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On the existence of dynamics in Wheeler–Feynman electromagnetism

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Abstract. Wheeler–Feynman electrodynamics (WF) is an action-at-a-distance theory about world-lines of charges that in contrary to the textbook formulation of classical electrodynamics is free of ultraviolet singularities and is capable of explaining the irreversible nature of radiation. In WF, the world-lines of charges obey the so-called Fokker–Schwarzschild–Tetrode (FST) equations, a coupled set of nonlinear and neutral differential equations that involve time-like advanced as well as retarded arguments of unbounded delay. Using a reformulation of this theory in terms of Maxwell–Lorentz electrodynamics without self-interaction that we have introduced in a preceding work, we are able to establish the existence of conditional solutions. These conditional solutions solve the FST equations on any finite time interval with prescribed continuations outside of this interval. As a byproduct, we also prove existence and uniqueness of solutions to the Synge equations on the time half-line for a given history of charge world-lines.

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Contents

1. Introduction 1087
2. Our setup and results 1090
3. Preliminaries 1097
3.1. Strong solutions to the Maxwell equations 1097
3.2. The ML-SI$_\varrho$ time evolution 1100
4. Proofs 1104
4.1. Weak uniqueness of WF$_\varrho$ and Synge solutions by ML-SI$_\varrho$ Cauchy data 1104
4.2. Existence and uniqueness of Synge solutions for given histories 1105
4.3. Existence of WF$_\varrho$ solutions on finite time intervals 1106
Acknowledgments 1122
References 1122

1. Introduction

Wheeler–Feynman electrodynamics (WF) describes the classical electromagnetic interaction of a number of N charges by *action-at-a-distance* [28]. The nature of the action-at-a-distance is such that two charges interact with each other's if and only if they are in each others light-cone. Hence, the force acting on one charge at a certain time instant depends on the respective future and past of all other charges. In contrast to Maxwell–Lorentz electrodynamics, the theory contains no fields and is free from ultraviolet divergences originating from ill-defined self-fields. Electrodynamics without fields was considered as early

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as 1845 by Gauss [14] and continued to be of interest, for example, [13,20,26], where the fundamental equations of WF, the so-called Fokker–Schwarzschild–Tetrode (FST) equations, were already discussed. The connection to physical phenomena was then made by Wheeler and Feynman [27,28] who showed that this alternative formulation of classical electrodynamics leads to a satisfactory description of radiation damping: Accelerated charges are supposed to radiate and to lose energy thereby. How can this be accounted for in a theory without fields? To answer this question, Wheeler and Feynman introduced the so-called *absorber condition*, which needs to be satisfied by the world-lines of all charges, and they argue that it is satisfied on thermodynamic scales. Under this absorber condition, it is straightforwardly seen that the net force acting on any selected charge can effectively be described by the sum of forces arising from the respective past of all other charges and the same radiation friction term that appears in Dirac's mass renormalization procedure [7]; see our short discussion in [4]. The advantage in Wheeler and Feynman's derivation of the radiation friction term is that it involves no divergences in the defining equations which in the case of Dirac's formal derivation provoke unphysical, so-called *run-away*, solutions. At the same time, Wheeler and Feynman's argument is able to explain the irreversible nature of radiation phenomena. These features make WF the most promising candidate for arriving at a mathematically well-defined theory of relativistic, classical electromagnetism.

However, mathematically, WF is completely opaque. It is not an initial value problem for differential equations because its fundamental equations of motion, the FST equations, contain time-like advanced and retarded state-dependent arguments for which no theory of existence or uniqueness of solutions is available. Apart from two exceptions discussed below, it is not even known whether in general there are solutions at all. In tensor notation, WF is given by the FST equations:

$$m\ddot{z}_i^\mu(\tau) = e_i \sum_{\substack{k=1,\ldots,N \\ k\neq i}} \frac{1}{2}\left[F[z_k]_+^{\mu\nu}(z_i(\tau)) + F[z_k]_-^{\mu\nu}(z_i(\tau))\right]\dot{z}_{i,\nu}(\tau), \qquad 1 \le i \le N, \tag{1}$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \qquad A[z_k]_\pm^\mu(x) := e_k \frac{\dot{z}_k^\mu(\tau_{k,\pm}(x))}{(x - z_k(\tau_{k,\pm}(x)))_\nu\, \dot{z}_k^\nu(\tau_{i,\pm}(x))}, \tag{2}$$

and the world-line parameters $\tau_{k,+}, \tau_{k,-} : \mathbb{M} \to \mathbb{R}$ are implicitly defined through

$$z_k^0(\tau_{k,+}(x)) = x^0 + \|\mathbf{x} - \mathbf{z}_k(\tau_{k,+}(x))\|, \qquad z_k^0(\tau_{k,-}(x)) = x^0 - \|\mathbf{x} - \mathbf{z}_k(\tau_{k,-}(x))\|. \tag{3}$$

Here, the world-lines of the charges $z_i : \tau \mapsto z_i^\mu(\tau)$ for $1 \le i \le N$ are parametrized by proper time $\tau \in \mathbb{R}$ and take values in Minkowski space $\mathbb{M} := (\mathbb{R}^4, g)$ equipped with the metric tensor $g = \mathrm{diag}(1, -1, -1, -1)$. We use Einstein's summation convention for Greek indices, that is, $x_\mu y^\mu := \sum_{\mu=0}^3 g_{\mu\nu} x^\mu y^\nu$, and the notation $x = (x^0, \mathbf{x})$ for an $x \in \mathbb{M}$ in order to distinguish the time component $x^0 \in \mathbb{R}$ from the spatial components $\mathbf{x} \in \mathbb{R}^3$. The overset dot denotes a differentiation with respect to the world-line parametrization τ. For simplicity, each particle has the same inertial mass $m \neq 0$ (all presented results however hold for charges having different masses, too). The coupling constant e_i denotes the charge of the i-th particle.

If one were to insist on using field theoretic language, then one may also say that Eq. (1) describe the interaction between the charges via their advanced and retarded Liénard–Wiechert fields $F[z_k]_+, F[z_k]_-$, $1 \le k \le N$. These fields are special solutions of the Maxwell equations of classical electrodynamics corresponding to a prescribed world-line z_k. The functional dependence on $\tau \mapsto z_k(\tau)$ is emphasized by the square bracket notation $[z_k]$. Given an $x \in \mathbb{M}$ and a time-like world-line $\tau \mapsto z_k(\tau)$, that is, one fulfilling $\dot{z}_{k,\mu}\dot{z}_k^\mu > 0$, the solutions $\tau_{k,+}(x)$, $\tau_{k,-}(x)$, are unique and given by the intersection of the forward and

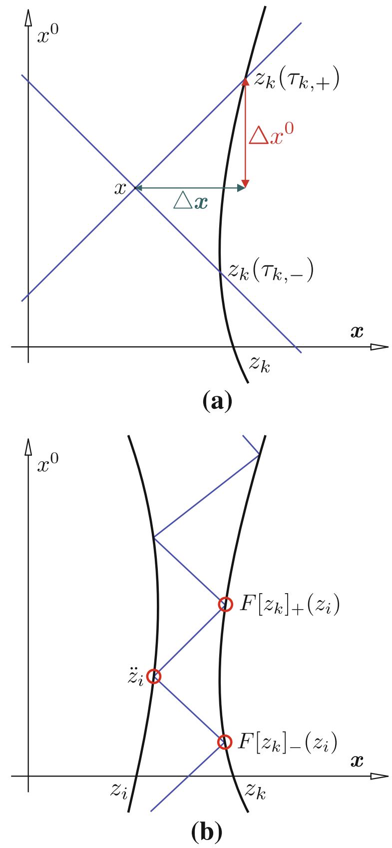

Fig. 1. **a** Solutions of equations (3) for $\triangle x^0 := z_k^0(\tau_{k,+}(x)) - x^0$ and $\triangle x := \|\mathbf{x} - \mathbf{z}_k(\tau_{k,+}(x))\|$. **b** Two WF world-lines z_i and z_k interacting via a ladder of light-cones (45° lines since in our units the speed of light equals one). Hence, the value of $\ddot{z}_i$ depends on both advanced and retarded data $F[z_k]_+(z_i)$ and $F[z_k]_+(z_i)$, respectively

backward light-cone of space-time point x and the world-line z_k, respectively; see Fig. 1a. The acceleration on the left-hand side of the FST equations depends through (3) on time-like advanced as well as retarded data [with respect to $z_i^0(\tau)$] of all the other world-lines; see Fig. 1b. The delay is unbounded, and by (2), the right-hand side of (1) again depends on the acceleration.

It is noteworthy that in early 1900 the mathematician and philosopher A.N. Whitehead [29] developed a philosophical view on nature which rejects "initial value problems" as fundamental descriptions of nature. He developed his own gravitational theory and motivated Synge's study of what is now referred to as Synge equations [19,24], that is,

$$m\ddot{z}_i^{\mu}(\tau) = e_i \sum_{\substack{k=1,\ldots,N \\ k\neq i}} F[z_k]_-^{\mu\nu}(z_i(\tau))\dot{z}_{i,\nu}(\tau). \tag{4}$$

The Synge equations share many difficulties with the FST equations but, as we shall show, are simpler to handle because they only depend on time-like retarded arguments. We would like to remark that independent of Whitehead's philosophy, it seems to be the case that often fields are introduced to formulate a physical law, even though it may have a delay character, as initial value problem. Maxwell–Lorentz electrodynamics is a prime example. However, these very fields are then often the source of singularities of the theory, quantum or classical. Whitehead's idea might therefore point toward a fruitful new reflection about the character of physical laws.

The books [9,11,18] provide a beautiful overview on the topic of delay differential equations. However, for the FST equations as well as similar types of delay differential equations with advanced and retarded arguments of unbounded delay, there are almost no mathematical results available. The problem one usually deals with in the field of differential equations without delay is extension of local solutions to a maximal domain and avoiding critical points by introducing a notion of typicality of initial conditions. For WF, the situation is dramatic. Because of the unbounded delay, the notion of local solutions does not make sense, so that the issue is not local versus global existence and also not explosion or running into singular points of the vector field. The issue is simply: *Do solutions exist?* and *What kind of data of the solutions is necessary and/or sufficient to characterize solutions uniquely?*

To put our work in perspective, we call attention to the following literature: Angelov studied existence of Synge solutions in the case of two equal point-like charges and three dimensions [2]. Under the assumption of an extra condition on the minimal distance between the charges to prevent collisions, he proved existence of Synge solutions on the positive time half-line. Uniqueness is only known in a special case in one dimension for two equal charges initially having sufficiently large opposite velocities and sufficiently large space-like separation. Under these conditions, Driver has shown [8] that the Synge solutions are uniquely characterized by initial positions and momenta. With regard to WF, two types of special solutions are known to exist: First, the explicitly known Schild solutions [21] composed of specially positioned charges revolving around each other on stable orbits, and second, the scattering solutions of two equal charges constrained on the straight line [3]. The latter result rests on the fact that the asymptotic behavior of world-lines on the straight line is well controllable [due to this special geometry the acceleration dependent term on the right-hand side of (1) vanishes]. Uniqueness of FST solutions was proven in one dimension with zero initial velocity and sufficiently large separation of two equal charges [10]. In a recent work [17], a well-defined analog of the formal Fokker variational principle for two charges restricted to finite intervals was proposed. It is shown that its minima, if they exist, fulfill the FST equations on these finite times intervals. Furthermore, there are conjectures about uniqueness of FST solutions, for example, [1,12,25,28]. While Driver's result [10] points to the possibility of uniqueness by initial positions and momenta, Bauer's [3] work suggests to specify asymptotic positions and momenta. Furthermore, a WF toy model for two charges in three dimensions was given in [5,6] for which a sufficient condition for a unique characterization of all its (sufficiently regular) solutions is the prescription of connected strips of time-like world-lines long enough such that at least for one point on each strip the right-hand side of the FST equation is well defined and the FST equation is fulfilled.

2. Our setup and results

Our focus is on the bare existence of solutions of WF, that is, on the above question: *Do solutions exist?* For that question, the issue that in a dynamical evolution of a system of point-like charges catastrophic events may happen is secondary (compare the famous n-body problem of classical gravitation [22]). More on target, such considerations would have to invoke a notion of typicality of world-lines, so that catastrophic events can be shown to be atypical. But that would require not only existence of solutions but also a classification of solutions. We are far from that. To avoid such issues at this early state of research, we regard WF_ϱ as introduced in [4] instead of WF, that is, we consider extended rigid charges described by the charge distributions ϱ_i, $1 \le i \le N$, where singularities do not even occur when charges pass through each other. It has to be emphasized that in contrary to textbook electrodynamics in WF the charges do not acquire electrodynamic mass, and as long as the world-lines do not cross or approach the speed of light, the limit back to point-particles can be carried out without obstacles.

