Relaxed parameter conditions for chemotactic collapse in logistic-type parabolic–elliptic Keller–Segel systems

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Abstract. We study the finite-time blow-up in two variants of the parabolic–elliptic Keller–Segel system with nonlinear diffusion and logistic source. In n -dimensional balls, we consider

$$\begin{cases} u_t = \nabla \cdot ((u+1)^{m-1} \nabla u - u \nabla v) + \lambda u - \mu u^{1+\kappa}, \\ 0 = \Delta v - \frac{1}{|\Omega|} \int_{\Omega} u + u \end{cases} \quad (\text{JL})$$

and

$$\begin{cases} u_t = \nabla \cdot ((u+1)^{m-1} \nabla u - u \nabla v) + \lambda u - \mu u^{1+\kappa}, \\ 0 = \Delta v - v + u, \end{cases} \quad (\text{PE})$$

where λ and μ are given spatially radial nonnegative functions and $m, \kappa > 0$ are given parameters subject to further conditions. In a unified treatment, we establish a bridge between previously employed methods on blow-up detection and relatively new results on pointwise upper estimates of solutions in both of the systems above and then, making use of this newly found connection, provide extended parameter ranges for m, κ leading to the existence of finite-time blow-up solutions in space dimensions three and above. In particular, for constant $\lambda, \mu > 0$, we find that there are initial data which lead to blow-up in (JL) if

$$\begin{aligned} 0 \leq \kappa < \min \left\{ \frac{1}{2}, \frac{n-2}{n} - (m-1)_+ \right\} & \quad \text{if } m \in \left[\frac{2}{n}, \frac{2n-2}{n} \right) \\ \text{or } 0 \leq \kappa < \min \left\{ \frac{1}{2}, \frac{n-1}{n} - \frac{m}{2} \right\} & \quad \text{if } m \in \left(0, \frac{2}{n} \right), \end{aligned}$$

and in (PE) if $m \in [1, \frac{2n-2}{n})$ and

$$0 \leq \kappa < \min \left\{ \frac{(m-1)n+1}{2(n-1)}, \frac{n-2-(m-1)n}{n(n-1)} \right\}.$$

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1. Introduction

How strong does a degrading term need to be in order to rule out chemotactic collapse in a Keller–Segel system? Or, phrased differently, when is chemotactic aggregation stronger than even superlinear dampening?

On the one hand, in the absence of any degrading terms the minimal Keller–Segel model, proposed in the 1970s [20] to model chemotaxis, that is, the directed movement of, for instance, cells or bacteria toward a chemical signal, and given by

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - v + u, \end{cases} \quad (1.1)$$

admits solutions blowing up in finite time.

While all solutions are global and bounded in one dimension [31], blow-up does occur in two- [12, 14, 32] and higher-dimensional [46] domains. For a broader introduction to (1.1) and similar systems, we refer to the surveys [1, 26].

On the other hand, adding logistic terms to (1.1), for instance, in order to model population dynamics [13, 33] or tumor invasion processes [3], leads to the system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \lambda u - \mu u^{1+\kappa}, \\ v_t = \Delta v - v + u, \end{cases}$$

where $\lambda, \mu > 0$ and $\kappa = 1$ are given parameters. Here, all solutions are global and bounded in two [30] and, provided that $\mu \geq \mu_0$ for some $\mu_0 > 0$ depending on the space dimension, also in higher dimensions [44]. (See also [38] for the corresponding parabolic–elliptic system.) Moreover, without any restriction on $\mu > 0$, in all space dimensions global weak solutions have been constructed, which in three-dimensional convex domains additionally become eventually smooth if λ is small enough [24].

In order to better understand the relative strengths of the possibly explosion-enhancing cross-diffusive chemotaxis effect and the damping force of the logistic terms, also systems with weaker damping (e.g., $\kappa < 1$) have been investigated with respect to the existence of classical, weak or generalized solutions (see, e.g., [29, 40, 41, 53, 54]).

For a more complete answer, however, it seems indispensable to also search for the opposite case, that of blow-up: What happens for $\kappa \in (0, 1)$ (or for $\kappa = 1$ and small $\mu > 0$)? For which values of κ can solutions blowing up in finite time be constructed?

Even beyond solutions that grow on smaller timescales in case of slow diffusion [23, 47], some partial results in this direction are available: The first blow-up result for a chemotaxis system with superlinear degradation apparently goes back to [45]. Following a simplification introduced in [19] by Jäger and Luckhaus, there it was shown that for the system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \lambda u - \mu u^{1+\kappa}, \\ 0 = \Delta v - \overline{M}(t) + u, \quad \overline{M}(t) = \frac{1}{|\Omega|} \int_{\Omega} u \end{cases} \quad (1.2)$$

in a ball in $\mathbb{R}^n$, $n \geq 5$, finite-time blow-up is possible provided that $\lambda, \mu > 0$ and $\kappa < \frac{1}{2} + \frac{1}{2n-2}$. Moreover, chemotactic collapse may even happen in the physically (most) relevant space dimension three. In [49], it was shown that the parabolic–elliptic system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \lambda u - \mu u^{1+\kappa}, \\ 0 = \Delta v - v + u \end{cases} \quad (1.3)$$

in a ball in $\mathbb{R}^n$, $n \geq 3$, admits solutions blowing up in finite time provided $\lambda, \mu > 0$ and

$$\kappa < \kappa_{\text{pe}}(n) := \begin{cases} \frac{1}{6}, & n \in \{3, 4\}, \\ \frac{1}{2n-2}, & n > 4. \end{cases}$$

This result has also recently been transferred to a setting where two species are attracted by the same signal in [39], where only limited adaptations were necessary to retrieve the same values for $\kappa_{\text{pe}}(n)$, and to a system with a weaker sensitivity function, [36].

In two-dimensional domains, however, known results seem to be limited to the case of space-dependent functions μ . That is, if one replaces the first equation in (1.2) with $u_t = \Delta u - \nabla \cdot (u \nabla v) + \lambda u - \mu_1 |x|^\alpha u^{1+\kappa}$, then solutions blowing up in finite-time have been constructed if again the domain is a ball, $\lambda, \mu_1, \alpha > 0$ and $\kappa < \frac{\alpha}{2}$. Phrased differently, given any $\kappa > 0$, there exist blow-up solutions even in 2D—provided α is large enough [10].

There is another effect that can hinder blow-up and is often included in the model, be it for reasons of biological modelling, for example, of tumor cells, cf. [35] or [11, 21], or from a purely mathematical motivation: nonlinear, porous-medium type diffusion (i.e., the replacement of Δu by, e.g., $\nabla \cdot ((u+1)^{m-1} \nabla u)$). If sufficiently strong, it can prevent blow-up even on its own (see [15, 17, 34, 37] for boundedness results in case of $m > 2 - \frac{2}{n}$) or at least in combination with logistic sources, [27, 42, 55–57].

However, for the regime of slightly weaker diffusion, the occurrence of blow-up may still be possible. And indeed, in the absence of logistics ($\kappa = 0$, $\mu = \lambda$), $m < \frac{2n-2}{n}$ for Ω being a ball in $\mathbb{R}^n$ allows for some unbounded solutions (i.e., blow-up after either finite or infinite time, see [15, 43]), with finite-time blow-up having been detected in [6]. As to blow-up for different combinations of diffusion and sensitivity terms, refer to [4, 5, 50] or to [16, 18] for the case of degenerate diffusion.

Parabolic–elliptic analogues were investigated in [7, 25, 52], revealing blow-up after finite or infinite time for different parameter ranges.

If logistics and nonlinear diffusion both are incorporated into the model, at least in space dimensions $n \geq 5$ sometimes blow-up is possible: In [28], it was shown that the techniques of [45] can be applied for diffusion rates $m \in [1, \frac{2n-4}{n})$ and dampening exponents $\kappa \in (0, \frac{mn}{2(n-1)})$, resulting in finite-time blow-up for some radial solutions of the system

$$\begin{cases} u_t = \nabla \cdot ((u+1)^{m-1} \nabla u - (u \nabla v)) + \lambda(|x|)u - \mu(|x|)u^{1+\kappa}, & \text{in } \Omega \times (0, T_{\max}) \\ 0 = \Delta v - \bar{M}(t) + u, \quad \bar{M}(t) := \frac{1}{|\Omega|} \int_{\Omega} u, & \text{in } \Omega \times (0, T_{\max}) \\ \partial_{\nu} u = \partial_{\nu} v = 0, & \text{on } \partial\Omega \times (0, T_{\max}) \\ u(\cdot, 0) = u_0, & \text{in } \Omega, \end{cases} \quad (\text{JL})$$

in which the signal's equation is simplified analogously to the system from the famous work [19] by Jäger and Luckhaus. In [28, Remark 1.2], it was conjectured that blow-up should occur for $m < \frac{2n-2}{n}$ but the answer was left open to further research. For an earlier extension of the same methods from [45] to systems with nonlinear diffusion and logistics, combined with superlinear sensitivity functions, see [59].

The recent advances of [49] in the linear-diffusion case with logistic raise some hope that also in a nonlinear setting, the discovery of blow-up is also possible in the slightly less simplified parabolic–elliptic system

$$\begin{cases} u_t = \nabla \cdot ((u+1)^{m-1} \nabla u - (u \nabla v)) + \lambda(|x|)u - \mu(|x|)u^{1+\kappa}, & \text{in } \Omega \times (0, T_{\max}) \\ 0 = \Delta v - v + u, & \text{in } \Omega \times (0, T_{\max}) \\ \partial_{\nu} u = \partial_{\nu} v = 0, & \text{on } \partial\Omega \times (0, T_{\max}) \\ u(\cdot, 0) = u_0, & \text{in } \Omega \end{cases} \quad (\text{PE})$$

and, more importantly, even in the physically more relevant case of $n = 3$, for instance.

Main results The present article is dedicated to this question. Aiming for blow-up, we study (JL) and (PE) in a ball $\Omega \subset \mathbb{R}^n$, $n \geq 3$, for sufficiently smooth nonnegative functions λ, μ and a parameter $\kappa \geq 0$. We refer to the introduction of [10] for a motivation for logistic source terms with spatial dependence.

