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CORRECTION

Correction to: On the dissipative effect of a magnetic field in a Mindlin-Timoshenko plate model

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In obtaining in Theorem 4.1 of the original paper, the strong asymptotic decay of the semigroup $\exp(-t\mathcal{C}^{-1}\mathcal{A})$ associated with $Pr\ (P)$, the argumentation following (4.5) on p. 1060 is incorrect. Also, with regard to the polynomial decay of the semigroup associated with the model, the conclusion (4.15) on p. 1061 from (4.14) requires substantiation. Whereas the strong asymptotic stability of the original model $Pr\ (P)$ can be achieved by rectification of the argumentation, it turns out, with regard to the polynomial stability, that the conclusion (4.15) can only be accomplished when $[\psi, \phi]$, the vector of shear angles, is divergence-free. This requires the analysis of a modification of $Pr\ (P)$ in which the constitutive equations include the equation $\nabla \cdot [\psi, \phi] = 0$ in the domain Ω. For the semigroup associated with the modified model, $Pr\ (P')$, we are able to establish a decay rate of $(\frac{1}{t})^{\frac{1}{2}}$, which is a faster decay rate than the rate $(\frac{1}{t})^{\frac{1}{4}}$ achieved for $Pr\ (P)$ in the original paper.[1]

In this situation, we consider the initial-boundary-value problem $Pr\ (P')$

$$
\begin{aligned}
\frac{\rho b^3}{12}[\psi,\phi]_{tt} - (1-\nu)D\nabla\cdot &\begin{bmatrix} \psi_x & \frac{1}{2}(\psi_y+\phi_x) \\ \frac{1}{2}(\psi_y+\phi_x) & \phi_y \end{bmatrix} \\
&- \nu D\nabla\cdot \begin{bmatrix} \psi_x+\phi_y & 0 \\ 0 & \psi_x+\phi_y \end{bmatrix} \\
+ K\left(\nabla w + [\psi,\phi]\right) - \frac{b^3}{12}&\left((\nabla\times[h_1^1,h_1^2])\times[H_0^1,H_0^2]\right) + \frac{b^3}{12}\nabla p = 0 \\
\rho b w_{tt} - K\nabla\cdot(\nabla w + [\psi,\phi]) &- b\nabla h_1^3\cdot[H_0^1,H_0^2] = 0 \\
\nabla\cdot[\psi,\phi] &= 0 \qquad \text{in } \Omega \\
\frac{b^3}{12}\left([h_{1,t}^1,h_{1,t}^2] + \nabla\times(\nabla\times[h_1^1,h_1^2]) \right.&\left. - \nabla\times([\psi_t,\phi_t]\times[H_0^1,H_0^2])\right) = 0 \\
b(h_{1,t}^3 - \Delta h_1^3 - \nabla w_t\cdot[H_0^1,H_0^2]) &= 0 \\
\nabla\cdot[h_1^1,h_1^2] &= 0 \\
w = \psi = \phi &= 0
\end{aligned}
$$

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[1] The author gratefully acknowledges an enquiry by Marcio V. Ferreira from the University of Santa Maria, Brazil, which inspired the revision of Theorem 4.1 in her original paper.

$$[h_1^1, h_1^2] \cdot \mathbf{n} = 0, \mathbf{n} \times (\nabla \times [h_1^1, h_1^2]) = 0, \frac{\partial h_1^3}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega$$
$$\psi|_{t=0} = \psi^0, \psi_t|_{t=0} = \psi^1, \phi|_{t=0} = \phi^0, \phi_t|_{t=0} = \phi^1,$$
$$w|_{t=0} = w^0, w_t|_{t=0} = w^1, \left[h_1^1, h_1^2, h_1^3\right]|_{t=0} = \left[h_1^{1,0}, h_1^{2,0}, h_1^{3,0}\right]$$

in which $(\nabla\times[h_1^1, h_1^2])\times[H_0^1, H_0^2] = [H_0^2(\nabla\times[h_1^1, h_1^2]), -H_0^1(\nabla\times[h_1^1, h_1^2])]$. As in [7] Ω is a simply connected domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, say of class $\mathcal{C}^2$, and $\mathbf{n}$ is the unit outward normal vector. For the meaning of the physical parameters in Pr (P'), the reader is referred to the original paper [7].

A moment of reflection shows that the system of coupled partial differential (PDEs) in $[\psi, \phi], w$ and $[h_1^1, h_1^2], h_1^3$, is, as a consequence of the condition $\nabla \cdot [\psi, \phi] = 0$ in Ω, reduced to the system

$$\frac{\rho b^3}{12}[\psi_{tt}, \phi_{tt}] - D\left(\frac{1-\nu}{2}\right)\Delta[\psi, \phi] + K(\nabla w + [\psi, \phi])$$
$$-\frac{b^3}{12}\left((\nabla \times [h_1^1, h_1^2]) \times [H_0^1, H_0^2]\right) + \frac{b^3}{12}\nabla p = 0$$
$$\rho b w_{tt} - K\Delta w - b\nabla h_1^3 \cdot [H_0^1, H_0^2] = 0 \quad \text{in } \Omega$$
$$\nabla \cdot [\psi, \phi] = 0$$
$$\frac{b^3}{12}\left([h_{1,t}^1, h_{1,t}^2] + \nabla \times \left(\nabla \times [h_1^1, h_1^2]\right) - [H_0^1, H_0^2] \cdot \nabla[\psi_t, \phi_t]\right) = 0$$
$$b\left(h_{1,t}^3 - \Delta h_1^3 - \nabla w_t \cdot [H_0^1, H_0^2]\right) = 0$$
$$\nabla \cdot [h_1^1, h_1^2] = 0$$

with $[H_0^1, H_0^2] \cdot \nabla[\psi_t, \phi_t] = ([H_0^1, H_0^2] \cdot \nabla)[\psi_t, \phi_t]$—the replacement of the dissipative term $\nabla \times ([\psi_t, \phi_t] \times [H_0^1, H_0^2])$ in Pr (P') by $[H_0^1, H_0^2] \cdot \nabla[\psi_t, \phi_t]$ will be instrumental in establishing the polynomial stability of the model.

