_{loc}^2([0, \infty); W_{0,\sigma}^{1,2}(\Omega))
\end{aligned}$$

and such that the solutions $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ of (3.1) fulfill

$$n_\varepsilon \rightarrow n \quad \text{in} \quad L_{loc}^p(\overline{\Omega} \times [0, \infty)) \quad \text{for any } p \in [1, 2] \text{ and a.e. in } \Omega \times (0, \infty), \quad (6.1)$$

$$\nabla n_\varepsilon \rightharpoonup \nabla n \quad \text{in} \quad L_{loc}^{\frac{4}{3}}(\overline{\Omega} \times [0, \infty); \mathbb{R}^N), \quad (6.2)$$

$$\rho_\varepsilon f_\varepsilon(n_\varepsilon) n_\varepsilon \rightarrow n \quad \text{in} \quad L_{loc}^p(\overline{\Omega} \times [0, \infty)) \quad \text{for any } p \in [1, 2], \quad (6.3)$$

$$g_\varepsilon(n_\varepsilon) \rightarrow n \quad \text{in} \quad L_{loc}^p(\overline{\Omega} \times [0, \infty)) \quad \text{for any } p \in [1, 2], \quad (6.4)$$

$$c_\varepsilon \rightarrow c \quad \text{in} \quad L_{loc}^q(\overline{\Omega} \times [0, \infty)) \quad \text{for any } q \in [1, \infty) \text{ and a.e. in } \Omega \times (0, \infty), \quad (6.5)$$

$$c_\varepsilon \xrightarrow{*} c \quad \text{in} \quad L^\infty(\Omega \times (0, \infty)), \quad (6.6)$$

$$\nabla c_\varepsilon \rightharpoonup \nabla c \quad \text{in} \quad L_{loc}^4(\overline{\Omega} \times [0, \infty); \mathbb{R}^N), \quad (6.7)$$

$$u_\varepsilon \rightarrow u \quad \text{in } L^r_{loc}(\overline{\Omega} \times [0, \infty); \mathbb{R}^N) \text{ for any } r \in [1, 6) \text{ and a.e. in } \Omega \times (0, \infty), \quad (6.8)$$

$$u_\varepsilon \xrightarrow{*} u \quad \text{in } L^\infty((0, \infty); L^6(\Omega; \mathbb{R}^N)), \quad (6.9)$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L^2_{loc}(\overline{\Omega} \times [0, \infty); \mathbb{R}^{N \times N}), \quad (6.10)$$

as $\varepsilon = \varepsilon_j \searrow 0$.

Proof. A combination of an Aubin–Lions-type lemma ([29, Sec. 8, Corollary 4]) with the bounds presented in Lemmas 3.2, 4.3 and 5.1 ensures that

$$\{n_\varepsilon\}_{\varepsilon \in (0,1)} \text{ is relatively compact in } L^{\frac{4}{3}}_{loc}(\overline{\Omega} \times [0, \infty))$$

and that hence we can find a subsequence $(\varepsilon_j)_{j \in \mathbb{N}}$ with $\varepsilon_j \searrow 0$ as $j \rightarrow \infty$ and $n \in L^{\frac{4}{3}}_{loc}(\overline{\Omega} \times [0, \infty))$ such that $n_\varepsilon \rightarrow n$ in $L^{\frac{4}{3}}_{loc}(\overline{\Omega} \times [0, \infty))$ and a.e. in $\Omega \times (0, \infty)$. Additionally, the spatiotemporal bound of Lemma 4.3 also allows us to conclude that for some $z \in L^{\frac{4}{3}}_{loc}(\overline{\Omega} \times [0, \infty); \mathbb{R}^N)$ we have $\nabla n_\varepsilon \rightharpoonup z \in L^{\frac{4}{3}}_{loc}(\overline{\Omega} \times [0, \infty); \mathbb{R}^N)$ along another subsequence (still denoted by $(\varepsilon_j)_{j \in \mathbb{N}}$). Then, the established convergence of $n_\varepsilon \rightarrow n$ in $L^{\frac{4}{3}}_{loc}(\overline{\Omega} \times [0, \infty))$, however, necessarily entails that z coincides with ∇n due to the uniqueness of the weak limit and thereby verifies (6.2). Moreover, noting that $\Theta : [0, \infty) \rightarrow [0, \infty]$, $\Theta(s) := s \ln(s+1)$ is an increasing and convex function with