We extend the methods of [49] to nonlinear diffusion and show that they are applicable in (JL) as well as in (PE). At the same time, we would like to offer a different perspective on these, seeing them as a bridge connecting *pointwise upper estimates* of solutions to the occurrence of explosions. We therefore give our main result in the following form:

Theorem 1.1. Let $n \geq 3$, $\Omega := B_R(0)$, $R > 0$, $M_0 > 0$, $M_1 \in (0, M_0)$, $\alpha \geq 0$, $\mu_1 > 0$, $p \geq n$, $T > 0$ as well as $K > 0$ and suppose that λ, μ are such that

$$0 \leq \lambda \in C^{1+\beta}([0, R]) \quad \text{and} \quad 0 \leq \mu \in C^\beta([0, R]) \cap C^{1+\beta}((0, R]) \quad \text{for some } \beta \in (0, 1) \quad (1.4)$$

and comply with

$$\mu(r) \leq \mu_1 r^\alpha \quad \text{for all } r \in [0, R]. \quad (1.5)$$

Assume moreover that $\kappa \geq 0$ and $m > 0$ satisfy

$$m < 1 + \frac{n-2}{p}$$

and

$$0 \leq \kappa < \frac{\alpha}{p} + \min \left\{ \frac{n}{2p}, \frac{n-2}{p} - (m-1)_+ \right\} \quad \text{if } m \geq \frac{2}{p} \quad (1.6)$$

$$\text{or} \quad 0 \leq \kappa < \frac{\alpha}{p} + \min \left\{ \frac{n}{2p}, \frac{n-1}{p} - \frac{m}{2} \right\} \quad \text{if } m \in \left(0, \frac{2}{p}\right). \quad (1.7)$$

Then, we can find $r_1 \in (0, R)$ with the following property: If

$$(u, v) \in (C^0(\bar{\Omega} \times [0, T_{\max})) \times C^{2,1}(\bar{\Omega} \times (0, T_{\max})))^2$$

is a classical solution to (JL) or (PE) in $\bar{\Omega} \times [0, T_{\max})$ for some $T_{\max} \in (0, \infty]$ with

$$u_0 \in \bigcup_{\beta \in (0,1)} C^\beta(\bar{\Omega}) \quad \text{being nonnegative, radially symmetric and radially decreasing} \quad (1.8)$$

and

$$\int_{\Omega} u_0 = M_0 \quad \text{but} \quad \int_{B_{r_1}(0)} u_0 \geq M_1 \quad (1.9)$$

as well as

$$\sup_{t \in (0, \min\{T, T_{\max}\})} u(x, t) \leq K|x|^{-p} \quad \text{for all } x \in \Omega, \quad (1.10)$$

then (u, v) blows up in finite time in the sense that $T_{\max} < \infty$ and

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \quad (1.11)$$

If upper estimates as in (1.10) are known, this theorem shows that finite-time blow-up is possible in (PE) and (JL); that is, there are initial data such that (1.11) holds with some $T_{\max} < \infty$. This results in the following:

Theorem 1.2. Let $n \geq 3$, $\Omega := B_R(0)$, $R > 0$, $M_0 > 0$, $\alpha \geq 0$, $\mu_1 > 0$, $p \geq n$ and

$$m < \frac{2n-2}{n}.$$

Assume moreover that λ, μ satisfy (1.4) and (1.5).

(i) Suppose additionally $-\lambda', \mu' \geq 0$ in $(0, R)$. If

$$0 \leq \kappa < \frac{\alpha}{n} + \min \left\{ \frac{1}{2}, \frac{n-2}{n} - (m-1)_+ \right\} \quad \text{if } m \geq \frac{2}{n} \quad (1.12)$$

$$\text{or} \quad 0 \leq \kappa < \frac{\alpha}{n} + \min \left\{ \frac{1}{2}, \frac{n-1}{n} - \frac{m}{2} \right\} \quad \text{if } m \in \left(0, \frac{2}{n}\right), \quad (1.13)$$

then there exists $0 \leq u_0 \in C^0(\bar{\Omega})$ with $\int_{\Omega} u_0 = M_0$ leading to finite-time blow-up; that is, there is a classical solution (u, v) to (JL) fulfilling (1.11) for some finite $T_{\max}$.

Moreover, for $m = 1$, the condition (1.12)–(1.13) is equivalent to

$$0 \leq \kappa < \begin{cases} \frac{1}{3} + \frac{\alpha}{3}, & n = 3, \\ \frac{1}{2} + \frac{\alpha}{n}, & n \geq 4. \end{cases}$$

(ii) Provided that $m \geq 1$ and

$$0 \leq \kappa < \frac{\alpha[(m-1)n+1]}{n(n-1)} + \min \left\{ \frac{(m-1)n+1}{2(n-1)}, \frac{n-2-(m-1)n}{n(n-1)} \right\}, \quad (1.14)$$

an initial datum $0 \leq u_0 \in C^0(\bar{\Omega})$ with $\int_{\Omega} u_0 = M_0$ and a solution (u, v) of (PE) satisfying (1.11) for some $T_{\max} < \infty$ can be constructed.

Furthermore, for $m = 1$, the condition (1.14) reads

$$0 \leq \kappa < \begin{cases} \frac{1}{6} + \frac{\alpha}{6}, & n = 3, \\ \frac{1}{2(n-1)} + \frac{\alpha}{n(n-1)}, & n \geq 4. \end{cases}$$

Before we provide a more detailed comparison to the conditions on the existence of solutions blowing up in finite-time established in previous works, let us note the following.

- Remark 1.3.** (i) The condition $m < \frac{2n-2}{n}$ in Theorem 1.2 is optimal: Even without any degradation terms (i.e., $\lambda = \mu \equiv 0$), classical solutions to (JL) and (PE) are always global in time for $m > \frac{2n-2}{n}$ (cf. [52] and [25], respectively). In particular, for (JL) the upper bound on the admissible diffusion exponents in Theorem 1.2 (i) coincides with the conjectured critical exponent in [28, Remark 1.2].
- (ii) The assumption $p \geq n$ in Theorem 1.1 is a natural limitation. In fact, since (1.10) implies $\int_{\Omega} u^q \leq K^q \omega_{n-1} \int_0^R r^{n-1+pq} dr < C(q)$ for all $q < \frac{n}{p}$ and certain $C(q) > 0$, assuming that (1.10) holds for some $p < n$ and a large class of initial data, these initial data would automatically be uniformly bounded in $L^{\frac{n+p}{2p}}(\Omega)$, say, by C' . However, as can be seen by applying Hölder's inequality, their mass on $B_{r_1}(0)$ would then be bounded by $C'|B_{r_1}(0)|^{\frac{n-p}{n+p}}$, which converges to 0 for $r_1 \searrow 0$. Thus, it would not be clear if one of these initial data could still fulfill (1.9) for the value of r_1 given by Theorem 1.1.
- (iii) To the best of our knowledge, Theorem 1.2 provides the first detection of finite-time blow-up for Keller–Segel systems with nonlinear diffusion and superlinear damping terms in space dimensions 3 and 4. For (PE) and $m \neq 1$, it is furthermore the first such result in higher dimensions.
- (iv) The finite-time blow-up result for (PE) also constitutes a partial answer to the second part of Open Problem (i) in [58].

Now, let us take a more in-depth look at the new ranges for the parameter κ in some different spatial dimensions under the assumption of $\alpha = 0$ for some special values of m . In this setting, earlier works have established a certain κ^* (provided in Table 1) for which blow-up has been proven for $\kappa < \kappa^*$.

Evidently, the findings of [45] and [28] (and also of the related [59]) only cover higher dimensions. For $n \geq 5$ and $m \in [1, \frac{2n-4}{n})$, however, these results still provide better ranges than the one we could attain with our method. To the best of our knowledge, for larger values of m or for small space dimensions, however, our results provide the first proof of finite-time blow-up in (JL).

Regarding (PE), for the linear diffusion case we are able to match the range previously established in [49], while also providing first results for the nonlinear diffusion setting in higher dimensions.

When comparing the parameter ranges across the two different systems for $m = 1$ and $n \in \{2, 3\}$, we see that our results for (JL) yield a wider regime for κ than the corresponding results obtained in [49] for (PE). Indeed, $\frac{1}{3} > \frac{1}{6}$ and $\frac{1}{2} > \frac{1}{6}$. In general, known results for (JL) are stronger than for (PE). However, lacking global existence results for $\kappa < 2$, it is yet unclear whether blow-up is actually more prominent in (JL) or just easier to detect.

TABLE 1. *Supremum of the range of κ for which finite-time blow-up has been detected*

Work System	[45] (JL)	[28] (JL)	[49] (PE)	present article (JL)	(PE)
$n = 3, m = 1$			$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$
$n = 4, m = 1$			$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$
$n \geq 5, m = 1$	$\frac{n}{2(n-1)}$	$\frac{n}{2(n-1)}$	$\frac{1}{2(n-1)}$	$\frac{1}{2}$	$\frac{1}{2(n-1)}$
$n \geq 5, m \in (1, \frac{2n-4}{n})$		$\frac{nm}{2(n-1)}$		$\frac{1}{2}$	*
$n \geq 3, m \in (1, \frac{2n-2}{n})$				*	*

Here, * means that for the prescribed values of n and m , Theorem 1.2 asserts the existence of solutions blowing up in finite time for certain $\kappa > 0$, for whose precise values we refer to Theorem 1.2

Main ideas As is meanwhile well established in the context of finite-time blow-up proofs for chemotaxis systems and has first been proposed by Jäger and Luckhaus in [19], we consider the mass accumulation function

$$w(s, t) := \int_0^{\frac{1}{s^n}} \rho^{n-1} u(\rho, t) \, d\rho, \quad s \in [0, R^n], t \in [0, T_{\max}),$$

which transforms (JL) into the scalar equation

$$w_t = n^2 s^{2-\frac{2}{n}} w_{ss} + n w w_s - n \bar{m}(t) s w_s + n \int_0^s \lambda(\sigma^{\frac{1}{n}}) w_s(\sigma, t) \, d\sigma - n \int_0^s \mu(\sigma^{\frac{1}{n}}) w_s^{\kappa+1}(\sigma, t) \, d\sigma$$

(and (PE) at least into a system that is easier to handle than (PE) itself). The main difficulty for detecting finite-time blow-up lies in the fact that the term $+n w w_s$, stemming from the cross-diffusion in (JL), has to counter the quite different terms $n^2 s^{2-\frac{2}{n}} w_{ss}$ and $-n \int_0^s \mu(\sigma^{\frac{1}{n}}) w_s^2(\sigma, t) \, d\sigma$ originating from the diffusion and logistic terms, respectively.

Following [49], our approach consists of showing that for certain initial data, $\gamma \in (0, 1)$ and $s_0 \in (0, R^n)$, the function

$$\phi(s_0, \cdot) : [0, T_{\max}) \rightarrow \mathbb{R}, \quad t \mapsto \int_0^{s_0} s^{-\gamma} (s_0 - s) w(s, t) \, ds$$

cannot exist globally in time, which due to the blow-up criterion asserted in Lemma 2.1 implies the desired finite-time blow-up result (1.11). That is, in Sect. 3 we show that ϕ is a supersolution to the ODI $\phi' = a\phi^2 - b$ for certain $a, b > 0$ and in Sect. 4, we conclude the existence of initial data leading to finite-time blow-up of ϕ and hence u .

