A note on the article “Anomalous relaxation model based on the fractional derivative with a Prabhakar-like kernel” [Z. Angew. Math. Phys. (2019) 70: 42]

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Abstract. Inspired by the article “Anomalous relaxation model based on the fractional derivative with a Prabhakar-like kernel” (Z. Angew. Math. Phys. (2019) 70:42) whose authors Zhao and Sun studied the integro-differential equation with the kernel given by the Prabhakar function $e_{\alpha,\beta}^{-\gamma}(t, \lambda)$, we provide the solution to this equation which is complementary to that obtained up to now. Our solution is valid for effective relaxation times whose admissible range extends the limits given in Zhao and Sun (Z Angew Math Phys 70:42, 2019, Theorem 3.1) to all positive values. For special choices of parameters entering the equation itself and/or characterizing the kernel, the solution comprises to known phenomenological relaxation patterns, e.g., to the Cole–Cole model (if $\gamma = 1, \beta = 1 - \alpha$) or to the standard Debye relaxation.

Mathematics Subject Classification. 26A33.

1. Introduction

In the recently published article [1], its authors Dazhi Zhao and HongGuang Sun studied the linear integro-differential equation

$$\int_0^t e_{\alpha,\beta}^{-\gamma}(t-t', \lambda) \frac{d}{dt'} f(t') dt' = -M(\tau, \alpha) f(t) \quad (1.1)$$

where the kernel $k(t; \alpha) = e_{\alpha,\beta}^{-\gamma}(t; \lambda)$ is given by the Prabhakar function whose parameters satisfy $0 < \gamma \leq 1$ and $\alpha, \beta > 0, \alpha + \beta = 1$. For this range of parameters recall that the Laplace transform of $k(t; \alpha)$, namely $K(s, \alpha) = s^{-\alpha\gamma-\beta}(s^\alpha - \lambda)^\gamma$, satisfies the condition $\lim_{s \rightarrow \infty} [sK(s, \alpha)]^{-1} = 0$, which according to [1, Eq. (2) *et seq.*] permits to qualify the integro-differential operator in Eq. (1.1) as the so-called generalized Caputo (GC) derivative. Here, $M(\tau, \alpha)$ stands for $\Lambda(\tau, \alpha)/N(\alpha)$ where $N(\alpha) = (1 - \alpha)^{-1}$ normalizes the integral in Eq. (1.1) and $\Lambda(\tau, \alpha)$ is a function of the effective relaxation time τ .

Considering Eq. (1.1) as a model of the anomalous relaxation and solving it, the authors of [1] showed that the model extends the Cole–Cole relaxation pattern and contains as the limiting case $\alpha \rightarrow 1$ the standard Debye relaxation. Here, we would like to emphasize that just mentioned two cases do not exhaust possible mutual relations which link the relaxation phenomena and using Eq. (1.1) for modeling their time behavior. An instructive example is an application of Eq. (1.1)-like equation to describe the Havriliak–Negami relaxation, the most widely used “asymmetric” generalization of the Debye and Cole–Cole approaches. In the review paper [2], the authors presented a detailed analysis of equations describing the time behavior of the Havriliak–Negami relaxation function $\Psi_{\alpha,\gamma}(t)$. They came to the conclusion that it is governed by a non-homogenous equation

$${}^C({}_0D_t^\alpha + \tau^{-\alpha})^\gamma \Psi_{\alpha,\gamma}(t) = -\tau^{-\alpha\gamma}, \quad \Psi_{\alpha,\gamma}(0) = 1,$$

where the pseudo-differential operator ${}^C({}_0D_t^\alpha + \tau^{-\alpha})^\gamma$ is a Caputo-like counterpart of the operator $({}_0D_t^\alpha + \tau^{-\alpha})^\gamma$, the latter understood as an infinite binomial series of the Riemann–Liouville fractional derivatives.

