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Modeling and analysis of a contact problem for a viscoelastic rod

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Abstract. We consider a nonlinear viscoelastic rod which is in contact with a foundation along its length and is in contact with an obstacle at its end. The rod is acted up by body forces and, as a result, its mechanical state evolves. Our aim in this paper is twofold. The first one is to construct an appropriate mathematical model which describes the evolution of the rod. The second one is to prove the weak solvability of the problem. To this end, we use arguments on second-order inclusions with multivalued pseudomonotone operators.

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1. Introduction

Vibration processes and objects or media arise in engineering and everyday life. The vibration of a bridge, of a window, of a spring or an automotive platform with a low-power active suspension represents only four simple examples, among others. Vibration produces sounds and could lead to resonance phenomena which can have destructive effects on the mechanical systems. For this reason, there is a considerable interest in the modeling, analysis and numerical simulation of such processes. The literature in the field is extensive.

In the engineering literature, the vibrations are very often studied by considering mechanical systems based on finite collections of masses, springs, dampers, and rods in frictional or frictionless contact. The analysis of such systems leads often to nonlinear differential equations or inclusions of second order. The nonlinearity arises either from the nonlinearity of the constitutive laws involved into the model or from the contact conditions.

In this work, we study the evolution of a simple mechanical system, consisting of a vibrating rod in contact with a support along its length, the so-called foundation. The interaction between the rod and the foundation is modeled with specific interface conditions. The rod is clamped at one end and, in addition, is in contact with an obstacle, at the other end. The contact is modeled with a subdifferential condition involving a possible nonconvex potential. We derive a mathematical model describing the above physical setting which leads to a new and nonstandard problem, expressed in terms of a second-order differential inclusion. Solving this inclusion, which involves strongly nonlinearities, represents the main trait of novelty of this paper. In this way, we show how one can apply the rapidly developing theory of differential inclusions to describe contact processes with rods. We do it in a simple setting that avoids some complications related to higher dimensions, making the mathematical approach more transparent. Moreover, these simple settings are of importance since they allow for easier experimental measurements and identification of the system parameters. These parameter functions then may be used in more realistic applications.

Mathematical models describing the evolution of rods in contact with obstacle have been studied by many authors. For instance, a dynamic unilateral problem for rods was considered in [22], where the existence of the weak solutions was established and numerical simulations have been provided. Dynamic contact of two rods was modeled, analyzed and numerically simulated in [16]. The dynamic impact of two thermoelastic rods can be found in [2] and the quasistatic contact in [3]. In both papers, the heat exchange between the tips was assumed to depend on the distance or the gap when the rods were separated. Contact problems with thermo-viscoelastic rods have been discussed in [17,18]. Quasistatic contact of a elastic-perfectly-plastic rod was studied in [23]. This was, to the best of our knowledge, the first result for contact of a material with such a constitutive law. We also refer to [1] for a survey of static and quasistatic frictional contact problems in elasticity.

The rest of the paper is organized as follows. In Sect. 2, we introduce some basic definitions and preliminaries that will be used in the sequel. In Sect. 3, we describe the physical setting and construct our mathematical model of contact. It is in a form of a nonlinear inclusion in which the unknown is the displacement field. Finally, in Sect. 4 we provide the existence result for the inclusion under consideration. In this way, we prove that the contact problem has a weak solution. The proof of the existence theorem is based on time discretization technique, the so-called Rothe method, that we present in Sect. 5. It could also be obtained by using an abstract result recently obtained in [5]. However, for the convenience of the reader, we made the choice to present a direct proof for our main existence result, Theorem 4.9. We refer to [14,15] for additional results and methods concerning the discretization of nonlinear evolutionary problems.

2. Preliminaries

In this section, we briefly present the notation and some preliminary material to be used later in this paper. More details on the material presented below can be found in the books [10,11,19,20,26].

First, we precise that all linear spaces used in this paper are assumed to be real. Unless it is stated otherwise, below in this section we denote by X a normed space with the norm $\|\cdot\|_X$, by X^* its topological dual, and $\langle\cdot,\cdot\rangle_{X^*\times X}$ will represent the duality pairing of X and X^*. The symbol 2^{X^*} is used to represent the set of all subsets of X^*. We start with definition of the generalized directional derivative and the subdifferential in sense of Clarke.

Definition 2.1. Let $\varphi\colon X\to\mathbb{R}$ be a locally Lipschitz function. The Clarke generalized directional derivative of φ at the point $x\in X$ in the direction $v\in X$ is defined by

$$\varphi^0(x;v)=\limsup_{y\to x,\lambda\downarrow 0}\frac{\varphi(y+\lambda v)-\varphi(y)}{\lambda}.$$

The Clarke subdifferential of φ at x is a subset of X^* given by

$$\partial\varphi(x)=\{\,\zeta\in X^*\mid \varphi^0(x;v)\geq\langle\zeta,v\rangle_{X^*\times X}\quad\text{for all}\quad v\in X\,\}.$$

In what follows we introduce the notion of coercivity.

Definition 2.2. Let X be a real Banach space and $A\colon X\to 2^{X^*}$ be a multivalued operator. We say that A is coercive if either the domain $D(A)$ of A is bounded or $D(A)$ is unbounded and

$$\lim_{\|v\|_X\to\infty\ v\in D(A)}\frac{\inf\{\langle v^*,v\rangle_{X^*\times X}\mid v^*\in Av\}}{\|v\|_X}=+\infty.$$

We now proceed with the definition of a pseudomonotone operator in both single valued and multivalued case.

Definition 2.3. Let X be a real, reflexive Banach space. A single valued operator $A\colon X \to X^*$ is called pseudomonotone, if for any sequence $\{v_n\}_{n=1}^\infty \subset X$ such that $v_n \to v$ weakly in X and $\limsup\limits_{n\to\infty} \langle Av_n, v_n - v\rangle \leqslant 0$, we have $\langle Av, v-y\rangle \leqslant \liminf\limits_{n\to\infty} \langle Av_n, v_n - y\rangle$ for every $y \in X$.

Definition 2.4. Let X be a real, reflexive Banach space. A multivalued operator $A\colon X \to 2^{X^*}$ is called pseudomonotone if the following conditions hold:

1) A has values which are nonempty, weakly compact and convex,
2) A is upper semicontinuous from every finite dimensional subspace of X into X^* furnished with weak topology,
3) if $\{v_n\}_{n=1}^\infty \subset X$ and $\{v_n^*\}_{n=1}^\infty \subset X^*$ are two sequences such that $v_n \to v$ weakly in X, $v_n^* \in A(v_n)$ for all $n \geqslant 1$ and $\limsup\limits_{n\to\infty} \langle v_n^*, v_n - v\rangle \leqslant 0$, then for every $y \in X$ there exists $u(y) \in A(v)$ such that $\langle u(y), v - y\rangle \leqslant \liminf\limits_{n\to\infty} \langle v_n^*, v_n - y\rangle$.

We now recall two important results concerning properties of pseudomonotone operators.

Proposition 2.5. *Let X be a reflexive Banach space and $A_1, A_2\colon X \to 2^{X^*}$ be pseudomonotone operators. Then $A_1 + A_2\colon X \to 2^{X^*}$ is a pseudomonotone operator.*

Theorem 2.6. *Let X be a reflexive Banach space and let $A\colon X \to 2^{X^*}$ be a coercive, bounded and pseudomonotone multivalued operator. Then A is surjective, i.e., $R(A) = X^*$.*

Let X be a Banach space and $T > 0$. We introduce the space $BV(0,T;X)$ of functions of bounded total variation on $[0,T]$. Let π denote any finite partition of $[0,T]$ by a family of disjoint subintervals $\{\sigma_i = (a_i, b_i)\}$ such that $[0,T] = \cup_{i=1}^n \overline{\sigma}_i$. Let $\mathcal{F}$ denote the family of all such partitions. Then for a function $x\colon [0,T] \to X$ and for $1 \leq q < \infty$, we define a seminorm

$$\|x\|^q_{BV^q(0,T;X)} = \sup_{\pi \in \mathcal{F}} \left\{ \sum_{\sigma_i \in \pi} \|x(b_i) - x(a_i)\|^q_X \right\}$$

and the space

$$BV^q(0,T;X) = \{x\colon [0,T] \to X \mid \|x\|_{BV^q(0,T;X)} < \infty\}.$$

For $1 \leq p \leq \infty$, $1 \leq q < \infty$ and Banach spaces X, Z such that $X \subset Z$, we introduce a vector space

$$M^{p,q}(0,T;X,Z) = L^p(0,T;X) \cap BV^q(0,T;Z).$$

Then the space $M^{p,q}(0,T;X,Z)$ is also a Banach space with the norm given by $\|\cdot\|_{L^p(0,T;X)} + \|\cdot\|_{BV^q(0,T;Z)}$.

Next we recall a compactness result, which will be used in the sequel. For its proof, we refer to [15].

Proposition 2.7. *Let $1 \leqslant p, q < \infty$. Let $X_1 \subset X_2 \subset X_3$ be real Banach spaces such that X_1 is reflexive, the embedding $X_1 \subset X_2$ is compact and the embedding $X_2 \subset X_3$ is continuous. Then the embedding $M^{p,q}(0,T;X_1,X_3) \subset L^p(0,T;X_2)$ is compact.*

The following version of Aubin–Celina convergence theorem (see [4]) will be used in what follows.

Proposition 2.8. *Let X and Y be Banach spaces, and $F\colon X \to 2^Y$ be a multifunction such that*

(a) the values of F are nonempty closed and convex subsets of Y;

(b) F is upper semicontinuous from X into Y endowed with a weak topology.

Let $x_n, x\colon (0,T) \to X$, $y_n, y\colon (0,T) \to Y$, $n \in \mathbb{N}$, be measurable functions such that $x_n(t) \to x(t)$ for a.e. $t \in (0,T)$ and $y_n \to y$ weakly in $L^1(0,T;Y)$. If $y_n(t) \in F(x_n(t))$ for all $n \in \mathbb{N}$ and a.e. $t \in (0,T)$, then $y(t) \in F(x(t))$ for a.e. $t \in (0,T)$.

Finally we will need the following discrete Gronwall lemma ([13, Chap. 7]).

Lemma 2.9. *Let $T > 0$ be given. For a positive integer N we define $k = T/N$. Assume that $\{g_n\}_{n=1}^N$ and $\{e_n\}_{n=1}^N$ are two sequences of nonnegative numbers satisfying*

$$e_n \leq \overline{c} g_n + \overline{c} \sum_{j=1}^{n} k e_j, \quad n = 1, \ldots, N$$

for a positive constant $\overline{c}$ independent of N or k. Then there exists a positive constant c, independent of N or k, such that

$$\max_{1 \leq n \leq N} e_n \leq c \max_{1 \leq n \leq N} g_n.$$

3. The model

We consider a viscoelastic rod which occupies, in its reference configuration, the interval $(0, L)$ with $L > 0$. The rod is attached at its end $x = 0$ and is in contact with an obstacle at $x = L$. In addition, it is contact with a reactive foundation along its length that opposes its deformation. The rod is acted up by time-dependent body forces of density f. The physical setting is depicted in Fig. 1.

