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A class of fractional differential hemivariational inequalities with application to contact problem

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Abstract. In this paper, we study a class of generalized differential hemivariational inequalities of parabolic type involving the time fractional order derivative operator in Banach spaces. We use the Rothe method combined with surjectivity of multivalued pseudomonotone operators and properties of the Clarke generalized gradient to establish existence of solution to the abstract inequality. As an illustrative application, a frictional quasistatic contact problem for viscoelastic materials with adhesion is investigated, in which the friction and contact conditions are described by the Clarke generalized gradient of nonconvex and nonsmooth functionals, and the constitutive relation is modeled by the fractional Kelvin–Voigt law.

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1. Introduction

The fractional calculus, as a natural generalization of the classical integer order calculus, provides a precise description of some physical phenomena for viscoelastic materials, for example, fractional Kelvin–Voigt constitutive laws and fractional Maxwell model [16,42,51]. Recent advances in the fractional calculus concern the fractional derivative modeling in applied science, see [2,9,38], the theory of fractional differential equations, see [21], numerical approaches for the fractional differential equations, see [26,55] and the references therein. Another hot issue is the theory of hemivariational inequalities which is based on properties of the Clarke generalized gradient, defined for locally Lipschitz functions. This theory has started with the works of Panagiotopoulos, see [39,40], and has been substantially developed during the last 30 years. The mathematical results on hemivariational inequalities have found numerous applications to mechanics, physics and engineering, see [4,14,33,35,37,46,49,50] and the references therein. In this paper, we combine these hot issues and initiate a study of a class of differential hemivariational inequalities of parabolic type involving the time fractional order derivative operator in Banach spaces.

Let $(V, \| \cdot \|_V)$, $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be reflexive Banach spaces. We consider the generalized fractional differential hemivariational inequality of the following form

Problem 1. *Find $u \in AC(0,T;V)$ and $\beta \in W^{1,2}(0,T;Y)$ such that*

$$\begin{cases} \langle A({}_0^C D_t^\alpha u(t)) + B(u(t)), v \rangle + J^0(\beta(t), Mu(t); Mv) \geq \langle f(t), v \rangle \\ \qquad\qquad\qquad \text{for all } v \in V, \text{ a.e. } t \in (0,T) \\ u(0) = u_0 \\ \beta'(t) = g(t, Mu(t), \beta(t)) \text{ for a.e. } t \in (0,T) \\ \beta(0) = \beta_0. \end{cases}$$

Here, A, $B \colon V \to V^*$, $M \colon V \to X$, $g \colon (0,T) \times X \times Y \to Y$, $f \in L^\infty(0,T;V^*)$, $\alpha \in (0,1)$, $u_0 \in V$ and $\beta_0 \in Y$. The notation ${}_0^C D_t^\alpha u(t)$ stands the α-order time fractional derivative of u in the sense of Caputo defined by

$$ {}_0^C D_t^\alpha u(t) = {}_0 I_t^{1-\alpha} u'(t) \quad \text{for a.e. } t \in (0,T), $$

where operator ${}_0 I_t^{1-\alpha} u'(t)$ is the $(1-\alpha)$-order time fractional integral of u' in the sense of Riemann–Liouville, i.e.,

$$ {}_0 I_t^{1-\alpha} u'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) \, \mathrm{d}s \quad \text{for a.e. } t \in (0,T). $$

Note that for $\alpha = 1$ the formula becomes a little bit different, see formula (2.1.10) of [21]. The symbol $J^0(y,x;z)$ denotes the Clarke generalized directional derivative of a locally Lipschitz functional $J \colon Y \times X \to \mathbb{R}$ with respect to its second variable, at a point x in the direction z, for each $y \in Y$ fixed. Moreover, ∂J stands for the Clarke generalized gradient of J with respect to the last variable.

Now, we give a definition of a solution to Problem 1.

Definition 2. A pair of functions (u, β) with $u \in AC(0,T;V)$ and $\beta \in W^{1,2}(0,T;Y)$ is called a solution to Problem 1, if there exists a function $\xi \in L^2(0,T;X^*)$ such that

$$\begin{cases} A({}_0^C D_t^\alpha u(t)) + B(u(t)) + M^*\xi(t) = f(t) \text{ for a.e. } t \in (0,T) \\ \beta'(t) = g(t, Mu(t), \beta(t)) \text{ for a.e. } t \in (0,T) \\ u(0) = u_0 \text{ and } \beta(0) = \beta_0 \end{cases}$$

with $\xi(t) \in \partial J(\beta(t), Mu(t))$ for a.e. $t \in (0,T)$.

Systems consisting of variational inequalities and differential equations were introduced initially by Aubin and Cellina [1] in 1984. From another point of view, they were firstly considered and systematically studied in a framework of finite-dimensional spaces by Pang and Stewart [41] in 2008. They named this complex system a differential variational inequality ((DVI), for short). They also indicated the applications of DVI to several areas involving both dynamics and constraints in the inequality form, for example, mechanical impact problems, electrical circuits with ideal diodes, the Coulomb frictional problem in contact mechanics, economical dynamics and related models such as dynamic traffic networks. Since then, many scientists have contributed to the development of (DVI). In 2013 Liu et al. [24] employed the topological degree theory for multivalued maps and the method of guiding functions to establish the existence and global bifurcation behavior for periodic solutions to a class of differential variational inequalities in finite-dimensional spaces. In 2014 Chen and Wang [8], using the idea of (DVI), have solved the dynamic Nash equilibrium problem with shared constraints, which involves a dynamic decision process with multiple players. Subsequently, Ke et al. [20] in 2015 investigated a class of fractional differential variational inequalities with decay term in finite-dimensional spaces, for details on this topic in finite-dimensional spaces, we refer to [7,13,22,23,25,32,48] and the references therein. It should be pointed out that all results in the aforementioned papers were considered only in finite-dimensional spaces. Being motivated by many applied problems in engineering, operations research, economics, and

physics, recently, Liu et al. [27], Liu et al. [28], and Liu et al. [31] have provided existence results for a class of differential mixed variational inequalities in Banach spaces exploiting the semigroup theory, theory of measure of noncompactness, the Filippov implicit function lemma, and a fixed point theorem for condensing set-valued operators, etc. Very recently, Liu et al. [29,30] have initiated the study on differential hemivariational inequalities in Banach spaces. There are only a few applications of (DVI) in infinite-dimensional spaces which were discussed to support these theoretical findings. Furthermore, until now, fractional differential hemivariational inequalities have not been studied in both finite and infinite-dimensional spaces. For this reason, in this work, we will fill in this gap and develop new mathematical tools and methods for fractional differential hemivariational inequalities.

Main novelties of the paper can be summarized as follows. First, for the first time, we apply the Rothe method, see [17,53], to study a system of a fractional hemivariational inequality of parabolic type driven by a nonlinear evolution equation. Until now, there are a few contributions devoted to the Rothe method for hemivariational inequalities, see [4,18,19], and all of them investigated only a single hemivariational inequality by using the Rothe method.

Second, the main results of the present paper can be applied to a special form of Problem 1 in which the locally Lipschitz functional J is assumed to be independent of the function β. In this case, Problem 1 reduces to the following parabolic hemivariational inequality involving the time fractional order derivative operator in the sense of Caputo: find $u \in AC(0,T;V)$ such that $u(0) = u_0$ and

$$\langle A(^C_0 D^\alpha_t u(t)) + B(u(t)) - f(t), v \rangle + J^0(Mu(t); Mv) \geq 0 \tag{1}$$

for all $v \in V$ and a.e. $t \in (0,T)$. This problem has been recently studied by Zeng and Migórski [54].

Third, the current paper initiates the study of a quasistatic contact problem for a viscoelastic body with adhesion and the fractional Kelvin–Voigt constitutive law, in which the friction and contact conditions are both described by the Clarke generalized gradient of nonconvex and nonsmooth functionals involving adhesion.

Fourth, for our problem, if we are restricted to the case $\alpha = 1$, then Problem 1 reduces to the following differential hemivariational inequality of parabolic type: find $u \in AC(0,T;V)$ and $\beta \in W^{1,2}(0,T;Y)$ such that

$$\begin{cases} \langle A(u'(t)) + B(u(t)), v \rangle + J^0(\beta(t), Mu(t); Mv) \geq \langle f(t), v \rangle \\ \qquad\qquad \text{for all } v \in V, \text{ a.e. } t \in (0,T) \\ \beta'(t) = g(t, Mu(t), \beta(t)) \text{ for a.e. } t \in (0,T) \\ \beta(0) = \beta_0 \text{ and } u(0) = u_0. \end{cases} \tag{2}$$

In this situation, the corresponding contact problem, see Problem 17, becomes a frictional viscoelastic contact problem with adhesion described by the classical Kelvin–Voigt constitutive law. It is obvious that the contact problem under consideration has the form of a differential hemivariational inequality.

The paper is organized as follows. In Sect. 2, we recall notation and auxiliary materials. Section 3 establishes a result on solvability to a class of fractional differential hemivariational inequality by using the Rothe method and a surjectivity theorem for multivalued pseudomonotone operators. Finally, in Sect. 4, we consider a quasistatic fractional viscoelastic contact model with adhesion, and then apply the theoretical results from Sect. 3 to obtain the weak solvability to the contact problem.

2. Preliminaries

In this section we recall the basic notation and preliminary results which are needed in the sequel, see [10,12,21,26,42,52]. We start by recalling important and useful properties of the fractional integral and the Caputo derivative operators, for more details, we refer to [21,42].

Proposition 3. *Let X be a Banach space and α, $\beta > 0$. Then, the following statements hold*

(a) *for $y \in L^1(0, T; X)$, we have $_0I_t^\alpha {_0}I_t^\beta y(t) = {_0}I_t^{\alpha+\beta} y(t)$ for a.e. $t \in (0, T)$,*

(b) *for $y \in AC(0, T; X)$ and $\alpha \in (0, 1]$, we have*

$$_0I_t^\alpha {_0^C}D_t^\alpha y(t) = y(t) - y(0) \quad \text{for a.e. } t \in (0, T),$$

(c) *for $y \in L^1(0, T; X)$, we have $_0^C D_t^\alpha {_0}I_t^\alpha y(t) = y(t)$ for a.e. $t \in (0, T)$.*

We now recall definitions and results from nonlinear analysis which can be found in [10–12,33,52]. Let X be a reflexive Banach space and $\langle \cdot, \cdot \rangle$ denote the duality of X and X^*. A single-valued operator $A \colon X \to X^*$ is pseudomonotone if A is bounded (it maps bounded sets in X into bounded sets in X^*) and for every sequence $\{x_n\} \subseteq X$ converging weakly to $x \in X$ such that $\limsup \langle A x_n, x_n - x \rangle \leq 0$, we have

$$\langle Ax, x - y \rangle \leq \liminf_{n \to \infty} \langle A x_n, x_n - y \rangle \quad \text{for all} \quad y \in X.$$

Obviously, an operator $A \colon X \to X^*$ is pseudomonotone if and only if it is bounded, and $x_n \to x$ weakly in X, and $\limsup \langle A x_n, x_n - x \rangle \leq 0$ entails

$$\lim \langle A x_n, x_n - x \rangle = 0 \quad \text{and} \quad A x_n \to Ax \quad \text{weakly in} \quad X^*.$$

Furthermore, if $A \in \mathcal{L}(X, X^*)$ is nonnegative, then it is pseudomonotone. Moreover, the notion of pseudomonotonicity of a multivalued operator is recalled below.