For our mathematical analysis, it is convenient to express WF_ϱ in coordinates where it takes the form

$$\begin{aligned}\partial_t \mathbf{q}_{i,t} &= \mathbf{v}(\mathbf{p}_{i,t}) := \frac{\mathbf{p}_{i,t}}{\sqrt{m^2+\mathbf{p}_{i,t}^2}}\\ \partial_t \mathbf{p}_{i,t} &= \sum_{\substack{k=1,\ldots,N\\ k\neq i}} \int \mathrm{d}^3x\ \varrho_i(\mathbf{x}-\mathbf{q}_{i,t})\left(\mathbf{E}_t[\mathbf{q}_k,\mathbf{p}_k](\mathbf{x}) + \mathbf{v}(\mathbf{q}_{i,t})\wedge \mathbf{B}_t[\mathbf{q}_k,\mathbf{p}_k](\mathbf{x})\right)\end{aligned} \tag{5}$$

for $1 \le i \le N$ and

$$\begin{pmatrix}\mathbf{E}_t^{(e_+,e_-)}[\mathbf{q}_i,\mathbf{p}_i](\mathbf{x})\\ \mathbf{B}_t^{(e_+,e_-)}[\mathbf{q}_i,\mathbf{p}_i](\mathbf{x})\end{pmatrix} = \sum_{\pm} 4\pi e_\pm \int \mathrm{d}s \int \mathrm{d}^3y\ K_{t-s}^{\pm}(\mathbf{x}-\mathbf{y}) \begin{pmatrix}-\nabla\varrho_i(\mathbf{y}-\mathbf{q}_{i,s}) - \partial_s\left[\mathbf{v}(\mathbf{p}_{i,s})\varrho_i(\mathbf{y}-\mathbf{q}_{i,s})\right]\\ \nabla\wedge\left[\mathbf{v}(\mathbf{p}_{i,s})\varrho_i(\mathbf{y}-\mathbf{q}_{i,s})\right]\end{pmatrix} \tag{6}$$

where as in (5) most of the time we drop the superscript $^{(e_+,e_-)}$. Here, $K_t^{\pm}(\mathbf{x}) := \frac{\delta(\|\mathbf{x}\|\pm t)}{4\pi\|\mathbf{x}\|}$ are the advanced and retarded Green's functions of the d'Alembert operator. The partial derivative with respect to time t is denoted by ∂_t, the gradient by ∇, the divergence by $\nabla\cdot$, and the curl by $\nabla\wedge$. At time t the ith charge for $1 \le i \le N$ is situated at position $\mathbf{q}_{i,t}$ in space $\mathbb{R}^3$, has momentum $\mathbf{p}_{i,t} \in \mathbb{R}^3$ and carries the classical mass $m \in \mathbb{R}\setminus\{0\}$. The geometry of the rigid charges is described by the smooth charge densities ϱ_i of compact support, that is, $\varrho_i \in \mathcal{C}_c^\infty(\mathbb{R}^3,\mathbb{R})$, for $1 \le i \le N$.

Using the notation $\mathbf{E}_t := (F^{0i}(t,\cdot))_{1\le i\le 3}$ and $\mathbf{B}_t := (F^{23}(t,\cdot), F^{31}(t,\cdot), F^{12}(t,\cdot))$ and replacing ϱ_i by the three-dimensional Dirac delta distribution $\delta^{(3)}$ times e_i, one retrieves from (5) the FST equations (1) for $e_+ = \frac{1}{2} = e_-$ and the Synge equations (4) for $e_+ = 0, e_- = 1$. As discussed in Theorem 3.10, the expression (6) for the choices for $e_+ = 1, e_- = 0$ and $e_+ = 0, e_- = 1$ is the advanced and retarded Liénard–Wiechert field, respectively. The square brackets $[\mathbf{q}_i, \mathbf{p}_i]$ emphasize that these fields are functionals of the charge world-line $t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})$ and no dynamical degrees of freedom in their own.

The first idea to come to grips with existence of solutions is to adapt fixed point arguments from ordinary differential equations. That is not practical because of two difficulties. The first difficulty is that in general in WF one cannot separate the second-order derivative from lower order derivatives; see (23), (24) and (25) for a more explicit expression of (6). Therefore, one cannot rewrite the FST equations in terms of an integral equation which is normally employed in the fixed points arguments. The second difficulty is that the time-like advanced and retarded arguments introduced by (6) are of unbounded delay so that WF dynamics makes only sense for charge world-lines which are globally defined in time. One would thus have to find an appropriately normed space of functions on $\mathbb{R}$ on which the fixed point map can be controlled—which has not been found yet. One may circumvent this problem by introducing a notion of conditional solutions where outside a chosen time interval $[-T, T]$ the world-lines are prescribed by hand. The fixed point argument—if that could be formulated—would then run on the time interval $[-T, T]$ only. If successful, one may then try to construct a bonafide global solution by letting $T \to \infty$. In this work, we show how one can formulate a fixed point procedure on intervals $[-T, T]$ for arbitrary large $T > 0$, that is, we show how one can circumvent the first difficulty albeit gaining conditionally solutions only. The extension to global solutions would require good control on the asymptotic behavior (as e.g., in [3] in the case of the motion on the straight line), which we do not pursue here. We stress, however, that the extension to infinite time intervals is an interesting and worthwhile task, joining the results of this paper with the removal technique for $T \to \infty$ introduced in [3].

The key idea to define a fixed point map on time intervals $[-T, T]$ is a reformulation of the WF functional differential equations into a system of nonlinear partial differential equations without delay, namely the Maxwell–Lorentz equations without self-interaction (abbrev. ML-SI) introduced in [4, (4)–(7)]. Relying on the notation in [4, (13)], the relation between WF and ML-SI can be expressed as an equality of sets of charge world-lines:

$$\mathrm{WF} = \left\{\text{world-lines of ML-SI} \upharpoonright \{F_0 \equiv 0\}\right\}. \tag{7}$$

On the left-hand side, we consider the set of world-lines of the charges that fulfill WF. On the right, we have the set charge world-lines corresponding to solutions of ML-SI restricted to the subset for which $F_0 = F - \frac{1}{2}(F_+ + F_-)$ vanishes, that is, the electrodynamic fields F coincide with the WF fields (6).

In the case of rigid charges, we shall use the relation (7) in the following way: Consider charge world-lines $t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1\le i\le N}$ which solve WF_ϱ. By definition, the fields (6) fulfill the Maxwell equations which implies that the map

$$t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N} := (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_t[\mathbf{q}_i, \mathbf{p}_i], \mathbf{B}_t[\mathbf{q}_i, \mathbf{p}_i])_{1\le i\le N}$$

is a solution of ML-SI_ϱ, that is, the *Maxwell* equations:

$$\begin{aligned}
\partial_t \mathbf{E}_{i,t} &= \nabla \wedge \mathbf{B}_{i,t} - 4\pi \mathbf{v}(\mathbf{p}_{i,t})\varrho_i(\cdot - \mathbf{q}_{i,t}) \quad \nabla \cdot \mathbf{E}_{i,t} = 4\pi \varrho_i(\cdot - \mathbf{q}_{t,i}) \\
\partial_t \mathbf{B}_{i,t} &= -\nabla \wedge \mathbf{E}_{i,t} \qquad \nabla \cdot \mathbf{B}_{i,t} = 0
\end{aligned} \tag{8}$$

together with the *Lorentz* equations (without self-interaction):

$$\begin{aligned}
\partial_t \mathbf{q}_{i,t} &= \mathbf{v}(\mathbf{p}_{i,t}) := \frac{\mathbf{p}_{i,t}}{\sqrt{m^2 + \mathbf{p}_{i,t}^2}} \\
\partial_t \mathbf{p}_{i,t} &= \sum_{\substack{k=1,\ldots,N \\ k\neq i}} \int \mathrm{d}^3x \; \varrho_i(\mathbf{x} - \mathbf{q}_{i,t}) \left[\mathbf{E}_{k,t}(\mathbf{x}) + \mathbf{v}_{i,t} \wedge \mathbf{B}_{k,t}(\mathbf{x})\right].
\end{aligned} \tag{9}$$

On the other hand, global existence and uniqueness of solutions of ML-SI_ϱ for initial data $p := (\mathbf{q}_i^0, \mathbf{p}_i^0)_{1\le i\le N} \in \mathbb{R}^{6N}$ and sufficiently regular initial fields $F := (\mathbf{E}_i^0, \mathbf{B}_i^0)_{1\le i\le N}$, for example, at time $t_0 \in \mathbb{R}$, has been shown in [4]; the needed definitions and results are summarized in the Sect. 3.2. For any $(p, F) \in D_w(A^\infty)$, the particular solution is then denoted by

$$t \mapsto M_L[p, F](t, t_0) := (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N}. \tag{10}$$

In this sense, we say that sufficiently regular WF_ϱ charge world-lines give rise to ML-SI_ϱ solutions.

Changing the point of view, we now fix some Newtonian Cauchy data p and ask our

Crucial Question: Do fields F exist such that the corresponding ML-SI_ϱ solution

$$t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N} =: M_L[p, F](t, t_0)$$

fulfills

$$F = (\mathbf{E}_t[\mathbf{q}_i, \mathbf{p}_i], \mathbf{B}_t[\mathbf{q}_i, \mathbf{p}_i])_{1\le i\le N}|_{t=t_0} \quad ? \tag{11}$$

Condition (11) expresses that the initial fields F equal the WF_ϱ fields (6) at initial time $t = t_0$. Equivalently, it ensures that the time-evolved fields $t \mapsto (\mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N}$ of the ML-SI_ϱ solution equal the WF_ϱ fields $t \mapsto (\mathbf{E}_t[\mathbf{q}_i, \mathbf{p}_i], \mathbf{B}_t[\mathbf{q}_i, \mathbf{p}_i])_{1\le i\le N}$ for all times t because their difference is a solution to the homogeneous Maxwell equations [i.e. (8) for $\varrho_i = 0$] which is zero; compare (7). Given the equality of fields for all times, Eqs. (9) turn into the WF_ϱ equations (5), and hence, the charge world-lines of the ML-SI_ϱ solution fulfilling (11) solve the WF_ϱ equations. In other words, the subset of sufficiently regular solutions of ML-SI_ϱ that correspond to initial conditions fulfilling (11) have WF_ϱ charge world-lines. We shall show that any once differentiable charge world-line $t \mapsto (\mathbf{p}_t, \mathbf{q}_t)$ with bounded momenta and accelerations produces WF_ϱ fields (6) that are regular enough to serve as initial conditions for ML-SI_ϱ. This covers all physically interesting WF_ϱ solutions, including the known Schild solutions. The advantage gained from this change of viewpoint is that ML-SI_ϱ is given in terms of an initial value problem. Therefore, instead of working directly with the WF_ϱ functional equations, it will be more convenient to formulate a fixed point procedure for ML-SI_ϱ to find initial fields for which (11) holds.

We now give an overview of our main results for which we need a precise definition of the considered classes of charge world-lines:

Definition 2.1. (*Charge world-lines*)

(i) We call any map

$$(\mathbf{q},\mathbf{p}) \in \mathcal{C}^1(\mathbb{R},\mathbb{R}^3\times\mathbb{R}^3),\ t\mapsto(\mathbf{q}_t,\mathbf{p}_t)$$

a charge world-line and denote with $\mathbf{q}_t$ and $\mathbf{p}_t$ the position and momentum of the charge, respectively. Its velocity at time t is given by $\mathbf{v}(\mathbf{p}_t) := \frac{\mathbf{p}_t}{\sqrt{m^2+\mathbf{p}_t^2}}$.

(ii) We collect all time-like charge world-lines in the set

$$\mathcal{T}^1_{\circleddash} := \left\{ (\mathbf{q},\mathbf{p}) \in \mathcal{C}^1(\mathbb{R},\mathbb{R}^3\times\mathbb{R}^3) \;\middle|\; \|\mathbf{v}(\mathbf{p}_t)\|<1 \quad \text{for all } t\in\mathbb{R} \right\},$$

(iii) and all strictly time-like charge world-lines in the set

$$\mathcal{T}^1_{\circleddash!}(I) := \left\{ (\mathbf{q},\mathbf{p}) \in \mathcal{T}^1_{\circleddash} \;\middle|\; \exists v_{max}<1 \text{ such that } \sup_{t\in I}\|\mathbf{v}(\mathbf{p}_t)\| \le v_{max} \right\}.$$

where we use the abbreviation $\mathcal{T}^1_{\circleddash!} := \mathcal{T}^1_{\circleddash!}(\mathbb{R})$.

The symbol $\circleddash$ refers to "time-like" whereas the symbol $\circleddash!$ refers to "strictly time-like". Furthermore, we use the notation

$$(\mathbf{q},\mathbf{p}) = (\widetilde{\mathbf{q}},\widetilde{\mathbf{p}}) :\Leftrightarrow \forall t\in\mathbb{R} : (\mathbf{q}_t,\mathbf{p}_t) = (\widetilde{\mathbf{q}}_t,\widetilde{\mathbf{p}}_t)$$

and define the Cartesian products $\mathcal{T}^N_{\circleddash} := (\mathcal{T}^1_{\circleddash})^N$ and $\mathcal{T}^N_{\circleddash!} := (\mathcal{T}^1_{\circleddash!})^N$.

Furthermore, we define the class of charge world-lines that fulfill the WF_ϱ equations (5)–(6).

Definition 2.2. (*Class of Solutions*) We define $\mathcal{T}^N_{(e_+,e_-)}$ to consist of elements $(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N} \in \mathcal{T}^N_{\circleddash!}$ which fulfill:

(i) There exists an $a_{max}<\infty$ such that for all $1\le i\le N$, $\sup_{t\in\mathbb{R}}\|\partial_t\mathbf{v}(\mathbf{p}_{i,t})\| \le a_{max}$.

(ii) $(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}$ solve the Eqs. (5)–(6) for all times $t\in\mathbb{R}$ and the particular choice of e_+,e_-.

Our first results is as follows:

Theorem 2.3. (*Weak Uniqueness of Solutions*) *For $e_+,e_-\in\mathbb{R}$, $(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}\in\mathcal{T}^N_{(e_+,e_-)}$ and $t\in\mathbb{R}$ we define*

$$\varphi_t^{(e_+,e_-)}[(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}] = (\mathbf{q}_{i,t},\mathbf{p}_{i,t},\mathbf{E}_t^{(e_+,e_-)}[\mathbf{q}_i,\mathbf{p}_i],\mathbf{B}_t^{(e_+,e_-)}[\mathbf{q}_i,\mathbf{p}_i])_{1\le i\le N}. \tag{12}$$

The following statements are true:

(i) For any $t_0\in\mathbb{R}$ we have $\varphi_{t_0}^{(e_+,e_-)}[(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}]\in D_w(A^\infty)$.

(ii) For all $t,t_0\in\mathbb{R}$ also $\varphi_t^{(e_+,e_-)}[(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}] = M_L\left[\varphi_{t_0}^{(e_+,e_-)}[(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}]\right](t,t_0)$ holds.