We are interested in the description of the dynamic evolution of the rod in the physical setting above and in providing the analysis of the corresponding mathematical model. To this end, we denote in what follows by x and t the spatial and the time variables, respectively. Note that $x \in [0, L]$ and $t \in [0, T]$, where T represents the length of the time interval of interest. Moreover, for a function $G = G(x, t)$, we use the subscripts x and t for the derivatives with respect to x and t, i.e.,

$$G_x = \frac{\partial G}{\partial x}, \quad G_t = \frac{\partial G}{\partial t}, \quad G_{xx} = \frac{\partial^2 G}{\partial x^2}, \quad G_{xt} = \frac{\partial^2 G}{\partial x \partial t}, \quad G_{tt} = \frac{\partial^2 G}{\partial t^2}.$$

Everywhere in this paper, we denote by $u = u(x, t)$ and $\sigma = \sigma(x, t)$ the displacement and the stress function, respectively. We also denote by $\varepsilon = \varepsilon(x, t)$ the deformation function defined by $\varepsilon = u_x$.

We turn now to the construction of our mathematical model, which gathers the equation of motion, the constitutive law, the boundary conditions and the initial conditions, that we describe in what follows.

First, the equation of motion of the rod is given by

$$\rho(x) u_{tt}(x, t) = \sigma_x(x, t) + \mathcal{F}(x, t) \quad \text{for all } x \in [0, L],\ t \in [0, T]. \tag{3.1}$$

Here $\rho = \rho(x)$ represents the density of mass in the reference configuration and $\mathcal{F} = \mathcal{F}(x, t)$ represents the total force acting on the rod, i.e., the sum of the applied force and the reaction of the foundation. We assume that the reaction of the foundation has an additive decomposition of the form $\psi + \xi$, where the functions ψ and ξ will be described below. Therefore

$$\mathcal{F}(x, t) = f(x, t) + \psi(x, t) + \xi(x, t) \quad \text{for all } x \in [0, L],\ t \in [0, T]. \tag{3.2}$$

Next we assume that

$$\psi(x, t) = -g(u_t(x, t)) \quad \text{for all } x \in [0, L],\ t \in [0, T], \tag{3.3}$$

$$\xi(x, t) = -h(u(x, t)) \quad \text{for all } x \in [0, L],\ t \in [0, T]. \tag{3.4}$$

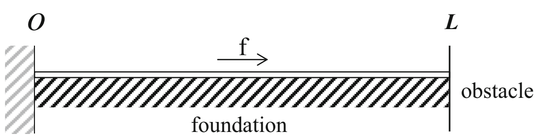

Fig. 1. The rod in contact

where g and h are given nonlinear functions, assumed to be positive for positive argument and negative for negative argument. This restriction guarantees that the forces ψ and ξ are opposite to the velocity and the displacement fields, respectively. Note that assumption (3.3) shows that the force ψ depends only on the velocity field u_t which mimics the behavior of a nonlinear viscous damper. Therefore it could be used to model the friction between the rod and the foundation. In contrast, assumption (3.4) shows that the force ξ depends only on the displacement field u, which mimics the behavior of a nonlinear elastic spring. It could be used to model the adhesion between the rod and the foundation.

We now gather the Eqs. (3.1)–(3.4) and assume, for simplicity, that $\rho \equiv 1$. As a result, we obtain the balance equation

$$\begin{aligned} &u_{tt}(x,t) + g(u_t(x,t)) + h(u(x,t)) = \sigma_x(x,t) + f(x,t) \\ &\qquad\qquad\qquad \text{for all } x \in [0,L],\ t \in [0,T]. \end{aligned} \tag{3.5}$$

The next step is to prescribe the constitutive law. We assume that the rod is viscoelastic and its behavior is described with the equation

$$\sigma = \eta|\varepsilon_t|^{p-2}\varepsilon_t + E\varepsilon. \tag{3.6}$$

Here $\eta > 0$ is a viscosity coefficient, $E > 0$ represents the Young modulus and $p \geq 2$ is a given coefficient. Note that (3.6) represents a nonlinear version of the well-known Kelvin–Voigt linear viscoelastic constitutive law

$$\sigma = \widetilde{\eta}\varepsilon_t + E\varepsilon. \tag{3.7}$$

Actually, (3.6) could be recovered from (3.7) by assuming that the viscosity coefficient $\widetilde{\eta}$ depends on the strain rate ε_t, i.e., $\widetilde{\eta} = \widetilde{\eta}(\varepsilon_t) = \eta|\varepsilon_t|^{p-2}$. Such kind of dependence is justified from physical point of view since it can be observed to various materials like polymers and pastes, as explained in [24], for instance. In addition, it makes the resulting boundary value problem more difficult from mathematical point of view, since it introduces a strong nonlinearity into the model.

We now replace $\varepsilon = u_x$ in (3.6) to see that

$$\sigma = \eta|u_{xt}|^{p-2}u_{xt} + Eu_x, \tag{3.8}$$

then we substitute this equality in the balance Eq. (3.5) to find that

$$\begin{aligned} &u_{tt}(x,t) - (\eta|u_{xt}|^{p-2}u_{xt})_x - Eu_{xx} + g(u_t(x,t)) + h(u(x,t)) = f(x,t) \\ &\quad \text{for all } x \in [0,L],\ t \in [0,T]. \end{aligned} \tag{3.9}$$

We now describe the boundary conditions. First of all, since the rod is fixed in $x = 0$, the displacement field vanishes there, i.e.,

$$u(0,t) = 0 \quad \text{for all } t \in [0,T]. \tag{3.10}$$

Next we assume that the rod is in contact at $x = L$ with an obstacle and we model the contact with a subdifferential inclusion of the form

$$-\sigma(L,t) \in \partial j(u_t(L,t)) \quad \text{for all } t \in [0,T]. \tag{3.11}$$

Here j is a prescribed possible nonconvex function and ∂j represents its Clarke subdifferential. Examples of contact conditions which can be cast in the form (3.11), can be found in [13,19,25], for instance. Here we restrict ourselves to recall that they include the so-called normal damped response condition and various viscous-type contact conditions. We combine now (3.8) and (3.11) to deduce that

$$-\Big(\eta|u_{xt}(L,t)|^{p-2}u_{xt}(L,t) + Eu_x(L,t)\Big) \in \partial j(u_t(L,t)) \text{ for all } t \in [0,T]. \tag{3.12}$$

Finally, we prescribe the initial displacement and the initial velocity of the rod, i.e.,

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x) \quad \text{for all } t \in [0,T], \tag{3.13}$$

where u_0 and v_0 are given functions.

We are now in a position to formulate the mathematical model which describes the dynamic evolution of the rod, in the physical setting described above.

Problem P. *Find a displacement field* $u\colon [0,L]\times[0,T]\to\mathbb{R}$ *which satisfies the balance equation* (3.9), *the boundary conditions* (3.10), (3.12) *and the initial conditions* (3.13).

The existence of weak solution of Problem P will be provided in Sect. 4. It is based on technique used recently in [5]. Here we mention that the main difficulty in the analysis of Problem P arises in the nonlinearities of this problem, which appear both in the second-order Eq. (3.9) and in the multivalued boundary condition (3.12). We also note that, if the displacement function u represents a solution to Problem P, then the corresponding stress field can be easily computed by using the constitutive law (3.8).

4. Main result

In this section, we state our main result in the study of Problem P, Theorem 4.9. Here and below, we take $2\le p<\infty$ and $1<q\le 2$ satisfying $\frac{1}{p}+\frac{1}{q}=1$. We use notation $\mathbb{R}_+$ for a set of nonnegative real numbers.
We impose the following assumptions on the functions g, h, j and f.

$\underline{H(g)}$ $g\colon\mathbb{R}\to\mathbb{R}$ is such that

(i) g is continuous,
(ii) $\underline{g}:=\inf\limits_{s\in\mathbb{R}} g(s)s>-\infty$,
(iii) $|g(s)|\leqslant c_g(1+|s|^{p-1})$ for all $s\in\mathbb{R}$ with $c_g>0$.

$\underline{H(h)}$ $h\colon\mathbb{R}\to\mathbb{R}$ is such that

(i) $|h(s)|\le c_h(1+|s|^{\frac{2}{q}})$ for all $s\in\mathbb{R}$ with $c_h>0$,
(ii) $|h(r)-h(s)|\le\overline{h}\,(\max\{|r|,|s|\})\,|s-r|^{\frac{1}{q}}$ for all $r,s\in\mathbb{R}$, where $\overline{h}\colon\mathbb{R}\to\mathbb{R}$ is a nondecreasing function.

$\underline{H(j)}$ $j\colon\mathbb{R}\to\mathbb{R}$ is such that

(i) j is locally Lipschitz
(ii) $|\xi|\leqslant c_j(1+|s|^{p-1})$ for all $\xi\in\partial j(s)$ with $c_j>0$.

$\underline{H(f)}$ $f\in L^q([0,T]\times[0,L])$.
$\underline{H_0}$ $u_0\in V$, $v_0\in H$.

A typical example which satisfies assumption $H(h)$ is the function

$$h(s)=|s|^\delta s,$$

where $\delta\leqslant 1-\frac{2}{p}$.

We introduce the spaces

$$W=\{\,v\in W^{1,p}(0,L)\mid v(0)=0\,\},\quad V=\{\,v\in H^1(0,L)\mid v(0)=0\,\}$$

and $H=L^2(0,L)$ with the norms defined by

$$\|v\|_W^p=\int\limits_0^L|v_x|^p\,\mathrm{d}x,\quad \|v\|_V^2=\int\limits_0^L|v_x|^2\,\mathrm{d}x,\quad \|v\|_H^2=\int\limits_0^L|v|^2\,\mathrm{d}x.$$

We denote by $\langle\cdot,\cdot\rangle_{W^*\times W}$ and $\langle\cdot,\cdot\rangle_{V^*\times V}$ the duality in spaces W and V, respectively. The inner product in H is denoted by $(\cdot,\cdot)_H$. Identifying H with its dual, we remark that the above spaces form the evolution system

$$W \subset V \subset H \subset V^* \subset W^*$$

with all embeddings dense and continuous. We also recall that the embeddings $W \subset H$ and $V \subset H$ are compact. It is well known that $V \subset C(0, L)$ and the following inequality holds:

$$\|v\|_{C(0,L)} \le \sqrt{L}\|v\|_V \quad \text{for all } v \in V. \tag{4.1}$$

Here $C(0, L)$ denotes the space of continuous functions on $[0, L]$ with norm $\|v\|_{C(0,L)} = \max\{\, |v(x)| \mid x \in [0, L] \,\}$ for all $v \in C(0, L)$. We consider operators $A\colon W \to W^*$, $B\colon V \to V^*$ and $C\colon V \to W^*$ defined by

$$\langle Au, v\rangle_{W^*\times W} = \eta \int_0^L |u_x|^{p-2} u_x v_x \,\mathrm{d}x + \int_0^L g(u)v \,\mathrm{d}x \quad \text{for all}\, u, v \in W,$$

$$\langle Bu, v\rangle_{V^*\times V} = E \int_0^L u_x v_x \mathrm{d}x \quad \text{for all}\, u \in V,\ v \in V,$$

$$\langle Cu, v\rangle_{W^*\times W} = \int_0^L h(u)v \,\mathrm{d}x \quad \text{for all } u \in V,\ v \in W.$$

We also define the functional $F\colon [0, T] \to W^*$ by

$$\langle F(t), v\rangle_{W^*\times W} = \int_0^L f(t)v \,\mathrm{d}x \quad \text{for all } v \in W.$$

Moreover we define the spaces $\mathcal{W} = L^p(0, T; W)$, $\mathcal{V} = L^p(0, T; V)$, $\mathcal{H} = L^2(0, T; H)$ and $\mathcal{U} = L^p(0, T)$. We note that their dual spaces are $\mathcal{W}^* = L^q(0, T; W^*)$, $\mathcal{V}^* = L^q(0, T; V^*)$ and $\mathcal{U}^* = L^q(0, T)$, respectively.