Let us briefly discuss how we deal with the two most problematic terms stemming from the degradation and diffusion terms, respectively. As we will see in Lemma 3.5, in order to handle the former, we essentially need to control

$$- \int_0^{s_0} s^{\frac{\alpha}{n}} (s_0 - s) w_s^{1+\kappa}(s, t) \, ds. \quad (1.15)$$

At this point, the assumption (1.10) comes into play, which due to $w_s(s, t) = \frac{u(s^{\frac{1}{n}}, t)}{n}$ for $(s, t) \in [0, R^n] \times [0, T_{\max})$ implies $w(s, t) \leq \frac{C}{n} s^{\frac{n}{n}}$ for $(s, t) \in [0, R^n] \times [0, T_{\max})$. Thus, as a starting point, we can apply

this estimate to w_s^κ in (1.15) and then integrate by parts. Moreover, by (3.11), the term arising from the diffusion can be estimated against (some positive multiple of)

$$- \int_0^{s_0} s^{1-\frac{2}{n}-\gamma} (s_0 - s) (nw_s + 1)^m ds.$$

For $m \geq 1$, we can proceed as above; that is, we apply the pointwise upper bound to w_s^{m-1} and then integrate by parts, while for $m < 1$ we can follow at least two different paths: For $m \in (0, \frac{2}{p})$, we apply this bound to w^m and do not integrate by parts and for $m \geq \frac{2}{p}$, we estimate $(nw_s + 1)^m \leq nw_s + 1$ and integrate by parts without using the pointwise upper estimate for w_s at all. The fact that depending on the value of m we employ two different methods here is the reason for the different conditions in (1.6) and (1.7).

At last, we show that pointwise upper estimates of the form (1.10) are indeed available both for (JL) and (PE). While for the former system we make use of the comparison principle applied to u_r in Lemma 5.1, for the latter we resort to the recent study on blow-up profiles [9] to obtain the desired bounds in Lemma 5.2.

2. Preliminaries

We henceforth always assume $n \geq 3$ and $\Omega := B_R(0) \subset \mathbb{R}^n$ for some $R > 0$. Furthermore, in Sects. 2–4, we also fix $m > 0$, $\kappa \geq 0$, $\alpha \geq 0$, $M_0 > 0$, $M_1 \in (0, M_0)$, $\lambda_1 > 0$ as well as functions λ, μ complying with $\lambda \leq \lambda_1$, (1.4) and (1.5). To simplify the notation, we also fix an initial datum u_0 satisfying (1.8) with $\int_\Omega u_0 = M_0$, but emphasize that all constants below, unless otherwise stated, are independent of u_0 .

By (u, v) , we will refer to a solution to either of the systems (JL) or (PE), and we also set $\overline{M}(t) := |\Omega|^{-1} \int_\Omega u(\cdot, t)$ for $t \in [0, T_{\max})$.

Lemma 2.1. *Suppose that $u_0: \overline{\Omega} \rightarrow [0, \infty)$ is Hölder continuous. Then, for each of the systems (JL) and (PE) there exist $T_{\max} \in (0, \infty]$ and a classical solution (u, v) , uniquely determined by*

$$u \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \quad (2.1)$$

$$v \in \bigcap_{q>n} C^0([0, T_{\max}); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$$

and, in case of (JL),

$$\int_\Omega v(\cdot, t) = 0 \quad \text{for all } t \in (0, T_{\max}).$$

Moreover, $u \geq 0$ in $\Omega \times (0, T_{\max})$ and if $T_{\max} < \infty$, then

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$

If, finally, u_0 is radially symmetric, then so are $u(\cdot, t)$ and $v(\cdot, t)$ for any $t \in (0, T_{\max})$.

Proof. Local existence can be proved by a standard fixed point argument, which is explained in more detail in [7] or [38], for instance, while nonnegativity of u follows by the maximum principle and preservation of radial symmetry is a consequence of uniqueness. $\square$

As a first basic observation, we note that, at least locally in time, the mass of u can be controlled by the parameters we fixed above—and thus, independently of the precise choice of u_0 .

Lemma 2.2. *For all $t \in (0, T_{\max})$, we have*

$$\int_{\Omega} u(\cdot, t) \leq M_0 e^{\lambda_1 t}.$$

Proof. Due to $\lambda \leq \lambda_1$ and nonnegativity of μ , integrating the first equation over Ω gives

$$\frac{d}{dt} \int_{\Omega} u = \int_{\Omega} \lambda u - \int_{\Omega} \mu u^{1+\kappa} \leq \lambda_1 \int_{\Omega} u \quad \text{in } (0, T_{\max}),$$

so that the statement follows by an ODE comparison argument. $\square$

3. Proving finite-time blow-up

Following [2, 19, 49], we define

$$w(s, t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho, t) d\rho, \quad s \in [0, R^n], t \in [0, T_{\max})$$

and, given $s_0 \in (0, R^n)$ and $\gamma \in (0, 1)$, introduce the functions

$$\phi(s_0, \cdot) : [0, T_{\max}) \rightarrow \mathbb{R}, \quad t \mapsto \int_0^{s_0} s^{-\gamma} (s_0 - s) w(s, t) ds \quad (3.1)$$

(cf. [49, equation (4.1)]) and

$$\psi(s_0, \cdot) : [0, T_{\max}) \rightarrow \mathbb{R}, \quad t \mapsto \int_0^{s_0} s^{-\gamma} (s_0 - s) w(s, t) w_s(s, t) ds.$$

If, by the usual slight abuse of notation, we identify the radially symmetric function $u \in C^0(\overline{\Omega} \times [0, T_{\max}))$ with $u \in C^0([0, R] \times [0, T_{\max}))$ and write u_r for its radial derivative, we can compute the spatial derivatives of w :

Lemma 3.1. *We have*

$$w \in C^{1,0}([0, R^n] \times [0, T_{\max})) \cap C^{2,1}([0, R^n] \times (0, T_{\max})) \cap C^{3,0}((0, R^n] \times (0, T_{\max})), \quad (3.2)$$

and

$$w_s(s, t) = \frac{1}{n} u(s^{\frac{1}{n}}, t), \quad w_{ss}(s, t) = \frac{1}{n^2} s^{\frac{1}{n}-1} u_r(s^{\frac{1}{n}}, t) \quad \text{for } s \in (0, R^n], t \in (0, T_{\max}) \quad (3.3)$$

and, with K and T from (1.10),

$$w_s(s, t) \leq \frac{K}{n} s^{-\frac{p}{n}} \quad \text{for } (s, t) \in (0, R^n] \times (0, T). \quad (3.4)$$

Proof. For the regularity, we rely on (2.1); the final estimate (3.4) results from (3.3) and (1.10). $\square$

For ϕ , which we later want to show to blow up, the following differential inequality holds:

Lemma 3.2. *For any choice of $\gamma \in (0, 1)$ and $s_0 \in (0, R^n)$, the function ϕ of (3.1) belongs to $C^0([0, T_{\max})) \cap C^1((0, T_{\max}))$ and fulfills*

$$\phi'(s_0, t) \geq n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma} (s_0 - s) (n w_s + 1)^{m-1} w_{ss}(s, t) ds$$

$$\begin{aligned}
& +n \int_0^{s_0} s^{-\gamma}(s_0-s)w(s,t)w_s(s,t) \, ds \\
& -\overline{M}(t) \int_0^{s_0} s^{1-\gamma}(s_0-s)w_s(s,t) \, ds \\
& -n^\kappa \mu_1 \int_0^{s_0} s^{-\gamma}(s_0-s) \left(\int_0^s \sigma^{\frac{\alpha}{n}} w_s^{1+\kappa}(\sigma,t) \, d\sigma \right) \, ds \\
& =: I_1(s_0,t) + I_2(s_0,t) + I_3(s_0,t) + I_4(s_0,t)
\end{aligned} \tag{3.5a}$$

for all $t \in (0, T_{\max})$ in the case of (JL). For (PE), the same estimate holds with

$$I_3(s_0,t) = -n \int_0^{s_0} s^{-\gamma}(s_0-s)w_s(s,t)z(s,t) \, ds, \quad t \in (0, T_{\max}), \tag{3.5b}$$

where

$$z(s,t) := \int_0^{\frac{1}{s^{\frac{1}{n}}}} \rho^{n-1} v(\rho,t) \, d\rho \quad \text{for } s \in [0, R^n] \text{ and } t \in [0, T_{\max}).$$

Proof. The regularity of ϕ follows from (3.2). Written in radial coordinates, the differential equations in (JL) read

$$\begin{cases} u_t(r,t) &= r^{1-n} (r^{n-1} ((u(r,t)+1)^{m-1} u_r(r,t) - u(r,t)v_r(r,t)))_r + \lambda(r)u(r,t) - \mu(r)u^{1+\kappa}(r,t), \\ 0 &= r^{1-n} (r^{n-1} v_r(r,t))_r - \overline{M}(t) + u(r,t) \end{cases} \tag{3.6}$$

for $(r,t) \in [0, R) \times (0, T_{\max})$, where the second equation can be transformed into

$$v_r(r,t) = \overline{M}(t)r^{1-n} \int_0^r \rho^{n-1} d\rho - r^{1-n} \int_0^r \rho^{n-1} u(\rho,t) d\rho = \frac{r}{n} \overline{M}(t) - r^{1-n} w(r^n,t), \tag{3.7}$$

for $r \in [0, R)$, $t \in (0, T_{\max})$, and the first equation results in

$$\begin{aligned}
w_t(s,t) &= \int_0^{\frac{1}{s^{\frac{1}{n}}}} \rho^{n-1} u_t(\rho,t) \, d\rho \\
&= \int_0^{\frac{1}{s^{\frac{1}{n}}}} \rho^{n-1} r^{1-n} (r^{n-1} ((u(r,t)+1)^{m-1} u_r(r,t) - u(r,t)v_r(r,t)))_r \Big|_{r=\rho} d\rho \\
&\quad + \int_0^{\frac{1}{s^{\frac{1}{n}}}} \rho^{n-1} \lambda(\rho) u(\rho,t) \, d\rho - \int_0^{\frac{1}{s^{\frac{1}{n}}}} \rho^{n-1} \mu(\rho) u^{1+\kappa}(\rho,t) \, d\rho \\
&= s^{\frac{n-1}{n}} u_r(s^{\frac{1}{n}},t) \left(u(s^{\frac{1}{n}},t) + 1 \right)^{m-1} - s^{\frac{n-1}{n}} u(s^{\frac{1}{n}},t) v_r(s^{\frac{1}{n}},t) \\
&\quad + \int_0^{\frac{1}{s^{\frac{1}{n}}}} \rho^{n-1} \lambda(\rho) u(\rho,t) \, d\rho - \int_0^{\frac{1}{s^{\frac{1}{n}}}} \rho^{n-1} \mu(\rho) u^{1+\kappa}(\rho,t) \, d\rho
\end{aligned}$$

for $(s, t) \in (0, R^n) \times (0, T_{\max})$. If we insert (3.3) into (3.7) and use nonnegativity of λ , we obtain that

$$w_t \geq n^2 s^{2-\frac{2}{n}} (nw_s + 1)^{m-1} w_{ss} + nww_s - \overline{M}(t)sw_s - n^\kappa \mu \int_0^s \sigma^{\frac{\alpha}{n}} w_s^{1+\kappa}(\sigma, t) d\sigma \quad (3.8)$$

in $(0, R^n) \times (0, T_{\max})$, which by multiplication with $s^{-\gamma}(s_0 - s)$ and integration implies the statement for (JL).