The model Pr (P'), with observance of the above comment, is, in the presence of the term $\nabla p, p$ the unknown pressure, together with the equation $\nabla \cdot [\psi, \phi] = 0$, more intricate than Pr (P) and serves as a model for the magnetoelastic interaction of an elastically incompressible, electrically conducting Mindlin–Timoshenko plate and an instationary magnetic field. To see this, we note that by the Hecky–Mindlin hypothesis (see [7, p.1051]), i.e.

$$[[u^1, u^2](x, y, z, t), u^3(x, y, t)] = [z[\psi, \phi](x, y, t), w(x, y, t)]$$

$u^i, i = 1, 2, 3$, the components of the displacement $\mathbf{u}$ in the Lamé system, incompressibility of the plate would require

$$0 = \nabla \cdot [u^1, u^2, u^3] = \nabla_{x,y,z} \cdot [z\psi, z\phi, w] = z\nabla_{x,y} \cdot [\psi, \phi], \ |z| \leq \frac{b}{2}$$

Hence, $\nabla \cdot [\psi, \phi] = 0$ expresses the incompressibility of the plate material.[2]

It is clear that, since Pr (P') is different from Pr (P), the question of unique solvability has to be investigated afresh. To consider the problem in the framework of semigroup theory, we will need abstract spaces and operators. For the sake of clarity, we provide all the spaces and operators even where they coincide with spaces and operators used in the treatment of the original Pr (P).

We shall use the following spaces:

$H^m(\Omega), m \geq 1$, denotes the usual Sobolev spaces of order m endowed with norms $\|\cdot\|_{m,\Omega}$ or equivalent norms as will be indicated. $H^0(\Omega)$ denotes the Hilbert space $L^2(\Omega)$ with norm $\|\cdot\|_{0,\Omega}$. Since no confusion can arise, $\|\cdot\|_{m,\Omega}$ will also denote the norm in $(H^m(\Omega))^n, n = 2, 3$.

$H_\sigma := \{\mathbf{f}: \mathbf{f} \in (L^2(\Omega))^2, \nabla \cdot \mathbf{f} = 0 \text{ in } \Omega, \mathbf{f} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$. $H_\sigma$is equipped with the usual $\|\cdot\|_{0,\Omega}$ norm.

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[2] The author is indebted to Michael Pokojovy from the University of Texas at El Paso, Texas, USA, for an informative discussion on this and other aspects of the work.

$V_\sigma^1 := \{\mathbf{u}: \mathbf{u} = [u,v,w] \in (H_0^1(\Omega))^3, \nabla\cdot[u,v] = 0 \text{ in } \Omega\}$. The space V_σ^1 with elements $[\psi,\phi,w]$ is endowed with the scalar product

$$\begin{aligned}(1-\nu)D&\left((\psi_x,\hat{\psi}_x)_{0,\Omega} + (\phi_y,\hat{\phi}_y)_{0,\Omega}\right)\\ &+ \left(\frac{1-\nu}{2}\right)D\left((\psi_y+\phi_x,\hat{\psi}_y+\hat{\phi}_x)_{0,\Omega}\right)\\ &+ K\left((\psi,\hat{\psi})_{0,\Omega} + (\phi,\hat{\phi})_{0,\Omega} + (w_x,\hat{w}_x)_{0,\Omega} + (w_y,\hat{w}_y)_{0,\Omega}\right)\\ &= a'\left([\psi,\phi,w];[\hat{\psi},\hat{\phi},\hat{w}]\right)\end{aligned}$$

and the norm $\|[\psi,\phi,w]\|_{V_\sigma^1} = (a'([\psi,\phi,w]))^{\frac{1}{2}}\ \forall\ [\psi,\phi,w],[\hat{\psi},\hat{\phi},\hat{w}] \in V_\sigma^1$, which is equivalent to the usual $\|\cdot\|_{1,\Omega}$ norm.

$V_\sigma^2 := \{[h_1^1,h_1^2,h_1^3]: [h_1^1,h_1^2,h_1^3] \in ((H^1(\Omega))^2 \cap H_\sigma)\times H^1(\Omega)\}. V_\sigma^2$ is endowed with the $\|\cdot\|_{1,\Omega}$ norm.

$\mathcal{H}_\sigma := V_\sigma^1 \times (L^2(\Omega))^3 \times (H_\sigma \times L^2(\Omega))$. The energy space $\mathcal{H}_\sigma$ will be equipped with the usual product norm.

The following operators will be needed. We agree that when the action, in a possibly different space, of an operator A_i is different from the action of the corresponding operator in [7], we will use the notation A_i':