We now introduce a notion of weak solution of Problem P.

Definition 4.1. A function $u \in \mathcal{W}$ is said to be a weak solution of Problem P if $u_t \in \mathcal{W}$, $u_{tt} \in \mathcal{W}^*$ and there exists a function $\xi : [0, T] \to \mathbb{R}$ such that

$$\begin{aligned}
&\langle u_{tt}(t) + Au_t(t) + Bu(t) + Cu(t), v\rangle_{W^*\times W} + \xi(t)v(L) = \langle F(t), v\rangle_{W^*\times W}\\
&\quad \text{for a.e. } t \in (0, T), \quad \text{for all } v \in W,\\
&\xi(t) \in \partial j(u_t(L, t)) \quad \text{for a.e. } t \in (0, T),\\
&u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x) \quad \text{for a.e. } x \in (0, L).
\end{aligned}$$

We remark that the weak formulation used in Definition 4.1 can be obtained from equation in Problem P by multiplying it by a test function $v \in W$ and using an integration by parts formula.

In what follows we will deal with the existence of weak solutions of Problem P. To this end, we define the multifunction $M\colon \mathbb{R} \to 2^{\mathbb{R}}$ by $M(s) = \partial j(s)$ for all $s \in \mathbb{R}$. We also define the operator $\gamma\colon W \to \mathbb{R}$ given by $\gamma v = v(L)$ for all $v \in W$. We recall that $W \subset C(0, L)$ and $v(L)$ is understood as a value of a continuous representant of $v \in W$ at L. Thus operator γ is well defined. We use notation $\|\gamma\| := \|\gamma\|_{\mathcal{L}(W,\mathbb{R})}$. Next we formulate the following auxiliary problem.

Problem $\mathcal{P}$. Find $u \in \mathcal{W}$ with $u_t \in \mathcal{W}$ and $u_{tt} \in \mathcal{W}^*$ such that

$$\begin{aligned}
&u_{tt}(t) + Au_t(t) + Bu(t) + Cu(t) + \gamma^* M(\gamma u_t(t)) \ni F(t) \ \text{ a.e. } t \in [0, T],\\
&u(0) = u_0, \quad u_t(0) = v_0.
\end{aligned}$$

Remark 4.2. By the definition of operator M, it follows that each solution of Problem $\mathcal{P}$ is also a weak solution of Problem P.

Applying Poincare and Jensen inequalities, we find that

$$\int_0^L |v|^p \mathrm{d}x \le L^p \int_0^L |v_x|^p \mathrm{d}x \quad \text{for all } v \in W. \tag{4.2}$$

Next we state and prove several properties of the operators A, B, C, M and γ which will be used in Sect. 5.

Lemma 4.3. *If the assumptions $H(g)$ hold, then the operator $A\colon W \to W^*$ satisfies*

(i) $\|Au\|_{W^*} \le c_A(1 + \|u\|_W^{p-1})$ *for all $u \in W$, where $c_A = \eta + c_g$,*
(ii) $\langle Au, u\rangle_{W^*\times W} \ge \eta\|u\|_W^p + L\,\underline{g}$ *for all $u \in W$,*
(iii) A is pseudomonotone.

Proof. Condition (i) follows from $H(g)(iii)$ and (4.2). Condition (ii) follows directly from the definition of A and $H(g)(ii)$. Finally for the proof of (iii), we refer to the proof of Proposition 27.9 in [26]. $\square$

Lemma 4.4. *The operator $B\colon V \to V^*$ is linear, bounded, symmetric and strongly positive, i.e.,*

$$\langle Bu, u\rangle_{V^*\times V} = E\|u\|_V^2, \quad \|Bu\|_{V^*} = E\|u\|_V \quad \textit{for all } u \in V. \tag{4.3}$$

The proof of Lemma 4.4 follows directly from the definition of B.

Lemma 4.5. *If the assumptions $H(h)$ hold, then the operator $C\colon V \to W^*$ satisfies*

(i) $\|Cv\|_{W^*} \leqslant \beta_C(1 + \|v\|_V^{\frac{2}{q}})$ *for all $v \in V$ with $\beta_C > 0$,*
(ii) there exists a nondecreasing function $\overline{C}\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|Cv - Cw\|_{W^*} \leqslant \overline{C}(\max\{\|v\|_V, \|w\|_V\})\|v - w\|_H^{\frac{1}{q}}$$

for all $v, w \in V$.
(iii) C is strongly continuous.

Proof. We start with the proof of (i). Using $H(h)(i)$ and Hölder inequality, and (4.2), we calculate

$$\begin{aligned} |\langle Cv, w\rangle_{W^*\times W}| &\le \int_0^L |h(v)||w|\,\mathrm{d}x \le \int_0^L c_h\left(1 + |v|^{\frac{2}{q}}\right)|w|\,\mathrm{d}x \\ &\le c_h L\left(L^{\frac{1}{q}} + \|v\|_V^{\frac{2}{q}}\right)\|w\|_W \quad \text{for all } v \in V,\ w \in W. \end{aligned}$$

Thus it follows that $\|Cv\|_{W^*} \le c_h L(L^{\frac{1}{q}} + \|v\|_V^{\frac{2}{q}})$ and (i) holds with $\beta_C = c_h L \max\{1, L^{\frac{1}{q}}\}$. Now we prove (ii). Let $u, v \in V$ and $w \in W$. Applying again Hölder inequality and (4.2), we get

$$|\langle Cu - Cv, w\rangle_{W^*\times W}| \le \int_0^L |h(u) - h(v)||w|\mathrm{d}x \le$$

$$\left(\int_0^L (\overline{h}(\max\{|u|, |v|\}))^q |u - v|\mathrm{d}x\right)^{\frac{1}{q}} L\|w\|_W,$$

and, in a consequence, it follows that

$$\|Cu - Cv\|_{W^*} \le \left(\int_0^L (\overline{h}(\max\{|u|, |v|\}))^q |u - v|\mathrm{d}x\right)^{\frac{1}{q}} L. \tag{4.4}$$

Since function $\overline{h}$ is nondecreasing, using (4.1), we obtain

$$\begin{aligned}\overline{h}(\max\{|u(x)|,|v(x)|\}) &\le \overline{h}(\max\{\|u\|_{C(0,L)},\|v\|_{C(0,L)}\})\\ &\le \overline{h}(\sqrt{L}\max\{\|u\|_V,\|v\|_V\}) \quad \text{for all } x\in[0,L].\end{aligned}$$

Moreover using again Hölder inequality, we obtain

$$\left(\int_0^L |u-v|\,\mathrm{d}x\right)^{\frac{1}{q}} \le L^{\frac{1}{2q}}\|u-v\|_H^{\frac{1}{q}}.$$

Thus condition (ii) holds with the function $\overline{C}$ given by $\overline{C}(s)=L^{\frac{1}{2q}+1}\overline{h}(\sqrt{L}s)$ for all $s\in\mathbb{R}_+$. It remains to show (iii). Let $v_n\to v$ weakly in V. It is enough to show that $Cv_n\to Cv$ in W^* for a subsequence. Since the embedding $V\subset H$ is compact then, for a subsequence again denoted v_n, we have $v_n\to v$ in H. Thus by (ii), it follows that $Cv_n\to Cv$ in W^*. This completes the proof. □

Lemma 4.6. *If the assumptions $H(j)$ hold, then the operator M satisfies*

(i) for all $u\in\mathbb{R}$, $M(u)$ is a nonempty, closed and convex set,
(ii) M is upper semicontinuous,
(iii) $|\xi|\leqslant c_j(1+|s|^{p-1})$ for all $w\in\mathbb{R}$, $\xi\in M(s)$.

The proof of Lemma 4.6 follows directly from the properties of Clarke subdifferential (see [10]).

Lemma 4.7. *The operator $\gamma\colon W\to\mathbb{R}$ is linear and strongly continuous.*

Proof. The linearity of γ is obvious. We also observe, that for all $v\in C(0,L)$, we have $|\gamma v|=|v(L)|\le \max_{x\in[0,L]}|v(x)|=\|v\|_{C(0,L)}$, which means, that $\gamma\in C(0,L)^*$. Let $v_n\to v$ weakly in W. Since the embedding $W\subset C(0,L)$ is continuous, we also have $v_n\to v$ weakly in $C(0,L)$, so, in particular $\gamma v_n\to\gamma v$ in $\mathbb{R}$, which completes the proof. □

Lemma 4.8. *The operator $\gamma^*M(\gamma(\cdot))\colon W\to W^*$ is pseudomonotone.*

The proof of Lemma 4.8 exploits Lemma 4.7 and follows the lines of the proof of Proposition 1.6 in [6] and, therefore, we omit it.

We now impose the following additional assumption on the constants of the problem.

$\underline{H_{const}}$ $\quad\eta>c_j\|\gamma\|^p$.

Our existence result in the study of Problem P that we state here and prove in Sect. 5 is the following.

Theorem 4.9. *Let the assumptions $H(g)$, $H(h)$, $H(j)$, $H(f)$ and H_0 hold. Moreover assume that either $p=2$ or H_{const} holds. Then Problem P admits a weak solution.*

5. The Rothe method

In this section, we study a time semidiscrete scheme corresponding to Problem $\mathcal{P}$. We provide the existence result for the approximate problem, and we study the convergence of its solution to the solution of Problem $\mathcal{P}$, when the discretization parameter converges to 0. In this way, we will prove Theorem 4.9. The technique presented below is referred as the Rothe method and was already used in many references, including [5,7,8].