¹ Next, using results of [3], they argued that the operator ${}^C({}_0D_t^\alpha + \tau^{-\alpha})^\gamma$ may be represented in terms of an integro-differential operator involving the Prabhakar function in the kernel, the object usually nick-named the Prabhakar derivative. Adjusted to our notation, the suitable equations [2, Eq. (B.23)] read

$$\begin{aligned} {}^C({}_0D_t^\alpha + \tau^{-\alpha})^\gamma \Psi_{\alpha,\gamma}(t) &\equiv e_{\alpha,1-\alpha\gamma}^{-\gamma}(t; \lambda) \star \frac{d}{dt} \Psi_{\alpha,\gamma}(t) \\ &= \int_0^t e_{\alpha,1-\alpha\gamma}^{-\gamma}(t-u; \lambda) \Psi'_{\alpha,\gamma}(u) du, \end{aligned}$$

where $\star$ denotes the convolution operator. This justifies the condition $\beta = 1 - \alpha\gamma$ to appear in Eq. (1.1) as meaningful for understanding properties of physically admissible relaxation models. In [4] it has been also shown that the nonlinear heat conduction equations with memory involving Prabhakar derivative can be characterized by Eq. (1.1) in which $\beta = 1 - \alpha\gamma$.

The Laplace transform method applied to Eq. (1.1) results in $F(s) = f(0+)H(s, \alpha)$, where

$$H(s, \alpha) = \frac{K(s, \alpha)}{sK(s, \alpha) + M(\tau, \alpha)}, \quad (1.2)$$

in which the inverse Laplace transform of $F(s)$, denoted as $f(t)$, satisfies $\lim_{t \rightarrow \infty} f(t) < \infty$. In what follows

$$f(0+) \equiv 1 \quad (1.3)$$

will be used throughout, since this constraint neither harms nor restricts our further considerations. In [1] the authors used the fact that the inverse Laplace transform of the geometric series (which results after pulling out $K(s, \alpha)$ in the nominator and denominator of Eq. (1.2) and subsequently reducing it) may be performed termwise. This leads to their main result formulated as [1, Theorem 3.3]

$$f(t) = \sum_{r \geq 0} (-1)^r M^r(\tau, \alpha) e_{\alpha,1+r(1-\beta)}^{r\gamma}(t; \lambda), \quad (1.4)$$

for $|M(\tau, \alpha)/[sK(s, \alpha)]| < 1$, bearing in mind Eq. (1.3). The aim of our note is to show that just given restriction is not mandatory to solve Eq. (1.1) as we can consider the inverse Laplace transform of Eq. (1.2), namely the function $f(t)$, also for $|M(\tau, \alpha)/[sK(s, \alpha)]| > 1$.

The note is organized as follows: We begin with a few less known remarks on the properties of the Prabhakar function with negative upper index, next show how to find the solution for $|M(\tau, \alpha)/[sK(s, \alpha)]| > 1$ and complete the paper with remarks concerning relations between the standard Cole–Cole model and the solution to Eq. (1.1). We also comment how the results of [1] and this work are viewed in the light of general approach proposed in [5].

2. The Prabhakar function

The Prabhakar function [6]

$$e_{\alpha,\beta}^\gamma(t, \lambda) \stackrel{\text{def}}{=} t^{\beta-1} E_{\alpha,\beta}^\gamma(\lambda t^\alpha) \quad (2.1)$$

is expressed by the three parameters Mittag-Leffler function $E_{\alpha,\beta}^\gamma(\lambda t^\alpha)$ defined by the series [6, p. 7, Eq. (1.3)]

$$E_{\alpha,\beta}^\gamma(x) = \sum_{r \geq 0} \frac{(\gamma)_r x^r}{r! \Gamma(\alpha r + \beta)}, \quad \Re(\alpha) > 0; \beta, \mu \in \mathbb{C};$$

¹For a comprehensive information about ${}^C({}_0D_t^\alpha + \tau^{-\alpha})^\gamma$, see [2, Section 3.3, Appendix B].

here $(\gamma)_r = \Gamma(\gamma+r)/\Gamma(\gamma)$ stands for the familiar Pochhammer symbol. If $\gamma = -n$, n positive integer, the three parameter Mittag-Leffler function is given through hypergeometric type polynomial

$$E_{\alpha,\beta}^{-n}(x) = \frac{1}{\Gamma(\beta)} \sum_{k=0}^n \frac{(-n)_k}{(\beta)_{\alpha k}} \frac{x^k}{k!} = \frac{1}{\Gamma(\beta)} {}_1\Psi_1 \left[\begin{matrix} (-n, 1 \\ (\beta, \alpha) \end{matrix} \middle| x \right]. \quad (2.2)$$

For positive integer α they are the biorthogonal polynomials pairs discussed in [6–8]; the polynomials with general values of $\alpha > 0$ are mentioned in [4]. Here ${}_1\Psi_1$ stands for the confluent Fox-Wright generalized hypergeometric function, see, for instance, [9, p. 21]. The particular case of Eq. (2.2) for $n = 1$ reads

$$E_{\alpha,\beta}^{-1}(x) = \frac{1}{\Gamma(\beta)} + \frac{x}{\Gamma(\alpha + \beta)}. \quad (2.3)$$

This expression will be used in *Remark* which closes the next section and enables a comment on the relation between Eq. (1.1) and the Cole–Cole relaxation model.