(iii) For any $t_0\in\mathbb{R}$ the following map is injective:

$$i_{t_0}^{(e_+,e_-)} : \mathcal{T}^N_{(e_+,e_-)} \to D_w(A^\infty),\ (\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}\mapsto\varphi_{t_0}^{(e_+,e_-)}[(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}].$$

Hence, for any choice of the coupling parameters e_+,e_- we know that: (i) The charge world-lines in $\mathcal{T}^N_{(e_+,e_-)}$ produce sufficiently regular initial fields for ML-SI_ϱ. (ii) The expression (12) coincides with a ML-SI_ϱ solution. (iii) Each solution of (5)–(6) can be identified by positions, momenta and fields $\mathbf{E}_t^{(e_+,e_-)}[\mathbf{q}_i,\mathbf{p}_i],\mathbf{B}_t^{(e_+,e_-)}[\mathbf{q}_i,\mathbf{p}_i])_{1\le i\le N}$ at an initial time t_0.

This gives us a good handle on the existence and uniqueness of the Synge solutions. We define the initial data:

Definition 2.4. (*Synge Histories*) For $t_0\in\mathbb{R}$ we define the set $\mathfrak{H}(t_0)$ to consist of elements $(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}\in\mathcal{T}^N_{\circleddash!}$ which fulfill:

(i) There exists an $a_{max} < \infty$ such that for all $1 \leq i \leq N$, $\sup_{t\in\mathbb{R}} \|\partial_t \mathbf{v}(\mathbf{p}_{i,t})\| \leq a_{max}$.
(ii) $(\mathbf{q}_i, \mathbf{p}_i)_{1\leq i\leq N}$ solve the Eqs. (5)–(6) for $e_+ = 0, e_- = 1$ at time $t = t_0$.

Furthermore, $\mathfrak{H}(t_0)^+$ denotes the set $\mathfrak{H}(t_0)$ equipped with

$$(\mathbf{q}_i, \mathbf{p}_i)_{1\leq i\leq N} \overset{\mathfrak{H}^+(t_0)}{=} (\widetilde{\mathbf{q}}_i, \widetilde{\mathbf{p}}_i)_{1\leq i\leq N} :\Leftrightarrow \forall t \in [t_0, \infty) : (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1\leq i\leq N} = (\widetilde{\mathbf{q}}_{i,t}, \widetilde{\mathbf{p}}_{i,t})_{1\leq i\leq N}$$

while $\mathfrak{H}(t_0)^-$ denotes the set $\mathfrak{H}(t_0)$ equipped with

$$(\mathbf{q}_i, \mathbf{p}_i)_{1\leq i\leq N} \overset{\mathfrak{H}^-(t_0)}{=} (\widetilde{\mathbf{q}}_i, \widetilde{\mathbf{p}}_i)_{1\leq i\leq N} :\Leftrightarrow \forall t \in (-\infty, t_0] : (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1\leq i\leq N} = (\widetilde{\mathbf{q}}_{i,t}, \widetilde{\mathbf{p}}_{i,t})_{1\leq i\leq N}.$$

Given a history $(\mathbf{q}_i^-, \mathbf{p}_i^-)_{1\leq i\leq N} \in \mathfrak{H}^-(t_0)$ one can simply compute the retarded Liénard–Wiechert fields $(\mathbf{E}_t^{(0,1)})[\mathbf{q}_i^-, \mathbf{p}_i^-], \mathbf{B}_t^{(0,1)})[\mathbf{q}_i^-, \mathbf{p}_i^-])_{1\leq i\leq N}$ at time $t = t_0$ and use them as initial fields for ML-SI$_\varrho$. The charge world-lines of the time-evolved ML-SI$_\varrho$ solutions then obey the Synge equations for times $t \geq t_0$. This way we shall prove:

Theorem 2.5. *(Existence and Uniqueness of Synge Solutions) Let* $e_+ = 0, e_- = 1$, $t_0 \in \mathbb{R}$ *and* $(\mathbf{q}_i^-, \mathbf{p}_i^-)_{1\leq i\leq N} \in \mathfrak{H}(t_0)^-$.

(i) (existence) There exists an extension $(\mathbf{q}_i^+, \mathbf{p}_i^+)_{1\leq i\leq N} \in \mathfrak{H}(t_0)^+$ *such that the concatenation*

$$(\mathbf{q}_i, \mathbf{p}_i)_{1\leq i\leq N} : t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1\leq i\leq N} := \begin{cases} (\mathbf{q}_{i,t}^-, \mathbf{p}_{i,t}^-)_{1\leq i\leq N} & \text{for } t \leq t_0 \\ (\mathbf{q}_{i,t}^+, \mathbf{p}_{i,t}^+)_{1\leq i\leq N} & \text{for } t > t_0 \end{cases} \tag{13}$$

is an element of $\mathcal{T}_{\odot!}^N((-\infty, T])$ *for all* $T \in \mathbb{R}$ *and solves the Eqs. (5)–(6) for all* $t \geq t_0$.

(ii) (uniqueness) Let $(\widetilde{\mathbf{q}}_i, \widetilde{\mathbf{p}}_i)_{1\leq i\leq N} \in \mathcal{T}_{\odot!}^N((-\infty, T])$ *for any* $T \in \mathbb{R}$ *and suppose further that it solves the Eqs. (5)–(6) for all times* $t \geq t_0$. *Then* $(\widetilde{\mathbf{q}}_i, \widetilde{\mathbf{p}}_i)_{1\leq i\leq N} \overset{\mathfrak{H}^-(t_0)}{=} (\mathbf{q}_i^-, \mathbf{p}_i^-)_{1\leq i\leq N}$ *implies* $(\widetilde{\mathbf{q}}_{i,t}, \widetilde{\mathbf{p}}_{i,t})_{1\leq i\leq N} = (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1\leq i\leq N}$ *for all* $t \in \mathbb{R}$.

Given Theorem 2.3, this existence and uniqueness result is not hard to prove, and the reason for this is that we only ask for solutions on the half-line $[t_0, \infty)$. In contrast to WF$_\varrho$, the notion of local solutions again makes sense since the histories simply act as prescribed external potentials. However, if we ask for solutions on whole $\mathbb{R}$, we again face the problem as in WF$_\varrho$, that is, by the unboundedness of the delay the notion of local solutions loses its meaning (a conceivable way around this without necessarily sacrificing uniqueness is to give initial conditions for $t_0 \to -\infty$).

We now come to the main part of this work where we discuss the existence of WF$_\varrho$ solutions. From now on we shall keep the choice $e_+ = \frac{1}{2}$, $e_- = \frac{1}{2}$ fixed, although all the results hold also for any choices of $-1 \leq e_+, e_- \leq 1$. We take on the mentioned idea of conditional solutions: For given initial positions and momenta of the charges at $t = 0$, we look for WF$_\varrho$ solutions on time intervals $[-T, T]$ for an arbitrary large but fixed $T > 0$. To be able to regard only the time interval $[-T, T]$ of the WF$_\varrho$ dynamics, we need to prescribe how the charge world-lines continue for times $|t| > T$ because due to the delay the dynamics within $[-T, T]$ will of course depend also on the world-lines at times $|t| > T$. This is done by specifying the advanced Liénard–Wiechert fields at time T as well as the retarded Liénard–Wiechert fields and time $-T$ corresponding to each continuation of the charge world-line for times $|t| > T$. We shall refer to these fields as *boundary fields* and denote them by $X_{i,+T}^+$ and $X_{i,-T}^-$. The set of WF$_\varrho$ equations for $1 \leq i \leq N$ with respect to the boundary fields $X_{i,\pm T}^\pm$ turn into

$$\partial_t \mathbf{q}_{i,t} = \mathbf{v}(\mathbf{p}_{i,t}) := \frac{\mathbf{p}_{i,t}}{\sqrt{m^2 + \mathbf{p}_{i,t}^2}}$$
$$\partial_t \mathbf{p}_{i,t} = \sum_{\substack{k=1,\ldots,N\\ k\neq i}} \int \mathrm{d}^3x \; \varrho_i(\mathbf{x} - \mathbf{q}_{i,t}) \left(\mathbf{E}_t^X[\mathbf{q}_k, \mathbf{p}_k](\mathbf{x}) + \mathbf{v}(\mathbf{q}_{i,t}) \wedge \mathbf{B}_t^X[\mathbf{q}_k, \mathbf{p}_k](\mathbf{x})\right) \tag{14}$$

and

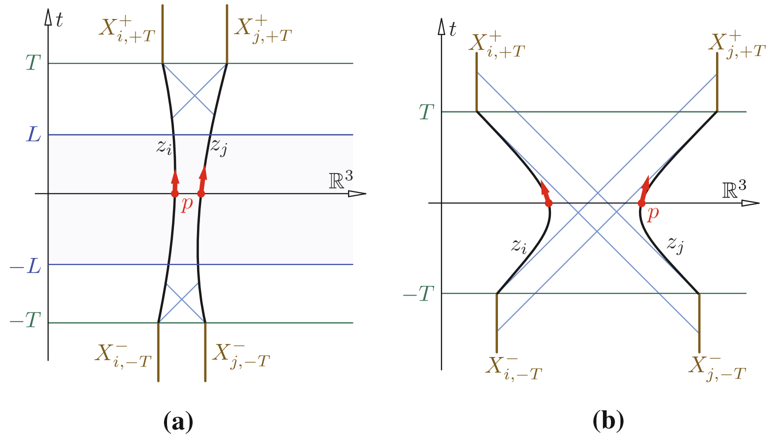

FIG. 2. Two WF world-lines z_i and z_j on time interval $[-T,T]$ with Newtonian Cauchy data p. The *straight lines* for times $|t|>T$ are the prescribed asymptotes which generate the advanced and retarded Liénard–Wiechert fields $X^+_{i,+T}$ and $X^-_{i,-T}$. In **a** one observes true WF interaction between the charge world-lines on $[-T,T]$ within the time interval $[-L,L]$. In the extreme case **b** the charge world-lines on $[-T,T]$ interact only with the given asymptotes (apart from the connection conditions at $\pm T$)

$$(\mathbf{E}^X_{i,t},\mathbf{B}^X_{i,t}) = \frac{1}{2}\sum_{\pm} M_{\varrho_i}[X^{\pm}_{i,\pm T},(\mathbf{q}_i,\mathbf{p}_i)](t,\pm T), \quad \text{for } 1\le i\le N \tag{15}$$

where the $M_\varrho[F^0,(\mathbf{q},\mathbf{p})](t,t_0)$ denotes the solution of the Maxwell equations for initial fields F^0 at time t_0 corresponding to a prescribed world-line $t\mapsto(\mathbf{q}_t,\mathbf{p}_t)$ with a charge distribution ϱ; see Definition 3.9 below. Note that the above set of equations is a natural restriction of the WF_ϱ dynamics onto the time interval $[-T,T]$ because, first, for the choice

$$X^{\pm}_{i,\pm T} = 4\pi\int \mathrm{d}s\int \mathrm{d}^3y\; K^{\pm}_{\pm T-s}(\mathbf{x}-\mathbf{y})\begin{pmatrix} -\nabla\varrho_i(\mathbf{y}-\mathbf{q}_{i,s})-\partial_s\left[\mathbf{v}(\mathbf{p}_{i,s})\varrho_i(\mathbf{y}-\mathbf{q}_{i,s})\right] \\ \nabla\wedge\left[\mathbf{v}(\mathbf{p}_{i,s})\varrho_i(\mathbf{y}-\mathbf{q}_{i,s})\right]\end{pmatrix} \tag{16}$$

they turn into the WF_ϱ set of equations (5)–(6). And second, it is well known that for large T the boundary fields, should they have sufficient space-like decay, are forgotten by the Maxwell time evolution M in the pointwise sense; see Remark 3.11 below. Based on this behavior, one may expect to be able to study also unconditional existence of WF_ϱ solutions by considering the limit $T\to\infty$ for a convenient choice of controllable boundary fields.

For simplicity of our introductory discussion, let us choose

$$X^{\pm}_{\pm T} := (\mathbf{E}^C_i(\cdot-\mathbf{q}_{i,\pm T}),0)_{1\le i\le N},$$
$$(\mathbf{E}^C_i,0) := M_{\varrho_i}[t\mapsto(0,0)](0,-\infty) = \left(\int \mathrm{d}^3z\;\varrho_i(\cdot-\mathbf{z})\frac{\mathbf{z}}{\|\mathbf{z}\|^3},0\right),$$

that is, the Coulomb fields corresponding to a charge at rest at $\mathbf{q}_{i,\pm T}$. With this prescription, the conditional WF_ϱ equations (14)–(15) are equivalent to WF_ϱ dynamics for charges initially being held at rest for times $t\le -T$ and then instantaneously stopped at times $t\ge T$ by external mechanical forces; see Fig. 2a. The presented results, however, admit not only this particular case but a large class of boundary fields which also allow a continuous continuation of the momentum of the charges at times $t=\pm T$.

In view of our discussion of (7), it seems natural to implement the following fixed point map in order to find solutions to the conditional WF_ϱ equations (14)–(15) for initial positions and momenta $p\in\mathbb{R}^{6N}$ of the charges:

INPUT: $F = (\mathbf{E}^0_i, \mathbf{B}^0_i)_{1\le i\le N}$ for any fields such that $(p, F) \in D_w(A^\infty)$.

(i) Compute the ML-SI$_\varrho$ solution $[-T, T] \ni t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N} := M_L[p, F](t, 0)$.

(ii) Compute the advanced and retarded fields

$$(\widetilde{\mathbf{E}}_{i,t}, \widetilde{\mathbf{B}}_{i,t}) = \frac{1}{2}\sum_{\pm} M_{\varrho_i}[X^{\pm}_{i,\pm T}[p, F], (\mathbf{q}_i, \mathbf{p}_i)](t, \pm T)$$

corresponding to the charge world-lines $t \mapsto (\mathbf{q}_i, \mathbf{p}_i)$ computed in (i) with prescribed initial fields $X^{\pm}_{i,\pm T}[p, F]$ at times $\pm T$.