$A_1': D(A_1') \to (L^2(\Omega))^3, (A_1'[\psi,\phi,w],[\hat{\psi},\hat{\phi},\hat{w}])_{0,\Omega} := a'([\psi,\phi,w];[\hat{\psi},\hat{\phi},\hat{w}])$
$D(A_1') = \{[\psi,\phi,w] \in V_\sigma^1; A_1'[\psi,\phi,w] \in (L^2(\Omega))^3\}$
$A_2: D(A_2) \to (L^2(\Omega))^2,\ A_2[h_1^1,h_1^2] := -\frac{b^3}{12}((\nabla\times[h_1^1,h_1^2])\times[H_0^1,H_0^2])$
$D(A_2) = \{\mathbf{f}: \mathbf{f} \in (H_1(\Omega))^2 \cap H_\sigma, (\nabla\times\mathbf{f})\times[H_0^1,H_0^2] \in (L^2(\Omega))^2\}$
$A_3: D(A_3) \to L^2(\Omega),\ A_3h_1^3 := -b([H_0^1,H_0^2]\cdot\nabla h_1^3)$
$D(A_3) = \{g: g \in H^1(\Omega), \nabla g\cdot[H_0^1,H_0^2] \in L^2(\Omega)\}$
$A_4': D(A_4') \to H_\sigma,\ A_4'[\psi_t,\phi_t] := -\frac{b^3}{12}([H_0^1,H_0^2]\cdot\nabla[\psi_t,\phi_t])$
$D(A_4') = \{\mathbf{j} \in (L^2(\Omega))^2; [H_0^1,H_0^2]\cdot\nabla\mathbf{j} \in H_\sigma\}$
$A_5: D(A_5) \to H_\sigma,\ A_5[h_1^1,h_1^2] := \frac{b^3}{12}(\nabla\times(\nabla\times[h_1^1,h_1^2]))$
$D(A_5) = \{\mathbf{m} \in (H^1(\Omega))^2 \cap H_\sigma, \nabla\times(\nabla\times\mathbf{m}) \in H_\sigma, \mathbf{n}\times(\nabla\times\mathbf{m}) = 0 \text{ on } \partial\Omega\}$[3]
$A_6: D(A_6) \to L^2(\Omega),\ A_6w_t := -b[H_0^1,H_0^2]\cdot\nabla w_t$
$D(A_6) = \{o \in L^2(\Omega): \nabla o\cdot[H_0^1,H_0^2] \in L^2(\Omega)\}$
$A_7: D(A_7) \to L^2(\Omega),\ A_7h_1^3 := -b\Delta h_1^3$
$D(A_7) = \{q: q \in H^1(\Omega), \Delta q \in L^2(\Omega), \frac{\partial q}{\partial\mathbf{n}} = 0 \text{ on } \partial\Omega\}$

We now define the operator $\mathcal{A}': \mathcal{D}(\mathcal{A}') \to \mathcal{H}_\sigma$ by

$$\mathcal{A}'\mathbf{U} = \begin{bmatrix} 0 & -1 & 0\\ A_1' & 0 & [A_2,A_3]\\ 0 & [A_4',A_6] & [A_5,A_7]\end{bmatrix}\mathbf{U}$$

$\mathbf{U} = [[\psi,\phi,w],[\psi_t,\phi_t,w_t],[h_1^1,h_1^2,h_1^3]] \in \mathcal{D}(\mathcal{A}')$ where $\mathcal{D}(\mathcal{A}') \subset \mathcal{H}_\sigma$ is defined by

$$\mathcal{D}(\mathcal{A}') = \left\{\mathbf{U}: \mathbf{U} \in \mathcal{H}_\sigma, [h_1^1,h_1^2,h_1^3] \in V_\sigma^2, \mathcal{A}'\mathbf{U} \in \mathcal{H}_\sigma\right\} \tag{3.1}$$

Note that the description of $\mathcal{D}(\mathcal{A}')$ is made precise by the definitions of the operators $A_i', i = 1,4$ and $A_j, j = 2,3,5,6,7$, while in addition we have $[\psi_t,\phi_t,w_t] \in V_\sigma^1$ which furnishes $\nabla\cdot[\psi_t,\phi_t] = 0$.

[3] For the "generalized"s sense in which the boundary conditions are satisfied in the definition of A_5 and A_7 below, see e.g. [4, p. 381].

Finally, we define the canonical operator $\mathcal{C}$ in $\mathcal{H}_\sigma$ by

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho b[\frac{b^2}{12}, \frac{b^2}{12}, 1] & 0 \\ 0 & 0 & b\left[\frac{b^2}{12}, \frac{b^2}{12}, 1\right] \end{bmatrix}$$

From the point of view of weak formulation, the interactive system of PDEs for the incompressible magnetoelastic Mindlin–Timoshenko plate can now be considered in the form of the abstract evolution problem $Pr\ (AEP)'$

$$\mathcal{C}\mathbf{U}' + \mathcal{A}'\mathbf{U} = 0$$
$$\mathbf{U}(0) = \mathbf{U}^0 \in \mathcal{H}_\sigma \qquad Pr\ (AEP)'$$

With a view to asserting the existence of a unique C_0 semigroup of contractions with generator $-\mathcal{C}^{-1}\mathcal{A}'$ for $Pr\ (AEP)'$, the only novelty in showing the dissipativity of $-\mathcal{C}^{-1}\mathcal{A}'$ by considering the inner product $(\mathcal{A}'\mathbf{U}, \mathcal{C}\mathbf{U})_{\mathcal{H}_\sigma}$ is the term

$$\begin{aligned} &\frac{12}{b^3}\left(A_2[h_1^1, h_1^2], [\psi_t, \phi_t]\right)_{0,\Omega} \\ &= -\left(\left(\nabla \times [h_1^1, h_1^2]\right) \times [H_0^1, H_0^2], [\psi_t, \phi_t]\right)_{0,\Omega} \\ &= \left([h_1^1, h_1^2], \nabla \times ([\psi_t, \phi_t] \times [H_0, H_0^2])\right)_{0,\Omega} \\ &= \left([h_1^1, h_1^2], [H_0^1, H_0^2] \cdot \nabla[\psi_t, \phi_t]\right)_{0,\Omega} \\ &= -\frac{12}{b^3}(A_4'[\psi_t, \phi_t], [h_1^1, h_1^2])_{0,\Omega} \end{aligned}$$

by implementing $\nabla \cdot [\psi_t, \phi_t] = 0$ in the identity $\nabla \times ([\psi_t, \phi_t] \times [H_0, H_0^2]) = [H_0^1, H_0^2] \cdot \nabla[\psi_t, \phi_t] - [H_0^1, H_0^2]\nabla \cdot [\psi_t, \phi_t]$.