Definition 4. A multivalued operator $T \colon X \to 2^{X^*}$ is pseudomonotone if

(a) for every $v \in X$, the set $Tv \subset X^*$ is nonempty, closed and convex;

(b) T is upper semicontinuous from each finite-dimensional subspace of X to X^* endowed with the weak topology;

(c) for any sequences $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ such that $u_n \to u$ weakly in X, $u_n^* \in Tu_n$ for all $n \geq 1$ and $\limsup \langle u_n^*, u_n - u \rangle \leq 0$, we have that for every $v \in X$, there exists $u^*(v) \in Tu$ such that

$$\langle u^*(v), u - v \rangle \leq \liminf_{n \to \infty} \langle u_n^*, u_n - v \rangle.$$

Let $j \colon X \to \mathbb{R}$ be a locally Lipschitz function. We denote by $j^0(u; v)$ the generalized (Clarke) directional derivative of j at the point $u \in X$ in the direction $v \in X$ defined by

$$j^0(u; v) = \limsup_{\lambda \to 0^+, w \to u} \frac{j(w + \lambda v) - j(w)}{\lambda}.$$

The generalized gradient of $j \colon X \to \mathbb{R}$ at $u \in X$ is defined by

$$\partial j(u) = \{\, \xi \in X^* \mid j^0(u; v) \geq \langle \xi, v \rangle \quad \text{for all} \quad v \in X \}.$$

The following result provides an example of a multivalued pseudomonotone operator which is a superposition of the Clarke subgradient with a compact operator, its proof can be found in [14, Proposition 5.6].

Lemma 5. *Let V and X be two reflexive Banach spaces, $\gamma \colon V \to X$ be a linear, continuous, and compact operator. We denote by $\gamma^* \colon X^* \to V^*$ the adjoint operator of γ. Let $j \colon X \to \mathbb{R}$ be a locally Lipschitz function such that*

$$\|\partial j(v)\|_{X^*} \leq c_j (1 + \|v\|_X) \quad \text{for all} \quad v \in X,$$

with $c_j > 0$. Then the multivalued operator $G \colon V \to 2^{V^*}$ defined by

$$G(v) = \gamma^* \partial j(\gamma(v)) \quad \text{for all} \quad v \in V,$$

is pseudomonotone.

Furthermore, we recall the following surjective result, which can be found in [12, Theorem 1.3.70] or [52].

Theorem 6. *Let X be a reflexive Banach space and $T: X \to 2^{X^*}$ be pseudomonotone and coercive. Then T is surjective, i.e., for every $f \in X^*$, there exists $u \in X$ such that $Tu \ni f$.*

From Theorem 6, we have the following corollary.

Corollary 7. *Let V be a reflexive Banach space. Assume that*
  (i) *$A: V \to V^*$ is a pseudomonotone and strongly monotone operator, i.e., there exists $c_A > 0$ such that $\langle Av - Au, v - u \rangle \geq c_A \|v - u\|^2$ for all $v, u \in V$.*
 (ii) *$U: V \to 2^{V^*}$ is a pseudomonotone operator such that there exist $c_U > 0$ and $c^* > 0$ satisfying $\|U(v)\|_{V^*} \leq c_U \|v\| + c^*$ for all $v \in V$.*
If $c_U < c_A$, then $A + U$ is surjective in V^*.

Proof. Since A and U are pseudomonotone, it follows from [33, Proposition 3.59(ii)] that $A + U$ is pseudomonotone as well. Having in mind Theorem 6, it remains to prove that $A + U$ is coercive. Indeed, we have

$$\langle Av + U(v), v \rangle = \langle Av - A0, v \rangle + \langle A0, v \rangle + \langle U(v), v \rangle$$

$$\geq c_A \|v\|^2 - (\|A0\|_{V^*} + \|U(v)\|_{V^*})\|v\| \geq (c_A - c_U)\|v\|^2 - (\|A0\|_{V^*} + c^*)\|v\|$$

for all $v \in V$. The smallness condition $c_U < c_A$ guarantees that $A + U$ is coercive. Therefore, from Theorem 6, we conclude that $A + U$ is surjective, which completes the proof of the corollary. $\qquad\square$

Lemma 8. *Let X and Y be reflexive Banach spaces, $\beta_0 \in Y$, and $u \in L^2(0, T; X)$. Suppose that $F: (0, T) \times X \times Y \to Y$ satisfies the following conditions*
  (i) *$t \mapsto F(t, x, y)$ is measurable on $(0, T)$ for all $x \in X$ and $y \in Y$.*
 (ii) *$(x, y) \mapsto F(t, x, y)$ is Lipschitz continuous, i.e., there is a constant $L_F > 0$ such that for all (x_1, y_1), $(x_2, y_2) \in X \times Y$ and a.e. $t \in (0, T)$, we have*
$$\|F(t, x_1, y_1) - F(t, x_2, y_2)\|_Y \leq L_F (\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y).$$
(iii) *$t \mapsto F(t, 0, 0)$ belongs to $L^2(0, T; Y)$.*
Then there exists $\beta \in W^{1,2}(0, T; Y)$ a unique solution to the Cauchy problem

$$\begin{cases} \beta'(t) = F(t, u(t), \beta(t)) & \text{for a.e. } t \in (0, T), \\ \beta(0) = \beta_0, \end{cases} \tag{3}$$

Moreover, given $u_i \in L^2(0, T; X)$ and denoting by $\beta_i \in W^{1,2}(0, T; Y)$ the unique solution corresponding to u_i, for $i = 1, 2$, we have

$$\|\beta_1(t) - \beta_2(t)\|_Y \leq c_\beta \int_0^t \|u_1(s) - u_2(s)\|_X \, ds \quad \text{for all } t \in [0, T] \text{ with } c_\beta > 0. \tag{4}$$

Proof. Given $u \in L^2(0, T; X)$ we consider the function $\mathcal{F}_u: (0, T) \times Y \to Y$ defined by

$$\mathcal{F}_u(t, y) = F(t, u(t), y) \quad \text{for all } y \in Y, \text{ a.e. } t \in (0, T).$$

Hypothesis (ii) implies that

$$\|F(t, x, y)\|_Y \leq \|F(t, x, y) - F(t, 0, 0)\|_Y + \|F(t, 0, 0)\|_Y$$

$$\leq L_F(\|x\|_X + \|y\|_Y) + \|F(t, 0, 0)\|_Y \quad \text{for all } (x, y) \in X \times Y.$$

Combining the latter with hypotheses (i) and (iii), we deduce that the function $t \mapsto F(t, u(t), y)$ belongs to $L^2(0, T; Y)$ for all $u \in L^2(0, T; X)$ and $y \in Y$, thus is, $t \mapsto \mathcal{F}_u(t, y) \in L^2(0, T; Y)$ for all $y \in Y$. On the other hand, by hypothesis (ii), for all $y_1, y_2 \in Y$, we get

$$\|\mathcal{F}_u(t, y_1) - \mathcal{F}_u(t, y_2)\|_Y = \|F(t, u(t), y_1) - F(t, u(t), y_2)\|_Y \leq L_F \|y_1 - y_2\|_Y$$

for a.e. $t \in (0, T)$, i.e., $\mathcal{F}_u(t, \cdot)$ is Lipschitz continuous for a.e. $t \in (0, T)$. Therefore, all conditions of [15, Theorem 9.9, p.198] are verified. By applying this theorem, we conclude that there exists a unique function $\beta \in W^{1,2}(0, T; Y)$ such that (3) holds.

We now prove inequality (4). In fact, it is clear that, for any $u \in L^2(0, T; X)$ fixed, the unique function $\beta \in W^{1,2}(0, T; Y)$ has the form

$$\beta(t) = \beta_0 + \int_0^t F(s, u(s), \beta(s)) \mathrm{d}s \quad \text{for all } t \in [0, T].$$

For $u_i \in L^2(0, T; X)$, let $\beta_i \in W^{1,2}(0, T; Y)$ be the unique solution corresponding to u_i, for $i = 1, 2$. So, we have

$$\|\beta_1(t) - \beta_2(t)\|_Y \leq \int_0^t \Big\| F(s, u_1(s), \beta_1(s)) - F(s, u_2(s), \beta_2(s)) \Big\|_Y \mathrm{d}s$$

$$\leq L_F \int_0^t \|u_1(s) - u_2(s)\|_X \, \mathrm{d}s + L_F \int_0^t \|\beta_1(s) - \beta_2(s)\|_Y \, \mathrm{d}s \quad \text{for all } t \in [0, T].$$

The Gronwall inequality (see, e.g., [47, Lemma 2.31, p.49]) entails

$$\|\beta_1(t) - \beta_2(t)\|_Y \leq L_F(1 + T L_F e^{L_F T}) \int_0^t \|u_1(s) - u_2(s)\|_X \mathrm{d}s \quad \text{for all } t \in [0, T].$$

This means that (4) holds with constant $c_\beta = L_F(1 + T L_F e^{L_F T})$, which completes the proof of the lemma. $\qquad\square$

We conclude this section by recalling the generalized discrete version of the Gronwall inequality which proof can be found in [43, Lemma 2].

Lemma 9. *Let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be nonnegative sequences satisfying*

$$u_n \leq v_n + \sum_{k=1}^{n-1} w_k u_k \quad \text{for } n \geq 1.$$

Then, we have

$$u_n \leq v_n + \sum_{k=1}^{n-1} v_k w_k \exp\left(\sum_{j=k+1}^{n-1} w_j \right) \quad \text{for } n \geq 1.$$

Moreover, if $\{u_n\}$ and $\{w_n\}$ are such that

$$u_n \leq \alpha + \sum_{k=1}^{n-1} w_k u_k \quad \text{for } n \geq 1,$$

where $\alpha > 0$ is a constant, then for all $n \geq 1$, it holds

$$u_n \leq \alpha \exp\left(\sum_{k=1}^{n-1} w_k \right).$$

3. Fractional differential hemivariational inequality

In this section, we focus our attention to the abstract differential hemivariational inequality involving fractional derivative operator, Problem 1, and provide a result on existence of solutions for this inequality. The method of proof relies on a surjectivity result for multivalued pseudomonotone operators and the Rothe method.

To provide readers with better convenience, we now introduce the standard notation following [11, 12, 52]. Let V be a reflexive and separable Banach space with dual space V^*. Subsequently, we use the symbols $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ to stand for the duality pairing between V^* and V, and a norm in V, respectively. Let $0 < T < +\infty$. We use the standard Bochner–Lebesgue function space $\mathcal{V} = L^2(0, T; V)$. Recall that since V is reflexive, it is obvious that both $\mathcal{V}$ and its dual space $\mathcal{V}^* = L^2(0, T; V^*)$ are reflexive Banach spaces. The notation $\langle \cdot, \cdot \rangle_{\mathcal{V}^* \times \mathcal{V}}$ stands for the duality between $\mathcal{V}$ and $\mathcal{V}^*$. Let X and Y be other separable and reflexive Banach spaces, $\mathcal{X} = L^2(0, T; X)$ and $\mathcal{X}^* = L^2(0, T; X^*)$. In the rest of the paper, we denote by C a constant whose value may change from line to line.