5.1. Discrete problem

We divide the time interval $[0,T]$ by means of a sequence $\{t_k\}_{k=0}^{N_n} \subset [0,T]$ defined as follows

$$t_k = k\tau_n, \text{ where } \tau_n = T/N_n \text{ for } k = 0,\ldots,N_n.$$

In the above notation, N_n denotes the number of time steps in nth division of $[0,T]$, so we have $N_n \to \infty$ and $\tau_n \to 0$, as $n \to \infty$. For the convenience we will omit the subscript n and write N, τ instead of N_n, τ_n. We approximate the initial condition u_0 and v_0 by elements of W. Namely, let $\{u_\tau^0\}, \{v_\tau^0\} \subset W$ be sequences such that $u_\tau^0 \to u_0$ in V and $v_\tau^0 \to v_0$ in H, as $\tau \to 0$, and $\|v_\tau^0\|_W \le c/\sqrt{\tau}$ for some constant $c > 0$.

For a given $\tau > 0$ we formulate the following Rothe problem.

Problem $\mathcal{P}_\tau$. Find a sequence $\{w_\tau^k\}_{k=0}^N \subset W$ such that $w_\tau^0 = v_\tau^0$ and

$$\begin{aligned}&\left\langle \frac{1}{\tau}(w_\tau^k - w_\tau^{k-1}) + Aw_\tau^k + Bu_\tau^k + Cu_\tau^k, v\right\rangle_{W^*\times W} + \xi_\tau^k \gamma v \\ &= \left\langle F_\tau^k, v\right\rangle_{W^*\times W} \text{ for all } v \in W \text{ and } k = 1,\ldots,N,\end{aligned} \tag{5.1}$$

where $\xi_\tau^k \in M(\gamma w_\tau^k)$, $F_\tau^k = \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} F(t)\,\mathrm{d}t$ for $k = 1,\ldots,N$ and the sequence $\{u_\tau^k\}_{k=1}^N$ is defined by

$$u_\tau^k = u_\tau^0 + \tau \sum_{i=1}^k w_\tau^i \text{ for } k = 1,\ldots,N. \tag{5.2}$$

In what follows we will study the existence of solution to Problem $\mathcal{P}_\tau$. To this end, we define an auxiliary multivalued operator $\mathcal{T}\colon \mathbb{R}_+ \times W \times W \times W \to 2^{W^*}$ by

$$\begin{aligned}\mathcal{T}(r,y,z,w) &:= \frac{1}{r}w + Aw + rBw + C(y + rz + rw) + \gamma^* M(\gamma w) \\ &\quad\text{for all } r \in \mathbb{R}_+,\ y,z,w \in W.\end{aligned} \tag{5.3}$$

The significance of operator $\mathcal{T}$ in the study of Problem $\mathcal{P}_\tau$ is explained below.

Remark 5.1. It is easy to observe, that Problem $\mathcal{P}_\tau$ is equivalent with finding a sequence $\{w_\tau^k\}_{k=0}^N \subset W$ such that $w_\tau^0 = v_\tau^0$, w_τ^1 satisfies

$$\mathcal{T}(\tau, u_\tau^0, 0, w_\tau^1) \ni F_\tau^k + \frac{1}{\tau}w_\tau^0 - Bu_\tau^0 \tag{5.4}$$

and, for $k = 2,\ldots,N$, w_τ^k satisfies

$$\mathcal{T}\left(\tau, u_\tau^0, \sum_{i=1}^{k-1} w_\tau^i, w_\tau^k\right) \ni F_\tau^k + \frac{1}{\tau}w_\tau^{k-1} - Bu_\tau^0 - \tau\sum_{i=1}^{k-1} Bw_\tau^i. \tag{5.5}$$

The following lemmata provide properties of operator $\mathcal{T}$.

Lemma 5.2. *Let the assumptions $H(g)$, $H(h)$ and $H(j)$ hold. Moreover assume that either $p = 2$ or H_{const} holds. Then, there exists $\tau_0 > 0$ such that for all $0 < \tau < \tau_0$, operator $\mathcal{T}(\tau, y, z, \cdot)\colon W \to 2^{W^*}$ is coercive for all $y, z \in W$.*

Proof. Let $y, z \in W$ be fixed. In the whole proof, we will denote by c a positive function, which may change from line to line and may have a various set of arguments. Suppose that $u \in \mathcal{T}(\tau, y, z, w)$, where

$w \in W$ is given. Then we have, $u = \frac{1}{\tau}w + Aw + \tau Bw + \tau C(y + \tau z + \tau w) + \gamma^* \xi$, where $\xi \in M(\gamma w)$. In order to show the coercivity of $\mathcal{T}$, we calculate

$$\begin{aligned} \langle u, w\rangle_{W^*\times W} = \frac{1}{\tau}\|w\|_H^2 + \langle Aw, w\rangle_{W^*\times W} + \tau \langle Bw, w\rangle_{V^*\times V} \\ + \langle C(y+\tau z+\tau w), w\rangle_{W^*\times W} + \xi\gamma w. \end{aligned} \tag{5.6}$$

Using Lemma 4.5 (i) and Young inequality, we estimate

$$\begin{aligned} |\langle C(y+\tau z+\tau w), w\rangle_{W^*\times W}| &\le \|C(y+\tau z+\tau w)\|_{W^*}\|w\|_W \\ &\le \beta_C \left(1 + \|y+\tau z+\tau w\|_V^{\frac{2}{q}}\right)\|w\|_W \le \varepsilon\|w\|_W^p + c(\varepsilon)(1+\|y+\tau z+\tau w\|_V^2) \\ &\le \varepsilon\|w\|_W^p + c(\varepsilon)\tau^2\|w\|_V^2 + c(\varepsilon, \|y\|_V, \|z\|_V) \end{aligned} \tag{5.7}$$

Moreover by Lemma 4.6 (iii), we have

$$|\xi\gamma w| \le c_M|\gamma w| + c_M|\gamma w|^p \le (c_M+\varepsilon)|\gamma w|^p + c(\varepsilon). \tag{5.8}$$

It is known that $W \subset C(0, L) \subset H$, where the first embedding is compact and the last one is continuous. Thus using the Ehrling lemma (cf. Lemma 7.6 of [21]), we claim, that for all $\varepsilon > 0$

$$|\gamma w| = |w(L)| \le \|w\|_{C(0,L)} \le \varepsilon\|w\|_W + c(\varepsilon)\|w\|_H. \tag{5.9}$$

Now we consider two cases.

Case 1. $p = 2$. Then combining (5.8) and (5.9), we get

$$|\xi\gamma w| \le \varepsilon\|w\|_W^2 + c(\varepsilon)\|w\|_H^2 + c(\varepsilon). \tag{5.10}$$

Thus by Lemma 4.3 (ii), Lemma 4.4, (5.6), (5.7) and (5.10), we have

$$\begin{aligned} \langle u, w\rangle_{W^*\times W} \ge \left(\frac{1}{\tau} - c(\varepsilon)\right)\|w\|_H^2 + (\eta - 2\varepsilon)\|w\|_W^2 \\ + \tau(E - c(\varepsilon)\tau)\|w\|_V^2 + L\,\underline{g} - c(\varepsilon, \|y\|_V, \|z\|_V). \end{aligned}$$

Let us take $\varepsilon = \frac{1}{4}\eta$ and $\tau_0 = \min\{c(\frac{1}{4}\eta)^{-1}, Ec(\frac{1}{4}\eta)^{-1}\}$. Then it follows that operator $\mathcal{T}$ is coercive for $\tau \le \tau_0$.

Case 2. $p > 0$ and H_{const} holds. From Lemma 4.3 (ii), Lemma 4.4, (5.6)–(5.8) we get

$$\begin{aligned} \langle u, w\rangle_{W^*\times W} \ge \frac{1}{\tau}\|w\|_H^2 + (\eta - (c_M+\varepsilon)\|\gamma\|^p - \varepsilon)\|w\|_W^p \\ + \tau(E - c(\varepsilon)\tau)\|w\|_V^2 + L\,\underline{g} - c(\varepsilon, \|y\|_V, \|z\|_V). \end{aligned}$$

We take $\varepsilon = \overline{\varepsilon} := (\eta - c_M\|\gamma\|^p)/(\|\gamma\|^p + 1)$. The assumption H_{const} implies that $\overline{\varepsilon} > 0$. Let us define $\tau_0 = Ec(\overline{\varepsilon})^{-1}$. Now we observe, that $\mathcal{T}$ is coercive for $\tau < \tau_0$. □

Lemma 5.3. *Let the assumptions $H(g)$, $H(h)$ and $H(j)$ hold. Then operator $\mathcal{T}$ is bounded with respect to the last variable.*

Proof. The boundedness of $\mathcal{T}$ follows directly from Lemma 4.3 (i), Lemmas 4.4, 4.5 (i) and 4.6 (iii). □

Lemma 5.4. *Let the assumptions $H(g)$, $H(h)$ and $H(j)$ hold. Then operator $\mathcal{T}$ is pseudomonotone with respect to the last variable.*

Proof. We examine the pseudomonotonicity of each components of $\mathcal{T}$. The operator $W \ni w \to \frac{1}{\tau}w \in W^*$ is pseudomonotone, since it is linear and monotone. The pseudomonotonicity of A is provided by Lemma 4.3 (iii). Operator τB is pseudomonotone, since it is linear and monotone. By Lemma 4.5 (iii), and the continuity of embedding $W \subset V$, we claim, that the operator C is strongly continuous from W to W^* and, in a consequence, it is pseudomonotone. Finally the multivalued term of $\mathcal{T}$ is pseudomonotone due to Lemma 4.8. Thus from Proposition 2.5, it follows, that $\mathcal{T}$ is pseudomonotone. □

Corollary 5.5. *Let the assumptions $H(g)$, $H(h)$ and $H(j)$ hold. Moreover assume that either $p = 2$ or H_{const} holds. Then there exists $\tau_0 > 0$ such that for all $0 < \tau < \tau_0$ the mapping $T(\tau, y, z, \cdot): W \to 2^{W^*}$ is surjective for all $y, z \in W$, i.e., for every $f \in W^*$, there exists $w \in W$ such that*

$$\mathcal{T}(\tau, y, z, w) \ni f.$$

Proof. The proof is a consequence of Lemmas 5.2–5.4 and Theorem 2.6. □

Now we are in a position to formulate an existence result for Problem $\mathcal{P}_\tau$.

Theorem 5.6. *Let the assumptions $H(g)$, $H(h)$, $H(j)$ and H_0 hold. Moreover assume that either $p = 2$ or H_{const} holds. Then there exists $\tau_0 > 0$ such that for all $0 < \tau < \tau_0$ Problem $\mathcal{P}_\tau$ has a solution.*

Proof. We have to provide the existence of a sequence $\{w_\tau^k\}_{k=0}^N$, that is a solution of Problem $\mathcal{P}_\tau$. First, we define $w_\tau^0 = v_\tau^0$. By Corollary 5.5, we know that for $\tau > 0$ small enough, operator $\mathcal{T}$ is surjective with respect to the last variable, and, in a consequence, there exists w_τ^1 that satisfies (5.4). Then we proceed by induction. Suppose that elements w_τ^j, $j = 0, \ldots, k-1$ are already found for a fixed $k = 2, \ldots, N$. Using again surjectivity of $\mathcal{T}$, we deduce that there exists $w_\tau^k \in W$ that satisfies (5.5). Proceeding recursively, we provide existence of the entire sequence $\{w_\tau^k\}_{k=0}^N$. Applying Remark 5.1, we state that it is a solution of Problem $\mathcal{P}_\tau$. □

5.2. A-priori estimates

In this subsection, we provide a priori estimates for the solution of Problem $\mathcal{P}_\tau$.