It follows that $(\mathcal{A}'\mathbf{U}, \mathcal{C}\mathbf{U})_{\mathcal{H}_\sigma} = \frac{b^3}{12}\|\nabla \times [h_1^1, h_1^2]\|_{0,\Omega}^2 + b\|h_1^3\|_{0,\Omega}^2$. The surjectivity of $\lambda\mathbf{I} + \mathcal{C}^{-1}\mathcal{A}'$ for $\lambda > 0$ may be achieved by proceeding analogously as in the analysis of the classical system of magnetoelasticity [9]. We can now invoke the Lumer–Phillips generation theorem to conclude that $-\mathcal{C}^{-1}\mathcal{A}'$ is the infinitesimal generator of a C_0 semigroup of contractions in $\mathcal{H}_\sigma$. Theorem 3.1 in [7] is now replaced by

Theorem 3.1. *Given* $\mathbf{U}^0 \in \mathcal{H}_\sigma$ *there exists a unique weak solution of* $Pr\ (AEP)'$ *in* $\mathcal{H}_\sigma$*, given by* $\mathbf{U}(t) = \exp(-t\mathcal{C}^{-1}\mathcal{A}')\mathbf{U}^0$*, such that*

$$\mathbf{U} \in C([0, \infty); \mathcal{H}_\sigma)$$

Moreover, $\mathbf{U}^0 \in \mathcal{D}(\mathcal{A}')$ *furnishes a unique strong solution such that*

$$\mathbf{U} \in C([0, \infty); \mathcal{D}(\mathcal{A}')) \cap C^1([0, \infty); \mathcal{H}_\sigma)$$

Finally, if $\mathbf{U}^0 \in \mathcal{D}((\mathcal{A}')^k)$*, then*

$$\mathbf{U} \in \bigcap_{j=0}^{k} C^{k-j}([0, \infty); \mathcal{D}((\mathcal{A}')^j))$$

Redirecting our attention to $Pr\ (P')$, we need to comment on the existence of a function p such that $[\mathbf{U}, p], \mathbf{U}$ the solution constructed in Theorem 3.1, solves $Pr\ (P')$. First, we can invoke the work of Leray (see e.g. [10, p. 23] and the references therein) to see that, to establish the existence of solutions $[\mathbf{U}, p]$ of the variational form of $Pr\ (P')$, it is sufficient to find only $\mathbf{U}$—the existence of p is then achieved by making crucial use of $(\nabla p, [\psi_t, \phi_t])_{0,\Omega} = 0$ [10, p. 14]. Since the existence of a weak solution of $Pr\ (P')$ is in turn equivalent to the existence of a C_0 semigroup solution $\exp(-t\mathcal{C}^{-1}\mathcal{A}')\mathbf{U}^0$ in $\mathcal{H}_\sigma$ of $Pr\ (AEP)'$ [1], we have, by constructing the semigroup $\exp(-t\mathcal{C}^{-1}\mathcal{A}')$ in $\mathcal{H}_\sigma$, accomplished the existence of a weak solution $[\mathbf{U}, p] = [[\psi, \phi, w], [\psi_t, \phi_t, w_t], [h_1^1, h_1^2, h_1^3], p]$ of $Pr\ (P')$, with $\mathbf{U}$ unique and p unique up to the addition of a constant. Regularity of $[\mathbf{U}, p]$ can be achieved by assuming more regularity of the data and the domain (cf. [10, pp. 267–269]). For example, if $\partial\Omega$ is of class $C^{1,1}$, then the domain $\mathcal{D}(\mathcal{A}')$, as defined in (3.1), can be replaced by the equivalent domain

$$\mathfrak{D}(\mathcal{A}') = \begin{cases} [\psi, \phi, w] \in (H^2(\Omega))^3 \cap V_\sigma^1, [\psi_t, \phi_t, w_t] \in V_\sigma^1 \\ [h_1^1, h_1^2] \in (H^2(\Omega))^2 \cap H_\sigma, \mathbf{n} \times (\nabla \times [h_1^1, h_1^2]) = 0 \text{ on } \partial\Omega \\ h_1^3 \in H^2(\Omega), \frac{\partial h_1^3}{\partial \mathbf{n}} \text{ on } \partial\Omega \end{cases}$$

In this case, we have for $\mathbf{U}^0 \in \mathfrak{D}(\mathcal{A}')$ a strong solution $[\mathbf{U}, p]$ of Pr (P') such that

$$\mathbf{U} \in C([0, \infty); \mathfrak{D}(\mathcal{A}')) \cap C^1([0, \infty); \mathcal{H}_\sigma)$$
$$p \in C([0, \infty); H^1(\Omega))$$

We now present the corrected version of Theorem 4.1 in [7].