For (PE), in (3.6), $-\overline{M}(t)$ has to be replaced by $-v$, so that after essentially the same computation, in (3.8) the term $-\overline{M}(t)sw_s$ is substituted by $-nw_s \int_0^{\frac{1}{n}} \rho^{n-1} v(\rho, t) d\rho = -nw_s z$. $\square$

In the remaining part of this section, we further estimate the terms of the right-hand side of (3.5), aiming to show that $\phi(s_0, \cdot)$ fulfills a certain superlinear ODE. These results will then be combined in Sect. 4; ultimately, the consolidation of the lemmata will show that at least for certain values of γ and s_0 , $\phi(s_0, \cdot)$ cannot exist globally.

In order to streamline the arguments below, let us first state two elementary lemmata.

Lemma 3.3. *For all $\alpha > -1$ and $\beta > -1$ and any $s_0 \geq 0$, we have*

$$\int_0^{s_0} s^\alpha (s_0 - s)^\beta ds = B(\alpha + 1, \beta + 1) s_0^{\alpha+\beta+1}.$$

Proof. This is an evident consequence of the properties of the beta function. $\square$

Lemma 3.4. *Let $\gamma \in (0, 1)$ and $s_0 \in (0, R^n)$. Then,*

$$w(s, t) \leq \sqrt{2} s^{\frac{\gamma}{2}} (s_0 - s)^{-\frac{1}{2}} \sqrt{\psi(s_0, t)}$$

holds true for all $(s, t) \in (0, s_0) \times (0, T_{\max})$.

Proof. This inequality, in its essence based on the fundamental theorem of calculus, is a direct consequence of [49, Lemma 4.2], which for every $t \in (0, T_{\max})$ can be applied to $\varphi = w(\cdot, t)$, because $\varphi(0) = 0$, $\varphi' \geq 0$ in $(0, s_0)$ and $\varphi \in C^1([0, s_0])$. $\square$

The estimate (3.4), which originates in the crucial assumption (1.10), will come into play at two different places. The first of these is the following lemma, where said upper estimate is the most important ingredient for controlling the term arising from the logistic source, namely I_4 in (3.5).

Lemma 3.5. *Let $\gamma \in (0, 1)$ and $p \geq n$ satisfy $\frac{p\kappa}{n} - \frac{\alpha}{n} < \frac{\gamma}{2}$. Whenever (1.10) is fulfilled for some $K > 0$, $T > 0$ and $s_0 \in (0, R^n)$, then I_4 from Lemma 3.2 satisfies*

$$I_4(s_0, t) \geq -K^\kappa C s_0^{\frac{3-\gamma}{2} - \frac{p\kappa}{n} + \frac{\alpha}{n}} \sqrt{\psi(s_0, t)} \text{ for all } t \in (0, \min\{T, T_{\max}\}),$$

with $C = \frac{\mu_1 \sqrt{2}((\frac{p\kappa}{n} - \frac{\alpha}{n}) + 1)}{1-\gamma} B(\frac{\alpha}{n} - \frac{p\kappa}{n} + \frac{\gamma}{2}, \frac{1}{2})$.

Proof. This can be proved analogously to [49, Lemma 4.5]: Firstly, Fubini's theorem asserts

$$\begin{aligned} I_4(s_0, t) &= -c_1 \int_0^{s_0} \sigma^{\frac{\alpha}{n}} w_s^{1+\kappa}(\sigma, t) \left(\int_\sigma^{s_0} s^{-\gamma} (s_0 - s) ds \right) d\sigma \\ &\geq -c_1 \int_0^{s_0} \sigma^{\frac{\alpha}{n}} (s_0 - \sigma) w_s^{1+\kappa}(\sigma, t) \left(\int_0^{s_0} s^{-\gamma} ds \right) d\sigma \end{aligned}$$

$$= -c_2 s_0^{1-\gamma} \int_0^{s_0} s^{\frac{\alpha}{n}} (s_0 - s) w_s^{1+\kappa}(s, t) ds$$

for all $t \in (0, T_{\max})$, where I_4 is as in (3.5), $c_1 := n^\kappa \mu_1 > 0$ and $c_2 := \frac{c_1}{1-\gamma} > 0$. Next, we use (3.4) and integrate by parts to see that

$$\begin{aligned} \int_0^{s_0} s^{\frac{\alpha}{n}} (s_0 - s) w_s^{1+\kappa}(s, t) ds &\leq \frac{K^\kappa}{n^\kappa} \int_0^{s_0} s^{\frac{\alpha}{n} - \frac{p\kappa}{n}} (s_0 - s) w_s(s, t) ds \\ &= \frac{K^\kappa}{n^\kappa} \left(\frac{p\kappa}{n} - \frac{\alpha}{n} \right) \int_0^{s_0} s^{\frac{\alpha}{n} - \frac{p\kappa}{n} - 1} (s_0 - s) w(s, t) ds \\ &\quad + \frac{K^\kappa}{n^\kappa} \int_0^{s_0} s^{\frac{\alpha}{n} - \frac{p\kappa}{n}} w(s, t) ds + \frac{K^\kappa}{n^\kappa} \left[s^{\frac{\alpha}{n} - \frac{p\kappa}{n}} (s_0 - s) w(s, t) \right]_0^{s_0} \\ &\leq K^\kappa c_3 s_0 \int_0^{s_0} s^{\frac{\alpha}{n} - \frac{p\kappa}{n} - 1} w(s, t) ds \end{aligned}$$

holds for all $t \in (0, \min\{T, T_{\max}\})$ with $c_3 := \frac{(\frac{p\kappa}{n} - \frac{\alpha}{n})_+ + 1}{n^\kappa} > 0$. Here, we apply Lemma 3.4 and obtain

$$\int_0^{s_0} s^{\frac{\alpha}{n} - \frac{p\kappa}{n} - 1} w(s, t) ds \leq \sqrt{2} \int_0^{s_0} s^{\frac{\alpha}{n} - \frac{p\kappa}{n} - 1 + \frac{\gamma}{2}} (s_0 - s)^{-\frac{1}{2}} ds \sqrt{\psi(s_0, t)}$$

for all $t \in (0, \min\{T, T_{\max}\})$. Finally, we note that according to Lemma 3.3

$$\int_0^{s_0} s^{\frac{\alpha}{n} - \frac{p\kappa}{n} - 1 + \frac{\gamma}{2}} (s_0 - s)^{-\frac{1}{2}} ds = B \left(\frac{\alpha}{n} - \frac{p\kappa}{n} + \frac{\gamma}{2}, \frac{1}{2} \right) s_0^{\frac{\alpha}{n} - \frac{p\kappa}{n} + \frac{\gamma}{2} - \frac{1}{2}},$$

since $\frac{\alpha}{n} - \frac{p\kappa}{n} + \frac{\gamma}{2} > 0$ by assumption. The statement follows by combining the estimates above. $\square$

We now turn our attention to the integral involving the effects of nonlinear diffusion. This is the second place where (at least for certain m) we make use of the assumption (1.10).

Lemma 3.6. *Suppose that (1.10) holds for some $p \geq n$, $K > 0$ and $T > 0$ and let I_1 be as in (3.5).*

(i) *Assume that*

$$0 < m < 1 + \frac{n-2}{p} \quad \text{and} \quad 1 - \frac{2}{n} - \frac{p}{n}(m-1)_+ < \gamma < 2 - \frac{4}{n} - \frac{2p}{n}(m-1)_+. \quad (3.9)$$

Then, there is $C > 0$ such that for any $s_0 \in (0, R^n)$ and all $t \in (0, \min\{T, T_{\max}\})$, we have

$$I_1(s_0, t) \geq -C s_0^{\frac{3-\gamma}{2} - \frac{2}{n} - \frac{p}{n}(m-1)_+} \sqrt{\psi(s_0, t)} - C s_0^{3-\gamma-\frac{2}{n}}.$$

(ii) *Assume that*

$$0 < m < \min \left\{ 1, \frac{2(n-1)}{p} \right\} \quad \text{and} \quad 0 < \gamma < 2 - \frac{2}{n} - \frac{pm}{n}. \quad (3.10)$$

Then, there is $C > 0$ such that for any $s_0 \in (0, R^n)$ and all $t \in (0, \min\{T, T_{\max}\})$, we have

$$I_1(s_0, t) \geq -C s_0^{3-\gamma-\frac{2}{n}-\frac{p}{n}m} - C s_0^{3-\gamma-\frac{2}{n}}.$$

Proof. Direct calculation gives for every $s_0 \in (0, R^n)$

$$\begin{aligned}
 I_1(s_0, t) &= n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma} (s_0 - s) (nw_s + 1)^{m-1} w_{ss} \, ds \\
 &= \frac{n}{m} \int_0^{s_0} s^{2-\frac{2}{n}-\gamma} (s_0 - s) ((nw_s + 1)^m)_s \, ds \\
 &= -\frac{n}{m} \left(2 - \frac{2}{n} - \gamma\right) \int_0^{s_0} s^{1-\frac{2}{n}-\gamma} (s_0 - s) (nw_s + 1)^m \, ds \\
 &\quad + \frac{n}{m} \int_0^{s_0} s^{2-\frac{2}{n}-\gamma} (nw_s + 1)^m \, ds + \frac{n}{m} \left[s^{2-\frac{2}{n}-\gamma} (s_0 - s) (nw_s + 1)^m \right]_0^{s_0}
 \end{aligned}$$

in $(0, T_{\max})$. The last two terms therein are positive, since (3.9) and (3.10) both entail $2 - \frac{2}{n} - \gamma > 0$, leading to

$$I_1(s_0, t) \geq -\frac{n}{m} \left(2 - \frac{2}{n} - \gamma\right) \int_0^{s_0} s^{1-\frac{2}{n}-\gamma} (s_0 - s) (nw_s + 1)^m \, ds \quad \text{for } (s_0, t) \in (0, R^n) \times (0, T_{\max}). \quad (3.11)$$