3. Alternative solution of Eq. (1.1)

As previously mentioned, the case when $|M(\tau, \alpha)/[sK(s, \alpha)]| > 1$ has not been included in considerations presented in [1]. To fill this gap we shall proceed in an analogous way and formulate

Theorem. For $|M(\tau, \alpha)/[sK(s, \alpha)]| > 1$ the solution of Eq. (1.1) becomes

$$f(t) = \frac{1}{M(\tau, \alpha)} \sum_{r \geq 0} \frac{(-1)^r}{M^r(\tau, \alpha)} e_{\alpha, 1-(1+r)(1-\beta)}^{-(1+r)\gamma}(t; \lambda). \quad (3.1)$$

Proof. First we pull out $M(\tau, \alpha)$ in the denominator of $H(s, \alpha)$ given by Eq. (1.2). Thus, it can be rewritten in the form

$$H(s, \alpha) = \frac{K(s, \alpha)}{M(\tau, \alpha)} \left[1 + \frac{sK(s, \alpha)}{M(\tau, \alpha)} \right]^{-1}. \quad (3.2)$$

Next, after applying the series expansion of $(1+x)^{-1} = \sum_{r \geq 0} (-x)^r$ for $|x| < 1$, Eq. (3.2) with $x = sK(s, \alpha)/M(\tau, \alpha)$ can be expressed as

$$H(s, \alpha) = \sum_{r \geq 0} (-1)^r M^{-1-r}(\tau, \alpha) s^r K^{1+r}(s, \alpha). \quad (3.3)$$

The condition $|x| < 1$ means that $|M(\tau, \alpha)/sK(s, \alpha)| > 1$. Substituting the explicit form of $K(s, \alpha)$ given below Eq. (1.1) into Eq. (3.3), we obtain Eq. (3.1), as $f(0+) = 1$. That finishes the proof. $\square$

Example. Taking the same values of parameters $M(\tau, \alpha)$ and $\gamma = 1$ as in [1, p. 42, Example 3.4] the constraint

$$|M(\tau, \alpha)/[sK(s, \alpha)]| > 1$$

used to get (3.1) gives different, but complementary restriction on τ from that found in [1]. Namely, we get $\tau < (1-\alpha)^2/(b\alpha)$ while in [1] one finds $\tau > (1-\alpha)^2/(b\alpha)$; both conditions merged together cover the admissible range of τ . To provide numerical estimations, we take $b = 1$, $\alpha = 0.5$ and $\alpha = 0.7$ which leads to $\tau < 1/2$ and $\tau < 9/70$, respectively. This means that with growing α our solution (3.1) works for shorter and shorter characteristic relaxation times τ 's, while for α close to 0 it covers almost all range of τ . $\square$

For the values of parameters listed in the example above, i.e., $\gamma = 1$, $M = (1 - \alpha)/\tau$, $\lambda = -b\alpha/(1 - \alpha)$, and $K(s, \alpha) = s^{-1}(s^\alpha - \lambda)$, Eq. (3.3) reads

$$\begin{aligned} H(s, \alpha) &= \frac{s^\alpha - \lambda}{sM(\tau, \alpha)} \sum_{r \geq 0} \left[-\frac{s^\alpha - \lambda}{M(\tau, \alpha)} \right]^r \\ &= \frac{s^{\alpha-1}}{s^\alpha + M(\tau, \alpha) - \lambda} - \frac{\lambda s^{-1}}{s^\alpha + M(\tau, \alpha) - \lambda} \end{aligned} \quad (3.4)$$

which is satisfied for $\tau < (1 - \alpha)^2/(b\alpha)$. The same results can be obtained by using Eq. (1.4), i.e., [1, Theorem 3.1], but, now, for $\tau > (1 - \alpha)^2/(b\alpha)$. This suggests that to have Eq. (3.4) satisfied we do not need to put any additional constraint on τ except of its positivity. Indeed, Eq. (1.2) valid for $\tau > 0$ is equal to Eq. (3.4). Hence, from the Laplace transform of the three parameters Mittag-Leffler function (recalling that $f(0+) = 1$), we conclude