Theorem 4.1. *Assume that Ω is a simply connected domain of $\mathbb{R}^2$ with sufficiently smooth boundary $\partial\Omega$. Then we have:*
(i) *The semigroup* $\exp(-t\mathcal{C}^{-1}\mathcal{A})$ *associated with Pr (P) is strongly asymptotically stable;*
(ii) *The semigroup* $\exp(-t\mathcal{C}^{-1}\mathcal{A}')$ *associated with Pr (P') satisfies:*
For any positive integer k, there exists a constant $C_k > 0$ such that

$$\| \exp(-t\mathcal{C}^{-1}\mathcal{A}')\mathbf{U}^0\|_{\mathcal{H}} \leq C_k \left(\frac{1}{t}\right)^{\frac{k}{2}} \|\mathbf{U}^0\|_{\mathcal{D}((\mathcal{A}')^k)} \forall \mathbf{U}^0 \in \mathcal{D}((\mathcal{A}')^k)$$

Proof. We present the proof only for $k = 1$, i.e. for $\mathbf{U}^0 \in \mathcal{D}(\mathcal{A}')$ we establish a polynomial decay rate of order $(\frac{1}{t})^{\frac{1}{2}}$ of the semigroup solution $\exp(-t\mathcal{C}^{-1}\mathcal{A}')\mathbf{U}^0$ of Pr (P'). The decay rate can then be improved with higher regularity of the data, i.e. $\mathbf{U}^0 \in \mathcal{D}((\mathcal{A}')^k)$ will yield a decay rate of order $(\frac{1}{t})^{\frac{k}{2}}$.

To validate (ii) for the case $k = 1$, we use as our main tool the resolvent criterion for polynomial stability of semigroups established by Borichev and Tomilov [3]. Thus, to achieve the polynomial decay at a rate of $(\frac{1}{t})^{\frac{1}{2}}$ of the semigroup $\exp(-t\mathcal{C}^{-1}\mathcal{A}')$ associated with Pr (P'), we need to show that the following conditions are satisfied:

$$(H_1)\colon i\mathbb{R} \cap \sigma(-\mathcal{C}^{-1}\mathcal{A}') = \emptyset$$
$$(H_2)\colon \sup_{|\beta|\geq 1} \frac{1}{\beta^\ell} \|(i\beta + \mathcal{C}^{-1}\mathcal{A}')^{-1}\|_{\mathcal{L}(\mathcal{H}_\sigma)} \leq M \text{ for some } \ell > 0 \tag{4.1}$$

where, in our case, $\ell = 2$ will emerge as the "optimal" (smallest possible) choice.

With regard to (H_1), we note that, with our definition of the domain of $\mathcal{A}'$, it is evident that $\mathcal{R}(\lambda; -\mathcal{C}^{-1}\mathcal{A}')$ is compact. Thus, the validity of (H_1) in which we may take $\sigma = \sigma_p$, the point spectrum, will furnish the strong stability of $\exp(-t\mathcal{C}^{-1}\mathcal{A}')$ by invoking Benchimol's spectral criterion [2]. It will turn out that the restriction $\nabla \cdot [\psi, \phi] = 0$ is redundant, as the solenoidality property of the shear angles vector will emerge as a "side" condition (see ff.). Thus, in the validation of (H_1) the operator $\mathcal{A}'$ may be replaced by $\mathcal{A}$.

We first validate (H_2): Proceeding by a contradiction argument, we assume that (H_2) does not hold. Thus, we assume that $\sup_{|\beta|\geq 1} \frac{1}{\beta^\ell}\|(i\beta + \mathcal{C}^{-1}\mathcal{A}')^{-1}\|_{\mathcal{L}(\mathcal{H}_\sigma)} \leq M$ does not hold. Then there exists a sequence $\{\beta_n\}, \beta_n \in \mathbb{R}^+, \beta_n \to \infty$ and $\{\mathbf{U}_n\} \subset \mathcal{D}(\mathcal{A}'), n = 1, 2, ..,$ such that

$$\|\mathbf{U}_n\|_{\mathcal{H}_\sigma} := \|[\psi_n, \phi_n, w_n], [\psi_{1n}, \phi_{1n}, w_{1n}], [h_{1n}^1, h_{1n}^2, h_{1n}^3]\|_{\mathcal{H}_\sigma} = 1 \ \forall n \tag{4.2}$$

and, by taking account of the canonical nature of the operator $\mathcal{C}$,

$$\beta_n^\ell (i\beta_n\mathcal{C} + \mathcal{A}')\mathbf{U}_n \to 0 \text{ in } \mathcal{H}_\sigma \text{ as } n \to \infty. \tag{4.3}$$

This yields

$$
\begin{aligned}
&\beta_n^\ell (i\beta_n[\psi_n,\phi_n,w_n]-[\psi_{1n},\phi_{1n},w_{1n}])\to 0 \text{ in } V_\sigma\\
&\beta_n^\ell (i\beta_n\rho\left[\frac{b^3}{12}\psi_{1n},\frac{b^3}{12}\phi_{1n},bw_{1n}\right]+A_1'[\psi_n,\phi_n,w_n]\\
&\quad+\left[A_2[h_{1n}^1,h_{1n}^2],A_3h_{1n}^3\right])\to 0 \text{ in } (L^2(\Omega))^3\\
&\beta_n^\ell (i\beta_n\left[\frac{b^3}{12}h_{1n}^1,\frac{b^3}{12}h_{1n}^2,bh_{1n}^3\right]+[A_4'[\psi_{1n},\phi_{1n}],A_6w_{1n}]\\
&\quad+\left[A_5[h_{1n}^1,h_{1n}^2],A_7h_{1n}^3\right])\to 0 \text{ in } (L^2(\Omega))^3
\end{aligned}
\tag{4.4}
$$