OUTPUT: $S^{p,X^\pm}_T[F] := (\widetilde{\mathbf{E}}_{i,t}, \widetilde{\mathbf{B}}_{i,t})_{1\le i\le N}|_{t=0}$.

Note that the boundary fields $X^{\pm}_{i,\pm T} = X^{\pm}_{i,\pm T}[p, F]$ need to depend on the ML-SI$_\varrho$ initial values (p, F). Otherwise, it would not be possible to continuously connect the charge world-lines with the prescribed continuation of the charge world-lines at times $t = \pm T$. The precise definition of $S^{p,X^\pm}_T$ is given in Definition 4.11 below. By construction, any fixed point F^* of this map $S^{p,X^\pm}_T$ gives rise to a ML-SI$_\varrho$ solutions $t \mapsto M_L[p, F^*](t, 0)$ whose charge world-lines fulfill the conditional WF$_\varrho$ equations (14)–(15); see Definition 4.10 and Theorem 4.12 below. We prove:

Theorem 2.6. (*Existence of Conditional WF$_\varrho$Solutions*) *Let $p \in \mathbb{R}^{6N}$ be given. For each finite $T > 0$ the map $S^{p,X^\pm}_T$ has a fixed point.*

The essential ingredient in the proof of this result is the good nature of the ML-SI$_\varrho$ dynamics which implies Lemma 4.17 below. Here we rely heavily on the work done in [4].

We close with a discussion of these fixed points. Recall that the Synge solutions on the time half-line $[t_0, \infty)$ for times sufficiently close to t_0 give rise to interaction with the given past world-lines on $(-\infty, t_0]$ only. For such small times, one simply solves an external field problem. Not until larger times the interaction becomes truly retarded in the sense that the future charge world-lines interact with their just generated histories for times $t \ge t_0$. However, in an extreme situation, a charge could approach the speed of light so fast that the time coordinate of the intersection of its backward light-cone with another charge world-line is bounded, say, by $T^{max} \in \mathbb{R}$. This means that this charge will never interact with the part $t \ge T^{max}$ of the other charge world-lines. If $T^{max} \le t_0$, one ends up solving a purely external field problem without seeing any truly retarded interaction. Such a scenario is of course so special that one would not expect it for all Synge solutions (recall that by Theorem 2.5 one has existence and uniqueness on the time half-line for any sufficiently regular set of past world-lines). For the WF$_\varrho$ equations, however, we only have solutions on time intervals $[-T, T]$ yet and, therefore, one should be more curious as the described scenario in the case of the Synge equations could happen in the case of the FST equations in the past as well as in the future of t_0. If the WF$_\varrho$ solution on $[-T, T]$ behaves as badly as described above or the initial position are too far apart from each other in the space-like sense, we might end up solving only an external field problem as the charge world-lines on $[-T, T]$ only "see" the prescribed boundary fields; see Fig. 2b. The following result makes sure that given T at least for some solutions this is not the case because on an interval $[-L, L]$ with $0 < L \le T$ they interact exclusively with all other charge world-lines on $[-T, T]$ and not with the given boundary fields; that is, the case as shown in Fig. 2a.

We prove:

Theorem 2.7. (*True WF$_\varrho$ Interaction*) *Choose $a, b, T > 0$. Then:*

(i) The absolute values of the velocities $\mathbf{v}(\mathbf{p}_{i,t})$ of all charges of any ML-SI$_\varrho$ solution with any initial data (p, F) such that

$$\|p\| \le a, \qquad \max_{1\le i\le N} \|\varrho_i\|_{L^2_w} + \max_{1\le i\le N} \|w^{-1/2}\varrho_i\|_{L^2} \le b, \qquad F \in \text{Range}\, S^{p,X^\pm}_T$$

have an upper bound $v^{a,b}_T$ with $0 \le v^{a,b}_T < 1$.

(ii) Let $R>0$ be the smallest radius such that the support of ϱ_i lies within a ball around the origin with radius R, that is, $\operatorname{supp}\varrho_i \subseteq B_R(0)$, *for all* $1 \le i \le N$, *and further* $\triangle q_{max}(p) := \max_{1\le i,j\le N} \|\mathbf{q}_i^0 - \mathbf{q}_j^0\|$. *For sufficiently small R there exist $p = (\mathbf{q}_i^0, \mathbf{p}_i^0)_{1\le i\le N}$ such that*

$$L := \frac{(1-v_T^{a,b})T - \triangle q_{max} - 2R}{1+v_T^{a,b}} > 0 \tag{17}$$

and any fixed point F^* of $S_T^{p,X^\pm}$ gives rise to a ML-SI$_\varrho$ solution $t \mapsto M_L[p,F^*](t,0)$ whose charge world-lines for $t \in [-L, L]$ solve the WF$_\varrho$ equations (5)–(6).

The form of L in (17) is a direct consequence of the geometry as displayed in Fig. 2a and the nature of the free Maxwell time evolution, see Lemma 4.21, which can be seen from a direct computation using harmonic analysis. The proof further employs a very rough Grönwall estimate coming from the ML-SI$_\varrho$ dynamics to estimate the velocities of the charges during the time interval $[-T, T]$, see Lemma 4.22. The conditions for the above result are therefore quite restrictive but merely technical. Any uniform velocity estimate, for example, as given in [3] for two charges of equal sign restricted to a straight line, makes this result redundant as then T can just be chosen arbitrarily large to ensure an arbitrary large L, and hence charge world-lines that fulfill the WF$_\varrho$ equations on arbitrary large intervals. We expect such a bound also without the restriction to a straight line. However, even without such a uniform velocity bound the result above already ensures that in Theorem 2.6 we do see truly advanced and retarded WF$_\varrho$ interaction between the charges. Furthermore, we remark that for the charge world-lines found in (ii) above, one can already define the WF conservation laws [28] which we expect to be an important ingredient in order to control a limit procedure $T \to \infty$ to yield global WF$_\varrho$ solutions.

3. Preliminaries

In the proofs of the main results, we will frequently rely on explicit expressions of the time-evolved electric and magnetic fields appearing in the Maxwell equations as well as in the ML-SI$_\varrho$ time evolution. The ML-SI$_\varrho$ equations are (8)–(9), while the Maxwell equations for a given charge–current density $t \mapsto (\rho_t, \mathbf{j}_t)$ have the form

$$\begin{aligned} \partial_t \mathbf{E}_t &= \nabla \wedge \mathbf{B}_t - 4\pi \mathbf{j}_t \quad \nabla \cdot \mathbf{E}_t = 4\pi\rho_t \\ \partial_t \mathbf{B}_t &= -\nabla \wedge \mathbf{E}_t \quad \nabla \cdot \mathbf{B}_t = 0. \end{aligned} \tag{18}$$

Although the presented results on the Maxwell equations are well known in the physics community, we only found some of them in the mathematical literature; for example, [23]. Therefore, we give a mathematical review in Sect. 3.1. The proofs of all the claims are published separately in [5]. Furthermore, ML-SI$_\varrho$ was studied in [4]. In order to be self-contained, we given an overview of the needed results in Sect. 3.2.

Notation. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{R}^3$ vectors and vector-valued functions have bold letters. We denote the ball of radius $R>0$ around the center $\mathbf{x} \in \mathbb{R}^3$ by $B_R(\mathbf{x}) \subset \mathbb{R}^3$ and its boundary by $\partial B_R(\mathbf{x})$. We denote by $C \in$ `Bounds` any function $x \mapsto C(x) \in \mathbb{R}_+$ that depends continuously and nondecreasing on its argument x. Furthermore, let $\mathcal{C}^n(V,W)$ be the set of n-times continuously differentiable functions $V \to W$. $\mathcal{C}^\infty(V,W) := \bigcap_{n\in\mathbb{N}_0} \mathcal{C}^n(V,W)$. $\mathcal{C}_c^n(V,W) \subset \mathcal{C}^n(V,W)$ and $\mathcal{C}_c^\infty(V,W) \subset \mathcal{C}^\infty(V,W)$ are the respective subsets of functions with compact support. Where unambiguous we sometime drop the reference to V and W.

3.1. Strong solutions to the Maxwell equations

We review the solution theory of the Maxwell equations (18) omitting the proofs which can be found in, for example, [5]. The class of charge–current densities we treat is defined by:

Definition 3.1. (*Charge–Current Densities*) We shall call any pair of maps $\rho : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}, (t,\mathbf{x}) \mapsto \rho_t(\mathbf{x})$ and $\mathbf{j} : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3, (t,\mathbf{x}) \mapsto \mathbf{j}_t(\mathbf{x})$ a charge–current density whenever:

(i) For all $\mathbf{x}\in\mathbb{R}^3$: $\rho_{(\cdot)}(\mathbf{x}) \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ and $\mathbf{j}_{(\cdot)}(\mathbf{x}) \in \mathcal{C}^1(\mathbb{R},\mathbb{R}^3)$.

(ii) For all $t\in\mathbb{R}$: $\rho_t, \partial_t\rho_t \in \mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R})$ and $\mathbf{j}_t, \partial_t\mathbf{j}_t \in \mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R}^3)$.

(iii) For all $(t,\mathbf{x})\in\mathbb{R}\times\mathbb{R}^3$: $\partial_t\rho_t(\mathbf{x}) + \nabla\cdot\mathbf{j}_t(\mathbf{x}) = 0$ which is referred to as continuity equation.

We denote the set of such pairs $(\rho,\mathbf{j})$ by $\mathcal{D}$.

We are interested in solutions to the Maxwell equations (18) in the following sense:

Definition 3.2. (*Strong Solution Sense*) We define the space of fields

$$\mathcal{F}^1 := \mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R}^3) \oplus \mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R}^3).$$

Let $t_0\in\mathbb{R}$ and $F^0\in\mathcal{F}^1$. Then any mapping $F:\mathbb{R}\to\mathcal{F}^1, t\mapsto F_t := (\mathbf{E}_t,\mathbf{B}_t)$ that solves (18) in the pointwise sense for initial value $F_t|_{t=t_0} = F^0$ is called a strong solution to the Maxwell equations with t_0 initial value F^0.

Explicit formulas of those solutions are constructed with the help of:

Definition 3.3. (*Green's Functions of the d'Alembert*) We set

$$K_t^\pm(\mathbf{x}) := \frac{\delta(\|\mathbf{x}\| \pm t)}{4\pi\|\mathbf{x}\|}$$

where δ denotes the one-dimensional Dirac delta distribution.

Note that for every $f\in\mathcal{C}^\infty(\mathbb{R}^3)$

$$K_t^\pm * f(\mathbf{x}) = \begin{cases} 0 & \text{for } \pm t > 0 \\ t \fint_{\partial B_{|t|}(\mathbf{x})} d\sigma(y) f(\mathbf{y}) := \frac{t}{4\pi t^2}\int_{\partial B_{|t|}(\mathbf{x})} d\sigma(y) f(\mathbf{y}) & \text{otherwise} \end{cases}$$

holds, where $d\sigma$ denotes the surface element on $\partial B_{|t|}(\mathbf{x})$. We introduce the notation $\triangle = \nabla\cdot\nabla$ and $\square = \partial_t^2 - \triangle$.

Lemma 3.4. *(Green's Functions Properties) Let $f\in\mathcal{C}^\infty(\mathbb{R}^3)$. Then:*

(i) The following identities holds:

$$K_t^\pm * f = \mp t \fint_{\partial B_{\mp t}(0)} d\sigma(y) f(\cdot - \mathbf{y})$$

$$\partial_t K_t^\pm * f = \mp \fint_{\partial B_{\mp t}(0)} d\sigma(y)\, f(\cdot - \mathbf{y}) \mp \frac{t^2}{3} \fint_{B_{\mp t}(0)} d^3y\, \triangle f(\cdot - \mathbf{y})$$

$$\partial_t^2 K_t^\pm * f = K_t^\pm * \triangle f = \triangle K_t^\pm * f. \tag{19}$$

(ii) Set $K_t = \sum_\pm \mp K_t^\pm$. The mapping $(t,\mathbf{x}) \mapsto [K_t * f](\mathbf{x})$ can uniquely be extended at $t=0$ to become a $C^\infty(\mathbb{R}\times\mathbb{R}^3)$ function such that for all $n\in\mathbb{N}$

$$\lim_{t\to 0\mp} \begin{pmatrix} \partial_t^{2n} K_t * f \\ \partial_t^{2n+1} K_t * f \end{pmatrix} = \begin{pmatrix} 0 \\ \triangle^n f \end{pmatrix} \tag{20}$$

and $\square K_t * f = 0$ for all $t\in\mathbb{R}$.

Remark 3.5. In the future we will denote the unique extension of K_t by the same symbol K_t. It is called the *propagator* of the homogeneous wave equation.

Theorem 3.6. *(Maxwell Solutions) Let* $(\rho, \mathbf{j}) \in \mathcal{D}$.

(i) Given $(\mathbf{E}^0, \mathbf{B}^0) \in \mathcal{F}^1$ *fulfilling the Maxwell constraints* $\nabla \cdot \mathbf{E}^0 = 4\pi\rho_{t_0}$ *and* $\nabla \cdot \mathbf{B}^0 = 0$ *for any* $t_0 \in \mathbb{R}$, *the mapping* $t \mapsto F_t = (\mathbf{E}_t, \mathbf{B}_t)$ *defined by*

$$\begin{pmatrix} \mathbf{E}_t \\ \mathbf{B}_t \end{pmatrix} := \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t-t_0} * \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{B}^0 \end{pmatrix} + K_{t-t_0} * \begin{pmatrix} -4\pi \mathbf{j}_{t_0} \\ 0 \end{pmatrix} + 4\pi \int_{t_0}^{t} \mathrm{d}s\, K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_s \\ \mathbf{j}_s \end{pmatrix} \quad (21)$$

for all $t \in \mathbb{R}$ *is* $\mathcal{F}^1$ *valued, infinitely often differentiable and a solution to* (18) *with* t_0 *initial value* F^0.