Let the sequence $\{w_\tau^k\}_{k=0}^N$ be a solution of Problem $\mathcal{P}_\tau$, $\{\xi_\tau^k\}_{k=0}^N$ be a sequence that satisfies $\xi_\tau^k \in M(\gamma w_\tau^k)$ for $k = 1, \ldots, N$ and the sequence $\{u_\tau^k\}_{k=0}^N$ be defined by (5.2). Then we have the following result.

Lemma 5.7. *Let the assumptions $H(g)$, $H(h)$ and $H(j)$ hold. Moreover assume that either $p = 2$ or H_{const} holds. Then the sequences $\{w_\tau^k\}_{k=0}^N$, $\{u_\tau^k\}_{k=0}^N$ and $\{\xi_\tau^k\}_{k=0}^N$ satisfy*

$$\max_{1 \le n \le N} \|w_\tau^n\|_H^2 \le c, \tag{5.11}$$

$$\sum_{k=1}^N \tau \|w_\tau^k\|_W^p \le c, \tag{5.12}$$

$$\sum_{k=1}^N \tau |\xi_\tau^k|^q \le c, \tag{5.13}$$

$$\max_{1 \le n \le N} \|u_\tau^n\|_W^p \le c, \tag{5.14}$$

where the constant c does not depend on τ.

Proof. We take $v = w_\tau^k$ in (5.1) and obtain

$$\begin{aligned} &\left(w_\tau^k - w_\tau^{k-1}, w_\tau^k\right)_H + \tau \left\langle A w_\tau^k, w_\tau^k \right\rangle_{W^* \times W} + \tau \left\langle B u_\tau^k, w_\tau^k \right\rangle_{V^* \times V} \\ &\tau \left\langle C u_\tau^k, w_\tau^k \right\rangle_{W^* \times W} + \tau \xi_\tau^k \gamma w_\tau^k = \tau \left\langle F_\tau^k, w_\tau^k \right\rangle_{W^* \times W}. \end{aligned} \tag{5.15}$$

By a property of scalar product in Hilbert space, we have

$$\left(w_\tau^k - w_\tau^{k-1}, w_\tau^k\right)_H = \frac{1}{2}\|w_\tau^k\|_H^2 - \frac{1}{2}\|w_\tau^{k-1}\|_H^2 + \frac{1}{2}\|w_\tau^k - w_\tau^{k-1}\|_H^2. \tag{5.16}$$

By Lemma 4.3 (ii), we have

$$\tau \left\langle A w_\tau^k, w_\tau^k \right\rangle_{W^* \times W} \ge \tau \eta \|w_\tau^k\|_W^p + \tau L \underline{g}. \tag{5.17}$$

By the properties of operator B (see Lemma 4.4), we get

$$\begin{aligned}\tau \left\langle Bu_\tau^k, w_\tau^k\right\rangle_{V^*\times V} &= \left\langle Bu_\tau^k, u_\tau^k - u_\tau^{k-1}\right\rangle_{V^*\times V} = \frac{1}{2}\left\langle Bu_\tau^k, u_\tau^k\right\rangle_{V^*\times V} \\ &- \frac{1}{2}\left\langle Bu_\tau^{k-1}, u_\tau^{k-1}\right\rangle_{V^*\times V} + \frac{1}{2}\left\langle B(u_\tau^k - u_\tau^{k-1}), u_\tau^k - u_\tau^{k-1}\right\rangle_{V^*\times V} \\ &\geq \frac{1}{2}\left\langle Bu_\tau^k, u_\tau^k\right\rangle_{V^*\times V} - \frac{1}{2}\left\langle Bu_\tau^{k-1}, u_\tau^{k-1}\right\rangle_{V^*\times V} = E\left(\|u_\tau^k\|_V^2 - \|u_\tau^{k-1}\|_V^2\right).\end{aligned} \tag{5.18}$$

By Lemma 4.5 (i), we get

$$\begin{aligned}|\tau \left\langle Cu_\tau^k, w_\tau^k\right\rangle_{W^*\times W}| &\leq \tau\beta_C\left(1 + \|u_\tau^k\|_V^{\frac{2}{q}}\right)\|w_\tau^k\|_W \\ &\leq \tau\varepsilon\|w_\tau^k\|_W^p + \tau c_1(\varepsilon)\|u_\tau^k\|_V^2 + \tau c_2(\varepsilon).\end{aligned} \tag{5.19}$$

Moreover we claim that

$$\tau \left\langle F_\tau^k, w_\tau^k\right\rangle_{W^*\times W} \leq \tau\|F_\tau^k\|_{W^*}\|w_\tau^k\|_W \leq \tau\varepsilon\|w_\tau^k\|_W^p + \tau c_3(\varepsilon)\|F_\tau^k\|_{W^*}^q \tag{5.20}$$

Finally we estimate the term $\xi_\tau^k\gamma w_\tau^k$. To this end, we consider two cases. If $p = 2$, then, analogously to (5.10), we get

$$\tau|\xi_\tau^k\gamma w_\tau^k| \leq \tau\varepsilon\|w_\tau^k\|_W^2 + \tau c_4(\varepsilon)\|w_\tau^k\|_H^2 + \tau c_5(\varepsilon). \tag{5.21}$$

If $p > 2$, we use estimate analogous to (5.8) to obtain

$$\tau|\xi_\tau^k\gamma w_\tau^k| \leq \tau c_M\|\gamma\|^p\|w_\tau^k\|_W^p + \tau\varepsilon\|w_\tau^k\|_W^p + \tau c_6(\varepsilon). \tag{5.22}$$

We sum up the Eq. (5.15) for $k = 1, \ldots, n \leq N$ and use (5.16)–(5.20). Moreover we apply either (5.21) for $p = 2$ or (5.22) for $p > 2$. Thus for $p = 2$, we obtain

$$\begin{aligned}&\frac{1}{2}\|w_\tau^n\|_H^2 + E\|u_\tau^n\|_V^2 + \frac{1}{2}\sum_{k=1}^n\|w_\tau^k - w_\tau^{k-1}\|_H^2 + (\eta - 3\varepsilon)\sum_{k=1}^n\tau\|w_\tau^k\|_W^2 \\ &\leq c_4(\varepsilon)\sum_{k=1}^n\tau\|w_\tau^k\|_H^2 + c_1(\varepsilon)\sum_{k=1}^n\tau\|u_\tau^k\|_V^2 + c_3(\varepsilon)\sum_{k=1}^n\tau\|F_\tau^k\|_{W^*}^2 \\ &\quad + \frac{1}{2}\|w_\tau^0\|_H^2 + TL|\underline{g}| + T\left(c_2(\varepsilon) + c_5(\varepsilon)\right).\end{aligned} \tag{5.23}$$

On the other hand, when $p > 2$, we get

$$\begin{aligned}&\frac{1}{2}\|w_\tau^n\|_H^2 + E\|u_\tau^n\|_V^2 + \frac{1}{2}\sum_{k=1}^n\|w_\tau^k - w_\tau^{k-1}\|_H^2 + (\eta - c_M\|\gamma\|^p - 3\varepsilon)\sum_{k=1}^n\tau\|w_\tau^k\|_W^p \\ &\leq c_1(\varepsilon)\sum_{k=1}^n\tau\|u_\tau^k\|_V^2 + c_3(\varepsilon)\sum_{k=1}^n\tau\|F_\tau^k\|_{W^*}^q + \frac{1}{2}\|w_\tau^0\|_H^2 \\ &\quad + TL|\underline{g}| + T\left(c_2(\varepsilon) + c_6(\varepsilon)\right).\end{aligned} \tag{5.24}$$

If $p = 2$, we apply Lemma 2.9 to (5.23) with $\varepsilon = \frac{1}{6}\eta$. If $p > 2$, we use H_{const} and apply Lemma 2.9 to (5.24) with $\varepsilon = \frac{1}{2}(\eta - c_M\|\gamma\|^p) > 0$. In both cases, we obtain

$$\max_{1\leq n\leq N}\|w_\tau^n\|_H^2 + \max_{1\leq n\leq N}\|u_\tau^n\|_V^2 \leq c\left(1 + \sum_{k=1}^n\tau\|F_\tau^k\|_{W^*}^q\right) \leq c(1 + \|F\|_{\mathcal{W}^*}^q) \tag{5.25}$$

It follows from $H(f)$ that $F \in \mathcal{W}^*$, so the right-hand side of (5.25) remains bounded and we obtain (5.11). Now from (5.25), we conclude that the right-hand side of (5.23), as well as the right-hand side of

(5.24) are bounded, and in a consequence, (5.12) holds for all $p \geq 2$. The estimates (5.13) follows directly from $H(j)(ii)$ and (5.12). Finally using (5.2), we obtain for a fixed $k = 1, \ldots, N$

$$\begin{aligned}
\|u_\tau^k\|_W^p &= \left\| u_\tau^0 + \tau \sum_{i=1}^k w_\tau^i \right\|_W^p \leq 2^{p-1} \left(\|u_\tau^0\|_W^p + \left(\tau \sum_{i=1}^k \|w_\tau^i\|_W \right)^p \right) \\
&\leq 2^{p-1} \left(\|u_\tau^0\|_W^p + k^{p-1} \tau^{p-1} \tau \sum_{i=1}^k \|w_\tau^i\|_W^p \right) \\
&\leq 2^{p-1} \left(\|u_\tau^0\|_W^p + T^{p-1} \tau \sum_{i=1}^k \|w_\tau^i\|_W^p \right).
\end{aligned}$$

The last estimates, together with (5.12), give (5.14). □

5.3. Convergence of the Rothe method

In this subsection, we construct sequences of time-dependent piecewise constant and piecewise linear functions, whose values are determined by the solution of Problem $\mathcal{P}_\tau$. Next we study their convergence to the solution of Problem $\mathcal{P}$.

Let $\{w_\tau^k\}_{k=0}^N$, $\{u_\tau^k\}_{k=0}^N$ and $\{\xi_\tau^k\}_{k=0}^N$ be sequences described in the previous subsection. We define the functions $\overline{w}_\tau, \overline{u}_\tau \colon (0,T] \to W$, $\overline{\xi}_\tau \colon (0,T] \to \mathbb{R}$, $\overline{F}_\tau \colon (0,T] \to W^*$, $w_\tau, u_\tau, \colon [0,T] \to W$ by the formulas

$$\begin{aligned}
&\overline{w}_\tau(t) = w_\tau^k, \ \ \overline{u}_\tau(t) = u_\tau^k, \ \ \overline{\xi}_\tau(t) = \xi_\tau^k, \ \ \overline{F}_\tau(t) = F_\tau^k \quad \text{for } \ t \in ((k-1)\tau, k\tau]\,, \\
&w_\tau(t) = w_\tau^k + \left(\frac{t}{\tau} - k \right) (w_\tau^k - w_\tau^{k-1}) \ \text{ for } \ t \in [(k-1)\tau, k\tau]\,, \\
&u_\tau(t) = u_\tau^k + \left(\frac{t}{\tau} - k \right) (u_\tau^k - u_\tau^{k-1}) \ \text{ for } \ t \in [(k-1)\tau, k\tau]\,, \quad k = 1, \ldots, N.
\end{aligned}$$

Hereafter the convergence of all quantities subscribed with τ will be considered as $\tau \to 0$.