Let $u \in AC(0, T; V)$ be a solution to Problem 1 and $w = {}^C_0 D^\alpha_t u$. From Proposition 3(b), one has

$$u(t) = {}_0 I^\alpha_t w(t) + u_0$$

for a.e. $t \in (0, T)$. Therefore, Problem 1 can be rewritten as

Problem 10. *Find $w \in L^1(0, T; V)$ and $\beta \in W^{1,2}(0, T; Y)$ such that*

$$\begin{cases} \langle A(w(t)) + B({}_0 I^\alpha_t w(t) + u_0), v \rangle + J^0(\beta(t), \ M({}_0 I^\alpha_t w(t) + u_0); Mv) \geq \langle f(t), v \rangle \\ \qquad\qquad\qquad\qquad \text{for all } v \in V, \text{ a.e. } t \in (0, T) \\ \beta'(t) = g(t, M({}_0 I^\alpha_t w(t) + u_0), \ \beta(t)) \quad \text{for a.e. } t \in (0, T) \\ \beta(0) = \beta_0. \end{cases}$$

Observe that the above problem can be reformulated as the following inclusion problem driven by a fractional integral operator and a nonlinear differential equation.

Problem 11. *Find $w \in L^1(0, T; V)$ and $\beta \in W^{1,2}(0, T; Y)$ such that*

$$\begin{cases} A(w(t)) + B({}_0 I^\alpha_t w(t) + u_0) + M^* \partial J(\beta(t), \ M({}_0 I^\alpha_t w(t) + u_0)) \ni f(t) \\ \qquad\qquad\qquad\qquad\qquad \text{for a.e. } t \in (0, T) \\ \beta'(t) = g(t, M({}_0 I^\alpha_t w(t) + u_0), \ \beta(t)) \ \text{ for a.e. } t \in (0, T) \\ \beta(0) = \beta_0. \end{cases} \tag{5}$$

We now impose the following assumptions on the data of Problem 11.
$H(A)$: $A \in \mathcal{L}(V, V^*)$ is coercive, i.e., there exists a constant $m_A > 0$ such that

$$\langle Av, v \rangle \geq m_A \|v\|^2 \quad \text{for all } \ v \in V.$$

$H(B)$: $B \in \mathcal{L}(V, V^*)$.
$H(J)$: $J \colon Y \times X \to \mathbb{R}$ is such that

(i) $x \mapsto J(y, x)$ is locally Lipschitz for all $y \in Y$;
(ii) there exists a constant $c_J > 0$ such that

$$\|\partial J(y, x)\|_{X^*} \leq c_J (1 + \|x\|_X) \ \text{ for all } y \in Y \text{ and } x \in X;$$

(iii) $(y, x) \mapsto J^0(y, x; v)$ is upper semicontinuous from $Y \times X$ into $\mathbb{R}$ for all $v \in X$.
$H(M)$: $M \in \mathcal{L}(V, X)$ is compact.
$H(f)$: $f \in L^\infty(0, T; V^*)$.
$H(g)$: $g \colon (0, T) \times X \times Y \to Y$ is such that

(i) $t \mapsto g(t, x, y)$ is measurable on $(0, T)$ for all $x \in X$ and $y \in Y$;
(ii) $(x, y) \mapsto g(t, x, y)$ is Lipschitz continuous, i.e., there exists a constant $L_g > 0$ such that for all $(x_1, y_1), (x_2, y_2) \in X \times Y$ and a.e. $t \in (0, T)$, we have

$$\|g(t, x_1, y_1) - g(t, x_2, y_2)\|_Y \leq L_g(\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y);$$

(iii) $t \mapsto g(t, 0, 0)$ belongs to $L^2(0, T; Y)$.

In fact, hypothesis $H(J)$ guarantees that the subgradient operator ∂J of $J(y, \cdot)$ is upper semicontinuous.

Lemma 12. *Under hypothesis $H(J)$ the subgradient operator*

$$Y \times X \ni (y, x) \mapsto \partial J(y, x) \subset X^*$$

is upper semicontinuous from $Y \times X$ endowed with the norm topology to the subsets of X^* endowed with the weak topology.

Proof. From [11, Proposition 4.1.4], it is sufficient to show that for any weakly closed subset D of X^*, the weak inverse image $(\partial J)^-(D)$ of ∂J under D is closed in the norm topology, where $(\partial J)^-(D)$ is defined by

$$(\partial J)^-(D) = \big\{ (y, x) \in Y \times X \mid \partial J(y, x) \cap D \neq \emptyset \big\}.$$

Let $\{(y_n, x_n)\} \subset (\partial J)^-(D)$ be such that $(y_n, x_n) \to (y, x)$ in $Y \times X$, as $n \to \infty$ and $\{\xi_n\} \subset X^*$ be such that $\xi_n \in \partial J(y_n, x_n) \cap D$ for each $n \in \mathbb{N}$. The hypothesis $H(J)$(ii) implies that the sequence $\{\xi_n\}$ is bounded in X^*. Hence, from the reflexivity of X^*, without loss of generality, we may assume that $\xi_n \to \xi$ weakly in X^*. The weak closedness of D guarantees that $\xi \in D$. On the other hand, $\xi_n \in \partial J(y_n, x_n)$ reveals

$$\langle \xi_n, z \rangle_{X^* \times X} \leq J^0(y_n, x_n; z) \quad \text{for all} \quad z \in X.$$

Taking into account the upper semicontinuity of $(y, x) \mapsto J^0(y, x; z)$ for all $z \in X$ and passing to the limit, we have

$$\langle \xi, z \rangle_{X^* \times X} = \limsup_{n \to \infty} \langle \xi_n, z \rangle_{X^* \times X} \leq \limsup_{n \to \infty} J^0(y_n, x_n; z) \leq J^0(y, x; z)$$

for all $z \in X$. Hence $\xi \in \partial J(y, x)$, and consequently, we obtain $\xi \in \partial J(y, x) \cap D$, i.e., $(y, x) \in (\partial J)^-(D)$. This completes the proof of the lemma. $\qquad\square$

Let $N \in \mathbb{N}_+$ be fixed, $\tau = \frac{T}{N}$, $t_k = k\tau$, and f_τ^k be defined by

$$f_\tau^k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} f(s)\mathrm{d}s \quad \text{for} \quad k = 1, \ldots, N.$$

Consider the following discretized problem corresponding to Problem 11 called the Rothe problem.

Problem 13. *Find $\{w_\tau^k\}_{k=1}^N \subset V$, $\{\xi_\tau^k\}_{k=1}^N \subset X^*$ and $\beta_\tau \in W^{1,2}(0, T; Y)$ such that $w_\tau^0 = 0$, $\beta_\tau(0) = \beta_0$ and*

$$\beta_\tau'(t) = g(t, M\widehat{u}_\tau(t), \beta_\tau(t)) \text{ for a.e. } t \in (0, t_k) \tag{6}$$

$$Aw_\tau^k + B(u_\tau^k) + M^*\xi_\tau^k = f_\tau^k \tag{7}$$

with $\xi_\tau^k \in \partial J(\beta_\tau(t_k), Mu_\tau^k)$, for $k = 1, 2, \ldots, N$, where u_τ^k and $\widehat{u}_\tau(t)$ for $t \in (0, t_k)$ are defined by

$$u_\tau^k = u_0 + \frac{\tau^\alpha}{\Gamma(\alpha + 1)} \sum_{j=1}^k w_\tau^j \left[(k - j + 1)^\alpha - (k - j)^\alpha\right], \tag{8}$$

and

$$\widehat{u}_\tau(t) = \begin{cases} \sum_{i=1}^N \chi_{(t_{i-1}, t_i]}(t) u_\tau^{i-1}, & 0 < t \leq T, \\ u_0, & t = 0, \end{cases} \tag{9}$$

respectively. Here $\chi_{(t_{i-1}, t_i]}$ stands for the characteristic function of the interval $(t_{i-1}, t_i]$, i.e.,

$$\chi_{(t_{i-1}, t_i]}(t) = \begin{cases} 1, & t \in (t_{i-1}, t_i], \\ 0, & \text{otherwise.} \end{cases}$$

First, we shall show the existence of solution to Problem 13.

Lemma 14. *Let hypotheses $H(A)$, $H(B)$, $H(J)$, $H(g)$, $H(f)$ and $H(M)$ hold. Then, there exists $\tau_0 > 0$ such that, for all $\tau \in (0, \tau_0)$, Problem 13 has at least one solution.*

Proof. Given w_τ^0, w_τ^1, $\ldots$, w_τ^{n-1}, we will prove that there exist $w_\tau^n \in V$, $\xi_\tau^n \in X^*$ and a function $\beta_\tau \in W^{1,2}(0, t_n; Y)$ such that (6) and (7) hold.

From equality (8), we obtain elements $u_\tau^0, u_\tau^1, \ldots, u_\tau^{n-1}$. For this reason, the function $\widehat{u}_\tau$ in (9) is well-defined in $(0, t_n)$. It is clear that $\widehat{u}_\tau \in L^2(0, t_n; V)$ and all conditions of Lemma 8 are satisfied. Therefore, from this lemma, there exists a unique solution $\beta_\tau \in W^{1,2}(0, t_n; Y)$ such that Eq. (6) holds.

It remains to show that there exist elements $w_\tau^n \in V$ and $\xi_\tau^n \in X^*$ such that equality (7) holds. Denote

$$v_0 = u_0 + \frac{\tau^\alpha}{\Gamma(\alpha + 1)} \sum_{j=1}^{n-1} w_\tau^j \left[(n - j + 1)^\alpha - (n - j)^\alpha \right], \qquad c_0 = \frac{\tau^\alpha}{\Gamma(\alpha + 1)}.$$

To this end, we will show that the multivalued operator $V \ni v \mapsto Av + B(v_0 + c_0 v) + M^* \partial J(\beta_\tau(t_n), M(v_0 + c_0 v)) \subset V^*$ is surjective. Hypotheses $H(A)$ and $H(B)$ imply that operator $V \ni v \mapsto Av + B(v_0 + c_0 v) \in V^*$ is bounded, continuous, and fulfills the condition

$$\langle Av + B(v_0 + c_0 v) - Au - B(v_0 + c_0 u), v - u \rangle \geq \left(m_A - c_0 \|B\| \right) \|v - u\|^2$$

for all $v, u \in V$. For the mapping $v \mapsto J(\beta_\tau(t_n), v)$, by hypotheses $H(J)$, $H(M)$ and Lemma 5, we obtain that $V \ni v \mapsto M^* \partial J(\beta_\tau(t_n), M(v_0 + c_0 v)) \subset V^*$ is pseudomonotone and

$$\|M^* \partial J(\beta_\tau(t_n), M(v_0 + c_0 v))\| \leq c_0 c_J \|M\|^2 \|v\| + \|M\| c_J (1 + \|M\| \|v_0\|)$$

for all $v \in V$. Next, we choose

$$\tau_0 = \left(\frac{m_A \Gamma(1 + \alpha)}{\|B\| + c_J \|M\|^2} \right)^{\frac{1}{\alpha}}$$

to see that $v \mapsto Av + B(v_0 + c_0 v)$ is strongly monotone and $c_0 c_J \|M\|^2 + c_0 \|B\| < m_A$ for all $\tau \in (0, \tau_0)$. We are now in a position to apply Corollary 7 to deduce that operator $v \mapsto Av + B(v_0 + c_0 v) + M^* \partial J(\beta_\tau(t_n), M(v_0 + c_0 v))$ is surjective for all $0 < \tau < \tau_0$. Therefore, we conclude that there exist elements $w_\tau^n \in V$ and $\xi_\tau^n \in X^*$ such that equation (7) holds. This completes the proof of the lemma. $\square$

The following result provides estimates for the sequence of solutions of the Rothe problem, Problem 13.