(ii) c For all $t \in \mathbb{R}$ *we have* $\nabla \cdot \mathbf{E}_t = 4\pi\rho_t$ *and* $\nabla \cdot \mathbf{B}_t = 0$.

Remark 3.7. Clearly, one needs less regularity of the initial values in order to get a strong solution. With regard to WF_ϱ, however, we will only need to consider smooth initial values $\mathcal{F}^1$. The explicit formula of the solutions (after an additional partial integration) was already found in [15, (A.24),(A.25)] where it was derived with the help of the Fourier transform. (There seems to be a misprint in the matrix on the r.h.s of equation (A.24). In their notation $m_t * (E_0, B_0)$ should equal the first summand of the r.h.s. of (21). However, (A.20) from which it is derived is correct.)

For the rest of this paper, the charge–current densities $(\rho, \mathbf{j})$ we will consider are the ones generated by a moving rigid charge on time-like world-lines (recall Definition 2.1):

Definition 3.8. (*The Charge–Current Density of a Charge world-line*) For $\varrho \in \mathcal{C}_c^\infty(\mathbb{R}^3, \mathbb{R})$ and $(\mathbf{q}, \mathbf{p}) \in \mathcal{T}^1_\odot$ define

$$\rho_t(\mathbf{x}) := \varrho(\mathbf{x} - \mathbf{q}_t), \;\; \mathbf{j}_t(\mathbf{x}) := \frac{\mathbf{p}_t}{\sqrt{m^2 + \mathbf{p}_t^2}} \varrho(\mathbf{x} - \mathbf{q}_t)$$

for all $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3$ which we call the induced charge–current density of $(\mathbf{q}, \mathbf{p})$.

Clearly, $(\rho, \mathbf{j}) \in \mathcal{D}$ so that Theorem 3.6 applies:

Definition 3.9. (*Maxwell Time Evolution*) Given a charge world-line $(\mathbf{q}, \mathbf{p}) \in \mathcal{T}^1_\odot$ which induces $(\rho, \mathbf{j}) \in \mathcal{D}$ we denote the solution $t \mapsto F_t$ of the Maxwell equations (18) given by Theorem 3.6 and t_0 initial values $F^0 = (\mathbf{E}^0, \mathbf{B}^0) \in \mathcal{F}^1$ by

$$t \mapsto M_\varrho[F^0, (\mathbf{q}, \mathbf{p})](t, t_0) := F_t.$$

One finds the following special solutions:

Theorem 3.10. *(Liénard–Wiechert Fields) Let* $F^0 = (\mathbf{E}^0, \mathbf{B}^0) \in \mathcal{F}^1$ *such that* $\nabla \cdot \mathbf{E}^0 = 4\pi\rho_{t_0}$ *and* $\nabla \cdot \mathbf{B}^0 = 0$ *as well as*

$$\|\mathbf{E}^0(\mathbf{x})\| + \|\mathbf{B}^0(\mathbf{x})\| + \|\mathbf{x}\| \sum_{i=1}^{3} \left(\|\partial_{\mathbf{x}_i} \mathbf{E}^0(\mathbf{x})\| + \|\partial_{\mathbf{x}_i} \mathbf{B}^0(\mathbf{x})\| \right) = \underset{\|\mathbf{x}\| \to \infty}{\mathrm{O}} \left(\|\mathbf{x}\|^{-\epsilon} \right) \quad (22)$$

for some $\epsilon > 0$ *and all* $\mathbf{x} \in \mathbb{R}^3$ *are fulfilled. We distinguish two cases denoted by + or − and assume that for all* $t \in \mathbb{R}$, $(\mathbf{q}, \mathbf{p}) \in \mathcal{T}^1_{\odot!}([t, \infty))$ *or* $(\mathbf{q}, \mathbf{p}) \in \mathcal{T}^1_{\odot!}((-\infty, t])$ *holds, respectively. Then the pointwise limit*

$$\begin{aligned} M_\varrho[\mathbf{q}, \mathbf{p}](t, \pm\infty) &:= \text{pw-lim}_{t_0 \to \pm\infty} M_\varrho[F^0, (\mathbf{q}, \mathbf{p})](t, t_0) \\ &= 4\pi \int_{\pm\infty}^{t} \mathrm{d}s \left[K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_s \\ \mathbf{j}_s \end{pmatrix} \right] = \int \mathrm{d}^3 z\, \varrho(\mathbf{z}) \begin{pmatrix} \mathbf{E}_t^{LW\pm}(\cdot - \mathbf{z}) \\ \mathbf{B}_t^{LW\pm}(\cdot - \mathbf{z}) \end{pmatrix} \end{aligned} \quad (23)$$

exists in $\mathcal{F}^1$ *where*

$$\mathbf{E}_t^{LW\pm}(\mathbf{x} - \mathbf{z}) := \left[\frac{(\mathbf{n} \pm \mathbf{v})(1 - \mathbf{v}^2)}{\|\mathbf{x} - \mathbf{z} - \mathbf{q}\|^2 (1 \pm \mathbf{n} \cdot \mathbf{v})^3} + \frac{\mathbf{n} \wedge [(\mathbf{n} \pm \mathbf{v}) \wedge \mathbf{a}]}{\|\mathbf{x} - \mathbf{z} - \mathbf{q}\| (1 \pm \mathbf{n} \cdot \mathbf{v})^3} \right]^{\pm} \quad (24)$$

$$\mathbf{B}_t^{LW\pm}(\mathbf{x} - \mathbf{z}) := \mp [\mathbf{n} \wedge \mathbf{E}_t(\mathbf{x} - \mathbf{z})]^{\pm} \quad (25)$$

and

$$\begin{array}{lll} \mathbf{q}^\pm := \mathbf{q}_{t^\pm} & \mathbf{v}^\pm := \mathbf{v}(\mathbf{p}_{t^\pm}) & \mathbf{a}^\pm := \partial_t \mathbf{v}(\mathbf{p}_t)|_{t=t^\pm} \\ \mathbf{n}^\pm := \frac{\mathbf{x}-\mathbf{z}-\mathbf{q}^\pm}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}^\pm\|} & t^\pm = t \pm \|\mathbf{x}-\mathbf{z}-\mathbf{q}^\pm\|. & \end{array} \tag{26}$$

Remark 3.11. Condition (22) guarantees that in the limit $t_0 \to \pm\infty$, the initial value F^0 is forgotten by the time evolution of the Maxwell equations. The condition that $(\mathbf{q}, \mathbf{p})$ are strictly time-like is only sufficient for the limits $t_0 \to \pm\infty$ to exist but necessary to yield formulas (24) and (25); note the blowup of the denominators $(1 \pm \mathbf{n} \cdot \mathbf{v})$ for $\|\mathbf{v}\| \to 1$.

Theorem 3.12. *(Liénard–Wiechert Fields Solve the Maxwell Equations) Let $(\mathbf{q}, \mathbf{p}) \in \mathcal{T}^1_{\ominus!}$, then the Liénard–Wiechert fields $M_\varrho[\mathbf{q}, \mathbf{p}](t, \pm\infty)$ are a solution to the Maxwell equations (18) including the Maxwell constraints for all $t \in \mathbb{R}$.*

We immediately get a simple bound on the Liénard–Wiechert fields:

Corollary 3.13. *(Liénard–Wiechert Estimate) Let $(\mathbf{q}, \mathbf{p}) \in \mathcal{T}^1_{\ominus!}$. Furthermore, assume there exists an $a_{max} < \infty$ such that $\sup_{t\in\mathbb{R}} \|\partial_t \mathbf{v}(\mathbf{p}_t)\| \le a_{max}$. Then the Liénard–Wiechert fields (24) and (25) fulfill: For any multi-index $\alpha \in \mathbb{N}_0^3$ there exists a constant ${C_1}^{(\alpha)} < \infty$ such that for all $\mathbf{x} \in \mathbb{R}^3$, $t \in \mathbb{R}$*

$$\|D^\alpha \mathbf{E}_t^\pm(\mathbf{x})\| + \|D^\alpha \mathbf{B}_t^\pm(\mathbf{x})\| \le \frac{{C_1}^{(\alpha)}}{(1-v_{max})^3} \left(\frac{1}{1+\|\mathbf{x}-\mathbf{q}_t\|^2} + \frac{a_{max}}{1+\|\mathbf{x}-\mathbf{q}_t\|} \right)$$

holds.

3.2. The ML-SI$_\varrho$ time evolution

Next, we briefly summarize the results of [4] on the ML-SI$_\varrho$ equations (8)–(9):

Definition 3.14. (*Weighted Square Integrable Functions*) We define the class of weight functions

$$\mathcal{W} := \left\{ w \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}^+ \setminus \{0\}) \;\middle|\; \exists\, C_w \in \mathbb{R}^+, P_w \in \mathbb{N} : w(\mathbf{x}+\mathbf{y}) \le (1 + C_w\|\mathbf{x}\|)^{P_w} w(\mathbf{y}) \right\}. \tag{27}$$

For any $w \in \mathcal{W}$ and open $\Omega \subseteq \mathbb{R}^3$, we define the space of weighted square integrable functions $\Omega \to \mathbb{R}^3$ by

$$L^2_w(\Omega, \mathbb{R}) := \left\{ \mathbf{F} : \Omega \to \mathbb{R}^3 \text{ measurable} \;\middle|\; \int \mathrm{d}^3x \; w(\mathbf{x}) \|\mathbf{F}(\mathbf{x})\|^2 < \infty \right\}.$$

For regularity arguments, we need more conditions on the weight functions. For $k \in \mathbb{N}$, we define

$$\mathcal{W}^k := \left\{ w \in \mathcal{W} \;\middle|\; \exists\, C_\alpha \in \mathbb{R}^+ : |D^\alpha \sqrt{w}| \le C_\alpha \sqrt{w}, |\alpha| \le k \right\} \tag{28}$$

and

$$\mathcal{W}^\infty := \bigcap_{k\in\mathbb{N}} \mathcal{W}^k.$$

Remark 3.15. As computed in [4], $\mathcal{W} \ni w(\mathbf{x}) := (1 + \|\mathbf{x}\|^2)^{-1}$.

The space of initial values is then given by:

Definition 3.16. (*Phase Space*) We define

$$\mathcal{H}_w := \bigoplus_{i=1}^N \left(\mathbb{R}^3 \oplus \mathbb{R}^3 \oplus L^2_w(\mathbb{R}^3, \mathbb{R}^3) \oplus L^2_w(\mathbb{R}^3, \mathbb{R}^3) \right).$$

Any element $\varphi \in \mathcal{H}_w$ consists of the components $\varphi = (\mathbf{q}_i, \mathbf{p}_i, \mathbf{E}_i, \mathbf{B}_i)_{1\le i\le N}$, that is, positions $\mathbf{q}_i$, momenta $\mathbf{p}_i$ and electric and magnetic fields $\mathbf{E}_i$ and $\mathbf{B}_i$ for each of the $1 \le i \le N$ charges.

If not noted otherwise, any spatial derivative will be understood in the distribution sense, and the Latin indices $i, j, \ldots$ shall run over the charge labels $1, 2, \ldots, N$. For $w \in \mathcal{W}$, open set $\Omega \subseteq \mathbb{R}^3$ and $k \geq 0$ we define the following Sobolev spaces

$$\begin{aligned} H_w^k(\Omega, \mathbb{R}^3) &:= \left\{ \mathbf{f} \in L_w^2(\Omega, \mathbb{R}^3) \;\middle|\; D^\alpha \mathbf{f} \in L_w^2(\Omega, \mathbb{R}^3), |\alpha| \leq k \right\}, \\ H_w^{\triangle^k}(\Omega, \mathbb{R}^3) &:= \left\{ \mathbf{f} \in L_w^2(\Omega, \mathbb{R}^3) \;\middle|\; \triangle^j \mathbf{f} \in L_w^2(\Omega, \mathbb{R}^3) \text{ for } 0 \leq j \leq k \right\}, \\ H_w^{curl}(\Omega, \mathbb{R}^3) &:= \left\{ \mathbf{f} \in L_w^2(\Omega, \mathbb{R}^3) \;\middle|\; \nabla \wedge \mathbf{f} \in L_w^2(\Omega, \mathbb{R}^3) \right\} \end{aligned} \tag{29}$$

which are equipped with the inner products

$$\begin{aligned} \langle \mathbf{f}, \mathbf{g} \rangle_{H_w^k} &:= \sum_{|\alpha| \leq k} \langle D^\alpha \mathbf{f}, D^\alpha \mathbf{g} \rangle_{L_w^2(\Omega)}, \langle \mathbf{f}, \mathbf{g} \rangle_{H_w^\triangle(\Omega)} := \sum_{j=0}^{k} \left\langle \triangle^j \mathbf{f}, \triangle^j \mathbf{g} \right\rangle_{L_w^2(\Omega)} \\ \langle \mathbf{f}, \mathbf{g} \rangle_{H_w^{curl}(\Omega)} &:= \langle \mathbf{f}, \mathbf{g} \rangle_{L_w^2(\Omega)} + \langle \nabla \wedge \mathbf{f}, \nabla \wedge \mathbf{g} \rangle_{L_w^2(\Omega)}, \end{aligned}$$

respectively. We use the multi-index notation $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in (\mathbb{N}_0)^3$, $|\alpha| := \sum_{i=1}^3 \alpha_i$, $D^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$ where ∂_i denotes the derivative w.r.t. to the i-th standard unit vector in $\mathbb{R}^3$. In order to appreciate the structure of the ML equations, we will rewrite them using the following operators A and J:

Definition 3.17. (*Operator A*) For a $\varphi = (\mathbf{q}_i, \mathbf{p}_i, \mathbf{E}_i, \mathbf{B}_i)_{1 \leq i \leq N}$ we defined $\mathtt{A}$ and A by the expression

$$A\varphi = \Big(0, 0, \mathtt{A}(\mathbf{E}_i, \mathbf{B}_i)\Big)_{1 \leq i \leq N} := \Big(0, 0, -\nabla \wedge \mathbf{E}_i, \nabla \wedge \mathbf{B}_i)\Big)_{1 \leq i \leq N}.$$

on their natural domain

$$D_w(A) := \bigoplus_{i=1}^{N} \left(\mathbb{R}^3 \oplus \mathbb{R}^3 \oplus H_w^{curl}(\mathbb{R}^3, \mathbb{R}^3) \oplus H_w^{curl}(\mathbb{R}^3, \mathbb{R}^3) \right) \subset \mathcal{H}_w.$$

Furthermore, for any $n \in \mathbb{N}$ we define

$$D_w(A^n) := \left\{ \varphi \in D_w(A) \;\middle|\; A^k \varphi \in D_w(A) \quad \text{for } k = 0, \ldots, n-1 \right\}, \; D_w(A^\infty) := \bigcap_{n=0}^{\infty} D_w(A^n).$$

Definition 3.18. (*Operator J*) Together with $\mathbf{v}(\mathbf{p}_i) := \frac{\mathbf{p}_i}{\sqrt{\mathbf{p}_i^2 + m^2}}$ we define $J : \mathcal{H}_w \to D_w(A^\infty)$ by

$$\varphi \mapsto J(\varphi) := \left(\mathbf{v}(\mathbf{p}_i), \sum_{j \neq i}^{N} \int \mathrm{d}^3 x \; \varrho_i(\mathbf{x} - \mathbf{q}_i) \left(\mathbf{E}_j(\mathbf{x}) + \mathbf{v}(\mathbf{p}_i) \wedge \mathbf{B}_j(x) \right), -4\pi \mathbf{v}(\mathbf{p}_i) \varrho_i(\cdot - \mathbf{q}_i), 0 \right)_{1 \leq i \leq N}$$

for $\varphi = (\mathbf{q}_i, \mathbf{p}_i, \mathbf{E}_i, \mathbf{B}_i)_{1 \leq i \leq N}$.