Now, let us start by considering the case that (3.9) holds. First, we find that $w_s \geq 0$ implies for $m \geq 1$ that $(nw_s + 1)^m \leq 2^{m-1} (n^m w_s^m + 1)$ in $(0, T_{\max})$. By (3.4), this entails that $(nw_s + 1)^m \leq 2^{m-1} K^{m-1} n s^{-\frac{p}{n}(m-1)} w_s + 2^{m-1}$ on $(0, R^n) \times (0, \min\{T, T_{\max}\})$. On the other hand, for $m \in (0, 1)$ we have $(nw_s + 1)^m \leq nw_s + 1$ in $(0, R^n) \times (0, \min\{T, T_{\max}\})$. Thus, letting $c_1 := \max\{n, 2^{m-1}, 2^{m-1} n K^{(m-1)+}\}$ we find that from combining these two estimates, we have

$$(nw_s + 1)^m \leq c_1 s^{-\frac{p}{n}(m-1)+} w_s + c_1 \quad \text{in } (0, R^n) \times (0, \min\{T, T_{\max}\})$$

and hence, from (3.11),

$$\begin{aligned}
 I_1(s_0, t) &\geq -\frac{n}{m} \left(2 - \frac{2}{n} - \gamma\right) \int_0^{s_0} s^{1-\frac{2}{n}-\gamma} (s_0 - s) (nw_s + 1)^m \, ds \\
 &\geq -c_1 \left(2 - \frac{2}{n} - \gamma\right) \int_0^{s_0} s^{1-\frac{2}{n}-\frac{p}{n}(m-1)+-\gamma} (s_0 - s) w_s \, ds - c_1 \left(2 - \frac{2}{n} - \gamma\right) \int_0^{s_0} s^{1-\frac{2}{n}-\gamma} (s_0 - s) \, ds
 \end{aligned}$$

in $(0, \min\{T, T_{\max}\})$ and for every $s_0 \in (0, R^n)$. An integration by parts therefore yields

$$\begin{aligned}
 I_1(s_0, t) &\geq c_1 \left(2 - \frac{2}{n} - \gamma\right) \left(1 - \frac{2}{n} - \frac{p}{n}(m-1)+ - \gamma\right) \int_0^{s_0} s^{-\frac{2}{n}-\frac{p}{n}(m-1)+-\gamma} (s_0 - s) w \, ds \\
 &\quad - c_1 \left(2 - \frac{2}{n} - \gamma\right) \int_0^{s_0} s^{1-\frac{2}{n}-\frac{p}{n}(m-1)+-\gamma} w \, ds \\
 &\quad - c_1 \left(2 - \frac{2}{n} - \gamma\right) \left[s^{1-\frac{2}{n}-\frac{p}{n}(m-1)+-\gamma} (s_0 - s) w \, ds \right]_0^{s_0} \\
 &\quad - c_1 \left(2 - \frac{2}{n} - \gamma\right) \int_0^{s_0} s^{1-\frac{2}{n}-\gamma} (s_0 - s) \, ds \quad \text{in } (0, \min\{T, T_{\max}\}) \text{ and for every } s_0 \in (0, R^n).
 \end{aligned}$$

The third term on the right-hand side is nonnegative, and in the other terms, we use $s_0 - s \leq s_0$ and $s \leq s_0$ for all $s \in (0, s_0)$ as well as the conditions $\gamma < 2 - \frac{2}{n}$ and $\gamma > 1 - \frac{2}{n} - \frac{p}{n}(m-1)_+$ contained in (3.9) to see that

$$I_1(s_0, t) \geq -c_1 \left(2 - \frac{2}{n} - \gamma\right) \left(\gamma + \frac{2}{n} + \frac{p}{n}(m-1)_+\right) s_0 \int_0^{s_0} s^{-\frac{2}{n} - \frac{p}{n}(m-1)_+ - \gamma} w \, ds - c_1 s_0^{3-\gamma-\frac{2}{n}} \quad (3.12)$$

in $(0, \min\{T, T_{\max}\})$ and for $s_0 \in (0, R^n)$. To estimate further, we make use of Lemma 3.4, the fact that (3.9) entails $0 < 1 - \frac{2}{n} - \frac{p}{n}(m-1)_+ - \frac{\gamma}{2}$ and Lemma 3.3 to obtain

$$\begin{aligned} s_0 \int_0^{s_0} s^{-\frac{2}{n} - \frac{p}{n}(m-1)_+ - \gamma} w \, ds &\leq \sqrt{2} s_0 \left(\int_0^{s_0} s^{-\frac{2}{n} - \frac{p}{n}(m-1)_+ - \frac{\gamma}{2}} (s_0 - s)^{-\frac{1}{2}} \, ds \right) \sqrt{\psi(s_0, t)} \\ &= c_3 s_0^{\frac{3-\gamma}{2} - \frac{2}{n} - \frac{p}{n}(m-1)_+} \sqrt{\psi(s_0, t)} \quad \text{in } (0, T_{\max}) \end{aligned} \quad (3.13)$$

for $c_3 = \sqrt{2}B(1 - \frac{2}{n} - \frac{p}{n}(m-1)_+ - \frac{\gamma}{2}, \frac{1}{2}) > 0$. Collecting (3.12) and (3.13) proves the estimate of I_1 in the case that (3.9) holds.

To verify the asserted inequality in the case of (3.10), we return to (3.11) and note that due to $w_s \geq 0$ and $m \in (0, 1)$, we have $(nw_s + 1)^m \leq n^m w_s^m + 1$ on $(0, \min\{T, T_{\max}\})$. Here, we rely on (3.4) to conclude that $(nw_s + 1)^m \leq K^m s^{-\frac{pm}{n}} + 1$ in $(0, \min\{T, T_{\max}\})$ and hence

$$I_1(s_0, t) \geq -\frac{K^m n}{m} \left(2 - \frac{2}{n} - \gamma\right) \int_0^{s_0} s^{1-\frac{2}{n}-\frac{pm}{n}-\gamma} (s_0 - s) \, ds - \frac{n}{m} \left(2 - \frac{2}{n} - \gamma\right) \int_0^{s_0} s^{1-\frac{2}{n}-\gamma} (s_0 - s) \, ds$$

in $(0, \min\{T, T_{\max}\})$ and for $s_0 \in (0, R^n)$, where the conditions $\gamma < 2 - \frac{2}{n} - \frac{pm}{n}$ and $\gamma < 2 - \frac{2}{n}$ contained in (3.10) together with $s_0 - s \leq s_0$ entail

$$I_1(s_0, t) \geq -c_4 s_0^{3-\gamma-\frac{2}{n}-\frac{pm}{n}} - \frac{n}{m} s_0^{3-\gamma-\frac{2}{n}} \quad \text{in } (0, \min\{T, T_{\max}\})$$

for $c_4 = \frac{2-\frac{2}{n}-\gamma}{2-\frac{2}{n}-\frac{pm}{n}-\gamma} \cdot \frac{K^m n}{m} > 0$, completing the proof. $\square$

The arguments for estimating the remaining integrals in (3.5) rely on the following relation between ϕ and ψ , which was also obtained in [49, Lemma 3.4].

Lemma 3.7. *Let $\gamma \in (0, 1)$. For every $s_0 \in (0, R^n)$ and $t \in (0, T_{\max})$,*

$$\phi(s_0, t) \leq C s_0^{\frac{3-\gamma}{2}} \sqrt{\psi(s_0, t)},$$

where $C = \sqrt{2}B(1 - \frac{\gamma}{2}, \frac{1}{2})$.

Proof. By Lemma 3.4,

$$\phi(s_0, t) \leq s_0 \int_0^{s_0} s^{-\gamma} w(s, t) \, ds \leq s_0 \sqrt{2} \sqrt{\psi} \int_0^{s_0} s^{-\frac{\gamma}{2}} (s_0 - s)^{-\frac{1}{2}} \, ds,$$

and the claim follows from Lemma 3.3. $\square$

Lemma 3.8. *Let $\gamma \in (0, 1)$. Then, for all $s_0 \in (0, R^n)$ and all $t \in (0, T_{\max})$*

$$I_2(s_0, t) \geq C s_0^{\gamma-3} \phi^2(s_0, t)$$

with ϕ and I_2 as in (3.5) and $C = \frac{n}{2B^2(1-\frac{\gamma}{2}, \frac{1}{2})}$.

Proof. As $I_2 = n\psi$, this is a corollary of Lemma 3.7. $\square$

Lemma 3.9. *Let $\gamma \in (0, 1)$ and $T > 0$. Then, there is $C > 0$ such that for all $s_0 \in (0, R^n)$ and all $t \in (0, \min\{T, T_{\max}\})$*

$$I_3(s_0, t) \geq -Cs_0^{\frac{2}{n}+1-\gamma} - Cs_0^{\frac{2}{n}}\psi(s_0, t) \quad (3.14)$$

with I_3 as in (3.5).

Remark 3.10. For (JL), we can even obtain the stronger estimate

$$I_3(s_0, t) \geq -Ce^{\lambda_1 t} s_0^{\frac{3-\gamma}{2}} \sqrt{\psi(s_0, t)} \geq -Cs_0^{\frac{2}{n}+3-\gamma} e^{2\lambda_1 t} - Cs_0^{\frac{2}{n}}\psi(s_0, t) \quad (3.15)$$

in place of (3.14).

Proof of Lemma 3.9 and Remark 3.10. As their expressions for I_3 differ, we treat the cases of (JL) and (PE) separately, beginning with (JL): An integration by parts and Lemma 3.4 show

$$\begin{aligned} I_3(s_0, t) &= \overline{M}(t)(1-\gamma) \int_0^{s_0} s^{-\gamma}(s_0-s)w(s, t) \, ds - \overline{M}(t) \int_0^{s_0} s^{1-\gamma}w(s, t) \, ds - 0 \\ &\geq -\overline{M}(t)\sqrt{2} \int_0^{s_0} s^{1-\frac{\gamma}{2}}(s_0-s)^{-\frac{1}{2}} \, ds \cdot \sqrt{\psi(s_0, t)} \end{aligned}$$

for $t \in (0, T_{\max})$ and $s_0 \in (0, R^n)$, so that with some $c_1 > 0$, $c_2 > 0$,

$$I_3 \geq -\frac{\sqrt{2}M_0}{|\Omega|} e^{\lambda_1 t} c_1 s_0^{\frac{3-\gamma}{2}} \sqrt{\psi(s_0, t)} \geq -c_2 e^{2\lambda_1 t} s_0^{3-\gamma+\frac{2}{n}} - c_2 s_0^{\frac{2}{n}}\psi(s_0, t)$$

for $t \in (0, T_{\max})$, $s_0 \in (0, R^n)$ by Lemmata 2.2, 3.3 and Young's inequality. This proves (3.15) and, since $s_0^{3-\gamma+\frac{2}{n}} \leq R^{2n} s_0^{1-\gamma+\frac{2}{n}}$ for $s_0 \in (0, R^n)$, also (3.14).