Lemma 15. *Under assumptions $H(A)$, $H(B)$, $H(J)$, $H(g)$, $H(f)$, and $H(M)$, there exists $\tau_0 > 0$ and $C > 0$ independent of τ, such that for all $\tau \in (0, \tau_0)$, the solutions to Problem 13 satisfy*

$$\max_{k=1,2,\ldots,N} \|w_\tau^k\| \leq C, \tag{10}$$

$$\max_{k=1,2,\ldots,N} \|u_\tau^k\| \leq C, \tag{11}$$

$$\max_{k=1,2,\ldots,N} \|\xi_\tau^k\|_{X^*} \leq C, \tag{12}$$

where $\xi_\tau^k \in X^*$ is such that $\xi_\tau^k \in \partial J(\beta_\tau(t_k), M(u_\tau^k))$ and

$$A w_\tau^k + B(u_\tau^k) + M^* \xi_\tau^k = f_\tau^k,$$

for $k = 1, 2, \ldots, N$.

Proof. Taking $k = n$ in (7), we multiply equation (7) by w_τ^n to get

$$\langle A w_\tau^n, w_\tau^n \rangle + \langle B u_\tau^n, w_\tau^n \rangle + \langle \xi_\tau^n, M w_\tau^n \rangle_{X \times X^*} = \langle f_\tau^n, w_\tau^n \rangle.$$

From definition of u_τ^n (see (8)) and hypothesis $H(B)$, we have

$$
\langle Bu_\tau^n, w_\tau^n \rangle = \left\langle B\left(u_0 + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^n w_\tau^j \left[(n-j+1)^\alpha - (n-j)^\alpha\right]\right), w_\tau^n \right\rangle
$$

$$
\geq -\|Bu_0\|_{V^*}\|w_\tau^n\| - \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^{n-1} [(n-j+1)^\alpha - (n-j)^\alpha]\|B\|\|w_\tau^j\|\|w_\tau^n\| - \frac{\tau^\alpha\|B\|}{\Gamma(\alpha+1)}\|w_\tau^n\|^2.
$$

$$(13)$$

It follows from the growth condition $H(J)(ii)$ that

$$
\langle \xi_\tau^n, Mw_\tau^n \rangle_{X^* \times X} \geq -\|\xi_\tau^n\|_{X^*}\|Mw_\tau^n\|_X \geq -c_J(1 + \|Mu_\tau^n\|_X)\|Mw_\tau^n\|_X
$$

$$
\geq -c_J\|Mw_\tau^n\|_X \left(1 + \|Mu_0\|_X + \frac{\tau^\alpha\|M\|}{\Gamma(1+\alpha)} \sum_{j=1}^n \|w_\tau^j\|[(n-j+1)^\alpha - (n-j)^\alpha]\right)
$$

$$
\geq -(c_J\|M\| + c_J\|M\|^2\|u_0\|)\|w_\tau^n\| - \frac{\tau^\alpha c_J\|M\|^2}{\Gamma(1+\alpha)}\|w_\tau^n\|^2
$$

$$
- \frac{\tau^\alpha c_J\|M\|^2}{\Gamma(1+\alpha)} \sum_{j=1}^{n-1} \|w_\tau^j\|\|w_\tau^n\|[(n-j+1)^\alpha - (n-j)^\alpha].
$$

$$(14)$$

From the coercivity of operator A and inequalities (13) and (14), we get

$$
\langle f_\tau^n, w_\tau^n \rangle = \langle Aw_\tau^n, w_\tau^n \rangle + \langle Bu_\tau^n, w_\tau^n \rangle + \langle \xi_\tau^n, Mw_\tau^n \rangle_{X \times X^*}
$$

$$
\geq m_A\|w_\tau^n\|^2 - \frac{\tau^\alpha\|B\|}{\Gamma(\alpha+1)}\|w_\tau^n\|^2 - (\|Bu_0\|_{V^*} + c_J\|M\|^2\|u_0\| + c_J\|M\|)\|w_\tau^n\|
$$

$$
- \frac{\tau^\alpha\|B\|}{\Gamma(1+\alpha)} \sum_{j=1}^{n-1} \|w_\tau^j\|\,\|w_\tau^n\|\,[(n-j+1)^\alpha - (n-j)^\alpha] - \frac{\tau^\alpha c_J\|M\|^2}{\Gamma(1+\alpha)}\|w_\tau^n\|^2
$$

$$
- \frac{\tau^\alpha c_J\|M\|^2}{\Gamma(1+\alpha)} \sum_{j=1}^{n-1} \|w_\tau^j\|\,\|w_\tau^n\|\,[(n-j+1)^\alpha - (n-j)^\alpha],
$$

and subsequently

$$
\|f_\tau^n\|_{V^*} + \frac{\tau^\alpha(\|B\| + c_J\|M\|^2)}{\Gamma(1+\alpha)} \sum_{j=1}^{n-1} \|w_\tau^j\|\,[(n-j+1)^\alpha - (n-j)^\alpha]
$$

$$
+ \|Bu_0\|_{V^*} + c_J\|M\| + c_J\|M\|^2\|u_0\| \geq \left(m_A - \frac{\tau^\alpha(\|B\| + c_J\|M\|^2)}{\Gamma(1+\alpha)}\right)\|w_\tau^n\|.
$$

Taking $\tau_0 = \left(\frac{m_A\Gamma(1+\alpha)}{2(\|B\| + c_J\|M\|^2)}\right)^{\frac{1}{\alpha}}$, we deduce that $m_A - \frac{\tau^\alpha(\|B\| + c_J\|M\|^2)}{\Gamma(1+\alpha)} \geq \frac{m_A}{2}$ for all $\tau \in (0, \tau_0)$. Therefore, one has

$$
\frac{2\|f_\tau^n\|_{V^*}}{m_A} + \frac{2(c_J\|M\| + c_J\|M\|^2\|u_0\| + \|Bu_0\|_{V^*})}{m_A}
$$

$$
+ \frac{2\tau^\alpha(\|B\| + c_J\|M\|^2)}{m_A\Gamma(1+\alpha)} \sum_{j=1}^{n-1} \|w_\tau^j\|\,[(n-j+1)^\alpha - (n-j)^\alpha] \geq \|w_\tau^n\|.
$$

Next, from hypothesis $H(f)$, there exists a constant $m_f > 0$ such that $\|f_\tau^n\|_{V^*} \le m_f$ for all $\tau > 0$ and $n \in \mathbb{N}$. Setting

$$m_0 = \frac{2m_f}{m_A} + \frac{2(c_J\|M\| + c_J\|M\|^2\|u_0\| + \|Bu_0\|_{V^*})}{m_A},$$

we are in a position to apply the generalized discrete Gronwall inequality, Lemma 9, to see that

$$\|w_\tau^n\| \le m_0 \exp\Big(\frac{2(\|B\| + c_J\|M\|^2)\tau^\alpha}{m_A\Gamma(\alpha+1)} \sum_{j=1}^{n-1}[(n-j+1)^\alpha - (n-j)^\alpha]\Big)$$

$$= m_0 \exp\Big(\frac{2(\|B\| + c_J\|M\|^2)t_n^\alpha}{m_A\Gamma(1+\alpha)}\Big)$$

$$\le m_1 := m_0 \exp\Big(\frac{2(\|B\| + c_J\|M\|^2)T^\alpha}{m_A\Gamma(1+\alpha)}\Big).$$

Hence, the estimate (10) is verified.

Furthermore, by equality (8), the estimate (11) is easily obtained from the following inequality

$$\|u_\tau^n\| = \Big\|u_0 + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^{n} w_\tau^j [(n-j+1)^\alpha - (n-j)^\alpha]\Big\|$$

$$\le \|u_0\| + \frac{m_1}{\Gamma(\alpha+1)} \sum_{j=1}^{n}(t_{n-j+1}^\alpha - t_{n-j}^\alpha)$$

$$\le \|u_0\| + \frac{m_1}{\Gamma(\alpha+1)}t_n^\alpha$$

$$\le m_2 := \|u_0\| + \frac{m_1 T^\alpha}{\Gamma(\alpha+1)}.$$

Finally, the growth condition in $H(J)$(ii) ensures that

$$\|\xi_\tau^n\|_{X^*} \le c_J(1 + \|Mu_\tau^n\|_X) \le c_J\big(1 + \|M\|m_2\big).$$

Consequently, the condition (12) follows, which completes the proof of the lemma. $\square$

To state and prove our main result on the existence of solution to Problem 11, we define the piecewise constant interpolant functions $\overline{w}_\tau$, $\overline{u}_\tau : [0,T] \to V$, $f_\tau : [0,T] \to V^*$ and $\xi_\tau : [0,T] \to X^*$ by

$$\overline{w}_\tau(t) = w_\tau^k, \quad t \in (t_{k-1}, t_k],$$

$$\overline{u}_\tau(t) = u_\tau^k, \quad t \in (t_{k-1}, t_k],$$

$$f_\tau(t) = f_\tau^k, \quad t \in (t_{k-1}, t_k],$$

$$\xi_\tau(t) = \xi_\tau^k, \quad t \in (t_{k-1}, t_k]$$

for $k = 1, \dots, N$.

Theorem 16. *Assume that $H(A)$, $H(B)$, $H(J)$, $H(g)$, $H(f)$, and $H(M)$ hold. Let $\eta \in (0, \alpha)$ and $\{\tau_n\}$ be a sequence such that $\tau_n \to 0$, as $n \to \infty$. Then, for a subsequence still denoted by τ, we have*

$$\overline{w}_\tau \to w \quad \text{weakly in} \quad L^{\frac{1}{\eta}}(0,T;V),$$

$$\xi_\tau \to \xi \quad \text{weakly in} \quad \mathcal{X}^*,$$

$$\beta_\tau \to \beta \quad \text{in } C(0,T;Y),$$

as $\tau \to 0$, where $(w, \xi, \beta) \in L^{\frac{1}{\eta}}(0, T; V) \times \mathcal{X}^* \times W^{1,2}(0, T; Y)$ is a solution to Problem 11.

Proof. From the estimate (10), we have

$$\|\overline{w}_\tau\|_{L^{\frac{1}{\eta}}(0,T;V)}^{\frac{1}{\eta}} = \int_0^T \|\overline{w}_\tau(s)\|^{\frac{1}{\eta}} \, \mathrm{d}s = \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|w_\tau^i\|^{\frac{1}{\eta}} \, \mathrm{d}s = \tau \sum_{i=1}^N \|w_\tau^i\|^{\frac{1}{\eta}} \leq C.$$

Hence, we deduce that $\{\overline{w}_\tau\}$ is bounded in $L^{\frac{1}{\eta}}(0, T; V)$. Therefore, without loss of generality, we may assume that there exists $w \in L^{\frac{1}{\eta}}(0, T; V)$ such that

$$\overline{w}_\tau \to w \quad \text{weakly in} \quad L^{\frac{1}{\eta}}(0, T; V), \quad \text{as} \quad \tau \to 0. \tag{15}$$

For any $v^* \in V^*$ and $t \in [0, T]$, let $e(s) = (t - s)^{\alpha-1} v^* \chi_{[0,t]}(s)$ for $s \in (0, T)$. Obviously, $e \in L^{\frac{1}{\eta'}}(0, T; V^*)$, where $\eta' = 1 - \eta$. Now, we have

$$\left| \left\langle v^*, \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \overline{w}_\tau(s) \, \mathrm{d}s - \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} w(s) \, \mathrm{d}s \right\rangle \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t |\langle (t - s)^{\alpha-1} v^*, \overline{w}_\tau(s) - w(s) \rangle| \, \mathrm{d}s$$

$$\leq \frac{1}{\Gamma(\alpha)} |\langle e, \overline{w}_\tau - w \rangle_{L^{\frac{1}{\eta'}}(0,T;V^*) \times L^{\frac{1}{\eta}}(0,T;V)}| \to 0, \quad \text{as } \tau \to 0.$$