By taking the inner product in $\mathcal{H}_\sigma$ of (4.3) with $\mathbf{U}_n$, where $\mathcal{H}_\sigma$ is now the complexification of $\mathcal{H}_\sigma$, we obtain

$$
\begin{aligned}
&\beta_n^\ell\left(i\beta_n\rho b(\frac{b^2}{12}\|\psi_{1n}\|_{0,\Omega}^2+\frac{b^2}{12}\|\phi_{1n}\|_{0,\Omega}^2+\|w_{1n}\|_{0,\Omega}^2)+i\beta_n a'([\psi_n,\phi_n,w_n])\right)\\
&\quad+\beta_n^\ell\left((A_2[h_{1n}^1,h_{1n}^2],[\psi_{1n},\phi_{1n}])_{0,\Omega}+(A_3h_{1n}^3,w_{1n})_{0,\Omega}\right)\\
&\quad+\beta_n^\ell\left(i\beta_n b\left(\frac{b^2}{12}\|h_{1n}^1\|_{0,\Omega}^2+\frac{b^2}{12}\|h_{1n}^2\|_{0,\Omega}^2+\|h_{1n}^3\|_{0,\Omega}^2\right)\right)\\
&\quad+\beta_n^\ell\left((A_4'[\psi_{1n},\phi_{1n}],[h_{1n}^1,h_{1n}^2])_{0,\Omega}+(A_6w_{1n},h_{1n}^3)_{0,\Omega}\right)\\
&\quad+\beta_n^\ell\left((A_5[h_{1n}^1,h_{1n}^2],[h_{1n}^1,h_{1n}^2])_{0,\Omega}+(A_7h_{1n}^3,h_{1n}^3)_{0,\Omega}\right)\to 0
\end{aligned}
\tag{4.5}
$$

As in [7] it turns out that all the interactive terms contribute purely imaginary terms to (4.5). By taking real parts in (4.5), we obtain

$$
\beta_n^\ell\|\nabla\times[h_{1n}^1,h_{1n}^2]\|_{0,\Omega}^2+\beta_n^\ell\|\nabla h_{1n}^3\|_{0,\Omega}^2\to 0 \tag{4.6}
$$

which we combine into $\beta_n^\ell\|\nabla\times[h_{1n}^1,h_{1n}^2,h_{1n}^3]\|_{0,\Omega}^2\to 0$ as $n\to\infty$. By invoking a well-known imbedding theorem due to Duvaut and Lions [5, Theorem 6.1, p. 358] (see also [6, Lemma 1.6]), we conclude

$$
\begin{aligned}
&[h_{1n}^1,h_{1n}^2,h_{1n}^3]\to 0 \text{ in } (H^1(\Omega))^3\\
&[h_{1n}^1,h_{1n}^2,h_{1n}^3]\to 0 \text{ in } H_\sigma\times L^2(\Omega)
\end{aligned}
\tag{4.7}
$$

By next taking the inner product of the second equation of (4.4) with $[\psi_n,\phi_n,w_n]$ and using the definitions of A_1' and $A_i, i=2,3$, while combining terms, as indicated above, and dividing by $\beta_n^{\frac{\ell}{2}+1}$, we obtain

$$
\begin{aligned}
&\beta_n^{\frac{\ell}{2}-1}\left((i\beta_n\rho\left[\frac{b^3}{12}\psi_{1n},\frac{b^3}{12}\phi_{1n},bw_{1n}\right],[\psi_n,\phi_n,w_n])_{0,\Omega}+a'([\psi_n,\phi_n,w_n])\right.\\
&\quad\left.-\left(\left(\nabla\times\left[\frac{b^3}{12}h_{1n}^1,\frac{b^3}{12}h_{1n}^2,bh_{1n}^3\right]\right)\times\left[H_0^1,H_0^2\right],[\psi_n,\phi_n,w_n]\right)_{0,\Omega}\right)\to 0
\end{aligned}
\tag{4.8}
$$

Taking $\ell=2$ in (4.6) whence we have the convergence in $(L^2(\Omega))^3$ of the first component of the last inner product in (4.8), and availing ourselves of the boundedness of $[\psi_n,\phi_n,w_n]$ in $(L^2(\Omega))^3$ on the strength of (4.2), furnishes the convergence to zero of this inner product. Hence, we remain with

$$
(i\beta_n\rho\left[\frac{b^3}{12}\psi_{1n},\frac{b^3}{12}\phi_{1n},bw_{1n}\right],[\psi_n,\phi_n,w_n])_{0,\Omega}+a'([\psi_n,\phi_n,w_n])\to 0 \tag{4.9}
$$

Division of the third convergence relation of (4.4) by β_n renders the convergence relation

$$\beta_n\left(i\beta_n\left[\frac{b^3}{12}h^1_{1n},\frac{b^3}{12}h^2_{1n},bh^3_{1n}\right]+\nabla\times\left(\nabla\times[\frac{b^3}{12}h^1_{1n},\frac{b^3}{12}h^2_{1n},bh^3_{1n}]\right)\right.$$
$$\left.-[H^1_0,H^2_0]\cdot\nabla\left[\frac{b^3}{12}\psi_{1n},\frac{b^3}{12}\phi_{1n},bw_{1n}\right]\right)\longrightarrow 0 \text{ in } (L^2(\Omega))^3 \tag{4.10}$$

Taking the inner product with $[H^1_0,H^2_0]\cdot\nabla[\frac{12}{b^3}\psi_n,\frac{12}{b^3}\phi_n,\frac{1}{b}w_n]$ while taking account of (4.7), the boundedness of $[H^1_0,H^2_0]\cdot\nabla[\psi_n,\phi_n,w_n]$ in $(L^2(\Omega))^3$ on the strength of (4.2) and replacing $\beta_n[\psi_{1n},\phi_{1n},w_{1n}]$ by $i\beta^2_n[\psi_n,\phi_n,w_n]$ in the convergence relation provides

$$\beta_n\left(\nabla\times(\nabla\times[h^1_{1n},h^2_{1n},h^3_{1n}]),[H^1_0,H^2_0]\cdot\nabla[\psi_n,\phi_n,w_n]\right)_{0,\Omega}$$
$$-\ i\beta^2_n\|[H^1_0,H^2_0]\cdot\nabla[\psi_n,\phi_n,w_n]\|^2_{0,\Omega}\longrightarrow 0 \tag{4.11}$$