As to (PE), we follow [49, Lemma 4.8] but need to make some modifications to remove the additional condition $\gamma < 2 - \frac{4}{n}$ required there. To that end, we first fix $\tilde{\gamma} \in (\max\{\gamma - \frac{4}{n}, 0\}, \min\{\gamma, 2 - \frac{4}{n}\})$ and apply a variant of [49, Lemma 4.7] (for $\tilde{\gamma} \in (0, 2 - \frac{4}{n})$ instead of γ) to obtain $\Gamma = \Gamma(R, \lambda, \gamma, M_0) > 0$ such that

$$\begin{aligned} z(s, t) &\leq \Gamma s_0^{\frac{2}{n}-1} s + \Gamma s_0^{-\frac{1}{2}} s^{\frac{2}{n}+\frac{\tilde{\gamma}}{2}} \left(\int_0^{s_0} \sigma^{-\tilde{\gamma}}(s_0-\sigma)w(\sigma, t)w_s(\sigma, t) \, d\sigma \right)^{\frac{1}{2}} \\ &\leq \Gamma s_0^{\frac{2}{n}-1} s + \Gamma s_0^{-\frac{1}{2}} s^{\frac{2}{n}+\frac{\tilde{\gamma}}{2}} \left(\int_0^{s_0} s_0^{\gamma-\tilde{\gamma}} \sigma^{-\gamma}(s_0-\sigma)w(\sigma, t)w_s(\sigma, t) \, d\sigma \right)^{\frac{1}{2}} \\ &= \Gamma s_0^{\frac{2}{n}-1} s + \Gamma s_0^{-\frac{1}{2}+\frac{\gamma-\tilde{\gamma}}{2}} s^{\frac{2}{n}+\frac{\tilde{\gamma}}{2}} \sqrt{\psi(s_0, t)} \quad \text{for all } s_0 \in (0, R^n), t \in (0, \min\{T, T_{\max}\}). \end{aligned}$$

Combined with (4.5) of [49], Lemmata 2.2, 3.4 and 3.3, this shows that for any $s_0 \in (0, R^n)$, $t \in (0, \min\{T, T_{\max}\})$,

$$\begin{aligned} \frac{I_3(s_0, t)}{n(\gamma+1)} &\geq -s_0 \int_0^{s_0} s^{-\gamma-1} z(s, t) w(s, t) \, ds \\ &\geq -M_0 \Gamma e^{\lambda_1 t} s_0^{\frac{2}{n}} \int_0^{s_0} s^{-\gamma} \, ds - \Gamma s_0^{\frac{1}{2}+\frac{\gamma-\tilde{\gamma}}{2}} \int_0^{s_0} s^{\frac{2}{n}-1-\gamma+\frac{\tilde{\gamma}}{2}} w(s, t) \, ds \cdot \sqrt{\psi(s_0, t)} \end{aligned}$$

$$\begin{aligned}
&\geq -\frac{M_0 \Gamma e^{\lambda_1 T}}{1-\gamma} s_0^{1+\frac{2}{n}-\gamma} - \Gamma s_0^{\frac{1}{2}+\frac{\gamma-\tilde{\gamma}}{2}} \int_0^{s_0} s^{\frac{2}{n}-1-\frac{\gamma-\tilde{\gamma}}{2}} (s_0-s)^{-\frac{1}{2}} ds \cdot \psi(s_0, t) \\
&\geq -\frac{M_0 \Gamma e^{\lambda_1 T}}{1-\gamma} s_0^{1+\frac{2}{n}-\gamma} - \Gamma B \left(\frac{2}{n} - \frac{\gamma-\tilde{\gamma}}{2}, \frac{1}{2} \right) s_0^{\frac{2}{n}} \psi(s_0, t),
\end{aligned}$$

which again implies (3.14). $\square$

Combining these lemmata shows that $\phi(s_0, \cdot)$ is indeed a supersolution to a superlinear ODE as long as s_0 is sufficiently small and γ can be chosen in a suitable way.

Lemma 3.11. *Let $p \geq n$, $K > 0$ and $T > 0$. (i) Suppose $m \in [\frac{2}{p}, 1 + \frac{n-2}{p})$, $0 \leq \kappa < \frac{\alpha}{p} + \min\{\frac{n}{2p}, \frac{n-2}{p} - (m-1)_+\}$ and*

$$\max \left\{ 0, \frac{2p\kappa}{n} - \frac{2\alpha}{n}, 1 - \frac{2}{n} - \frac{p}{n}(m-1)_+ \right\} < \gamma < \min \left\{ 2 - \frac{4}{n} - \frac{2p}{n}(m-1)_+, 1 \right\}. \quad (3.16)$$

Then, there are $C_1, C_2 > 0$, $\theta \in (0, 2)$ and $s_1 \in (0, R^n)$ such that for every solution complying with (1.10), we have

$$\frac{\partial}{\partial t} \phi(s_0, t) \geq C_1 s_0^{\gamma-3} \phi^2(s_0, t) - C_2 s_0^{3-\gamma-\theta} \quad \text{for } t \in (0, \min\{T, T_{\max}\}) \text{ and } s_0 \in (0, s_1). \quad (3.17)$$

(ii) Suppose $m \in (0, \frac{2}{p})$, $0 \leq \kappa < \frac{\alpha}{p} + \min\{\frac{n}{2p}, \frac{n-1}{p} - \frac{m}{2}\}$ and

$$\max \left\{ 0, \frac{2p\kappa}{n} - \frac{2\alpha}{n} \right\} < \gamma < \min \left\{ 2 - \frac{2}{n} - \frac{pm}{n}, 1 \right\}. \quad (3.18)$$

Then, there are $C_1, C_2 > 0$, $\theta \in (0, 2)$ and $s_1 \in (0, R^n)$ such that (3.17) holds for every solution complying with (1.10) also in this case.

Proof. To verify part (i), we note that the prescribed conditions on κ and γ render Lemma 3.5 and the first part of Lemma 3.6 applicable, in addition to Lemmata 3.8 and 3.9. Inserting their respective results into (3.5) of Lemma 3.2, we find that there are $c_1, c_2 > 0$ such that

$$\begin{aligned}
\phi'(s_0, t) &\geq I_1(s_0, t) + I_2(s_0, t) + I_3(s_0, t) + I_4(s_0, t) \\
&\geq -c_1 s_0^{\frac{3-\gamma}{2}-\frac{2}{n}-\frac{p}{n}(m-1)_+} \sqrt{\psi(s_0, t)} - c_1 s_0^{3-\frac{2}{n}-\gamma} + c_2 s_0^{\gamma-3} \phi^2(s_0, t) \\
&\quad - c_1 s_0^{\frac{2}{n}+1-\gamma} - c_1 s_0^{\frac{2}{n}} \psi(s_0, t) - c_1 s_0^{\frac{3-\gamma}{2}+\frac{\alpha}{n}-\frac{p\kappa}{n}} \sqrt{\psi(s_0, t)}
\end{aligned}$$

is valid for all $t \in (0, \min\{T, T_{\max}\})$ and for every $s_0 \in (0, R^n)$. Young's inequality shows that thus for every $\eta > 0$, there is $c_3(\eta) > 0$ satisfying

$$\begin{aligned}
\frac{\partial}{\partial t} \phi(s_0, t) &\geq c_2 s_0^{\gamma-3} \phi^2(s_0, t) - c_1 s_0^{\frac{2}{n}} \psi(s_0, t) - \eta \psi(s_0, t) \\
&\quad - c_3(\eta) \left(s_0^{3-\gamma-\frac{4}{n}-\frac{2p}{n}(m-1)_+} + s_0^{3-\gamma-\frac{2}{n}} + s_0^{1+\frac{2}{n}-\gamma} + s_0^{3-\gamma-\frac{2p\kappa}{n}+\frac{2\alpha}{n}} \right)
\end{aligned}$$

for all $(0, \min\{T, T_{\max}\})$. If we employ Lemma 3.7, fix s_1 and η sufficiently small and use that $s_0 \leq s_1 \leq R^n$, we obtain (3.17) with

$$3 - \gamma - \theta = \min \left\{ 3 - \gamma - \frac{4}{n} - \frac{2p}{n}(m-1)_+, 3 - \gamma - \frac{2}{n}, 1 + \frac{2}{n} - \gamma, 3 - \gamma - \frac{2p\kappa}{n} + \frac{2\alpha}{n} \right\}.$$

Observing that $\frac{4}{n} + \frac{2p}{n}(m-1)_+ < \frac{4}{n} + \frac{2p}{n} \cdot \frac{n-2}{p} = 2$ due to $m < 1 + \frac{n-2}{p}$ and $\frac{2p\kappa}{n} - \frac{2\alpha}{n} < \frac{2\alpha+n}{n} - \frac{2\alpha}{n} = 1 < 2 - \frac{2}{n} < 2$ because of $\kappa < \frac{\alpha}{p} + \frac{n}{2p}$, we conclude

$$\theta = \max \left\{ \frac{4}{n} + \frac{2p}{n}(m-1)_+, \frac{2}{n}, 2 - \frac{2}{n}, \frac{2p\kappa}{n} - \frac{2\alpha}{n} \right\} \in (0, 2).$$

As to part (ii), similarly as before with Lemma 3.6, part (ii), in place of Lemma 3.6, part (i), we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \phi'(s_0, t) &\geq -c_1 s_0^{3-\gamma-\frac{2}{n}-\frac{pm}{n}} - c_1 s_0^{3-\gamma-\frac{2}{n}} + c_2 s_0^{\gamma-3} \phi^2(s_0, t) \\ &\quad - c_1 s_0^{\frac{2}{n}+1-\gamma} - c_1 s_0^{\frac{2}{n}} \psi(s_0, t) - c_1 s_0^{\frac{3-\gamma}{2}+\frac{\alpha}{n}-\frac{p\kappa}{n}} \sqrt{\psi(s_0, t)} \end{aligned}$$

for all $(0, \min\{T, T_{\max}\})$ and $s_0 \in (0, R^n)$ and conclude as before, with

$$\theta = \max \left\{ \frac{2}{n} + \frac{pm}{n}, \frac{2}{n}, 2 - \frac{2}{n}, \frac{2p\kappa}{n} - \frac{2\alpha}{n} \right\},$$

where again $\frac{2p\kappa}{n} - \frac{2\alpha}{n} < \frac{2\alpha+n}{n} - \frac{2\alpha}{n} < 2$ and $\frac{2}{n} + \frac{pm}{n} < \frac{2}{n} + \frac{2}{n} < 2$, so that $\theta \in (0, 2)$. $\square$

Let us close this section by verifying that the parameter choices of Lemma 3.11 are indeed feasible.

Lemma 3.12. *Under the conditions on m and κ in Lemma 3.11, part (i) or (ii), there is $\gamma > 0$ satisfying (3.16) or (3.18), respectively. Moreover, for every choice of m as indicated there, the respective ranges of κ are nonempty.*

Proof. (i) First, we note that due to $p \geq n > 2$, the interval $A := [\frac{2}{p}, 1 + \frac{n-2}{p})$ is not empty. For $m \in A$, due to $m < 1 + \frac{n-2}{p}$ and $\alpha \geq 0$, we then see that $B := [0, \frac{\alpha}{p} + \min\{\frac{n}{2p}, \frac{n-2}{p} - (m-1)_+\})$ is also not empty. Now, to check that for $m \in A$ and $\kappa \in B$, the condition (3.16) is not empty, we first note that $\kappa < \frac{\alpha}{p} + \frac{n-2}{p} - (m-1)_+$ and $m < 1 + \frac{n-2}{p}$ imply that

$$\frac{2p\kappa}{n} - \frac{2\alpha}{n} < 2 - \frac{4}{n} - \frac{2p}{n}(m-1)_+ \quad \text{and} \quad 1 - \frac{2}{n} - \frac{p}{n}(m-1)_+ < 2 - \frac{4}{n} - \frac{2p}{n}(m-1)_+.$$

Moreover, we find that $\kappa < \frac{n}{2p} + \frac{\alpha}{p}$ and $m < 1 + \frac{n-2}{p}$ entail that

$$\frac{2p\kappa}{n} - \frac{2\alpha}{n} < 1 \quad \text{and} \quad 0 < 1 - \frac{2}{n} - \frac{p}{n}(m-1)_+ < 1,$$

so that indeed $\max\{0, \frac{2p\kappa}{n} - \frac{2\alpha}{n}, 1 - \frac{2}{n} - \frac{p}{n}(m-1)_+\} < \min\{2 - \frac{4}{n} - \frac{2p}{n}(m-1)_+, 1\}$.