Therefore, we have

$$_0 I_t^\alpha \overline{w}_\tau(t) \to {}_0 I_t^\alpha w(t) \quad \text{weakly in} \quad V, \text{ as } \tau \to 0, \tag{16}$$

for all $t \in [0, T]$. Moreover, using estimate (10) again, one has

$$\|\overline{u}_\tau(t) - u_0 - {}_0 I_t^\alpha \overline{w}_\tau(t)\| = \left\| \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^n w_\tau^j [(n - j + 1)^\alpha - (n - j)^\alpha] \right.$$

$$\left. - \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \overline{w}_\tau(s) \, ds \right\| = \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_n} (t_n - s)^{\alpha-1} \overline{w}_\tau(s) \, ds \right.$$

$$\left. - \int_0^t (t - s)^{\alpha-1} \overline{w}_\tau(s) \, ds \right\| \leq \frac{1}{\Gamma(\alpha)} \left\| \int_t^{t_n} (t_n - s)^{\alpha-1} \overline{w}_\tau(s) \, ds \right\|$$

$$+ \frac{1}{\Gamma(\alpha)} \left\| \int_0^t [(t_n - s)^{\alpha-1} - (t - s)^{\alpha-1}] \overline{w}_\tau(s) \, ds \right\|$$

$$\leq \frac{C}{\Gamma(\alpha)} \left(\int_t^{t_n} (t_n - s)^{\alpha-1} \, ds + \int_0^t |(t - s)^{\alpha-1} - (t_n - s)^{\alpha-1}| \, ds \right)$$

$$\leq \frac{C}{\Gamma(\alpha+1)} [(t_n - t)^\alpha + t^\alpha + (t_n - t)^\alpha - t_n^\alpha] \tag{17}$$

for $t \in (t_{n-1}, t_n]$. So, we conclude

$$\overline{u}_\tau(t) - u_0 - {}_0 I_t^\alpha \overline{w}_\tau(t) \to 0 \quad \text{strongly in} \quad V, \quad \text{as } \tau \to 0$$

for all $t \in [0, T]$. Combining the latter and convergence (16), we obtain

$$\overline{u}_\tau(t) \to {}_0 I_t^\alpha w(t) + u_0 \quad \text{weakly in} \quad V, \quad \text{as} \quad \tau \to 0 \tag{18}$$

for all $t \in [0, T]$. Since the operator M is compact, we get

$$M(\overline{u}_\tau(t)) \to M(u_0 + {}_0 I_t^\alpha w(t)) \quad \text{strongly in} \quad X, \quad \text{as} \quad \tau \to 0 \tag{19}$$

for all $t \in [0, T]$.

Analogously, for functions $\widehat{u}_\tau$ given by (9) and $\overline{u}_\tau$, we have

$$\|\overline{u}_\tau(t) - \widehat{u}_\tau(t)\| = \frac{\tau^\alpha}{\Gamma(\alpha+1)} \Big\| \sum_{j=1}^n w_\tau^j \left[(n-j+1)^\alpha - (n-j)^\alpha \right]$$

$$- \sum_{j=1}^{n-1} w_\tau^j \left[(n-j)^\alpha - (n-j-1)^\alpha \right] \Big\| \leq \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^{n-1} |(n-j+1)^\alpha$$

$$- 2(n-j)^\alpha + (n-j-1)^\alpha| \cdot \|w_\tau^j\| + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \|w_\tau^n\| \leq \frac{\tau^\alpha C}{\Gamma(\alpha+1)}$$

$$+ \frac{\tau^\alpha C}{\Gamma(\alpha+1)} \sum_{j=1}^{n-1} |(n-j+1)^\alpha - 2(n-j)^\alpha + (n-j-1)^\alpha|$$

$$\leq \frac{\tau^\alpha C}{\Gamma(\alpha+1)} \left(1 + n^\alpha - (n-1)^\alpha \right)$$

for $t \in (t_{n-1}, t_n]$. This inequality together with convergence (18) and the compactness of M implies

$$M(\widehat{u}_\tau(t)) \to M(u_0 + {}_0I_t^\alpha w(t)) \quad \text{strongly in} \quad X, \quad \text{as} \quad \tau \to 0 \qquad (20)$$

for all $t \in [0, T]$.

Since $w \in L^{\frac{1}{\eta}}(0, T; V)$, it is obvious that function $t \mapsto M({}_0I_t^\alpha w(t))$ belongs to $AC(0, T; X)$. We denote $u = u_0 + {}_0I_t^\alpha w$. We are now in a position to apply Lemma 8 to deduce that there exists a unique solution $\beta \in W^{1,2}(0, T; Y)$ such that

$$\beta(t) = \beta_0 + \int_0^t g(s, M(u(s)), \beta(s)) \, \mathrm{d}s$$

for all $t \in [0, T]$. By hypothesis $H(g)$(ii) and Lemma 8, we have

$$\|\beta(t) - \beta_\tau(t)\|_Y \leq C \int_0^t \|M(u_0 + {}_0I_t^\alpha w(s)) - M(\widehat{u}(s))\|_X \, \mathrm{d}s$$

for all $t \in [0, T]$, thus is,

$$\max_{t \in [0, T]} \|\beta(t) - \beta_\tau(t)\|_Y \leq C \int_0^T \|M(u_0 + {}_0I_t^\alpha w(s)) - M(\widehat{u}(s))\|_X \, \mathrm{d}s.$$

We use convergence (20), estimate (10), and the Lebesgue-dominated convergence theorem, see, e.g., [33, Theorem 1.65], to conclude that β_τ converges to β in $C(0, T; Y)$.

On the other hand, estimate (12) guarantees that the sequence $\{\xi_\tau\}$ is bounded in $\mathcal{X}^*$. So, passing to a subsequence if necessary, there exists $\xi \in \mathcal{X}^*$ such that

$$\xi_\tau \to \xi \quad \text{weakly in} \quad \mathcal{X}^*, \quad \text{as} \quad \tau \to 0. \qquad (21)$$

By Lemma 12, we know that the mapping $(y, x) \mapsto \partial J(y, x)$ is upper semicontinuous from $Y \times X$ into X^* endowed with weak topology. Using this property, the relation

$$\xi_\tau^k \in \partial J(\beta_\tau(t_k), M(u_\tau^k)) \quad \text{for} \quad k = 1, 2, \dots, N,$$

and convergences $\beta_\tau \to \beta$ in $C(0, T; Y)$, (19) and (21), by [33, Theorem 3.13], we deduce that

$$\xi(t) \in \partial J(\beta(t), M(u_0 + {}_0I_t^\alpha w(t)))$$

for a.e. $t \in (0, T)$.

Subsequently, we consider the Nemytskii operators $\mathcal{A}$ and $\mathcal{B}$ corresponding to A and B, which are defined by

$$(\mathcal{A}v)(t) = Av(t) \quad \text{and} \quad (\mathcal{B}v)(t) = B(u_0 + {}_0I_t^\alpha v(t))$$

for $v \in \mathcal{V}$, a.e. $t \in (0,T)$, respectively. Since $A \in \mathcal{L}(V, V^*)$ and $\overline{w}_\tau \to w$ weakly in $L^{\frac{1}{\eta}}(0,T;V)$, as $\tau \to 0$, we obtain

$$\mathcal{A}\,\overline{w}_\tau \to \mathcal{A}\,w \quad \text{weakly in} \quad L^{\frac{1}{\eta}}(0,T;V^*) \subset \mathcal{V}^*, \quad \text{as} \quad \tau \to 0.$$

Using hypothesis $H(B)$ and convergence (16), one has

$$B(u_0 + {}_0I_t^\alpha \overline{w}_\tau(t)) \to B(u_0 + {}_0I_t^\alpha w(t)) \quad \text{weakly in} \quad V^*, \text{ as } \tau \to 0,$$

for all $t \in [0,T]$. Furthermore, we use estimate (10) again to obtain

$$\langle B(u_0 + {}_0I_t^\alpha \overline{w}_\tau(t)), v(t) \rangle \leq \|B(u_0 + {}_0I_t^\alpha \overline{w}_\tau(t))\| \, \|v(t)\|$$

$$\leq \left(\frac{\|B\|C}{\Gamma(\alpha+1)} t^\alpha + T\|B\|\|u_0\| \right) \|v(t)\|$$

$$\leq \left(\frac{\|B\|C}{\Gamma(\alpha+1)} T^\alpha + T\|B\|\|u_0\| \right) \|v(t)\|.$$

Exploiting the Lebesgue-dominated convergence theorem again, we get from the above inequality

$$\lim_{\tau \to 0} \langle \mathcal{B}\,\overline{w}_\tau, v \rangle_{\mathcal{V}^* \times \mathcal{V}} = \lim_{\tau \to 0} \int_0^T \langle B(u_0 + {}_0I_t^\alpha \overline{w}_\tau(t)), v(t) \rangle \, dt$$

$$= \int_0^T \lim_{\tau \to 0} \langle B(u_0 + {}_0I_t^\alpha \overline{w}_\tau(t)), v(t) \rangle \, dt$$

$$= \int_0^T \langle B(u_0 + {}_0I_t^\alpha w(t)), v(t) \rangle \, dt = \langle \mathcal{B}w, v \rangle_{\mathcal{V}^* \times \mathcal{V}}$$

for all $v \in \mathcal{V}$.

From [5, Lemma 3.3], we know that $f_\tau \to f$ strongly in $\mathcal{V}^*$, as $\tau \to 0$. We also introduce the Nemytskii operator $\mathcal{M}: \mathcal{V} \to \mathcal{X}$ corresponding to M,

$$(\mathcal{M}v)(t) = M(v(t)) \quad \text{for} \quad v \in \mathcal{V}, \text{ a.e. } t \in (0,T).$$

To conclude, for all $v \in \mathcal{V}$, we obtain the following results

$$\langle \mathcal{A}\,\overline{w}_\tau, v \rangle_{\mathcal{V}^* \times \mathcal{V}} \to \langle \mathcal{A}w, v \rangle_{\mathcal{V}^* \times \mathcal{V}},$$

$$\langle \mathcal{B}\,\overline{w}_\tau, v \rangle_{\mathcal{V}^* \times \mathcal{V}} \to \langle \mathcal{B}w, v \rangle_{\mathcal{V}^* \times \mathcal{V}},$$

$$\langle \xi_\tau, \mathcal{M}v \rangle_{\mathcal{X}^* \times \mathcal{X}} \to \langle \xi, \mathcal{M}v \rangle_{\mathcal{X}^* \times \mathcal{X}},$$

$$\langle f_\tau, v \rangle_{\mathcal{V}^* \times \mathcal{V}} \to \langle f, v \rangle_{\mathcal{V}^* \times \mathcal{V}},$$

as $\tau \to 0$. The above convergences entail

$$0 \leq \limsup_{\tau \to 0} \langle \mathcal{A}\,\overline{w}_\tau, v \rangle_{\mathcal{V}^* \times \mathcal{V}} + \limsup_{\tau \to 0} \langle \mathcal{B}\,\overline{w}_\tau, v \rangle_{\mathcal{V}^* \times \mathcal{V}} + \limsup_{\tau \to 0} \langle \xi_\tau, \mathcal{M}v \rangle_{\mathcal{X}^* \times \mathcal{X}}$$

$$- \liminf_{\tau \to 0} \langle f_\tau, v \rangle_{\mathcal{V}^* \times \mathcal{V}} \leq \langle \mathcal{A}w + \mathcal{B}w - f, v \rangle_{\mathcal{V}^* \times \mathcal{V}} + \langle \xi, \mathcal{M}v \rangle_{\mathcal{X}^* \times \mathcal{X}}$$

for all $v \in \mathcal{V}$. This implies

$$\langle \mathcal{A}w + \mathcal{B}w + \mathcal{M}^*\xi, v \rangle_{\mathcal{V}^* \times \mathcal{V}} \geq \langle f, v \rangle_{\mathcal{V}^* \times \mathcal{V}} \quad \text{for all} \quad v \in \mathcal{V},$$

where $\xi(t) \in \partial J(\beta(t), M(u_0 + {}_0I_t^\alpha w(t)))$ for a.e. $t \in (0, T)$. We conclude that $(w, \beta) \in L^{\frac{1}{\eta}}(0, T; V) \times W^{1,2}(0, T; Y)$ is a solution of Problem 11, which completes the proof of the theorem. $\qquad\square$

4. A fractional viscoelastic contact problem with friction and adhesion

In this section, the abstract theoretical results of Sect. 3 will be used to study a frictional contact problem for a viscoelastic body with time fractional Kelvin–Voigt constitutive law and adhesion.