By Lemma 3.3 in [9], we know that

$$\overline{F}_\tau \to F \ \text{ in } \ \mathcal{W}^*. \tag{5.26}$$

We observe that $\overline{w}_\tau$ is the distributional derivative of u_τ, namely $u_{\tau t}(t) = \overline{w}_\tau(t)$ for a.e. $t \in (0,T)$. Moreover the distributional derivative of w_τ is given by $w_{\tau t}(t) = \frac{w_\tau^k - w_\tau^{k-1}}{\tau}$ for all $t \in ((k-1)\tau, k\tau)\,,\ k = 1, \ldots, N$.

We define the Nemytskii operators $\mathcal{A} \colon \mathcal{W} \to \mathcal{W}^*$, $\mathcal{B} \colon \mathcal{V} \to \mathcal{V}^*$, $\mathcal{C} \colon \mathcal{V} \to \mathcal{W}^*$, and $\overline{\iota} \colon \mathcal{W} \to \mathcal{U}$ given by $(\mathcal{A}v)(t) = A(v(t))$ for $v \in \mathcal{W}$, $(\mathcal{B}v)(t) = B(v(t))$, $(\mathcal{C}v)(t) = C(v(t))$ for $v \in \mathcal{V}$ and $(\overline{\gamma}v)(t) = \gamma v(t)$ for $v \in \mathcal{W}$. Since $\{w_\tau^k\}_{k=0}^N$ solves Problem $\mathcal{P}_\tau$, it follows that

$$\begin{aligned}
&(w_{\tau t}, v)_{\mathcal{H}} + \langle \mathcal{A}\overline{w}_\tau, v \rangle_{\mathcal{W}^* \times \mathcal{W}} + \langle \mathcal{B}\overline{u}_\tau, v \rangle_{\mathcal{W}^* \times \mathcal{W}} + \langle \mathcal{C}\overline{u}_\tau, v \rangle_{\mathcal{W}^* \times \mathcal{W}} \\
&\quad + \left\langle \overline{\gamma}^* \overline{\xi}_\tau, v \right\rangle_{\mathcal{W}^* \times \mathcal{W}} = \left\langle \overline{F}_\tau, v \right\rangle_{\mathcal{W}^* \times \mathcal{W}} \quad \text{for all } \ v \in \mathcal{W},
\end{aligned} \tag{5.27}$$

$$\overline{\xi}_\tau(t) \in M((\overline{\gamma}\, \overline{w}_\tau)(t)) \quad \text{for a.e. } \ t \in (0,T), \tag{5.28}$$

where $\overline{\gamma}^* \colon \mathcal{U}^* \to \mathcal{W}^*$ denotes the operator adjoint to $\overline{\gamma}$. Our goal is to obtain a solution of the original Problem $\mathcal{P}$, by passing to the limit in (5.27) and (5.28), as $\tau \to 0$. In the first step, we provide appropriate bounds for functions $w_\tau, \overline{w}_\tau, u_\tau, \overline{u}_\tau$ and $\overline{\xi}_\tau$.

Lemma 5.8. *Let the assumptions of Lemma 5.7 hold. Then there exists a positive constant c, that does not depend on τ, such that*

$$\|\overline{w}_\tau\|_{L^\infty(0,T;H)} \le c, \tag{5.29}$$
$$\|w_\tau\|_{C(0,T;H)} \le c, \tag{5.30}$$
$$\|\overline{w}_\tau\|_{\mathcal{W}} \le c, \tag{5.31}$$
$$\|w_\tau\|_{\mathcal{W}} \le c, \tag{5.32}$$
$$\|\overline{u}_\tau\|_{L^\infty(0,T;W)} \le c, \tag{5.33}$$
$$\|u_\tau\|_{L^\infty(0,T;W)} \le c, \tag{5.34}$$
$$\|\overline{\xi}_\tau\|_{\mathcal{U}^*} \le c, \tag{5.35}$$
$$\|w_{\tau t}\|_{\mathcal{W}^*} \le c, \tag{5.36}$$
$$\|\overline{w}_\tau\|_{M^{p,q}(0,T;W,W^*)} \le c. \tag{5.37}$$

Proof. The estimates (5.29) and (5.30) follow directly from (5.11). Moreover $\|\overline{w}_\tau\|^p_{\mathcal{W}} = \tau\sum_{k=1}^N \|w^k_\tau\|^p_W$, so from (5.12), we obtain (5.31). The simple calculation shows that $\|w_\tau\|^p_{\mathcal{W}} \le \tau\sum_{k=0}^N \|w^k_\tau\|^p_W$. Thus using (5.12) and the fact that $\|w^0_\tau\| \le C/\sqrt{\tau}$, we get (5.32). The estimates (5.33) and (5.34) follows from (5.14). In order to show (5.35), it is enough to observe, that $\|\overline{\xi}_\tau\|^q_{\mathcal{U}^*} = \tau\sum_{k=1}^N |\xi^k_\tau|^q$ and apply (5.13).
Now we pass to the proof of (5.36). To this end, we calculate from (5.27)

$$\begin{aligned}\|w_{\tau t}\|_{\mathcal{W}^*} &= \sup_{\substack{v\,\in\,\mathcal{W}\\ \|v\|_{\mathcal{W}}\,=\,1}} |\langle w_{\tau t}, w\rangle_{W^*\times W}| = \sup_{\substack{v\,\in\,\mathcal{W}\\ \|v\|_{\mathcal{W}}\,=\,1}} |(w_{\tau t}, v)_{\mathcal{H}}| \\ &\le \|\mathcal{A}\overline{w}_\tau\|_{\mathcal{W}^*} + \|\mathcal{B}\overline{u}_\tau\|_{\mathcal{W}^*} + \|\mathcal{C}\overline{u}_\tau\|_{\mathcal{W}^*} + \|\overline{\gamma}^*\overline{\xi}_\tau\|_{\mathcal{W}^*} + \|\overline{F}_\tau\|_{\mathcal{W}^*}\end{aligned} \tag{5.38}$$

Applying Lemma 4.3 (i), Lemma 4.4 and Lemma 4.5 (i) and using (5.31), (5.33) and (5.35), we estimate

$$\|\mathcal{A}\overline{w}_\tau\|_{\mathcal{W}^*} \le c_1(1 + \|\overline{w}_\tau\|^{p-1}_{\mathcal{W}}) \le c, \tag{5.39}$$

$$\begin{aligned}\|\mathcal{B}\overline{u}_\tau\|^2_{\mathcal{W}^*} &\le c_1\|\mathcal{B}\overline{u}_\tau\|^2_{\mathcal{V}^*} = c_1\int_0^T \|B\overline{u}_\tau(t)\|^2_{V^*}\,\mathrm{d}t \\ &\le c_1 E\int_0^T \|\overline{u}_\tau(t)\|^2_V\,\mathrm{d}t = c_1 E\|\overline{u}_\tau\|^2_{\mathcal{V}} \le c_2\|\overline{u}_\tau\|_{L^\infty(0,T;W)} \le c,\end{aligned} \tag{5.40}$$

$$\begin{aligned}\|\mathcal{C}\overline{u}_\tau\|^q_{\mathcal{W}^*} &= \int_0^T \|C\overline{u}_\tau(t)\|^q_{W^*}\,\mathrm{d}t \le c_1\int_0^T (1 + \|\overline{u}_\tau(t)\|^2_V)\,\mathrm{d}t \\ &\le c_2(1 + \|\overline{u}_\tau\|^2_{\mathcal{V}}) \le c_3(1 + \|\overline{u}_\tau\|_{L^\infty(0,T;W)}) \le c,\end{aligned} \tag{5.41}$$

$$\|\overline{\gamma}^*\overline{\xi}_\tau\|_{\mathcal{W}^*} \le \|\overline{\gamma}^*\|_{\mathcal{L}(\mathcal{U}^*,\mathcal{W}^*)}\|\overline{\xi}_\tau\|_{\mathcal{U}^*} \le c, \tag{5.42}$$

where the constants c_1, c_2, c_3, c may vary from line to line. Moreover from (5.26), it follows that $\overline{F}_\tau$ is bounded in $\mathcal{W}^*$. Thus applying (5.39)–(5.42) to (5.38), we obtain (5.36).

Next we pass to the proof of (5.37). Taking into account (5.31), it is enough to estimate the seminorm $\|\overline{w}_\tau\|_{BV^q(0,T;W^*)}$. Since the function $\overline{w}_\tau$ is piecewise constant, the seminorm will be measured by means of jumps between elements of sequence $\{w^k_\tau\}^N_{k=1}$. Namely, let $\{k_j\}^M_{j=1} \subset \{1,\dots,N\}$ be an increasing sequence of numbers such that $k_1 = 1$, $k_M = N$ and

$$\|\overline{w}_\tau\|^q_{BV^q(0,T;W^*)} = \sum_{j=1}^{M-1} \|w^{k_{j+1}}_\tau - w^{k_j}_\tau\|^q_{W^*}. \tag{5.43}$$

For a fixed $j = 1, \ldots, M-1$, we obtain

$$\begin{aligned} &\left\|w_\tau^{k_{j+1}} - w_\tau^{k_j}\right\|_{W^*}^q \\ &\quad = \left\|w_\tau^{k_{j+1}} - w_\tau^{k_{j+1}-1} + w_\tau^{k_{j+1}-1} - w_\tau^{k_{j+1}-2} + \cdots + w_\tau^{k_j+1} - w_\tau^{k_j}\right\|_{W^*}^q \\ &\quad \leq (k_{j+1} - k_j)^{q-1} \sum_{l=k_j}^{k_{j+1}-1} \left\|w_\tau^{l+1} - w_\tau^l\right\|_{W^*}^q \leq N^{q-1} \sum_{l=k_j}^{k_{j+1}-1} \left\|w_\tau^{l+1} - w_\tau^l\right\|_{W^*}^q. \end{aligned} \tag{5.44}$$

Combining (5.43) with (5.44), we get

$$\begin{aligned} \|\overline{w}_\tau\|_{BV^q(0,T;W^*)}^q &\leq N^{q-1} \sum_{j=1}^{M-1} \left(\sum_{l=k_j}^{k_{j+1}-1} \left\|w_\tau^{l+1} - w_\tau^l\right\|_{W^*}^q \right) \\ &= N^{q-1} \sum_{l=1}^{N-1} \|w_\tau^{l+1} - w_\tau^l\|_{W^*}^q = N^{q-1}\tau^{q-1}\tau \sum_{l=1}^{N-1} \left\| \frac{w_\tau^{l+1} - w_\tau^l}{\tau} \right\|_{W^*}^q \\ &= T^{q-1} \|w_{\tau t}\|_{\mathcal{W}^*}^q. \end{aligned}$$

Thus it follows from (5.36) that $\|\overline{w}_\tau\|_{BV^q(0,T;W^*)} \leq c$. Combining this with (5.31), we get (5.37), which completes the proof of the lemma. □

The next lemmata deal with the properties of Nemytskii operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\overline{\gamma}$.