$$\lim_{s \rightarrow \infty} \frac{\Theta(s)}{s} = \infty \quad \text{and} \quad \Theta(s) \leq 2s \ln(s^{\frac{1}{2}}) + 1 \quad \text{for all } s \geq 0,$$

we observe that according to Lemma 4.2 for any $T > 0$ there is $C > 0$ satisfying

$$\int_0^T \int_\Omega \Theta(n_\varepsilon^2) = \int_0^T \int_\Omega n_\varepsilon^2 \ln(n_\varepsilon^2 + 1) \leq 2 \int_0^T \int_\Omega n_\varepsilon^2 \ln n_\varepsilon + |\Omega|T \leq C(T),$$

which, by a result of de la Vallée–Poussin (e.g., [24, II.T22]), entails that $\{n_\varepsilon^2\}_{\varepsilon \in (0,1)}$ is equi-integrable. Thus, a combination of the equi-integrability with the a.e. convergence of n_ε and Vitali’s theorem yields (6.1) along a further subsequence. Then, since $|\rho_\varepsilon f_\varepsilon(n_\varepsilon)| \leq 1$ in $\Omega \times (0, \infty)$ for all $\varepsilon \in (0, 1)$ and $\rho_\varepsilon f_\varepsilon(n_\varepsilon) \rightarrow 1$ a.e. in $\Omega \times (0, \infty)$ as $\varepsilon = \varepsilon_j \searrow 0$, we find from (6.1) and arguments akin to, e.g., [41, Lemma A.4], that (6.3) holds as well. Likewise, we may also conclude (6.4) from (6.1). Working along similar lines for the second and third components, we can draw on the bounds of Lemmas 3.2, 3.6 and 5.1 to obtain an additional subsequence along which (6.5), (6.6) and (6.7) hold and iterating the arguments once more with the bounds of Lemmas 3.4 and 5.1 concerning u_ε , finally, also (6.8), (6.9) and (6.10). The claimed regularity properties of (n, c, u) and $c - c_*$ are clearly a direct consequence of (6.1), (6.2), (6.6), (6.7), (6.8), (6.10) and (1.3) and the fact that $c_\varepsilon - c_* = 0$ on $\partial\Omega \times [0, \infty)$. $\square$

Finally, it remains to be checked that the limit objects provided by Proposition 6.1 indeed satisfy the integral identities (2.1), (2.2) and (2.3) of Definition 2.1. This, however, is a straightforward procedure, as the convergence properties of Proposition 6.1 already cover everything we need to pass to the limit in the corresponding equations of (3.1).

Proof of Theorem 1.1. Since the regularity requirements imposed on a weak solution by Definition 2.1 are already covered by the properties obtained in Lemma 6.1, we only have to verify that the limit objects obtained in said lemma also satisfy the integral identities (2.1)–(2.3). We pick $\varphi \in C_0^\infty(\overline{\Omega} \times [0, \infty))$, $\hat{\varphi} \in C_0^\infty(\Omega \times [0, \infty))$ and $\psi \in C_0^\infty(\Omega \times [0, \infty); \mathbb{R}^N)$ with $\nabla \cdot \psi \equiv 0$ and then fix $T > 0$ such that $\varphi, \hat{\varphi}, \psi \equiv 0$ in $\Omega \times [T, \infty)$. Now, we test the first equation of (3.1) against φ and integrate by parts to obtain

$$\begin{aligned}
& - \int_0^T \int_{\Omega} n_{\varepsilon} \varphi_t - \int_{\Omega} n_0 \varphi(\cdot, 0) \\
& = - \int_0^T \int_{\Omega} \nabla n_{\varepsilon} \cdot \nabla \varphi + \int_0^T \int_{\Omega} \rho_{\varepsilon} f_{\varepsilon}(n_{\varepsilon}) n_{\varepsilon} (\nabla c_{\varepsilon} \cdot \nabla \varphi) \\
& \quad + \kappa \int_0^T \int_{\Omega} n_{\varepsilon} \varphi - \mu \int_0^T \int_{\Omega} n_{\varepsilon}^2 \varphi + \int_0^T \int_{\Omega} n_{\varepsilon} (u_{\varepsilon} \cdot \nabla \varphi)
\end{aligned} \tag{6.11}$$