The physical formulation of the fractional viscoelastic contact problem is provided below. We consider a viscoelastic body which occupies a domain $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$. The boundary $\Gamma = \partial\Omega$ is assumed to be Lipschitz continuous, and it is divided into three disjoint measurable parts Γ_D, Γ_N and Γ_C with $\operatorname{meas}(\Gamma_D) > 0$. The contact problem will be discussed in a finite time interval $(0, T)$.

For convenience of the reader, the description of basic notation is provided in Table 1.

The inner products and corresponding norms in $\mathbb{R}^d$ and $\mathbb{S}^d$ are denoted by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\|_{\mathbb{R}^d} = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{u} = (u_i), \quad \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\|_{\mathbb{S}^d} = (\boldsymbol{\tau} : \boldsymbol{\tau})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}), \quad \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

respectively. Also, we denote

$$\mathcal{Q} = \Omega \times (0, T), \quad \Sigma_D = \Gamma_D \times (0, T)$$
$$\Sigma_N = \Gamma_N \times (0, T), \quad \Sigma_C = \Gamma_C \times (0, T).$$

For simplicity, we do not indicate explicitly the dependence of various functions and operators on $\boldsymbol{x}$. The classical formulation of the mechanical contact problem is described as follows.

Problem 17. *Find a displacement field* $\boldsymbol{u}\colon \mathcal{Q} \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma}\colon \mathcal{Q} \to \mathbb{S}^d$ *and a bonding field* $\beta\colon \Sigma_C \to [0, 1]$ *such that*

$$\boldsymbol{\sigma}(t) = \mathscr{C}(\boldsymbol{\varepsilon}({}_0^C D_t^\alpha \boldsymbol{u}(t))) + \mathscr{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \qquad \text{in} \quad \mathcal{Q}, \tag{22}$$

$$\operatorname{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = 0 \qquad \text{in} \quad \mathcal{Q}, \tag{23}$$

$$\boldsymbol{u}(t) = 0 \qquad \text{on} \quad \Sigma_D, \tag{24}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_N(t) \qquad \text{on} \quad \Sigma_N, \tag{25}$$

$$-\sigma_\nu(t) \in \partial j_\nu(\beta(t), u_\nu(t)) \qquad \text{on} \quad \Sigma_C, \tag{26}$$

TABLE 1. *Symbol description*

Symbol	Description
$\boldsymbol{\nu} = (\nu_i)$	The unit outward normal vector
$\boldsymbol{x} \in \overline{\Omega} = \Omega \cup \Gamma$	A position vector
indices i, j, k, l	They run from 1 to d and the summation convention over repeated indices is used
$\mathbb{S}^d$	The space of second order symmetric tensors on $\mathbb{R}^d$
$\boldsymbol{u} = (u_i)$	A displacement vector
$\boldsymbol{\sigma} = (\sigma_{ij})$	A stress tensor
$\varepsilon(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$	A linearized (small) strain tensor
	$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$, $i, j = 1, \ldots, d$
$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$	The normal component of stress field $\boldsymbol{\sigma}$ on Γ
$\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$	The tangential component of stress field $\boldsymbol{\sigma}$ on Γ
$u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$	The normal component of the displacement field $\boldsymbol{u}$ on Γ
$\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}$	The tangential component of the displacement field $\boldsymbol{u}$ on Γ
$\operatorname{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})$	The divergence of $\boldsymbol{\sigma}$, $\sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j}$

$$- \, \boldsymbol{\sigma}_\tau(t) \in \partial j_\tau(\beta(t), \boldsymbol{u}_\tau(t)) \qquad \text{on} \quad \Sigma_C, \tag{27}$$

$$\beta'(t) = F(t, \boldsymbol{u}(t), \beta(t)) \qquad \text{on} \quad \Sigma_C, \tag{28}$$

$$\beta(0) = \beta_0 \qquad \text{on} \quad \Gamma_C, \tag{29}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{in} \quad \Omega. \tag{30}$$

We now give a brief description of equations and relations in Problem 17. The generalized fractional Kelvin–Voigt constitutive law of the Caputo type, see [54], for viscoelastic body is given in (22). Operators $\mathscr{C}$ and $\mathscr{E}$ stand here for the viscosity and elasticity operators, respectively. Note that since the contact process is assumed to be quasistatic, the acceleration term is negligible and we deal in $\mathcal{Q}$ with equilibrium equation (23), where $\boldsymbol{f}_0$ denotes the time dependent density of volume forces. Moreover, conditions (24) and (25) reveal the displacement and traction boundary conditions on parts Γ_D and Γ_N of the boundary, respectively, i.e., the body is fixed on Γ_D and it is subjected to the time dependent surface traction of density $\boldsymbol{f}_N$ on Γ_N.

The unknown function β is a surface internal variable, which is usually called the bonding field or the adhesion field. It describes the pointwise fractional density of active bonds on the contact surface. The evolution of the bounding field is driven by a nonlinear ordinary differential equation (28) depending on the displacement, and considered on contact surface Γ_C. Furthermore, if $\beta = 1$ at a point of the contact part, the adhesion is complete and all the bonds are active, and $\beta = 0$ means that all bonds are inactive and there is no adhesion. But, when $0 < \beta < 1$ then the adhesion is partial and a fracture β of the bonds is active. The function β_0 denotes the initial bonding field in (29). For more details on the adhesion phenomena, see [3,6,15,36].

The contact condition (26) with adhesion is called a multivalued normal compliance contact boundary condition, which is described by the subgradient of a nonconvex function j_ν, where j_ν is assumed to be locally Lipschitz with respect to the last variable. On the other hand, the general tangential contact condition (27) with adhesion, i.e., friction contact condition with adhesion, is governed by the subgradient of a nonconvex function j_τ. In fact, this contact condition without the bonding field has been treated in many papers, see, e.g., [15,34,35,44,45]. The initial displacement is given in (30). For more details on the mathematical theory of contact mechanics, we refer to [33,37,44,45].

Subsequently, we obtain the variational formulation of Problem 17. We will use the function spaces V, H and $\mathcal{H}$ defined by

$$V = \{ \, \boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D \}, \quad H = L^2(\Omega; \mathbb{R}^d) \quad \text{and} \quad \mathcal{H} = L^2(\Omega; \mathbb{S}^d). \tag{31}$$

The trace of an element $\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d)$ is denoted by the same symbol $\boldsymbol{v}$. It is obvious that $\mathcal{H}$ is endowed with the Hilbertian structure by the inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_\mathcal{H} = \int_\Omega \sigma_{ij}(\boldsymbol{x}) \, \tau_{ij}(\boldsymbol{x}) \, \mathrm{d}x \quad \text{for} \quad \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H},$$

and the associated norm $\| \cdot \|_\mathcal{H}$. For space V, we consider the inner product by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_\mathcal{H} \quad \text{for} \quad \boldsymbol{u}, \boldsymbol{v} \in V,$$

and the associated norm $\| \cdot \|_V$. Recall that, since $\mathrm{meas}\,(\Gamma_D) > 0$, we know that V is a real Hilbert space. From the Sobolev trace theorem, there exists $c_k > 0$ (the Korn constant) such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_C; \mathbb{R}^d)} \le c_k \|\gamma\| \, \|\boldsymbol{v}\|_V \quad \text{for all} \quad \boldsymbol{v} \in V,$$

where $\|\gamma\|$ denotes the norm of the trace operator $\gamma \colon V \to L^2(\Gamma_C; \mathbb{R}^d)$.

In the study of Problem 17, the viscosity operator $\mathscr{C} \colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ and elasticity operator $\mathscr{E} \colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfy the following hypotheses.

$$\begin{cases} \mathscr{C} \colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d \text{ is such that} \\ \text{(a) } \mathscr{C}(\boldsymbol{x}, \boldsymbol{\varepsilon}) = a(\boldsymbol{x})\boldsymbol{\varepsilon} \text{ for a.e. } \boldsymbol{x} \in \Omega \text{ and all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \\ \text{(b) } a(\boldsymbol{x}) = (a_{ijkl}(\boldsymbol{x})) \text{ with } a_{ijkl} \in L^\infty(\Omega), \\ \text{(c) } a_{ijkl}(\boldsymbol{x})\varepsilon_{ij}\varepsilon_{kl} \geq m_a \, \|\boldsymbol{\varepsilon}\|_{\mathbb{S}^d}^2 \text{ for a.e. } \boldsymbol{x} \in \Omega, \\ \quad \text{and all } \boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathbb{S}^d \text{ with } m_a > 0. \end{cases} \tag{32}$$

$$\begin{cases} \mathscr{E} \colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d \text{ is such that} \\ \text{(a) } \mathscr{E}(\boldsymbol{x}, \boldsymbol{\varepsilon}) = b(\boldsymbol{x})\boldsymbol{\varepsilon} \text{ for a.e. } \boldsymbol{x} \in \Omega \text{ and all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \\ \text{(b) } b(\boldsymbol{x}) = (b_{ijkl}(\boldsymbol{x})) \text{ with } b_{ijkl} \in L^\infty(\Omega). \end{cases} \tag{33}$$

The normal potential $j_\nu \colon \Gamma_C \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and tangential function $j_\tau \colon \Gamma_C \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ have the following properties.