Note that J is well defined because $\varrho_i \in \mathcal{C}_c^\infty(\mathbb{R}^3, \mathbb{R})$, $1 \leq i \leq N$. With these definitions, the Lorentz force law (9) and the Maxwell equations (8) without the Maxwell constraints take the form

$$\partial_t \varphi_t = A\varphi_t + J(\varphi_t). \tag{30}$$

The two main theorems are as follows:

Theorem 3.19. *(Global Existence and Uniqueness) For $w \in \mathcal{W}^1$, $n \in \mathbb{N}$ and $\varphi^0 \in D_w(A^n)$ the following holds:*

(i) (global existence) *There exists an n-times continuously differentiable mapping*

$$\varphi_{(\cdot)} : \mathbb{R} \to \mathcal{H}_w,\ \ t \mapsto \varphi_t = (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N}$$

which solves (30) for initial value $\varphi_t|_{t=0} = \varphi^0$. Furthermore, it holds $\frac{\mathrm{d}^j}{\mathrm{d}t^j}\varphi_t \in D_w(A^{n-j})$ for all $t \in \mathbb{R}$ and $0 \le j \le n$,

(ii) (uniqueness and growth) *Any once continuously differentiable function $\widetilde{\varphi} : \Lambda \to D_w(A)$ for some open interval $\Lambda \subseteq \mathbb{R}$ which fulfills $\widetilde{\varphi}_{t^*} = \varphi_{t^*}$ for an $t^* \in \Lambda$, and which also solves the Eq. (30) on Λ, has the property that $\varphi_t = \widetilde{\varphi}_t$ holds for all $t \in \Lambda$. In particular, given ϱ_i, $1 \le i \le N$ there exists $C_2 \in$ `Bounds` such that for $T > 0$ with $[-T,T] \subset \Lambda$ it holds*

$$\sup_{t\in[-T,T]} \|\varphi_t - \widetilde{\varphi}_t\|_{\mathcal{H}_w} \le C_{??}(T, \|\varphi_{t_0}\|_{\mathcal{H}_w}, \|\widetilde{\varphi}_{t_0}\|_{\mathcal{H}_w})\|\varphi_{t_0} - \widetilde{\varphi}_{t_0}\|_{\mathcal{H}_w}. \tag{31}$$

Furthermore, there is a $C_3 \in$ `Bounds` such that for all ϱ_i, $1 \le i \le N$,

$$\sup_{t\in[-T,T]} \|\varphi_t\|_{\mathcal{H}_w} \le C_3\left(T, \|w^{-1/2}\varrho_i\|_{L^2}, \|\varrho_i\|_{L^2_w}; 1 \le i \le N\right)\ \|\varphi^0\|_{\mathcal{H}_w}. \tag{32}$$

(iii) (constraints) *If the solution $t \mapsto \varphi_t = (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N}$ obeys the Maxwell constraints*

$$\nabla \cdot \mathbf{E}_{i,t} = 4\pi\varrho_i(\cdot - \mathbf{q}_{i,t}),\ \ \nabla \cdot \mathbf{B}_{i,t} = 0 \tag{33}$$

for $1 \le i \le N$ and one time instant $t \in \mathbb{R}$, then they are obeyed for all times $t \in \mathbb{R}$.

Theorem 3.20. *(Regularity) Assume the same conditions as in Theorem 3.19 hold and let $t \mapsto \varphi_t = (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N}$ be the solution to (30) for initial value $\varphi^0 \in D_w(A^n)$. In addition, let $w \in \mathcal{W}^2$ and $n = 2m$ for $m \in \mathbb{N}$. Then for all $1 \le i \le N$:*

(i) It holds for any $t \in \mathbb{R}$ that $\mathbf{E}_{i,t}, \mathbf{B}_{i,t} \in \mathcal{H}_w^{\triangle^m}$.

(ii) The electromagnetic fields regarded as maps $\mathbf{E}_i : (t,\mathbf{x}) \mapsto \mathbf{E}_{i,t}(\mathbf{x})$ and $\mathbf{B}_i : (t,\mathbf{x}) \mapsto \mathbf{B}_{i,t}(\mathbf{x})$ are in $L^2_{loc}(\mathbb{R}^4, \mathbb{R}^3)$ and both have a representative in $\mathcal{C}^{n-2}(\mathbb{R}^4, \mathbb{R}^3)$ within their equivalence class.

(iii) For $w \in \mathcal{W}^k$ for $k \ge 2$ and every $t \in \mathbb{R}$ we have also $\mathbf{E}_{i,t}, \mathbf{B}_{i,t} \in H^n_w$ and $C < \infty$ such that:

$$\sup_{\mathbf{x}\in\mathbb{R}^3} \sum_{|\alpha|\le k} \|D^\alpha \mathbf{E}_{i,t}(\mathbf{x})\| \le C\|\mathbf{E}_{i,t}\|_{H^k_w},\ \ \sup_{\mathbf{x}\in\mathbb{R}^3} \sum_{|\alpha|\le k} \|D^\alpha \mathbf{B}_{i,t}(\mathbf{x})\| \le C\|\mathbf{B}_{i,t}\|_{H^k_w}. \tag{34}$$

As shown in [4, Lemma 2.19], A on $D_w(A)$ is a closed operator that generates a γ-contractive group $(W_t)_{t\in\mathbb{R}}$:

Definition 3.21. (*Free Maxwell Time Evolution*) We denote by $(W_t)_{t\in\mathbb{R}}$ the γ-contractive group on $\mathcal{H}_w$ generated by A on $D_w(A)$.

Remark 3.22. The γ-contractive group $(W_t)_{t\in\mathbb{R}}$ comes with a standard bound $\|W_t\varphi\|_{\mathcal{H}_w} \le e^{\gamma|t|}\|\varphi\|_{\mathcal{H}_w}$ for all $\varphi \in \mathcal{H}_w$ for some $\gamma \ge 0$.

The above existence and uniqueness result implies

Definition 3.23. (*ML Time Evolution*) We define the nonlinear operator

$$M_L : \mathbb{R}^2 \times D_w(A) \to D_w(A),\ \ (t, t_0, \varphi^0) \mapsto M_L(t, t_0)[\varphi^0] = \varphi_t = W_{t-t_0}\varphi^0 + \int_{t_0}^{t} W_{t-s}J(\varphi_s)$$

which encodes the ML time evolution from time t_0 to time t.

Using the presented results in Sect. 3.1 on the Maxwell equations, we can give explicit expressions of the free Maxwell time evolution group $(W_t)_{t\in\mathbb{R}}$ and the ML-SI$_\varrho$ time evolution for initial fields fulfilling both the regularity requirements of $D_w(A)$ and of $\mathcal{F}^1$. The following short-hand notation will be convenient:

Notation 3.24. (Projectors $\mathtt{P}, \mathtt{Q}, \mathtt{F}$) *For any* $\varphi = (\mathbf{q}_i, \mathbf{p}_i, \mathbf{E}_i, \mathbf{B}_i)_{1\le i\le N} \in \mathcal{H}_w$ *we define*

$$\mathtt{Q}\varphi = (\mathbf{q}_i, 0, 0, 0)_{1\le i\le N}, \mathtt{P}\varphi = (0, \mathbf{p}_i, 0, 0)_{1\le i\le N}, \mathtt{F}\varphi = (0, 0, \mathbf{E}_i, \mathbf{B}_i)_{1\le i\le N}.$$

where we sometime neglect the zeros and write for example

$$(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N} = (\mathtt{Q}+\mathtt{P})\varphi or (\mathbf{q}_i, \mathbf{p}_i, 0, 0)_{1\le i\le N} = (\mathtt{Q}+\mathtt{P})(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N}.$$

Definition 3.25. (*Projection of* A, W_t, J *to Field Space* $\mathcal{F}_w$) For all $t \in \mathbb{R}$ and $\varphi \in \mathcal{H}_w$ we define

$$\mathcal{F}_w := F\mathcal{H}_w, \mathsf{A} := \mathtt{F}A\mathtt{F}, \mathsf{W}_t := \mathtt{F}W_t\mathtt{F}, \mathsf{J} := \mathtt{F}J(\varphi).$$

The natural domain of A is given by $D_w(\mathsf{A}) := \mathtt{F}D_w(A) \subset \mathcal{F}_w$. We shall also need $D_w(\mathsf{A}^n) := \mathtt{F}D_w(A^n) \subset \mathcal{F}_w$ for every $n \in \mathbb{N}$ and $D_w(\mathsf{A}^\infty) := \mathtt{F}D_w(A^\infty)$.

Note the distinction between roman and sans serif letters, for example, A and A. Clearly, $\mathcal{F}_w$ is a Hilbert space, the operator A on $D_w(\mathsf{A})$ is again closed and inherits the resolvent properties from A on $D_w(A)$. This implies $(\mathtt{Q}+\mathtt{P})W_t = \mathrm{id}_{\mathcal{P}}$ and $\mathtt{F}W_t = \mathsf{W}_t$ so that $(\mathsf{W}_t)_{t\in\mathbb{R}}$ is also a γ-contractive group generated by A on $D_w(\mathsf{A})$. Finally, note also that by the definition of J we have $\mathsf{J}(\varphi) = \mathsf{J}((\mathtt{Q}+\mathtt{P})\varphi)$ for all $\varphi \in \mathcal{H}_w$, that is, J does not depend on the field components $\mathtt{F}\varphi$.

We extend the space of fields $\mathcal{F}^1$, cf. Definition 3.2, to comprise N electric and magnetic fields:

Definition 3.26. (*Space of* N *Smooth Fields*) $\mathcal{F}^N := \bigoplus_{i=1}^N \mathcal{F}^1 = \bigoplus_{i=1}^N \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}^3) \oplus \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}^3)$.

The following corollary gives an explicit expression for the action of the group $(\mathsf{W}_t)_{t\in\mathbb{R}}$ using the results from in Sect. 3.1 about the free Maxwell equation.

Corollary 3.27. (Kirchoff's Formulas for $(W_t)_{t\in\mathbb{R}}$) *Let* $w \in \mathcal{W}^1$, $F \in D_w(\mathsf{A}^n) \cap \mathcal{F}^N$ *for some* $n \in \mathbb{N}$, *and*

$$(\mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N} := \mathsf{W}_t F, \qquad t \in \mathbb{R}.$$

Then

$$\begin{pmatrix} \widetilde{\mathbf{E}}_{i,t} \\ \widetilde{\mathbf{B}}_{i,t} \end{pmatrix} = \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_t * \begin{pmatrix} \mathbf{E}_{i,0} \\ \mathbf{B}_{i,0} \end{pmatrix} - \int_0^t \mathrm{d}s\, K_{t-s} * \begin{pmatrix} \nabla\nabla\cdot\mathbf{E}_{i,0} \\ \nabla\nabla\cdot\mathbf{B}_{i,0} \end{pmatrix}$$

fulfill $\mathbf{E}_{i,t} = \widetilde{\mathbf{E}}_{i,t}$ *and* $\mathbf{B}_{i,t} = \widetilde{\mathbf{B}}_{i,t}$ *for all* $t \in \mathbb{R}$ *and* $1 \le i \le N$ *in the* L^2_w *sense. Furthermore, for all* $t \in \mathbb{R}$ *it holds also that* $(\mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N} \in D_w(\mathsf{A}^n) \cap \mathcal{F}^N$.

Proof. A direct application of Lemma 3.4 and Definition 3.21. □

From this corollary, we can also express the inhomogeneous Maxwell time evolution, cf. Definition 3.9, in terms of $(\mathsf{W}_t)_{t\in\mathbb{R}}$ and J.