for all $\varepsilon \in (0, 1)$, where we made use of the prescribed boundary conditions and the fact that u_{ε} is solenoidal. According to (6.1) and (6.2), we immediately find that

$$\int_0^T \int_{\Omega} n_{\varepsilon} \varphi_t \rightarrow \int_0^T \int_{\Omega} n \varphi_t, \quad \int_0^T \int_{\Omega} \nabla n_{\varepsilon} \cdot \nabla \varphi \rightarrow \int_0^T \int_{\Omega} \nabla n \cdot \nabla \varphi$$

and

$$\int_0^T \int_{\Omega} n_{\varepsilon} \varphi \rightarrow \int_0^T \int_{\Omega} n \varphi \quad \text{as } \varepsilon = \varepsilon_j \searrow 0.$$

Drawing on (6.1), (6.3) combined with (6.7) and (6.1) together with (6.8), we also conclude

$$\int_0^T \int_{\Omega} n_{\varepsilon}^2 \varphi \rightarrow \int_0^T \int_{\Omega} n^2 \varphi, \quad \int_0^T \int_{\Omega} \rho_{\varepsilon} f_{\varepsilon}(n_{\varepsilon}) n_{\varepsilon} (\nabla c_{\varepsilon} \cdot \nabla \varphi) \rightarrow \int_0^T \int_{\Omega} n (\nabla c \cdot \nabla \varphi)$$

and

$$\int_0^T \int_{\Omega} n_{\varepsilon} (u_{\varepsilon} \cdot \nabla \varphi) \rightarrow \int_0^T \int_{\Omega} n (u \cdot \nabla \varphi) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0$$

so that passing to the limit in (6.11) immediately entails (2.1) since $T > 0$ was chosen such that $\varphi \equiv 0$ in $\Omega \times [T, \infty)$.

Similarly, multiplying the second equation of (3.1) by $\hat{\varphi}$ and integrating, we have

$$\begin{aligned}
& - \int_0^T \int_{\Omega} c_{\varepsilon} \hat{\varphi}_t - \int_{\Omega} c_0 \hat{\varphi}(\cdot, 0) \\
& = - \int_0^T \int_{\Omega} \nabla c_{\varepsilon} \cdot \nabla \hat{\varphi} - \int_0^T \int_{\Omega} g_{\varepsilon}(n_{\varepsilon}) c_{\varepsilon} \hat{\varphi} + \int_0^T \int_{\Omega} c_{\varepsilon} (u_{\varepsilon} \cdot \nabla \hat{\varphi})
\end{aligned}$$

for all $\varepsilon \in (0, 1)$. Therefore, utilizing (6.5), (6.7), (6.4) and (6.8), we may also pass to the limit in this equation and obtain (2.2). Finally, testing the third equation in (3.1) against ψ and integrating by parts yields

$$\begin{aligned}
& - \int_0^T \int_{\Omega} u_{\varepsilon} \cdot \psi_t - \int_{\Omega} u_0 \cdot \psi(\cdot, 0) \\
& = - \int_0^T \int_{\Omega} \nabla u_{\varepsilon} \cdot \nabla \psi + \int_0^T \int_{\Omega} n_{\varepsilon} (\nabla \phi \cdot \psi) \quad \text{for all } \varepsilon \in (0, 1),
\end{aligned}$$

where (6.8), (6.10) and (6.1) imply that we may, once more, pass to the limit and obtain (2.3), concluding the proof. $\square$

Acknowledgements

Tobias Black acknowledges support of the *Deutsche Forschungsgemeinschaft* in the context of the project *Emergence of structures and advantages in cross-diffusion systems (project no. 411007140)*. Chunyan Wu was supported by the China Scholarship Council (*CSC NO. 201906070104*) and the *Applied Fundamental Research Program of Sichuan Province (no. 2020YJ0264)* during her stay at the University of Paderborn.