$$\begin{cases} j_\nu \colon \Gamma_C \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ \text{(a) } j_\nu(\cdot, r, s) \text{ is measurable on } \Gamma_C \text{ for all } r, s \in \mathbb{R} \text{ and} \\ \quad j_\nu(\cdot, 0, 0) \in L^1(\Gamma_C), \\ \text{(b) } j_\nu(\boldsymbol{x}, r, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for all } r \in \mathbb{R} \text{ and a.e. } \boldsymbol{x} \in \Gamma_C, \\ \text{(c) } |\partial j_\nu(\boldsymbol{x}, r, s)| \leq c_\nu(1 + |s|) \text{ for all } r, s \in \mathbb{R} \text{ and a.e. } \boldsymbol{x} \in \Gamma_C \\ \quad \text{with } c_\nu > 0, \\ \text{(d) either } j_\nu(\boldsymbol{x}, r, \cdot) \text{ or } -j_\nu(\boldsymbol{x}, r, \cdot) \text{ is regular for a.e. } \boldsymbol{x} \in \Gamma_C \text{ and } r \in \mathbb{R}, \\ \text{(e) } (r, s) \mapsto j_\nu^0(\boldsymbol{x}, r, s; z) \text{ is upper semicontinuous for all } z \in \mathbb{R} \\ \quad \text{and a.e. } \boldsymbol{x} \in \Gamma_C, \text{ where } j_\nu^0 \text{ denotes the Clarke derivative} \\ \quad \text{of } s \mapsto j_\nu(\boldsymbol{x}, r, s) \text{ in direction } z. \end{cases} \tag{34}$$

$$\begin{cases} j_\tau \colon \Gamma_C \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \text{ is such that} \\ \text{(a) } j_\tau(\cdot, r, \boldsymbol{\xi}) \text{ is measurable on } \Gamma_C \text{ for all } (r, \boldsymbol{\xi}) \in \mathbb{R} \times \mathbb{R}^d \text{ and} \\ \quad j_\tau(\cdot, 0, \mathbf{0}) \in L^1(\Gamma_C), \\ \text{(b) } j_\tau(\boldsymbol{x}, r, \cdot) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for all } r \in \mathbb{R} \text{ and a.e. } \boldsymbol{x} \in \Gamma_C, \\ \text{(c) } \|\partial j_\tau(\boldsymbol{x}, r, \boldsymbol{\xi})\|_{\mathbb{R}^d} \leq c_\tau(1 + \|\boldsymbol{\xi}\|_{\mathbb{R}^d}) \text{ for all } (r, \boldsymbol{\xi}) \in \mathbb{R} \times \mathbb{R}^d \\ \quad \text{and a.e. } \boldsymbol{x} \in \Gamma_C \text{ with } c_\tau > 0, \\ \text{(d) either } j_\tau(\boldsymbol{x}, r, \cdot) \text{ or } -j_\tau(\boldsymbol{x}, r, \cdot) \text{ is regular for a.e. } \boldsymbol{x} \in \Gamma_C \text{ and } r \in \mathbb{R}, \\ \text{(e) } (r, \boldsymbol{\xi}) \mapsto j_\tau^0(\boldsymbol{x}, r, \boldsymbol{\xi}; \boldsymbol{\eta}) \text{ is upper semicontinuous for all } \boldsymbol{\eta} \in \mathbb{R}^d \\ \quad \text{and a.e. } \boldsymbol{x} \in \Gamma_C, \text{ where } j_\tau^0 \text{ denotes the Clarke derivative} \\ \quad \text{of } \boldsymbol{\xi} \mapsto j_\tau(\boldsymbol{x}, r, \boldsymbol{\xi}) \text{ in direction } \boldsymbol{\eta}. \end{cases} \tag{35}$$

In conditions (34)(c) and (35)(c), the symbols ∂j_ν and ∂j_τ stand for the Clarke generalized gradient of j_ν and j_τ with respect to their last variables, respectively. Subsequently, if we suppose that (34)(d) and (35)(d) hold, we mean that "either $j_\nu(\boldsymbol{x}, r, \cdot)$ and $j_\tau(\boldsymbol{x}, r, \cdot)$ are regular" or "either $-j_\nu(\boldsymbol{x}, r, \cdot)$ and $-j_\tau(\boldsymbol{x}, r, \cdot)$ are regular" for all $r \in \mathbb{R}$ and a.e. $\boldsymbol{x} \in \Gamma_C$. Note that examples of functions which satisfy conditions (34) and (35) can be found in [3, Example 18].

The initial conditions, densities of volume forces and surface tractions satisfy the following regularity hypotheses.

$$\boldsymbol{u}_0 \in V, \quad \beta_0 \in L^2(\Gamma_C), \quad \boldsymbol{f}_0 \in L^\infty(0,T; L^2(\Omega; \mathbb{R}^d)), \quad \boldsymbol{f}_N \in L^\infty(0,T; L^2(\Gamma_N; \mathbb{R}^d)). \tag{36}$$

The adhesive evolution rate function F satisfies the following condition.

$$\begin{cases} F \colon \Gamma_C \times (0,T) \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ \text{(a) } F(\cdot, \cdot, \boldsymbol{\xi}, r) \text{ is measurable on } \Gamma_C \times (0,T) \text{ for all } (\boldsymbol{\xi}, r) \in \mathbb{R}^d \times \mathbb{R}, \\ \text{(b) } |F(\boldsymbol{x}, t, \boldsymbol{\xi}_1, r_1) - F(\boldsymbol{x}, t, \boldsymbol{\xi}_2, r_2)| \leq L_F \left(\|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|_{\mathbb{R}^d} + |r_1 - r_2| \right) \\ \quad \text{for a.e. } (\boldsymbol{x}, t) \in \Gamma_C \times (0,T) \text{ and all } (\boldsymbol{\xi}_i, r_i) \in \mathbb{R}^d \times \mathbb{R}, \ i = 1, 2, \\ \quad \text{with } L_F > 0, \\ \text{(c) } F(\boldsymbol{x}, t, \boldsymbol{\xi}, 0) = 0, \ F(\boldsymbol{x}, t, \boldsymbol{\xi}, r) \geq 0 \text{ for } r \leq 0, \text{ and } F(\boldsymbol{x}, t, \boldsymbol{\xi}, r) \leq 0 \\ \quad \text{for } r \geq 1, \text{ for a.e. } (\boldsymbol{x}, t) \in \Gamma_C \times (0,T), \text{ and for all } \boldsymbol{\xi} \in \mathbb{R}^d. \end{cases} \tag{37}$$

We now focus on the variational formulation of the contact problem (22)–(30). We suppose in what follows that $(\boldsymbol{u}, \boldsymbol{\sigma})$ are smooth functions on $\mathcal{Q}$ which solve (22)–(30). For any $\boldsymbol{v} \in V$ fixed, we multiply equilibrium equation (23) by $\boldsymbol{v}$ and then use the Green formula, cf. [33, Theorem 2.25] to get

$$\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) \rangle_\mathcal{H} = \langle \boldsymbol{f}_0(t), \boldsymbol{v} \rangle_H + \int_\Gamma \boldsymbol{\sigma}(t) \boldsymbol{\nu} \cdot \boldsymbol{v} \, \mathrm{d}\Gamma$$

for a.e. $t \in (0,T)$. Recalling that

$$\int_\Gamma \boldsymbol{\sigma}(t) \boldsymbol{\nu} \cdot \boldsymbol{v} \, \mathrm{d}\Gamma = \int_{\Gamma_D} \boldsymbol{\sigma}(t) \boldsymbol{\nu} \cdot \boldsymbol{v} \, \mathrm{d}\Gamma + \int_{\Gamma_N} \boldsymbol{\sigma}(t) \boldsymbol{\nu} \cdot \boldsymbol{v} \, \mathrm{d}\Gamma + \int_{\Gamma_C} \boldsymbol{\sigma}(t) \boldsymbol{\nu} \cdot \boldsymbol{v} \, d\Gamma,$$

and applying boundary conditions (24) and (25), we have

$$\int_\Gamma \boldsymbol{\sigma}(t) \boldsymbol{\nu} \cdot \boldsymbol{v} \, \mathrm{d}\Gamma = \int_{\Gamma_N} \boldsymbol{f}_N(t) \cdot \boldsymbol{v} \, \mathrm{d}\Gamma + \int_{\Gamma_C} \boldsymbol{\sigma}(t) \boldsymbol{\nu} \cdot \boldsymbol{v} \, \mathrm{d}\Gamma$$

for a.e. $t \in (0,T)$, It follows from the Riesz representation principle that there exists an element $\boldsymbol{f} \in \mathcal{V}^*$ such that

$$\langle \boldsymbol{f}(t), \boldsymbol{v} \rangle = (\boldsymbol{f}_0(t), \boldsymbol{v})_H + (\boldsymbol{f}_N(t), \boldsymbol{v})_{L^2(\Gamma_N; \mathbb{R}^d)}$$

for all $\boldsymbol{v} \in V$, a.e. $t \in (0,T)$. From the decomposition formula, see (6.33) in [33], we obtain

$$\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) \rangle_\mathcal{H} = \langle \boldsymbol{f}(t), \boldsymbol{v} \rangle + \int_{\Gamma_C} \left(\sigma_\nu(t) v_\nu + \boldsymbol{\sigma}_\tau(t) \cdot \boldsymbol{v}_\tau \right) \mathrm{d}\Gamma \tag{38}$$

for a.e. $t \in (0,T)$. On the other hand, by contact conditions (26), (27), and the definition of the subgradient, we obtain

$$-\sigma_\nu(t) v_\nu \leq j_\nu^0(\beta(t), u_\nu(t); v_\nu), \quad -\boldsymbol{\sigma}_\tau(t) \cdot \boldsymbol{v}_\tau \leq j_\tau^0(\beta(t), \boldsymbol{u}_\tau(t); \boldsymbol{v}_\tau) \quad \text{on} \quad \Sigma_C. \tag{39}$$

Putting the fractional Kelvin–Voigt constitutive law (22), and inequalities (39) into (38), we have

$$\left\langle \mathscr{C}(\boldsymbol{\varepsilon}({}_0^C D_t^\alpha \boldsymbol{u}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) \right\rangle_\mathcal{H} + \left\langle \mathscr{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) \right\rangle_\mathcal{H}$$

$$+ \int_{\Gamma_C} j_\nu^0(\beta(t), u_\nu(t); v_\nu) + j_\tau^0(\beta(t), \boldsymbol{u}_\tau(t); \boldsymbol{v}_\tau) \, \mathrm{d}\Gamma \geq \langle \boldsymbol{f}(t), \boldsymbol{v} \rangle$$

for a.e. $t \in (0,T)$. Finally, using conditions (28)–(30) and the last inequality, we obtain the following variational formulation of Problem 17.

Problem 18. *Find $\boldsymbol{u} \in W^{1,2}(0,T;V)$ and $\beta \in W^{1,2}(0,T;L^2(\Gamma_C))$ such that*

$$
\begin{cases}
\left\langle \mathscr{C}(\boldsymbol{\varepsilon}(_0^C D_t^\alpha \boldsymbol{u}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) \right\rangle_{\mathcal{H}} + \left\langle \mathscr{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) \right\rangle_{\mathcal{H}} \\
\qquad + \displaystyle\int_{\Gamma_C} \left(j_\nu^0(u_\nu(t); v_\nu) + j_\tau^0(\boldsymbol{u}_\tau(t); \boldsymbol{v}_\tau) \right) d\Gamma \geq \langle \boldsymbol{f}(t), \boldsymbol{v} \rangle \\
\qquad\qquad \text{for all } \boldsymbol{v} \in V, \quad \text{a.e. } t \in (0,T), \\
\beta'(t) = F(t, \boldsymbol{u}(t), \beta(t)) \quad \text{on } \Sigma_C, \\
\beta(0) = \beta_0 \quad \text{in } \Gamma_C, \\
\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega.
\end{cases}
\tag{40}
$$

Note that Problem 18 represents a differential hemivariational inequality involving the Caputo time fractional derivative operator. Since $W^{1,2}(0,T;V) \subset C(0,T;V)$, it is clear that the initial condition has a meaning in the space V. We have the following existence result.

Theorem 19. *Assume hypotheses* (32)–(37). *Then Problem 18 has at least one solution* $(\boldsymbol{u}, \beta) \in W^{1,2}(0,T; V) \times W^{1,2}(0,T;L^2(\Gamma_C))$.