To the inner product in this relation, we apply integration by parts and write the resultant term, after discarding the boundary terms on the strength of the boundary conditions, as $(\beta^2_n(\nabla\times[h^1_{1n},h^2_{1n},h^3_{1n}]),$ $\frac{1}{\beta_n}(\nabla\times([H^1_0,H^2_0]\cdot\nabla[\psi_n,\phi_n,w_n])))_{0,\Omega}$. Observing that in this inner product the first component converges to zero in $(L^2(\Omega))^3$ in view of (4.6), while (4.2), the second convergence relation of (4.4) and elliptic regularity, renders the second component $\frac{1}{\beta_n}(\nabla\times([H^1_0,H^2_0]\cdot\nabla[\psi_n,\phi_n,w_n]))$ bounded in $(L^2(\Omega))^3$ uniformly in n, we obtain

$$\beta^2_n\|[H^1_0,H^2_0]\cdot\nabla[\psi_n,\phi_n,w_n]\|^2_{0,\Omega}\longrightarrow 0 \tag{4.12}$$

as $n\to\infty$. This furnishes

$$\beta^2_n\|[H^1_0,H^2_0]\cdot\nabla w_n\|^2_{0,\Omega}\to 0$$
$$\beta^2_n\|[H^1_0,H^2_0]\cdot\nabla[\psi_n,\phi_n]\|^2_{0,\Omega}\to 0 \tag{4.13}$$

From these relations, we wish to derive that $\beta^2_n\|w_n\|^2_{0,\Omega}\to 0$ and $\beta^2_n\|[\psi_n,\phi_n]\|^2_{0,\Omega}\to 0$. This will be achieved by using carefully chosen multipliers and making crucial use of the condition $\nabla\cdot[\psi,\phi]=0$. First, we take the inner product of $[H^1_0,H^2_0]\cdot\nabla w_n\equiv\nabla\cdot([H^1_0,H^2_0]w_n)$ with $([H^1_0,H^2_0]w_n)\cdot\mathbf{x},\mathbf{x}=[x,y]$ and apply integration by parts to obtain

$$2(\nabla\cdot([H^1_0,H^2_0]w_n),([H^1_0,H^2_0]w_n)\cdot\mathbf{x})_{0,\Omega}=-|[H^1_0,H^2_0]|^2\|w_n\|^2_{0,\Omega}$$

On the strength of the first relation in (4.13) and the boundedness of $([H^1_0,H^2_0]w_n)\cdot\mathbf{x}$ in $L^2(\Omega)$, the inner product on the left-hand side of the above equality converges to zero. Thus, we conclude that $\beta^2_n\|w_n\|^2_{0,\Omega}\to 0$ as $n\to\infty$, whence

$$\beta_n w_n\to 0 \text{ in } L^2(\Omega)$$
$$w_{1n}\to 0 \text{ in } L^2(\Omega) \tag{4.14}$$

as $n\to\infty$ by applying, in obtaining the second convergence relation, the first convergence relation of (4.4).

To obtain $\beta^2_n\|[\psi_n,\phi_n]\|^2_{0,\Omega}\to 0$ from the second convergence relation of (4.13), we note that $[H^1_0,H^2_0]\cdot\nabla[\psi_n,\phi_n]=[[H^1_0,H^2_0]\cdot\nabla\psi_n,[H^1_0,H^2_0]\cdot\nabla\phi_n]$.

Thus, by applying the multipliers $([H^1_0,H^2_0]\psi_n)\cdot\mathbf{x}$ and $([H^1_0,H^2_0]\phi_n)\cdot\mathbf{x}$ to the components of the vector in the last line of this identity, following the same procedure as before, we obtain

$$\beta_n[\psi_n,\phi_n]\quad\to 0 \text{ in } (L^2(\Omega))^2$$
$$[\psi_{1n},\phi_{1n}]\quad\to 0 \text{ in } (L^2(\Omega))^2 \tag{4.15}$$

as $n\to\infty$.

We now return to (4.9) writing the convergence relation as

$$i\left(\rho\left[\frac{b^3}{12}\psi_{1n}, \frac{b^3}{12}\phi_{1n}, bw_{1n}\right], \beta_n\left[\psi_n, \phi_n, w_n\right]\right)_{0,\Omega} + a'([\psi_n, \phi_n, w_n]) \to 0 \tag{4.16}$$

whence by (4.14)–(4.15) we can conclude

$$a'([\psi_n, \phi_n, w_n]) \to 0$$

as $n \to \infty$. This immediately yields

$$[\psi_n, \phi_n, w_n] \to 0 \text{ in } (H_0^1(\Omega))^3 \tag{4.17}$$

Combination of (4.7), (4.14), (4.15) and (4.17) provides a contradiction with (4.2). Thus, we have verified condition (H_2).

It remains to validate condition (H_1) in which we now replace $\mathcal{A}'$ by $\mathcal{A}$ as defined in [7]. Since $0 \in \rho(-\mathcal{C}^{-1}\mathcal{A})$, as is easily shown by using standard elliptic PDE theory (see e.g. [8, Theorem 2.1]) and $-\mathcal{C}^{-1}\mathcal{A}$ is compact, we need to consider only the possibility of purely imaginary eigenvalues. Assuming $i\beta \in \sigma(-\mathcal{C}^{-1}\mathcal{A}), \beta \neq 0$, there exists a non-trivial eigenfunction $\mathbf{U}, \mathbf{U} \in \mathcal{D}(\mathcal{A})$, such that

$$(i\beta\mathcal{C} + \mathcal{A})\mathbf{U} = 0$$

Taking $\mathbf{U} = [[\psi_0, \phi_0, w_0], [\psi_1, \phi_1, w_1], [h_1^1, h_1^2, h_1^3]]$, we have the system