Lemma 5.9. *If the assumptions $H(g)$ hold, then the operator $\mathcal{A}$ satisfies:*
$H(\mathcal{A})$: if a sequence $\{w_n\}_{n=1}^\infty \subset W$ is bounded in $M^{p,q}(0,T;W,W^*)$, $w_n \to w$ weakly in $\mathcal{W}$ and $\limsup_{n\to\infty} \langle \mathcal{A}w_n, w_n - w\rangle_{\mathcal{W}^* \times \mathcal{W}} \leq 0$, then $\mathcal{A}w_n \to \mathcal{A}w$ weakly in $\mathcal{W}^*$.

The proof of Lemma 5.9 follows the lines of the proof of Lemma 1 in [15] and exploits Lemma 4.3.
We remark that every operator, which satisfies $H(\mathcal{A})$, is said to be pseudomonotone with respect to the space $M^{p,q}(0,T;W,W^*)$.

Lemma 5.10. *The operator $\mathcal{B}$ is linear, bounded, symmetric and strongly positive.*

Lemma 5.10 follows directly from Lemma 4.4.

Lemma 5.11. *If the assumption $H(h)$ holds, then the operator $\mathcal{C}$ satisfies:*
$H(\mathcal{C})$: if a sequence $\{v_n\}$ is bounded in $L^\infty(0,T;V)$ and $v_n \to v$ in $L^1(0,T;H)$, then $\mathcal{C}v_n \to \mathcal{C}v$ in $\mathcal{W}^*$.

Proof. Let the sequence $\{v_n\}$ be bounded in $L^\infty(0,T;V)$ and let $v_n \to v$ in $L^1(0,T;H)$. It follows from Lemma 4.5 (ii) that

$$\|\mathcal{C}v_n - \mathcal{C}v\|_{\mathcal{W}^*}^q \leq \int_0^T \overline{C}^q(\max\{\|v_n(t)\|_V, \|v(t)\|_V\})\|v_n(t) - v(t)\|_H. \tag{5.45}$$

Since the function $\overline{C}$ is nondecreasing, and $\{v_n\}$ is bounded in $L^\infty(0,T;V)$, we conclude that for a.e. $t \in (0,T)$

$$\overline{C}(\max\{\|v_n(t)\|_V, \|v(t)\|_V\}) \leq \overline{C}(\max\{\|v_n\|_{L^\infty(0,T;V)}, \|v\|_{L^\infty(0,T;V)}\}) \leq c.$$

Combining it with (5.45), we have $\|\mathcal{C}v_n - \mathcal{C}v\|_{\mathcal{W}^*}^q \leq c\|v_n - v\|_{L^1(0,T;H)}$. Since $v_n \to v$ in $L^1(0,T;H)$, we found that $\|\mathcal{C}v_n - \mathcal{C}v\|_{\mathcal{W}^*} \to 0$, which completes the proof. □

Lemma 5.12. *The operator $\overline{\gamma}$ satisfies the following condition:*
$H(\overline{\gamma})$: if a sequence $\{v_n\}$ is bounded in $M^{p,q}(0,T;W,W^*)$, then, for a subsequence (still denoted by v_n), we have $\overline{\gamma}v_n \to \overline{\gamma}v$ in $\mathcal{U}$.

Proof. Let the sequence $\{v_n\}$ be bounded in $M^{p,q}(0,T;W,W^*)$. We recall that $W\subset C(0,L)\subset W^*$, where the first embedding is compact and the second one is continuous. Thus applying Proposition 2.7, we claim that for a subsequence (still denoted by v_n) we have

$$v_n \to v \ \text{ in } \ L^p(0,T;C(0,L)). \tag{5.46}$$

Thus we calculate

$$\begin{aligned}\|\overline{\gamma}v_n-\overline{\gamma}v\|_{\mathcal{U}}^p &= \int_0^T |\gamma v_n(t)-\gamma v(t)|^p\,\mathrm{d}t = \int_0^T |v_n(L,t)-v(L,t)|^p\,\mathrm{d}t\\ &\le \int_0^T \|v_n(t)-v(t)\|_{C(0,L)}^p\,\mathrm{d}t \le c\|v_n-v\|_{L^p(0,T;C(0,L))}^p.\end{aligned} \tag{5.47}$$

Combining (5.46) with (5.47), we get $\overline{\gamma}v_n\to\overline{\gamma}v$ in $\mathcal{U}$, which completes the proof. □

Now we are in a position to formulate the main theorem that guaranties the existence of solution of Problem $\mathcal{P}$.

Theorem 5.13. *Let $H(g)$, $H(h)$, $H(j)$ and H_0 hold. Moreover assume that either $p=2$ or H_{const} holds. Then there exists a solution of Problem $\mathcal{P}$.*

Proof. Let the assumptions of the theorem hold. Then for $\tau>0$ small enough, we can apply Theorem 5.6 to obtain the solution of Problem $\mathcal{P}_\tau$. Furthermore we construct the functions $w_\tau,\overline{w}_\tau,u_\tau,\overline{u}_\tau$ and $\overline{\xi}_\tau$ that have been introduced at the beginning of this subsection. By Lemma 5.8, and the reflexivity of spaces $\mathcal{W}$ and $\mathcal{U}$, for subsequences still subscribed with τ, we have the following convergence results

$$\overline{w}_\tau\to\overline{w} \quad \text{weakly} * \ \text{ in } \ L^\infty(0,T;H) \ \text{ and weakly in } \ \mathcal{W}, \tag{5.48}$$

$$w_\tau\to w \quad \text{weakly} * \ \text{ in } \ L^\infty(0,T;H) \ \text{ and weakly in } \ \mathcal{W}, \tag{5.49}$$

$$w_{\tau t}\to w_t \quad \text{weakly in } \ \mathcal{W}^*, \tag{5.50}$$

$$\overline{u}_\tau\to\overline{u} \quad \text{weakly} * \ \text{ in } \ L^\infty(0,T;W), \tag{5.51}$$

$$u_\tau\to u \quad \text{weakly} * \ \text{ in } \ L^\infty(0,T;W), \tag{5.52}$$

$$\overline{\xi}_\tau\to\overline{\xi} \quad \text{weakly in } \ \mathcal{U}^*, \tag{5.53}$$

where $\overline{w},w,\overline{u},u\in\mathcal{W}$ and $\overline{\xi}\in\mathcal{U}$. Note that symbol w_t in (5.50) denotes the distributional derivative of the function w from (5.49). In what follows we will show that the limits obtained in (5.48) and (5.49) coincide, and so do the limits obtained in (5.51) and (5.52). To this end, we calculate

$$\|\overline{w}_\tau-w_\tau\|_{\mathcal{W}^*}^p = \sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} (k\tau-t)^p \left\|\frac{w_\tau^k-w_\tau^{k-1}}{\tau}\right\|_{W^*}^p = \frac{\tau^p}{p+1}\|w_{\tau t}\|_{\mathcal{W}^*}^p.$$

Recalling (5.36), we find that $\overline{w}_\tau-w_\tau\to 0$ in $\mathcal{W}^*$. From (5.48) and (5.49), we have $\overline{w}_\tau-w_\tau\to\overline{w}-w$ weakly in $\mathcal{W}$. Since the embedding $\mathcal{W}\subset\mathcal{W}^*$ is continuous, we also have $\overline{w}_\tau-w_\tau\to\overline{w}-w$ weakly in $\mathcal{W}^*$. From uniqueness of weak limit, we have $\overline{w}=w$. Analogously, we calculate

$$\|\overline{u}_\tau-u_\tau\|_{\mathcal{W}}^p = \frac{\tau^p}{p+1}\|u_{\tau t}\|_{\mathcal{W}}^p = \frac{\tau^p}{p+1}\|\overline{w}_\tau\|_{\mathcal{W}}^p. \tag{5.54}$$

Thus it follows from (5.31), (5.51), (5.52) and the uniqueness of weak limit that $\overline{u}=u$. From (5.52), and from the fact that $u_{\tau t}=\overline{w}_\tau\to w$ weakly in $\mathcal{W}$, we have

$$w=u_t. \tag{5.55}$$

Our goal is to pass to the limit in (5.27) and (5.28), as $\tau \to 0$. From (5.50), we have

$$(w_{\tau t}, v)_{\mathcal{H}} = \langle w_{\tau t}, v \rangle_{\mathcal{W}^* \times \mathcal{W}} \to \langle w_t, v \rangle_{\mathcal{W}^* \times \mathcal{W}} = (w_t, v)_{\mathcal{H}}. \tag{5.56}$$

From $H(\mathcal{B})$, it is clear that $\mathcal{B}$ is linear and continuous and thus also weakly continuous. Therefore since $\overline{u}_\tau \to u$ weakly in $\mathcal{W}$, we get $\mathcal{B}\overline{u}_\tau \to \mathcal{B}u$ weakly in $\mathcal{W}^*$. Thus we have

$$\langle \mathcal{B}\overline{u}_\tau, v \rangle_{\mathcal{W}^* \times \mathcal{W}} \to \langle \mathcal{B}u, v \rangle_{\mathcal{W}^* \times \mathcal{W}}. \tag{5.57}$$

From (5.49) and (5.52), we have $u_\tau \to u$ weakly in $\mathcal{W}$ and $u_{\tau t} \to u_t$ weakly in $\mathcal{W}^*$. Thus by Lions–Aubin compactness theorem, it follows that $u_\tau \to u$ in $L^p(0,T;H)$. On the other hand, from (5.54), we get $\overline{u}_\tau \to u_\tau$ in $\mathcal{W}$ and, in a consequence, we find that $\overline{u}_\tau \to u$ in $L^p(0,T;H)$. Moreover (5.33) implies that $\overline{u}_\tau$ is bounded in $L^\infty(0,T;V)$. Thus it follows from Lemma 5.11 that

$$C\overline{u}_\tau \to Cu \text{ in } \mathcal{W}^*, \tag{5.58}$$

and, in particular, $C\overline{u}_\tau \to Cu$ weakly in $\mathcal{W}^*$, i.e., we get

$$\langle \mathcal{C}\overline{u}_\tau, v \rangle_{\mathcal{W}^* \times \mathcal{W}} \to \langle \mathcal{C}\overline{u}, v \rangle_{\mathcal{W}^* \times \mathcal{W}}. \tag{5.59}$$