Funding Open Access funding enabled and organized by Projekt DEAL.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Bellomo, N., Bellouquid, A., Tao, Y., Winkler, M.: Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues. *Math. Models Methods Appl. Sci.* **25**(9), 1663–1763 (2015)
- [2] Braukhoff, M.: Global (weak) solution of the chemotaxis-Navier–Stokes equations with non-homogeneous boundary conditions and logistic growth. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **34**(4), 1013–1039 (2017)
- [3] Braukhoff, M., Lankeit, J.: Stationary solutions to a chemotaxis-consumption model with realistic boundary conditions for the oxygen. *Math. Models Methods Appl. Sci.* **29**(11), 2033–2062 (2019)
- [4] Cao, X., Lankeit, J.: Global classical small-data solutions for a three-dimensional chemotaxis Navier-Stokes system involving matrix-valued sensitivities. *Calc. Var. Partial Differ. Equ.* **55**(4), Paper No. 107, (2016)
- [5] Chertock, A., Fellner, K., Kurganov, A., Lorz, A., Markowich, P.A.: Sinking, merging and stationary plumes in a coupled chemotaxis-fluid model: a high-resolution numerical approach. *J. Fluid Mech.* **694**, 155–190 (2012)
- [6] Dombrowski, C., Cisneros, L., Chatkaew, S., Goldstein, R.E., Kessler, J.O.: Self-concentration and large-scale coherence in bacterial dynamics. *Phys. Rev. Lett.* **93**(9), (2004)
- [7] Duan, R., Lorz, A., Markowich, P.: Global solutions to the coupled chemotaxis-fluid equations. *Commun. Partial Differ. Equ.* **35**(9), 1635–1673 (2010)
- [8] Evans, L. C.: *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, (2010)
- [9] Fuest, M., Lankeit, J., Mizukami, M.: Long-term behaviour in a parabolic-elliptic chemotaxis-consumption model. *J. Differ. Equ.* **271**, 254–279 (2021)
- [10] Fujikawa, H., Matsushita, M.: Fractal Growth of *Bacillus subtilis* on Agar Plates. *J. Phys. Soc. Japan* **58**(11), 3875–3878 (1989)