$$f(t) = E_\alpha(-[M(\tau, \alpha) - \lambda]t^\alpha) - \lambda t^\alpha E_{\alpha, 1+\alpha}(-[M(\tau, \alpha) - \lambda]t^\alpha), \quad (3.5)$$

which, after using the suitable property of the Mittag-Leffler functions (see [10, Eq. (4.2.3)]), implies

$$f(t) = \frac{M(\tau, \alpha)}{M(\tau, \alpha) - \lambda} E_\alpha(-[M(\tau, \alpha) - \lambda]t^\alpha) - \frac{\lambda}{M(\tau, \alpha) - \lambda}. \quad (3.6)$$

Thus, [1, Eq. (19)] can be treated as the approximation of exact solution given by Eq. (3.5) or Eq. (3.6).

Remark. Equation (1.1) for $\gamma = 1$ in which we applied Eqs. (2.1) and (2.3) can be written as

$${}^C D_t^{1-\beta} f(t) + \lambda \cdot {}^C D_t^{1-\alpha-\beta} f(t) = -M(\tau, \alpha) f(t),$$

where for an η suitable,

$${}^C D_t^\eta f(t) = \frac{1}{\Gamma(1-\eta)} \int_0^t (t-u)^{-\eta} f'(u) du$$

stands for the Caputo fractional derivative. For $\beta = 1 - \alpha$ we get

$${}^C D_t^\alpha f(t) + [M(\tau, \alpha) - \lambda] f(t) = \lambda f(0+) = \lambda, \quad (3.7)$$

whose solution coincides with (3.5), see, e.g., [11–13]. For $\lambda = 0$ Eq. (3.7) becomes the equation relevant for the Cole–Cole relaxation. Simultaneously, we have the relation $e_{\alpha, 1-\alpha}^{-1}(t; 0) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ easily seen from Eq. (2.3) for $\lambda = 0$. It implies that the Prabhakar derivative becomes Caputo fractional derivative and Eq. (1.1) tends to the evolution equation describing the Cole–Cole relaxation.

4. Conclusion

We would like to point out that our result is complementary to the result given in [1, Theorem 3.1] and extends it to the full range of $\tau > 0$. This places it within the general scheme developed by A. N. Kochubei [5] who investigated the Cauchy problem for evolution equations

$$(D_t^{GC} f)(t) = -M(\tau, \alpha) f(t). \quad (4.1)$$

governed by the integro-differential operator

$$(D_t^{GC} f)(t) = \frac{d}{d\tau} \int_0^t k(t-\tau, \alpha) f(\tau) d\tau - k(t) f(0).$$

In addition some requirements are put on the Laplace transform $K(s, \alpha)$ of the kernel $k(t, \alpha)$. Namely, it belongs to the Stieltjes class and satisfies the following asymptotic conditions: if $s \rightarrow 0$, then $K(s, \alpha) \rightarrow \infty$

and $sK(s, \alpha) \rightarrow 0$, while in the case $s \rightarrow \infty$, there hold $K(s, \alpha) \rightarrow 0$ and $sK(s, \alpha) \rightarrow \infty$. For instance, under this study all these conditions are satisfied and according to [5, Theorem 2] the solution $f(t)$ is continuous on $[0, \infty)$, infinitely differentiable and completely monotone on $(0, \infty)$.

Physical usefulness of Eq. (1.1) as a tool to develop a description of the anomalous relaxation patterns is rooted in its relation to the Cole–Cole and Debye models. The first case has been just discussed in the above. The Debye relaxation emerges when $K(s, \alpha)$ is a constant and consequently $k(t) = B(\alpha)\delta(t)$. It is seen from

$$H(s, \alpha) = \frac{B(\alpha)}{\Lambda(\tau, \alpha)} \left[1 + s \frac{B(\alpha)}{\Lambda(\tau, \alpha)} \right]^{-1}, \quad (4.2)$$

obtained either from (1.4) or (3.1). Calculating the inverse Laplace transform of (4.2), we obtain the solution of Eq. (4.1) in the form

$$f(t) = \exp \left[-\frac{\Lambda(\tau, \alpha)}{B(\alpha)} t \right],$$

which is the Debye relaxation function in time domain.

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