From (5.53) we easily obtain

$$\left\langle \overline{\gamma}^* \overline{\xi}_\tau, v \right\rangle_{\mathcal{W}^* \times \mathcal{W}} = \left\langle \overline{\xi}_\tau, \overline{\gamma} v \right\rangle_{\mathcal{U}^* \times \mathcal{U}} \to \left\langle \overline{\xi}, \overline{\gamma} v \right\rangle_{\mathcal{U}^* \times \mathcal{U}}. \tag{5.60}$$

Moreover (5.26) implies that $\overline{F}_\tau \to F$ weakly in $\mathcal{W}^*$, i.e., we have

$$\left\langle \overline{F}_\tau, v \right\rangle_{\mathcal{W}^* \times \mathcal{W}} \to \left\langle \overline{F}, v \right\rangle_{\mathcal{W}^* \times \mathcal{W}}. \tag{5.61}$$

It remains to show that

$$\langle \mathcal{A}\overline{w}_\tau, v \rangle_{\mathcal{W}^* \times \mathcal{W}} \to \langle \mathcal{A}w, v \rangle_{\mathcal{W}^* \times \mathcal{W}}. \tag{5.62}$$

Its enough to show that convergence (5.62) holds for a subsequence. From (5.27), it follows that

$$\begin{aligned}
&\limsup_{\tau\to 0} \langle \mathcal{A}\overline{w}_\tau, \overline{w}_\tau - w \rangle_{\mathcal{W}^* \times \mathcal{W}} \le \limsup_{\tau\to 0} \langle F_\tau, \overline{w}_\tau - w \rangle_{\mathcal{W}^* \times \mathcal{W}} \\
&\quad - \liminf_{\tau\to 0} (w_{\tau t}, \overline{w}_\tau - w)_{\mathcal{H}} - \liminf_{\tau\to 0} \langle \mathcal{B}\overline{u}_\tau, \overline{w}_\tau - w \rangle_{\mathcal{W}^* \times \mathcal{W}} \\
&\quad - \liminf_{\tau\to 0} \langle \mathcal{C}\overline{u}_\tau, \overline{w}_\tau - w \rangle_{\mathcal{W}^* \times \mathcal{W}} - \liminf_{\tau\to 0} \left\langle \overline{\xi}_\tau, \overline{\gamma}\, \overline{w}_\tau - \overline{\gamma} w \right\rangle_{\mathcal{U}^* \times \mathcal{U}}.
\end{aligned} \tag{5.63}$$

From (5.48) and (5.26), we have $\lim_{\tau\to 0} \langle F_\tau, \overline{w}_\tau - w \rangle_{\mathcal{W}^* \times \mathcal{W}} = 0$. Next we observe that

$$\begin{aligned}
&(w_{\tau t}, \overline{w}_\tau - w)_{\mathcal{H}} = (w_{\tau t} - w_t, w_\tau - w)_{\mathcal{H}} + \langle w_{\tau t}, \overline{w}_\tau - w_\tau \rangle_{\mathcal{W}^* \times \mathcal{W}} \\
&\quad + \langle w_t, w_\tau - w \rangle_{\mathcal{W}^* \times \mathcal{W}} = \frac{1}{2} \left(\| w_\tau(T) - w(T) \|_H^2 - \| w_\tau(0) - w(0) \|_H^2 \right) \\
&\quad + \langle w_{\tau t}, \overline{w}_\tau - w_\tau \rangle_{\mathcal{W}^* \times \mathcal{W}} + \langle w_t, w_\tau - w \rangle_{\mathcal{W}^* \times \mathcal{W}}.
\end{aligned} \tag{5.64}$$

Taking into account (5.49), (5.50) and the fact that the embedding $\{w \in \mathcal{W} \,| w' \in \mathcal{W}^*\} \subset C(0,T;H)$ is continuous, we claim that $w_\tau \to w$ weakly in $C(0,T;H)$. Thus, in particular, we have $w_\tau(0) \to w(0)$ weakly in H. On the other hand, by the assumptions, $w_\tau(0) = v_\tau^0 \to v_0$ strongly, and in consequence, also weakly in H. Thus from the uniqueness of the weak limit, we have

$$w(0) = v_0 \text{ and } w_\tau(0) \to w(0) \text{ strongly in } H. \tag{5.65}$$

A direct calculation shows that $\langle w_{\tau t}, \overline{w}_\tau - w_\tau \rangle_{\mathcal{W}^* \times \mathcal{W}} \ge 0$. Thus from (5.49), (5.64) and (5.65) we have $\liminf_{\tau\to 0} (w_{\tau t}, \overline{w}_\tau - w)_{\mathcal{H}} \ge 0$. We recall that $\overline{w}_\tau = u_{\tau t}$ and $w = u_t$ and calculate

$$\begin{aligned}
&\langle \mathcal{B}\overline{u}_\tau, \overline{w}_\tau - w \rangle_{\mathcal{W}^* \times \mathcal{W}} = \langle \mathcal{B}\overline{u}_\tau, u_{\tau t} - u_t \rangle_{\mathcal{W}^* \times \mathcal{W}} = \langle \mathcal{B}u_\tau - \mathcal{B}u, u_{\tau t} - u_t \rangle_{\mathcal{W}^* \times \mathcal{W}} \\
&\quad + \langle \mathcal{B}u, \overline{w}_\tau - w \rangle_{\mathcal{V}^* \times \mathcal{V}} + \langle \mathcal{B}\overline{u}_\tau - \mathcal{B}u_\tau, \overline{w}_\tau - w \rangle_{\mathcal{V}^* \times \mathcal{V}}.
\end{aligned} \tag{5.66}$$

From Lemma 4.4, it follows that

$$\langle \mathcal{B}u_\tau - \mathcal{B}u, u_{\tau t} - u_t\rangle_{\mathcal{W}^*\times\mathcal{W}} = \frac{1}{2}\langle B(u_\tau(T) - u(T)), u_\tau(T) - u(T)\rangle_{W^*\times W}$$
$$-\frac{1}{2}\langle B(u_\tau(0) - u(0)), u_\tau(0) - u(0)\rangle_{W^*\times W} \geq -\frac{1}{2}E\|u_\tau(0) - u(0)\|_V^2. \tag{5.67}$$

From (5.48), (5.52) and from the fact that $\overline{w} = w$, it follows that $u_\tau \to u$ weakly in $\{w \in \mathcal{W} \mid w_t \in \mathcal{W}\}$. Since the latter is continuously embedded in $C(0,T;W)$, we have $u_\tau \to u$ weakly in $C(0,T;W)$, and in particular $u_\tau(0) \to u(0)$ weakly in W. In a consequence, we also get $u_\tau(0) \to u(0)$ weakly in V. By assumption, we have $u_\tau(0) = u_\tau^0 \to u_0$ strongly and, in a consequence, also weakly in V. Since the weak limit is unique, we claim that

$$u(0) = u_0 \ \text{ and } \ u_\tau(0) \to u(0) \ \text{ in } \ V. \tag{5.68}$$

Since $\overline{w}_\tau \to \overline{w}$ weakly in $\mathcal{V}$, $\overline{u}_\tau \to u_\tau$ in $\mathcal{V}$ (see (5.54)) and $\mathcal{B}$ is continuous, we get

$$\lim_{\tau\to 0}\langle \mathcal{B}u, \overline{w}_\tau - w\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle \mathcal{B}\overline{u}_\tau - \mathcal{B}u_\tau, \overline{w}_\tau - w\rangle_{\mathcal{V}^*\times\mathcal{V}} = 0. \tag{5.69}$$

From (5.66) to (5.69), we see that $\liminf_{\tau\to 0}\langle \mathcal{B}\overline{u}_\tau, \overline{w}_\tau - w\rangle_{\mathcal{W}^*\times\mathcal{W}} \geq 0$. From (5.58) and (5.48), we get $\lim_{\tau\to 0}\langle \mathcal{C}\overline{u}_\tau, \overline{w}_\tau - w\rangle_{\mathcal{W}^*\times\mathcal{W}} = 0$. Finally, from (5.37) and Lemma 5.12, we find that for a subsequence, still denoted by $\overline{w}_\tau$, we have $\overline{\gamma}\,\overline{w}_\tau \to \zeta$ in $\mathcal{U}$, where $\zeta \in \mathcal{U}$. Since $\overline{\gamma}$ is linear and continuous, it is also weakly continuous. Thus from (5.48), we have $\overline{\gamma}\,\overline{w}_\tau \to \overline{\gamma}w$ weakly in $\mathcal{U}$. By the uniqueness of weak limit, we have $\zeta = \overline{\gamma}w$ and

$$\overline{\gamma}\,\overline{w}_\tau \to \overline{\gamma}w \ \text{ in } \mathcal{U}. \tag{5.70}$$

Combining it with (5.53), we obtain $\lim_{\tau\to 0}\left\langle \overline{\xi}_\tau, \overline{\gamma}\,\overline{w}_\tau - \overline{\gamma}w\right\rangle_{\mathcal{U}^*\times\mathcal{U}} = 0$. Summarizing, it follows from (5.63) that $\limsup_{\tau\to 0}\langle \mathcal{A}\overline{w}_\tau, \overline{w}_\tau - w\rangle_{\mathcal{W}^*\times\mathcal{W}} \leq 0$. Combining this with (5.37) and (5.48), we can use Lemma 5.9, to obtain (5.62). Now applying (5.56), (5.57), (5.59)–(5.62), we pass to the limit in (5.27) and obtain

$$(w_t, v)_{\mathcal{H}} + \left\langle \mathcal{A}w + \mathcal{B}u + \mathcal{C}u + \overline{\gamma}^*\overline{\xi} - f, v\right\rangle_{\mathcal{W}^*\times\mathcal{W}} = 0 \ \text{ for all } \ v \in \mathcal{W}. \tag{5.71}$$

Now we pass to the limit in (5.28). First we recall that the multifunction $M\colon \mathbb{R} \to 2^{\mathbb{R}}$ has nonempty, closed and convex values. Furthermore by Proposition 5.6.10 of [12], it is also upper semicontinuous from $\mathbb{R}$ furnished with the strong topology into $\mathbb{R}$ furnished with the weak topology (in fact, the strong and the weak topology on $\mathbb{R}$ coincide). By (5.70), it follows that for a subsequence, still denoted by $\overline{w}_\tau$, we have $\overline{\gamma}\,\overline{w}_\tau(t) \to \overline{\gamma}w(t)$ for a.e. $t \in (0,T)$. This, together with (5.53), allows to apply Proposition 2.8 to (5.28) and obtain

$$\overline{\xi}(t) \in M((\overline{\gamma}\,w)(t)) \ \text{ for a.e. } \ t \in (0,T). \tag{5.72}$$

Taking into account (5.55), (5.65), (5.68), (5.71) and (5.72), we conclude that u is a solution of Problem $\mathcal{P}$, which concludes the proof. □

Finally, using Remark 4.2 we deduce that u is a weak solution of Problem P, which completes the proof of Theorem 4.9.

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