- [11] Giga, Y.: Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system. *J. Differ. Equ.* **62**(2), (1986)
- [12] Henry, D.: *Geometric Theory of Semilinear Parabolic Equations*. Lecture Notes in Mathematics, vol. 840. Springer, Berlin (1981)
- [13] Horstmann, D.: From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I. *Jahresber. Deutsch. Math.-Verein.* **105**(3), 103–165 (2003)
- [14] Keller, E. F., Segel, L. A.: Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol.* **26**(3), 399–415 (1970)
- [15] Knosalla, P.: Global solutions of aerotaxis equations. *Applicationes Mathematicae* **44**(1), 135–148 (2017)
- [16] Ladyženskaja, O. A., Solonnikov, V. A., Ural'ceva, N. N.: *Linear and quasilinear equations of parabolic type*. Translations of mathematical monographs. Am. Math. Soc. (1968)
- [17] Lankeit, E., Lankeit, J.: Classical solutions to a logistic chemotaxis model with singular sensitivity and signal absorption. *Nonlinear Anal. Real World Appl.* **46**, 421–445 (2019)
- [18] Lankeit, J.: Chemotaxis can prevent thresholds on population density. *Discrete Contin. Dyn. Syst. Ser. B* **20**(5), 1499–1527 (2015)
- [19] Lankeit, J.: Long-term behaviour in a chemotaxis-fluid system with logistic source. *Math. Models Methods Appl. Sci.* **26**(11), 2071–2109 (2016)
- [20] Lankeit, J., Winkler, M.: Facing low regularity in chemotaxis systems. *Jahresber. Deutsch. Math.-Verein.* (2019)
- [21] Lee, H.G., Kim, J.: Numerical investigation of falling bacterial plumes caused by bioconvection in a three-dimensional chamber. *Eur. J. Mech. B Fluids* **52**, 120–130 (2015)
- [22] Lorz, A.: Coupled chemotaxis fluid model. *Math. Models Methods Appl. Sci.* **20**(06), 987–1004 (2010)
- [23] Matsushita, M., Fujikawa, H.: Diffusion-limited growth in bacterial colony formation. *Physica A* **168**(1), 498–506 (1990)
- [24] Meyer, P.A.: Probability and potentials. *Ann. Math. Stat.* **37**(6), 1856–1857 (1966)
- [25] Peng, Y., Xiang, Z.: Global solutions to the coupled chemotaxis-fluids system in a 3D unbounded domain with boundary. *Math. Models Methods Appl. Sci.* **28**(05), 869–920 (2018)
- [26] Peng, Y., Xiang, Z.: Global existence and convergence rates to a chemotaxis-fluids system with mixed boundary conditions. *J. Differ. Equ.* **267**(2), 1277–1321 (2019)
- [27] Porzio, M.M., Vespri, V.: Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations. *J. Differ. Equ.* **103**(1), 146–178 (1993)
- [28] Quittner, P., Souplet, P.: *Superlinear Parabolic Problems*. Basler Lehrbücher. Birkhäuser Verlag, Birkhäuser Advanced Texts, Basel (2007)
- [29] Simon, J.: Compact sets in the space $L^p(0, T; B)$. *Ann. Mat. Pura Appl.* **4**(146), 65–96 (1987)
- [30] Sohr, H.: *The Navier-Stokes equations*. Basler Lehrbücher. Birkhäuser Verlag, Birkhäuser Advanced Texts, Basel (2001)
- [31] Solonnikov, V. A.: Schauder estimates for the evolutionary generalized Stokes problem. In *Nonlinear equations and spectral theory*, volume 220 of *Amer. Math. Soc. Transl. Ser. 2*, pages 165–200. Amer. Math. Soc., Providence, RI, (2007)
- [32] Tao, Y., Winkler, M.: Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity. *J. Differ. Equ.* **252**(1), (2012)
- [33] Tuval, I., Cisneros, L., Dombrowski, C., Wolgemuth, C.W., Kessler, J.O., Goldstein, R.E.: Bacterial swimming and oxygen transport near contact lines. *Proc. Natl. Acad. Sci. USA* **102**(7), 2277–2282 (2005)
- [34] Vorotnikov, D.: Weak solutions for a bioconvection model related to *Bacillus subtilis*. *Commun. Math. Sci.* **12**(3), 545–563 (2014)
- [35] Wang, Y., Winkler, M., Xiang, Z.: Global mass-preserving solutions to a chemotaxis-fluid model involving Dirichlet boundary conditions for the signal (2020). Preprint
- [36] Wang, Y., Winkler, M., Xiang, Z.: Local energy estimates and global solvability in a three-dimensional chemotaxis-fluid system with prescribed signal on the boundary. *Commun. Partial. Differ. Equ.* **46**(6), 1058–1091 (2021)
- [37] Winkler, M.: Aggregation vs global diffusive behavior in the higher-dimensional Keller-Segel model. *J. Differ. Equ.* **248**(12), 2889–2905 (2010)
- [38] Winkler, M.: Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops. *Commun. Partial Differ. Equ.* **37**(2), 319–351 (2012)
- [39] Winkler, M.: Stabilization in a two-dimensional chemotaxis-Navier-Stokes system. *Arch. Ration. Mech. Anal.* **211**(2), 455–487 (2014)
- [40] Winkler, M.: Boundedness and large time behavior in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion and general sensitivity. *Calc. Var. Partial Differ. Equ.* **54**(4), 3789–3828 (2015a)
- [41] Winkler, M.: Large-data global generalized solutions in a chemotaxis system with tensor-valued sensitivities. *SIAM J. Math. Anal.* **47**(4), 3092–3115 (2015b)
- [42] Winkler, M.: Global weak solutions in a three-dimensional chemotaxis-Navier-Stokes system. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **33**(5), 1329–1352 (2016)

- [43] Winkler, M.: How far do chemotaxis-driven forces influence regularity in the Navier-Stokes system? *Trans. Amer. Math. Soc.* **369**(5), 3067–3125 (2017)
- [44] Winkler, M.: Does Leray’s structure theorem withstand buoyancy-driven chemotaxis-fluid interaction? (2020). To appear in *J. Eur. Math. Soc. (JEMS)*
- [45] Woodward, D., Tyson, R., Myerscough, M., Murray, J., Budrene, E., Berg, H.: Spatio-temporal patterns generated by *Salmonella typhimurium*. *Biophys. J.* **68**(5), 2181–2189 (1995)

Tobias Black
Institut für Mathematik
Universität Paderborn
Warburger Str. 100
33098 Paderborn
Germany
e-mail: tblack@math.upb.de

Chunyan Wu
School of Mathematical Sciences
University of Electronic Science and Technology of China
611731 Chengdu
China
e-mail: wcpde@163.com

(Received: October 26, 2020; revised: May 25, 2021; accepted: May 26, 2021)