(ii) We first note that $m < \frac{2}{p}$ ensures that $\frac{n-1}{p} - \frac{m}{2} > \frac{n-2}{p} > 0$, so that κ with $0 \leq \kappa < \frac{\alpha}{p} + \min\{\frac{n}{2p}, \frac{n-1}{p} - \frac{m}{2}\}$ exists. From $\kappa < \frac{\alpha}{p} + \frac{n}{2p}$, we conclude that $\frac{2p\kappa}{n} - \frac{2\alpha}{n} < \frac{2p}{n}(\frac{\alpha}{p} + \frac{n}{2p}) - \frac{2\alpha}{n} = 1$, and the condition that $\kappa < \frac{\alpha}{p} + \frac{n-1}{p} - \frac{m}{2}$ shows that $\frac{2p\kappa}{n} - \frac{2\alpha}{n} < \frac{2p}{n}(\frac{\alpha}{p} + \frac{n-1}{p} - \frac{m}{2}) - \frac{2\alpha}{n} = 2 - \frac{2}{n} - \frac{pm}{n}$. Together with the fact that $2 - \frac{2}{n} - \frac{pm}{n} > 2 - \frac{2}{n} - \frac{2}{n} \geq 0$, this shows that $\max\{0, \frac{2p\kappa}{n} - \frac{2\alpha}{n}\} < \min\{2 - \frac{2}{n} - \frac{pm}{n}, 1\}$. $\square$

4. First conclusion: Proof of Theorem 1.1

While Lemmata 3.11 and 3.12 already show that $\phi(s_0, \cdot)$ is (for certain s_0 and γ , at least) a supersolution to a superlinear ODE, we still need to show that $\phi(s_0, 0)$ can be arranged to be suitably large. We take care of this last step in the following lemma; Theorem 1.1 will then be proven directly thereafter.

Lemma 4.1. *Let $\gamma \in (0, 1)$, $s_0 \in (0, R)$, $M_1 \geq 0$ as well as $\eta \in (0, 1)$ and set $s_\eta := (1 - \eta)s_0$ as well as $r_1 := s_\eta^{\frac{1}{n}}$. If*

$$\int_{B_{r_1}(0)} u_0 \geq M_1,$$

then

$$\phi(s_0, 0) \geq \frac{\eta^2 M_1}{\omega_{n-1}} \cdot s_0^{2-\gamma}.$$

Proof. We use positivity and monotonicity of $w_0 := w(\cdot, 0)$, $w_0(s_\eta) = \frac{1}{\omega_{n-1}} \int_{B_{r_1}} u_0$ and $s_0 - s_1 = \eta s_0$ as well as the fact that $1 - (1 - \eta)^{1-\gamma} \geq \inf_{\xi \in (0, \eta)} (1 - \gamma)(1 - \xi)^{-\gamma} \eta = (1 - \gamma)\eta$ holds by the mean value theorem, to see that

$$\begin{aligned} \phi(s_0, 0) &= \int_0^{s_0} s^{-\gamma}(s_0 - s)w_0(s) \, ds \geq w_0(s_\eta) \int_{s_\eta}^{s_0} s^{-\gamma}(s_0 - s_\eta) \, ds \\ &\geq \frac{\eta M_1}{(1 - \gamma)\omega_{n-1}} s_0 \left(s_0^{1-\gamma} - s_\eta^{1-\gamma} \right) \geq \frac{\eta^2 M_1}{\omega_{n-1}} \cdot s_0^{2-\gamma}. \end{aligned}$$

□

□

These preparations now allow us to indeed prove Theorem 1.1.

Proof of Theorem 1.1. We observe that for $a, b, y_0 > 0$ with $\sqrt{\frac{a}{b}}y_0 > 1$, the solution

$$y(t) = \sqrt{\frac{b}{a}} \cdot \frac{1 + \frac{\sqrt{\frac{a}{b}}y_0 - 1}{\sqrt{\frac{a}{b}}y_0 + 1} e^{2\sqrt{abt}}}{1 - \frac{\sqrt{\frac{a}{b}}y_0 - 1}{\sqrt{\frac{a}{b}}y_0 + 1} e^{2\sqrt{abt}}} \quad \text{of} \quad \begin{cases} y' = ay^2 - b, \\ y(0) = y_0 \end{cases}$$

blows up at the finite time t with $e^{2\sqrt{abt}} = \frac{\sqrt{\frac{a}{b}}y_0 + 1}{\sqrt{\frac{a}{b}}y_0 - 1}$. If $\sqrt{\frac{a}{b}}y_0 > 2$, then $\frac{\sqrt{\frac{a}{b}}y_0 + 1}{\sqrt{\frac{a}{b}}y_0 - 1} < \frac{\sqrt{\frac{a}{b}}y_0 + \frac{1}{2}\sqrt{\frac{a}{b}}y_0}{\sqrt{\frac{a}{b}}y_0 - \frac{1}{2}\sqrt{\frac{a}{b}}y_0} = 3$ and blow-up hence occurs before time $\frac{1}{2\sqrt{ab}} \ln 3$. Given $m, n, p, \kappa, M_0, M_1, K, T$ from Theorem 1.1, we use Lemma 3.12 to find $\gamma \in (0, 1)$ such that Lemma 3.11 is applicable and let C_1 and C_2, θ and s_1 be as defined there. We furthermore set $C_3 := \frac{M_1}{4\omega_{n-1}}$ and introduce

$$a(s_0) = C_1 s_0^{\gamma-3}, \quad b(s_0) = C_2 s_0^{3-\gamma-\theta} \quad \text{and} \quad \phi_0(s_0) = C_3 s_0^{2-\gamma}.$$

By positivity of θ , $\sqrt{ab} \rightarrow \infty$ as $s_0 \rightarrow 0$; additionally, $\phi_0^{\frac{2}{a}} \rightarrow \infty$ as $s_0 \rightarrow 0$, because the exponent of s_0 in this expression is negative according to

$$2(2 - \gamma) + \gamma - 3 - (3 - \gamma - \theta) = -2 + \theta < 0.$$

We therefore can pick $s_0 \in (0, s_1)$ so small that $\frac{\ln 3}{2\sqrt{a(s_0)b(s_0)}} < T$ and $\phi_0 \sqrt{\frac{a(s_0)}{b(s_0)}} > 2$, and finally let $r_1 = (\frac{1}{2}s_0)^{\frac{1}{n}}$. Then, by Lemma 3.11 in conjunction with Lemma 4.1 for $\eta = \frac{1}{2}$, for every solution obeying (1.8), (1.9) and (1.10), the function $\phi := \phi(s_0, \cdot)$ from (3.1) satisfies

$$\phi'(t) \geq a\phi^2(t) - b \quad \text{for every } t \in (0, \min\{T, T_{\max}\}), \quad \phi(0) \geq \phi_0$$

and hence $\phi(t) \geq y(t)$ for all $t \in (0, \min\{T, T_{\max}\})$, which implies $T_{\max} < \frac{\ln 3}{2\sqrt{a(s_0)b(s_0)}} < T$ and (1.11). □

5. Pointwise upper estimates for u : Proof of Theorem 1.2

The goal of this section is to prove Theorem 1.2. To that end, we first derive estimates of the form (1.10) both for (JL) (Lemma 5.1) and (PE) (Lemma 5.2) and then apply Theorem 1.1.

Lemma 5.1. *Assume that λ, μ not only comply with (1.4) and (1.5) but additionally satisfy $-\lambda', \mu' \geq 0$. Let moreover $\kappa \geq 0$, $m > 0$, $T, M_0 > 0$ and suppose that u_0 satisfies (1.8) and $\int_{\Omega} u_0 = M_0$. Then, every solution $(u, v) \in (C^0(\bar{\Omega} \times [0, T)) \cap C^{2,1}(\bar{\Omega} \times (0, T)))^2$ of (JL) fulfills*

$$u(r, t) \leq \frac{M_0 n e^{\lambda t}}{\omega_{n-1}} \cdot r^{-n} \quad \text{for all } r \in (0, R) \text{ and } t \in (0, T).$$

Proof. As in [10, Lemma 3.7] (cf. also [48, Lemma 2.2]), we employ the comparison principle to see that u remains radially decreasing throughout evolution. To that end, we first note that by an approximation argument as in [48, Lemma 2.2], we may without loss of generality assume $\mu \in C^2([0, R])$ and $u_0 \in C^2(\bar{\Omega})$ with $\partial_\nu u_0 = 0$ on $\partial\Omega$. For henceforth fixed $T \in (0, T_{\max})$, these assumptions then assert that u_r belongs to $C^0([0, R] \times [0, T]) \cap C^{2,1}([0, R] \times (0, T))$: Indeed, elliptic regularity (cf. [8, Theorem 1.19.1]) asserts $\nabla v \in L^\infty(\Omega \times (0, T))$, which in conjunction with [22, Theorem IV.5.3] implies $u \in C^{1,0}(\bar{\Omega} \times [0, T]) \cap C^{3,1}(\bar{\Omega} \times (0, T))$.