Lemma 3.28. (The Maxwell Solutions in Terms of $(\mathsf{W}_t)_{t\in\mathbb{R}}$ and J) *Let times* $t, t_0 \in \mathbb{R}$ *be given,* $F = (F_i)_{1\le i\le N} \in D_w(\mathsf{A}^n) \cap \mathcal{F}^N$ *for some* $n \in \mathbb{N}$ *be given initial fields, and* $(\mathbf{q}_i, \mathbf{p}_i) \in \mathcal{T}^1_{\odot}$ *time-like charge world-lines for* $1 \le i \le N$. *In addition suppose the initial fields* $F_i = (\mathbf{E}_i, \mathbf{B}_i)$, $1 \le i \le N$, *fulfill the Maxwell constraints*

$$\nabla\cdot\mathbf{E}_i = 4\pi\varrho_i(\cdot - \mathbf{q}_{i,t_0}),\ \nabla\cdot\mathbf{B}_i = 0.$$

Then for all $t \in \mathbb{R}$

$$F_t := \mathsf{W}_{t-t_0}F + \int_{t_0}^t \mathrm{d}s\, \mathsf{W}_{t-s}\mathsf{J}(\varphi_s) \in D_w(\mathsf{A}^n) = \left(M_{\varrho_i}[F_i, (\mathbf{q}_i, \mathbf{p}_i)](t, t_0)\right)_{1\le i\le N}$$

holds in the L^2_w *sense where* $\varphi_s := (\mathtt{Q}+\mathtt{P})(\mathbf{q}_{i,s}, \mathbf{p}_{i,s})_{1\le i\le N}$ *for* $s \in \mathbb{R}$. *Furthermore,* $F_t \in D_w(\mathsf{A}^n) \cap \mathcal{F}^N$ *for all* $t \in \mathbb{R}$.

Proof. This can be computed by applying Corollary 3.27 twice and using one partial integration. □

4. Proofs

4.1. Weak uniqueness of WF$_\varrho$ and Synge solutions by ML-SI$_\varrho$ Cauchy data

Our first goal is to prove Theorem 2.3. Recall the Definition 2.2 where we defined what we mean by solutions to Eqs. (5)–(6) for particular choices of e_+ and e_-. Recall that $e_+ = \frac{1}{2}, e_- = \frac{1}{2}$ and $e_+ = 0, e_- = 1$ corresponds to the WF$_\varrho$ equations and the Synge equations, respectively.

Remark 4.1. (1) Note that Definition 2.2 is sensible because with $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N} \in \mathcal{T}^N_{\odot!}$, Eq. (6) for $1 \le i \le N$ coincide with

$$(\mathbf{E}^{(e_+,e_-)}_{i,t}, \mathbf{B}^{(e_+,e_-)}_{i,t}) = \sum_\pm e_\pm M_{\varrho_i}[\mathbf{q}_i, \mathbf{p}_i](t, \pm\infty)$$

by definition in (23). Theorem 3.10 guarantees that the right-hand side is well defined, and charge world-lines in $\mathcal{T}^1_{\odot!}$ are once continuously differentiable so that the left-hand side of (5) is also well defined. The bound on the acceleration will give us a bound on the WF$_\varrho$ fields in a suitable norm; cf. Lemma 4.4. (2) Furthermore, there is no doubt that $\mathcal{T}^N_{(e_+,e_-)}$ is nonempty because in the point-particle case the Schild solutions [21] as well as the solutions of Bauer's existence theorem [3] have smooth and strictly time-like charge world-lines with bounded accelerations.

As discussed in Sect. 3.2, the electric and magnetic fields live in the L^2_w space for a conveniently chosen weight $w \in \mathcal{W}^\infty$, cf. Definitions 3.14 and 3.16. In the following, we give an example weight w and show that with it the Liénard–Wiechert fields of charge world-lines in $\mathcal{T}^N_{\odot!}$ with uniformly bounded accelerations are admissible as ML-SI$_\varrho$ initial data; cf. Theorem 3.19 and Definition 3.17.

Definition 4.2. (*Example Weight*) We define the function

$$w : \mathbb{R}^3 \to \mathbb{R}^+ \setminus \{0\}, \quad \mathbf{x} \mapsto w(\mathbf{x}) := (1 + \|\mathbf{x}\|^2)^{-1}. \tag{35}$$

A straightforward computation given in [5] yields:

Lemma 4.3. *The function w is an element of $\mathcal{W}^\infty$.*

Lemma 4.4. (Regularity of the Liénard–Wiechert Fields) *Let $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N} \in \mathcal{T}^N_{\odot!}$ and assume there exists a constant $a_{max} < \infty$ such that for all $1 \le i \le N$, $\sup_{t\in\mathbb{R}} \|\partial_t \mathbf{v}(\mathbf{p}_{i,t})\| \le a_{max}$. Define $t \mapsto (\mathbf{E}_{i,t}, \mathbf{B}_{i,t}) := M_{\varrho_i}[\mathbf{q}_i, \mathbf{p}_i](t, \pm\infty)$. Then for all $t \in \mathbb{R}$*

$$(\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N} \in D_w(A^\infty),$$

holds true.

Proof. By Corollary 3.13, for $1 \le i \le N$ and each multi-index $\alpha \in \mathbb{N}^3_0$ there exists a constant ${C_1}^{(\alpha)} < \infty$ such that

$$\|D^\alpha \mathbf{E}^\pm_{i,t}(\mathbf{x})\| + \|D^\alpha \mathbf{B}^\pm_{i,t}(\mathbf{x})\| \le \frac{{C_1}^{(\alpha)}}{(1 - v_{max})^3} \left(\frac{1}{1 + \|\mathbf{x} - \mathbf{q}_t\|^2} + \frac{a_{max}}{1 + \|\mathbf{x} - \mathbf{q}_t\|} \right).$$

Hence, $w(\mathbf{x}) = \frac{1}{1+\|\mathbf{x}\|^2}$ ensures that

$$\left\|A^n(\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}^\pm_{i,t}, \mathbf{B}^\pm_{i,t})\right\|_{\mathcal{H}_w} \le \sum_{i=1}^N \sum_{|\alpha|\le n} \left(\|\mathbf{q}_{i,t}\| + \|\mathbf{p}_{i,t}\| + \int d^3x\, w(\mathbf{x}) \left(\|D^\alpha \mathbf{E}^\pm_{i,t}(\mathbf{x})\|^2 + \|D^\alpha \mathbf{B}^\pm_{i,t}(\mathbf{x})\|^2\right)\right)$$

is finite for all $n \in \mathbb{N}_0$, $t \in \mathbb{R}$. We conclude that $\varphi_t \in D_w(A^\infty)$ for all $t \in \mathbb{R}$. □

We prove the first main result:

Proof of Theorem 2.3 (Weak Uniqueness of Solutions). (i) Since $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N} \in \mathcal{T}^N_{(e_+,e_-)}$, Lemma 4.4 guarantees $\varphi_{t_0} \in D_w(A^\infty)$ for all $t_0 \in \mathbb{R}$.

ii) First, the charge world-lines $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N} \in \mathcal{T}^N_{(e_+,e_-)}$ are once continuously differentiable and fulfill the WF$_\varrho$ equations (5)–(6). Second, by Theorem 3.12 the fields $(\mathbf{E}^{(e_+,e_-)}_t[\mathbf{q}_i,\mathbf{p}_i], \mathbf{B}^{(e_+,e_-)}_t[\mathbf{q}_i,\mathbf{p}_i])$ given in (6) fulfill the Maxwell equations (8) including the Maxwell constraints for all $t \in \mathbb{R}$ and $1 \le i \le N$. Hence, using (i), the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi^{(e_+,e_-)}_t[(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}] = A\varphi^{(e_+,e_-)}_t[(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}] + J(\varphi^{(e_+,e_-)}_t[(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}]),$$

holds true [recall the notation in Sect. 3.2 before Eq. (30)]. Due to (i) also $\phi_t := M_L\left[\varphi^{(e_+,e_-)}_{t_0}[(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}]\right]$, $t \in \mathbb{R}$, is well defined; cf. Definition 3.23. Theorem 3.19 states that ϕ_t is the only solution of $\partial_t\phi_t = A\phi_t + J(\phi_t)$ which fulfills $\phi_{t_0} = \varphi^{(e_+,e_-)}_{t_0}[(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}]$. Hence, $\phi_t = \varphi^{(e_+,e_-)}_t[(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}]$ holds for all $t \in \mathbb{R}$.

(iii) Suppose $(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}, (\widetilde{\mathbf{q}}_i,\widetilde{\mathbf{p}}_i)_{1\le i\le N} \in \mathcal{T}^N_{(e_+,e_-)}$ and define $\varphi := i_{t_0}((\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N})$, $\widetilde{\varphi} := i_{t_0}((\widetilde{\mathbf{q}}_i,\widetilde{\mathbf{p}}_i)_{1\le i\le N})$ for some $t_0 \in \mathbb{R}$. According to (ii) we also set

$$\begin{aligned}\varphi_t &= (\mathbf{q}_{i,t},\mathbf{p}_{i,t},\mathbf{E}_{i,t},\mathbf{B}_{i,t})_{1\le i\le N} := M_L[\varphi](t,t_0),\\ \widetilde{\varphi}_t &= (\widetilde{\mathbf{q}}_{i,t},\widetilde{\mathbf{p}}_{i,t},\widetilde{\mathbf{E}}_{i,t},\widetilde{\mathbf{B}}_{i,t})_{1\le i\le N} := M_L[\widetilde{\varphi}](t,t_0)\end{aligned}$$

for all $t \in \mathbb{R}$. Now, $\varphi = \widetilde{\varphi}$ implies $\varphi_t = \widetilde{\varphi}_t$ for all $t \in \mathbb{R}$. Hence, $(\mathbf{q}_{i,t},\mathbf{p}_{i,t})_{1\le i\le N} = (\widetilde{\mathbf{q}}_{i,t},\widetilde{\mathbf{p}}_{i,t})_{1\le i\le N}$ for all $t \in \mathbb{R}$, that is, $(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N} = (\widetilde{\mathbf{q}}_i,\widetilde{\mathbf{p}}_i)_{1\le i\le N}$ by Definition 2.1. □

4.2. Existence and uniqueness of Synge solutions for given histories

We continue with the proof of our second main result:

Proof of Theorem 2.5 (Existence and Uniqueness of Synge Solutions). Set $e_+ = 0$ and $e_- = 1$. (i) By definition $(\mathbf{q}^-_i,\mathbf{p}^-_i)_{1\le i\le N} \in \mathcal{T}^N_{\odot!}$, so that due to Theorem 3.10 and (6) for all $t \le t_0$ we can define

$$\varphi^-_t = (\mathbf{q}^-_{i,t},\mathbf{p}^-_{i,t},\mathbf{E}_t[\mathbf{q}^-_i,\mathbf{p}^-_i],\mathbf{B}_t[\mathbf{q}^-_i,\mathbf{p}^-_i])_{1\le i\le N}$$

where the fields are given by the retarded Liénard–Wiechert fields of the history $(\mathbf{q}^-,\mathbf{p}^-) \in \mathfrak{H}^-(t_0)$, that is,

$$(\mathbf{E}_t[\mathbf{q}^-_i,\mathbf{p}^-_i],\mathbf{B}_t[\mathbf{q}^-_i,\mathbf{p}^-_i]) = M_{\varrho_i}[\mathbf{q}^-_i,\mathbf{p}^-_i](t,-\infty)).$$

Lemma 4.4 states $\varphi^-_{t_0} \in D_w(A^\infty)$. Hence, by Theorem 3.19, there is a unique mapping

$$t \mapsto (\mathbf{q}^+_{i,t},\mathbf{p}^+_{i,t},\mathbf{E}^+_{i,t},\mathbf{B}^+_{i,t})_{1\le i\le N} = \varphi^+_t := M_L[\varphi^-_{t_0}](t,t_0) \tag{36}$$

such that $\varphi^+_{t_0} = \varphi^-_{t_0}$. Let $(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}$ be the concatenation defined in (13). We consider now

$$\varphi_t = (\mathbf{q}_{i,t},\mathbf{p}_{i,t},\mathbf{E}_t[\mathbf{q}_i,\mathbf{p}_i],\mathbf{B}_t[\mathbf{q}_i,\mathbf{p}_i])_{1\le i\le N}$$

for all $t \in \mathbb{R}$ with the retarded Liénard–Wiechert fields of $(\mathbf{q}_i,\mathbf{p}_i)$ given by

$$(\mathbf{E}_t[\mathbf{q}_i,\mathbf{p}_i],\mathbf{B}_t[\mathbf{q}_i,\mathbf{p}_i]) := M_{\varrho_i}[\mathbf{q}_i,\mathbf{p}_i](t,-\infty), \tag{37}$$

which are well defined if $(\mathbf{q}_i,\mathbf{p}_i)$ would be in $\mathcal{T}^1_{\odot!}((-\infty,T])$ for all $T \in \mathbb{R}$. However, $(\mathbf{q}^-_i,\mathbf{q}^-_i)_{1\le i\le N} \in \mathcal{T}^N_{\odot!}$, and $(\mathbf{q}^+_i,\mathbf{p}^+_i)_{1\le i\le N}$ is continuously differentiable so that we only need to check that $(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}$ is continuously differentiable at $t = t_0$. Now according to the assumption, at time $t = t_0$, the history $(\mathbf{q}^-_i,\mathbf{p}^-_i)_{1\le i\le N}$ solves Eqs. (5)–(6) for $e_+ = 0$ and $e_- = 1$, and furthermore, Theorem 3.12 states that $(\mathbf{E}_t[\mathbf{q}^-_i,\mathbf{p}^-_i],\mathbf{B}_t[\mathbf{q}^-_i,\mathbf{p}^-_i])_{1\le i\le N}$ solve the Maxwell equations at $t = t_0$. Hence, we have

$$\lim_{t\nearrow t_0}\frac{\mathrm{d}}{\mathrm{d}t}\varphi^-_t = A\varphi^-_{t_0} + J(\varphi^-_{t_0}) = \lim_{t\searrow t_0}\frac{\mathrm{d}}{\mathrm{d}t}\varphi^+_t,$$

that is, $(\mathbf{q}_i, \mathbf{p}_i) \in \mathcal{T}^1_{\odot!}((-\infty, T])$ for $1 \leq i \leq N$ and any $T \in \mathbb{R}$ so that (37) is well defined. With the help of Theorem 3.19 for all $t \geq t_0$ we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t - \varphi_t^+) = A(\varphi_t - \varphi_t^+) + \left[J(\varphi_s) - J(\varphi_s^+)\right] = A(\varphi_t - \varphi_t^+) \tag{38}$$

because J does only depend on the charge world-lines. The only solution to this equation is $W_t(\varphi_{t_0} - \varphi_{t_0}^+) = 0$; cf. Definition 3.21. Hence, $(\mathbf{E}^+_{i,t}, \mathbf{B}^+_{i,t}) = (\mathbf{E}_t[\mathbf{q}_i, \mathbf{p}_i], \mathbf{B}_t[\mathbf{q}_i, \mathbf{p}_i])$ for $1 \leq i \leq N$ and all $t \geq t_0$, that is, the fields generated by the ML-SI$_\varrho$ time evolution equal the retarded Liénard–Wiechert fields corresponding to the charge world-lines generated by ML-SI$_\varrho$ time evolution. This implies that $(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N}$ solve the WF$_\varrho$ equations (5)–(6) for $e_+ = 0$ and $e_- = 1$ and all $t \geq t_0$.