$$\begin{aligned} i\beta\left[\psi_0, \phi_0, w_0\right] - \left[\psi_1, \phi_1, w_1\right] &= 0 \\ i\beta\rho\left[\frac{b^3}{12}\psi_1, \frac{b^3}{12}, \phi_1, bw_1\right] + A_1\left[\psi_0, \phi_0, w_0\right] + \left[A_2\left[h_1^1, h_1^2\right], A_3 h_1^3\right] &= 0 \\ i\beta\left[\frac{b^3}{12}h_1^1, \frac{b^3}{12}h_1^2, bh_1^3\right] + \left[A_4\left[\psi_1, \phi_1\right], A_6 w_1\right] + \left[A_5\left[h_1^1, h_1^2\right], A_7 h_1^3\right] &= 0 \end{aligned} \tag{4.18}$$

Since in the validation of (H_2) we did not use the condition $\beta_n \to \infty$, but rather its consequence, viz. β_n is bounded away from zero, we can proceed completely analogously as before, with the role of the sequences $\{\beta_n\}$ and $\{\mathbf{U}_n\}$ now taken over by, respectively, the eigenvalue $i\beta$ and the eigenfunction $\mathbf{U}$, to obtain $[h_1^1, h_1^2, h_1^3] = 0$ and the "side" conditions

$$\begin{aligned} \beta^2\|[H_0^1, H_0^2] \cdot \nabla w_0\|_{0,\Omega}^2 &= 0 \\ \beta^2\|\nabla \times ([\psi_0, \phi_0] \times [H_0^1, H_0^2])\|_{0,\Omega}^2 &= 0 \end{aligned} \tag{4.19}$$

Application of the multiplier $([H_0^1, H_0^2]w_0)\cdot\mathbf{x}$ to the first equation of (4.19), i.e. multiplication of $([H_0^1, H_0^2]\cdot \nabla w_0) = (\nabla \cdot ([H_0^1, H_0^2]w_0))$ by $([H_0^1, H_0^2]w_0) \cdot \mathbf{x}$ and integration over Ω, as before, furnishes

$$\begin{aligned} 0 &= 2\beta^2(\nabla \cdot ([H_0^1, H_0^2]w_0), ([H_0^1, H_0^2]w_0) \cdot \mathbf{x})_{0,\Omega} \\ &= -\beta^2|[H_0^1, H_0^2]|^2\|w_0\|_{0,\Omega}^2 \end{aligned}$$

whence we conclude $w_0 = 0 = w_1$. Since $\mathbf{U} \in \mathcal{D}(\mathcal{A})$, it follows that $\Delta w_0 = 0$. The third component of the second equation of (4.18) corresponding to the equation $i\beta\rho b w_1 - \nabla\cdot([\psi_0, \phi_0] + \nabla w_0) = 0$ in turn provides the additional "side" condition $\nabla \cdot [\psi_0, \phi_0] = 0$. Next, by applying the multipliers $([H_0^1, H_0^2]\psi_0) \cdot \mathbf{x}$ and $([H_0^1, H_0^2]\phi_0) \cdot \mathbf{x}$ to the second equation of (4.19), in which, on the strength of $\nabla \cdot [\psi_0, \phi_0] = 0$, the term in the norm sign satisfies $\nabla \times ([\psi_0, \phi_0] \times [H_0^1, H_0^2]) = [H_0^1, H_0^2] \cdot \nabla[\psi_0, \phi_0] = [[H_0^1, H_0^2] \cdot \nabla\psi_0, [H_0^1, H_0^2] \cdot \nabla\phi_0]$, we conclude, by proceeding componentwise, in exactly the same way as in the attainment of $w_0 = 0 = w_1$, that $\psi_0 = 0 = \psi_1$. Combining this with $[h_1^1, h_1^2, h_1^3] = 0$, we have a contradiction with the non-triviality of the eigenfunction $\mathbf{U}$, showing that $-\mathcal{C}^{-1}\mathcal{A}$ has no imaginary eigenvalues. Condition (H_1) is now verified, and the proof of Theorem 4.1 is complete. □

Remark. (i) The need for the additional condition $\nabla \cdot [\psi, \phi] = 0$ in Theorem 4.1 should be seen in the context of the fact that the interaction between the magnetic and elastic fields in $Pr\ (P)$ involves a uniform magnetic field $[H_0^1, H_0^2]$ in the dissipative term $\nabla \times ([\psi_t, \phi_t, w_t] \times [H_0^1, H_0^2]) = [H_0^1, H_0^2] \cdot \nabla[\psi_t, \phi_t, w_t] - [H_0^1, H_0^2]\nabla_{x,y,z} \cdot [\psi_t, \phi_t, w_t] = [H_0^1, H_0^2] \cdot \nabla[\psi_t, \phi_t, w_t] - [H_0^1, H_0^2]\nabla_{x,y} \cdot [\psi_t, \phi_t]$ in the PDE for the magnetic field $[h_1^1, h_1^2, h_1^3]$. The effect of this is manifested in the intricacy of proceeding from the "side" conditions (4.13) to the convergence of $[\psi_n, \phi_n, w_n]$ to zero in $(H_0^1(\Omega))^3$.

(ii) Since in the proof of Theorem 4.1, condition (H_1) could be verified without having to assume $\nabla \cdot [\psi, \phi] = 0$, as the condition emanated as a "side" condition in the verification of (H_1) by a contradiction argument, the property of strong asymptotical stability of the semigroup $\exp(-t\mathcal{C}^{-1}\mathcal{A})$ associated with $Pr\ (P)$ is attained without having to impose the solenoidality condition of the shear force vector. The author is indebted to Marcio Ferreira for this observation.

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