Thus, setting $f(s) := (s+1)^{n-1}$ for $s \geq 0$, and re-interpreting (JL) in the form of $u_t = \nabla \cdot (f(u)\nabla u) - \nabla u \cdot \nabla v - u(\bar{M} - u) + \lambda u - \mu u^{1+\kappa}$ as an equation for the radial function $u \in C^{1,0}([0, R] \times [0, T]) \cap C^{3,1}([0, R] \times (0, T))$, we may compute the radial derivative

$$\begin{aligned} u_{rt} &= \left((f(u)u_r)_r + \frac{n-1}{r} f(u)u_r - u_r v_r + u^2 - \bar{M}(t)u + \lambda u - \mu u^{1+\kappa} \right)_r \\ &= (f(u)u_r)_{rr} + a_1(f(u)u_r)_r + a_2 u_{rr} + b u_r + c \quad \text{in } (0, R) \times (0, T), \end{aligned} \quad (5.1)$$

where

$$\begin{aligned} a_1(r, t) &:= \frac{n-1}{r}, & a_2(r, t) &:= -v_r(r, t), & c(r, t) &:= \lambda'(r)u(r, t) - \mu'(r)u^{1+\kappa}(r, t) \quad \text{and} \\ b(r, t) &:= -\frac{n-1}{r^2} f(u(r, t)) - v_{rr}(r, t) + 2u(r, t) - \bar{M}(t) + \lambda(r) - (1+\kappa)\mu(r)u^\kappa(r, t) \end{aligned}$$

for $(r, t) \in (0, R) \times (0, T)$. Note that $-\lambda', \mu' \geq 0$ imply $c \leq 0$ in $(0, R) \times (0, T)$. Moreover, by the second equation in (JL), $(r^{n-1}v_r(r, t))_r \leq r^{n-1}\bar{M}(t)$ and hence $v_r(r, t) \leq \frac{r}{n}\bar{M}(t)$ for $(r, t) \in (0, R) \times (0, T)$. This implies

$$-v_{rr}(r, t) = u(r, t) - \bar{M}(t) + \frac{n-1}{r}v_r(r, t) \leq u(r, t) - \frac{1}{n}\bar{M}(t) \leq u(r, t) \quad \text{for } (r, t) \in (0, R) \times (0, T),$$

so that by setting $c_1 := \sup_{(r,t) \in (0,R) \times (0,T)} (3u(r, t) + \lambda(r))$, we obtain

$$b(r, t) \leq c_1 \quad \text{for } (r, t) \in (0, R) \times (0, T). \quad (5.2)$$

Using that $u \in C^2([0, R] \times [0, T])$, and, in particular, that $a_1 u_r = (n-1)\frac{u_r}{r} \leq (n-1)u_{rr}$ is bounded in $(0, R) \times (0, T)$, we can moreover introduce

$$c_2 := \sup_{(r,t) \in (0,R) \times (0,T)} ((f(u)_{rr} + a_1(f(u))_r) < \infty$$

and set $c_3 := c_1 + c_2 + 1$.

By (5.1), since $u_r(r, t) = 0$ for $r \in \{0, R\}$ and $t \in (0, T)$ (because u is radially symmetric and (u, v) solves (JL)) and as $u_{0r} \leq 0$ by assumption, the function $y: [0, R] \times [0, T] \rightarrow \mathbb{R}$, $(r, t) \mapsto u_r(r, t) - \varepsilon e^{c_3 t}$ belongs to $C^0([0, R] \times [0, T]) \cap C^{2,1}([0, R] \times (0, T))$ and fulfills

$$\begin{cases} y_t \leq (f(u)(y + \varepsilon e^{c_3 t}))_{rr} + a_1(f(u)(y + \varepsilon e^{c_3 t}))_r + a_2 y_r + b y + \varepsilon e^{\varepsilon t} b - c_3 \varepsilon e^{c_3 t} \\ \quad = (f(u)y)_{rr} + a_1(f(u)y)_r + a_2 y_r + b y + \varepsilon e^{c_3 t} [(f(u))_{rr} + a_1(f(u))_r + b - c_3] \\ \quad \leq (f(u)y)_{rr} + a_1(f(u)y)_r + a_2 y_r + b y - \varepsilon e^{c_3 t} & \text{in } (0, R) \times (0, T), \\ y < 0 & \text{on } \{0, R\} \times (0, T), \\ y(\cdot, 0) < 0, & \text{in } (0, R). \end{cases} \quad (5.3)$$

By the estimate for $y(\cdot, 0)$ in (5.3) and continuity of y , the time $t_0 := \sup\{t \in (0, T) : y \leq 0 \text{ in } [0, R] \times (0, t)\} \in (0, T]$ is well defined. Suppose $t_0 < T$, then there exists $r_0 \in [0, R]$ such that $y(r_0, t_0) = 0$ and $y(r, t) \leq 0$ for all $r \in [0, R]$ and $t \in [0, t_0]$; hence, $y_t(r_0, t_0) \geq 0$. As $f \geq 0$ in $[0, \infty)$, not only $y(\cdot, t_0)$ but also $h: (0, R) \rightarrow \mathbb{R}$, $r \mapsto f(u(r, t_0))y(r, t_0)$ attains its maximum 0 at r_0 . Since the second inequality in

(5.3) asserts $r_0 \in (0, R)$, we conclude $h_{rr}(r_0) \leq 0$, $h_r(r_0) = 0$ and $y_r(r_0, t_0) = 0$. However, from the first inequality in (5.3) and (5.2), we could then infer the contradiction

$$\begin{aligned} 0 &\leq y_t(r_0, t_0) \\ &\leq h_{rr}(r_0) + a_1(r_0, t_0)h_r(r_0) + a_2(r_0, t_0)y_r(r_0, t_0) + b(r_0, t_0)y(r_0, t_0) - \varepsilon e^{c_3 t_0} \\ &\leq -\varepsilon e^{c_3 t_0} < 0, \end{aligned}$$

so that $t_0 = T$, implying $y \leq 0$ in $[0, R] \times [0, T]$ and hence also $u_r \leq \varepsilon e^{c_3 T}$ in $[0, R] \times [0, T]$. Letting first $\varepsilon \searrow 0$ and then $T \nearrow T_{\max}$, this indeed gives $u_r \leq 0$ in $[0, R] \times [0, T_{\max})$.

This, together with Lemma 2.2, implies that

$$M_0 e^{\lambda_1 t} \geq \int_{\Omega} u(\cdot, t) = \omega_{n-1} \int_0^R \rho^{n-1} u(\rho, t) d\rho \geq \omega_{n-1} \int_0^r \rho^{n-1} u(\rho, t) d\rho = \frac{\omega_{n-1}}{n} r^n u(r, t)$$

for all $(r, t) \in (0, R) \times (0, T_{\max})$. $\square$

Arguments based on the comparison principle as used in Lemma 5.1 are apparently not expedient for the less simplified system (1.3). Therefore, we rely on the pointwise upper bounds gained in [9] instead and argue similar as in [49, Lemma 3.3], whose proof instead of [9] relies on its predecessor [51] dealing with linear diffusion only.

Lemma 5.2. *Let $m \in [1, 2 - \frac{2}{n})$, $\kappa \geq 0$, $T, L, M_0 > 0$ and suppose that λ, μ satisfy (1.4) and (1.5). For any $\varepsilon > 0$, there is $C > 0$ such that with $p := \frac{n(n-1)}{(m-1)n+1} + \varepsilon$, the following holds: Whenever $u_0 \in C^0(\overline{\Omega})$ complies with (1.8),*

$$\int_{\Omega} u_0 \leq M_0 \quad \text{as well as} \quad u_0(x) \leq L|x|^{-p} \quad \text{for all } x \in \Omega$$

and $(u, v) \in (C^0(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T)))^2$ is a classical solution of (PE), then

$$u(x, t) \leq C \cdot |x|^{-p} \quad \text{for all } x \in \Omega \text{ and } t \in (0, T).$$

Proof. Aiming to apply the pointwise upper bounds for quite general parabolic equations in divergence form obtained in [9], we first derive estimates for the second equation in (PE). Thus, writing the second equation in (PE) in radial coordinates, we see that

$$r^{n-1} v_r(r, t) = \int_0^r (\rho^{n-1} v_r(\rho, t))_{\rho} d\rho = \int_0^r \rho^{n-1} (v(\rho, t) - u(\rho, t)) d\rho \leq \frac{1}{\omega_{n-1}} \left(\int_{\Omega} v(\cdot, t) + \int_{\Omega} u(\cdot, t) \right) \quad (5.4)$$

for all $(r, t) \in (0, R) \times (0, T)$. Since integrating the second equation in (PE) reveals that $\int_{\Omega} v = \int_{\Omega} u$ in $(0, T_{\max})$, from (5.4) and Lemma 2.2, we infer that

$$r^{n-1} v_r(r, t) \leq \frac{2e^{\lambda_1 T} M_0}{\omega_{n-1}} =: c_1 \quad \text{for all } (r, t) \in (0, R) \times (0, T).$$

We now choose $\theta > n$ so large that $m-1 > \frac{1}{\theta} - \frac{1}{n}$ as well as $p = \frac{n(n-1)}{(m-1)n+1} + \varepsilon > \frac{n(n-1)}{(m-1)n+1-\frac{n}{\theta}}$. Then,

$$\int_{\Omega} |x|^{(n-1)\theta} |\nabla v(x, t)|^{\theta} dx \leq c_1^{\theta} |\Omega| =: c_2$$

, and since $\tilde{u}(x, t) := e^{-\lambda_1 t} u(x, t)$, $(x, t) \in \overline{\Omega} \times [0, T]$, solves

$$\begin{cases} \tilde{u}_t \leq \nabla \cdot ((e^{\lambda_1 t} \tilde{u} + 1)^{m-1} \nabla \tilde{u} - \tilde{u} \nabla v) & \text{in } \Omega \times (0, T), \\ ((e^{\lambda_1 t} \tilde{u} + 1)^{m-1} \nabla \tilde{u} - \tilde{u} \nabla v) \cdot \nu = 0 & \text{on } \partial\Omega \times (0, T), \\ \tilde{u}(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases}$$

classically, an application of [9, Theorem 1.1] (with $\alpha:=p$, $q:=1$, $K_{D,1}:=1$, $K_{D,2}:=\max\{e^{\lambda_1 T(m-1)}, 1\}$, $K_S:=1$, $K_f:=c_2$, $M:=M_0$, $\beta:=n-1$, $\mathbb{p}:=1$, $D(x, t, \tilde{u}):=(e^{-\lambda_1 t}\tilde{u}+1)^{m-1}$ and $S(x, t, \tilde{u}):=\tilde{u}$) yields $c_3 > 0$ such that $\tilde{u}(x, t) \leq c_3|x|^{-p}$ and hence $u(x, t) \leq c_3 e^{\lambda_1 T}|x|^{-p}$ for all $x \in \Omega$ and $t \in (0, T)$. $\square$

With these upper estimates at hand, we can now prove Theorem 1.2.

Proof of Theorem 1.2. Part (i) follows directly from Theorem 1.1 and Lemma 5.1.

Regarding part (ii), we first set $p_0:=\frac{n(n-1)}{(m-1)n+1} \geq n$ and note that the assumption $m \geq 1$ implies $m > \frac{2}{n} \geq \frac{2}{p_0}$. Thus, (1.6)–(1.7) reduces to (1.6) and if κ satisfies (1.14) then also (1.6) for some $p > p_0$ sufficiently close to p_0 .

For arbitrary $L > 0$ and $\varepsilon:=p-p_0 > 0$, we fix $C > 0$ as given by Lemma 5.2. With r_1 given by Theorem 1.1, we then choose u_0 satisfying (1.8), having compact support in $B_{r_1}(0)$ and fulfilling $\int_{\Omega} u_0 = M_0$ as well as $u_0(x) \leq L|x|^{-p}$ for all $x \in \Omega$. Note that this is indeed possible since $p \geq n$ implies $\int_{B_{r_1}(0)} L|x|^{-p} dx = \infty$. Finally, Theorem 1.1 asserts that the solution (u, v) emanating from u_0 blows up in finite time.

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