Proof. The proof is based on Theorem 16. We consider the spaces $X = L^2(\Gamma_C; \mathbb{R}^d)$, $Y = L^2(\Gamma_C)$, and define operators $A, B \colon V \to V^*$ by

$$
\langle A\boldsymbol{u}, \boldsymbol{v} \rangle_{V^* \times V} = \left\langle \mathscr{C}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}) \right\rangle_{\mathcal{H}} \quad \text{for} \quad \boldsymbol{u}, \boldsymbol{v} \in V,
\tag{41}
$$

$$
\langle B\boldsymbol{u}, \boldsymbol{v} \rangle_{V^* \times V} = \left\langle \mathscr{E}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}) \right\rangle_{\mathcal{H}} \quad \text{for} \quad \boldsymbol{u}, \boldsymbol{v} \in V,
\tag{42}
$$

respectively. For $\boldsymbol{u} \in X$ and $\beta \in Y$, denote the function $J \colon Y \times X \to \mathbb{R}$ by

$$
J(\beta, \boldsymbol{u}) = \int_{\Gamma_C} \left(j_\nu(\boldsymbol{x}, \beta, u_\nu) + j_\tau(\boldsymbol{x}, \beta, \boldsymbol{u}_\tau) \right) d\Gamma.
\tag{43}
$$

Combining hypotheses (34)(e), (35)(e) and [33, Corollary 4.15(vii)], we deduce that $J(\beta, \cdot)$ or $-J(\beta, \cdot)$ is regular on X for all $\beta \in Y$. On the other hand, [33, Lemma 3.39(3) and Corollary 4.15(vi)] imply

$$
J^0(\beta, \boldsymbol{u}) = \int_{\Gamma_C} \left(j_\nu^0(\boldsymbol{x}, \beta, u_\nu) + j_\tau^0(\boldsymbol{x}, \beta, \boldsymbol{u}_\tau) \right) d\Gamma,
\tag{44}
$$

$$
\partial J(\beta, \boldsymbol{u}) = \int_{\Gamma_C} \left(\partial j_\nu(\boldsymbol{x}, \beta, u_\nu) + \partial j_\tau(\boldsymbol{x}, \beta, \boldsymbol{u}_\tau) \right) d\Gamma
\tag{45}
$$

for all $\beta \in Y$ and $\boldsymbol{u} \in X$. Moreover, let $M = \gamma$ and $g \colon (0,T) \times X \times Y \to Y$ be defined by

$$
g(t, \boldsymbol{u}, \beta)(\boldsymbol{x}) = F(\boldsymbol{x}, t, \beta(\boldsymbol{x}), \boldsymbol{u}(\boldsymbol{x})) \quad \text{for all } \beta \in Y, \ \boldsymbol{u} \in X \ \text{ a.e. } \boldsymbol{x} \in \Gamma_C,
\tag{46}
$$

where $\gamma \colon V \to X$ is the trace operator.

Using these notation, Problem 18 can be reformulated as the following abstract fractional differential hemivariational inequality: find $\boldsymbol{u} \in W^{1,2}(0,T;V)$ and $\beta \in W^{1,2}(0,T;Y)$ such that

$$
\begin{cases}
\langle A(_0^C D_t^\alpha \boldsymbol{u}(t)), \boldsymbol{v} \rangle + \langle B(\boldsymbol{u}(t)), \boldsymbol{v} \rangle + J^0(\beta(t), M\boldsymbol{u}(t); M\boldsymbol{v}) \geq \langle \boldsymbol{f}(t), \boldsymbol{v} \rangle \\
\qquad\qquad\qquad\qquad \text{for all } \boldsymbol{v} \in V, \quad \text{a.e. } t \in (0,T), \\
\beta'(t) = g(t, \boldsymbol{u}(t), \beta(t)) \quad \text{for a.e. } t \in (0,T), \\
\beta(0) = \beta_0, \\
\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega.
\end{cases}
\tag{47}
$$

We will prove the existence of solution to problem (47) by using Theorem 16. To this end, we denote $\boldsymbol{w}(t) = {}_0^C D_t^\alpha \boldsymbol{u}(t)$ for a.e. $t \in (0,T)$. Applying Proposition 3(b), we obtain

$$
\boldsymbol{u}(t) = {}_0 I_t^\alpha \boldsymbol{w}(t) + \boldsymbol{u}_0 \quad \text{for a.e. } \quad t \in (0,T).
\tag{48}
$$

Thus, problem (47) can be reformulated as follows: find $\boldsymbol{w} \in L^1(0,T;V)$ and $\beta \in W^{1,2}(0,T;Y)$ such that

$$
\begin{cases}
\langle A\boldsymbol{w}(t) + B(\boldsymbol{u}_0 + {}_0I_t^\alpha\boldsymbol{w}(t)), \boldsymbol{v}\rangle + J^0(M(\boldsymbol{u}_0 + {}_0I_t^\alpha\boldsymbol{w}(t)); M\boldsymbol{v}) \geq \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle \\
\qquad\qquad\qquad\qquad \text{for all } \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T) \\
\beta'(t) = g(t, M(\boldsymbol{u}_0 + {}_0I_t^\alpha\boldsymbol{w}(t)), \beta(t)) \quad \text{for a.e. } t \in (0,T) \\
\beta(0) = \beta_0.
\end{cases} \tag{49}
$$

This means that $(\boldsymbol{u}, \beta) \in W^{1,2}(0,T;V) \times W^{1,2}(0,T;Y)$ is a solution to problem (47) if and only if $(\boldsymbol{w}, \beta) \in L^1(0,T;V) \times W^{1,2}(0,T;Y)$ solves problem (49).

To this end, we will verify that hypotheses $H(A)$, $H(B)$, $H(f)$, $H(J)$, $H(g)$ and $H(M)$ of Theorem 16 are satisfied. Obviously, from hypothesis (32), we can see that the operator A, see (41), is coercive with constant m_a and $A \in \mathcal{L}(V, V^*)$, i.e., $H(A)$ holds. Note that, since the elastic operator $\mathscr{E}$ satisfies properties (33), this yields that $B \in \mathcal{L}(V, V^*)$, i.e., $H(B)$ is verified. Moreover, hypotheses (34) and (35) combined with [33, Corollary 4.15(v)] imply that the conditions $H(J)$(i) and (ii) are satisfied with $c_J = \max\{\sqrt{3\,\mathrm{meas}(\Gamma_C)}, 1\}(c_\nu + c_\tau)$. The upper semicontinuity of j_ν and j_τ, and Fatou's lemma, see, e.g., [33, Theorem 1.64], guarantee that the function $(\beta, \boldsymbol{u}) \mapsto J^0(\beta, \boldsymbol{u}; \boldsymbol{v})$ is also upper semicontinuous from $Y \times X$ to $\mathbb{R}$, for all $\boldsymbol{v} \in X$. So, J has the property $H(J)$(iii). In addition, it follows from the regularity hypothesis (36) that $\boldsymbol{f}$ satisfies $H(f)$. From [12, Theorem 3.9.34], we infer that the trace operator γ satisfies $H(M)$. Finally, it is easy to verify that under hypothesis (37), operator g defined by (46) satisfies all conditions in $H(g)$.

Summing up, we have verified all hypotheses of Theorem 16. Therefore, applying this theorem, problem (49) has a solution $(\boldsymbol{w}, \beta) \in L^1(0,T;V) \times W^{1,2}(0,T;Y)$. Hence, we deduce that $(\boldsymbol{u}, \beta) \in W^{1,2}(0,T;V) \times W^{1,2}(0,T;Y)$ is a solution to problem (47), where $\boldsymbol{u}$ is defined by equality (48). Finally, we conclude that $(\boldsymbol{u}, \beta) \in W^{1,2}(0,T;V) \times W^{1,2}(0,T;Y)$ solves Problem 18. This completes the proof of the theorem. $\quad\square$

We say that a triple of functions $(\boldsymbol{u}, \boldsymbol{\sigma}, \beta)$ which satisfies (22) and (40) is called a weak solution to Problem 17. We conclude that, under assumptions of Theorem 19, Problem 17 has at least one weak solution. Moreover, the weak solution has the following regularity

$$
\boldsymbol{u} \in W^{1,2}(0,T;V), \ \boldsymbol{\sigma} \in L^2(0,T;L^2(\Omega, \mathbb{S}^d)), \quad \beta \in W^{1,2}(0,T;Y), \quad \text{and } \mathrm{Div}\,\boldsymbol{\sigma} \in \mathcal{V}^*,
$$

because $\boldsymbol{w} \in \mathcal{V}$ and $\boldsymbol{u}(t) = {}_0I_t^\alpha\boldsymbol{w}(t) + u_0$ for a.e. $t \in (0,T)$, which is such that (49) holds.

Note that if the nonconvex potential j_ν and tangential function j_τ are independent of the adhesion field β, then Problem 17 reduces to the fractional viscoelastic contact problem, which was studied by Zeng and Migórski [54]. On the other hand, if $\alpha = 1$ in Problem 17, then it reduces to the following viscoelastic contact problem with classical Kelvin–Voigt constitutive law and adhesion.

Problem 20. *Find a displacement field $\boldsymbol{u}\colon \mathcal{Q} \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}\colon \mathcal{Q} \to \mathbb{S}^d$ and a bonding field $\beta\colon \Sigma_C \to [0,1]$ such that*

$$
\begin{array}{lll}
\boldsymbol{\sigma}(t) = \mathscr{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}'(t))) + \mathscr{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) & \text{in} & \mathcal{Q}, \\
\mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = 0 & \text{in} & \mathcal{Q}, \\
\boldsymbol{u}(t) = 0 & \text{on} & \Sigma_D, \\
\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_N(t) & \text{on} & \Sigma_N, \\
-\sigma_\nu(t) \in \partial j_\nu(\beta(t), u_\nu(t)) & \text{on} & \Sigma_C, \\
-\boldsymbol{\sigma}_\tau(t) \in \partial j_\tau(\beta(t), \boldsymbol{u}_\tau(t)) & \text{on} & \Sigma_C, \\
\beta'(t) = F(t, \boldsymbol{u}(t), \beta(t)) & \text{on} & \Sigma_C, \\
\beta(0) = \beta_0 & \text{on} & \Gamma_C, \\
\boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} & \Omega.
\end{array}
$$

Clearly, Problem 20 has the following variational formulation, which is a classical differential hemivariational inequality.

Problem 21. *Find* $\boldsymbol{u} \in W^{1,2}(0,T;V)$ *and* $\beta \in W^{1,2}(0,T;L^2(\Gamma_C))$ *such that*

$$\left\langle \mathscr{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}'(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) \right\rangle_{\mathcal{H}} + \left\langle \mathscr{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) \right\rangle_{\mathcal{H}}$$

$$+ \int_{\Gamma_C} \left(j_\nu^0(u_\nu(t); v_\nu) + j_\tau^0(\boldsymbol{u}_\tau(t); \boldsymbol{v}_\tau) \right) \, \mathrm{d}\Gamma \geq \langle \boldsymbol{f}(t), \boldsymbol{v} \rangle$$

$$\text{for all} \quad \boldsymbol{v} \in V, \quad \text{a.e. } t \in (0,T),$$

$$\beta'(t) = F(t, \boldsymbol{u}(t), \beta(t)) \quad \text{on} \quad \Sigma_C.$$

$$\beta(0) = \beta_0 \quad \text{in} \quad \Gamma_C.$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega.$$

As a consequence of Theorem 19, we conclude the following result.

Corollary 22. *Assume that hypotheses* (32)–(37) *hold. Then Problem* 21 *has a solution* $(\boldsymbol{u}, \beta) \in W^{1,2}(0,T; V) \times W^{1,2}(0,T;L^2(\Gamma_C))$.

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