(ii) Since $(\widetilde{\mathbf{q}}_{i,t}, \widetilde{\mathbf{p}}_{i,t}) = (\mathbf{q}^-_{i,t}, \mathbf{p}^-_{i,t})_{1 \leq i \leq N}$ for all $t \leq t_0$, the claim follows from the uniqueness of the map (36). □

Remark 4.5. (1) Condition (ii) in Definition 2.4 is only needed to ensure continuity of the derivative of the charge world-lines at t_0. Theorem 3.10 can be generalized to piecewise $\mathcal{C}^1$ charge world-lines. Using this generalization, Theorem 2.5 can be proven without this condition, ensuring the existence of piecewise $\mathcal{C}^1$ Synge solutions for $t \geq t_0$. However, this condition is not restrictive in the sense that one had to fear $\mathfrak{H}(t_0)$ could be empty. Elements of $\mathfrak{H}(t_0)$ can be constructed with the following algorithm:

1. Choose positions and momenta $(\mathbf{q}^-_{i,t_0}, \mathbf{q}^-_{i,t_0})$ for $1 \leq i \leq N$ particles at time t_0.
2. For $1 \leq i \leq N$ choose $(\mathbf{q}^-_{i,t}, \mathbf{p}^-_{i,t})$ on time intervals from $-\infty$ up to the latest intersection of the backward light-cones of space-time points $(t_0, \mathbf{q}^-_{j,t_0})$, $j \neq i$, before time t_0.
3. Use the Synge equations to compute the acceleration for all $1 \leq i \leq N$ charges at t_0.
4. For $1 \leq i \leq N$ extend $(\mathbf{q}^-_{i,t}, \mathbf{p}^-_{i,t})$ up to time t_0 smoothly such that they connect to the chosen $(\mathbf{q}^-_{i,t_0}, \mathbf{q}^-_{i,t_0})$ with the correct acceleration computed in step 3.

(2) From the geometry of the Liénard–Wiechert fields, it is clear that the whole history $(\mathbf{q}^-_{i,t}, \mathbf{p}^-_{i,t})_{1 \leq i \leq N}$ for $t \leq t_0$ is sufficient for uniqueness but not necessary. The necessary data for the charge world-lines $(\mathbf{q}^-_{i,t}, \mathbf{p}^-_{i,t})_{1 \leq i \leq N}$ that identify a Synge solution for $t \geq t_0$ uniquely are the shortest world-line strips, so that the backward light-cone of each space-time point $(t, \mathbf{q}^-_{i,t_0})$ intersects all other charge world-lines $(\mathbf{q}^-_j, \mathbf{p}^-_j)$, $j \neq i$.

4.3. Existence of WF$_\varrho$ solutions on finite time intervals

We shall now prove the remaining main results Theorems 2.6 and Theorem 2.7. For the rest of this work, we keep the choice $e_+ = \frac{1}{2}$, $e_- = \frac{1}{2}$ fixed. The results, however, hold also for any choices of $0 \leq e_+, e_- \leq 1$. The strategy will be to use Schauder's fixed point theorem to prove the existence of a fixed point of $S_T^{X^\pm}$. Recall the distinction between roman and sans serif letters in Definition 3.25. We generalize the definition of $\mathcal{F}_w$:

Definition 4.6. (*Hilbert Spaces for the Fixed Point Theorem*) Given $n \in \mathbb{N}$ we define $\mathcal{F}^n_w$ to be the linear space of elements $F \in D_w(\mathsf{A}^n)$ equipped with the inner product

$$\langle F, G \rangle_{\mathcal{F}^n_w} := \sum_{k=0}^{n} \left\langle \mathsf{A}^k F, \mathsf{A}^k G \right\rangle_{\mathcal{F}_w}.$$

The corresponding norm is denoted by $\| \cdot \|_{\mathcal{F}^n_w}$ and we shall use the notation

$$\|F\|_{\mathcal{F}^n_w(B)} := \left(\sum_{k=0}^{n} \|A^k F\|^2_{L^2_w(B)} \right)^{1/2}$$

to denote the restriction of the norm to a subset $B \subset \mathbb{R}^3$.

Lemma 4.7. *For $n \in \mathbb{N}$, $\mathcal{F}_w^n$ is a Hilbert space.*

Proof. This is an immediate consequence of [4, Theorem 2.10] and relies on the fact that A is closed on $D_w(\mathsf{A})$. □

As explained in Sect. 2, we encode the continuation of the charge world-lines for times $|t| \geq T$ in terms of advanced and retarded Liénard–Wiechert fields $X^+_{i,+T}$ and $X^-_{i,-T}$, respectively. These fields are generated by the prescribed charge world-lines for times $|t| \geq T$ and evaluated at time T. They must depend on the charge world-lines within $[-T, T]$ because we want to impose certain regularity conditions at the connection times $t = \pm T$. Since these world-lines will be generated within the iteration of $S_T^{p,X^\pm}$ by the ML-SI$_\varrho$ time evolution, this dependence can be expressed simply by the dependence on the ML-SI$_\varrho$ initial data $(p, F) = \varphi \in D_w(A^\infty)$. We shall therefore use the notation $X^\pm_{i,\pm T}[\varphi]$ for the boundary fields.

Next, we introduce three classes of such boundary fields for our discussion, namely $\mathcal{A}_w^n \supset \widetilde{\mathcal{A}}_w^n \supset \mathcal{A}^{\mathrm{Lip}}$. The class $\mathcal{A}_w^n$ will allow to define what we mean by a conditional WF$_\varrho$ solution (see Definition 4.10 below). The existence of conditional WF$_\varrho$ solutions is then shown for the class $\widetilde{\mathcal{A}}_w^n$ with $n = 3$. The third class, $\mathcal{A}^{\mathrm{Lip}}$, is only needed for Remark 4.18 where we discuss uniqueness of the conditional WF$_\varrho$ solution for small enough T. We define

Definition 4.8. (*Boundary Fields Classes* $\mathcal{A}_w^n$, $\widetilde{\mathcal{A}}_w^n$ and $\mathcal{A}_w^{\mathrm{Lip}}$) For weight $w \in \mathcal{W}$ and $n \in \mathbb{N}$, we define $\mathcal{A}_w^n$ to be the set of maps

$$X : \mathbb{R} \times D_w(A) \to D_w(\mathsf{A}^\infty) \cap \mathcal{F}^N, \qquad (T, \varphi) \mapsto X_T[\varphi]$$

which have the following properties for all $p \in \mathcal{P}$ and $T \in \mathbb{R}$:

(i) There is a $C_4{}^{(n)} \in \mathtt{Bounds}$ such that for all $\varphi \in D_w(A)$ with $(\mathtt{Q} + \mathtt{P})\varphi = p$ it is true that $\|X_T[\varphi]\|_{\mathcal{F}_w^n} \leq C_4{}^{(n)}(|T|, \|p\|)$.

(ii) The map $F \mapsto X_T[p, F]$ as $\mathcal{F}_w^1 \to \mathcal{F}_w^1$ is continuous.

(iii) For $(\mathbf{E}_{i,T}, \mathbf{B}_{i,T})_{1\leq i\leq N} := X_T[\varphi]$ and $(\mathbf{q}_{i,T}, \mathbf{p}_{i,T})_{1\leq i\leq N} := (\mathtt{Q} + \mathtt{P})M_L[\varphi](T, 0)$ one has

$$\nabla \cdot \mathbf{E}_{i,T} = 4\pi\varrho_i(\cdot - \mathbf{q}_{i,T}),\ \ \nabla \cdot \mathbf{B}_{i,T} = 0.$$

The subset $\widetilde{\mathcal{A}}_w^n$ comprises maps $X \in \mathcal{A}_w^n$ that fulfill:

(iv) For balls $B_\tau := B_\tau(0) \subset \mathbb{R}^3$ with radius $\tau > 0$, $B_\tau^c := \mathbb{R}^3 \setminus B_\tau$, and any bounded set $M \subset D_w(\mathsf{A})$ it holds that

$$\lim_{\tau\to\infty} \sup_{F \in M} \|X_T[p, F]\|_{\mathcal{F}_w^n(B_\tau^c)} = 0.$$

Furthermore, $\mathcal{A}_w^{\mathrm{Lip}}$ comprise such maps $X \in\subset \mathcal{A}_w^1$ that fulfill:

(v) There is a $C_5 \in \mathtt{Bounds}$ such that for all $\varphi, \widetilde{\varphi} \in D_w(A)$ with $(\mathtt{Q} + \mathtt{P})\varphi = p = (\mathtt{Q} + \mathtt{P})\widetilde{\varphi}$ it is true that

$$\|X_T[\varphi] - X_T[\widetilde{\varphi}]\|_{\mathcal{F}_w^1} \leq |T| C_5(|T|, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w})\ \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}.$$

Remark 4.9. (1) Note also that $\mathcal{A}_w^{n+1} \subset \mathcal{A}_w^n$ as well as $\widetilde{\mathcal{A}}_w^{n+1} \subset \widetilde{\mathcal{A}}_w^n$ for $n \in \mathbb{N}$. (2) In Lemma 4.14, we shall show that these classes are not empty. In fact, the definitions are intended to allow Liénard–Wiechert fields generated by any once continuously differentiable asymptotes with strictly time-like and uniformly bounded accelerations.

With this definition, we can formalize the term "conditional WF$_\varrho$ solution" for given Newtonian Cauchy data and prescribed boundary fields which we have discussed in Sect. 2:

Definition 4.10. (*Conditional* WF$_\varrho$ *Solutions*) Let $T > 0$, $p \in \mathcal{P}$ and $X^\pm \in \mathcal{A}_w^1$ be given. The set $\mathcal{T}_T^{p,X^\pm}$ consists of elements $(\mathbf{q}_i, \mathbf{p}_i)_{1\leq i\leq N} \in \mathcal{T}_\odot^N$ that solve the conditional WF$_\varrho$ equations (14)–(15) for Newtonian Cauchy data $p = (\mathbf{q}_{i,t}, \mathbf{q}_{i,t})_{1\leq i\leq N}|_{t=0}$. We shall refer to elements in $\mathcal{T}_T^{p,X^\pm}$ as conditional WF$_\varrho$ solutions for initial value p and boundary fields $X_T^\pm$.

Furthermore, we define the potential fixed point map $S_T^{p,X^\pm}$ as discussed in Sect. 2 where we make use of the notation and results presented in Sects. 3.1 and 3.2.

Definition 4.11. (*Fixed Point Map* $S_T^{p,X^\pm}$) For any given finite $T>0$, $p\in\mathcal{P}$ and $X^\pm\in\mathcal{A}_w^1$, we define

$$S_T^{p,X^\pm}: D_w(\mathsf{A})\to D_w(\mathsf{A}^\infty), \qquad F\mapsto S_T^{p,X^\pm}[F]$$

by

$$S_T^{p,X^\pm}[F] := \frac{1}{2}\sum_\pm\left[\mathsf{W}_{\mp T}X^\pm_{\pm T}[p,F] + \int_{\pm T}^{t}\mathrm{d}s\ \mathsf{W}_{-s}\mathsf{J}(\varphi_s[p,F])\right]$$

where $s\mapsto\varphi_s[p,F]:=M_L[p,F](s,0)$ denotes the ML-SI$_\varrho$ solution, cf. Definition 3.23, for initial value $(p,F)\in D_w(A)$.

Next we make sure that this map is well defined and that its fixed points, if they exist, have corresponding charge world-lines in $\mathcal{T}_T^{p,X^\pm}$, that is, the conditional WF$_\varrho$ solutions.

Theorem 4.12. ($S_T^{p,X^\pm}$ and its Fixed Points) *For any finite $T>0$, $p\in\mathcal{P}$ and $X^\pm\in\mathcal{A}_w^1$ the following is true:*

(i) The map $S_T^{p,X^\pm}$ is well defined.

(ii) Given $F\in D_w(\mathsf{A})$, setting $(X^\pm_{i,\pm T})_{1\le i\le N}:=X^\pm_{\pm T}[p,F]$ and denoting the ML-SI$_\varrho$ charge world-lines

$$t\mapsto(\mathbf{q}_{i,t},\mathbf{q}_{i,t})_{1\le i\le N}:=(\mathsf{Q}+\mathsf{P})M_L[p,F](t,0) \tag{39}$$

by $(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}$ we have

$$S_T^{p,X^\pm}[F]=\frac{1}{2}\sum_\pm\left(M_{\varrho_i}[X^\pm_{i,\pm T},(\mathbf{q}_i,\mathbf{p}_i)](0,\pm T)\right)_{1\le i\le N}$$

as well as $S_T^{p,X^\pm}[F]\in D_w(\mathsf{A}^\infty)\cap\mathcal{F}^N$.

(iii) For any $F\in D_w(\mathsf{A})$ such that $F=S_T^{p,X^\pm}[F]$ the corresponding charge world-lines (39) are in $\mathcal{T}_T^{p,X^\pm}$.

Proof. (i) Let $F\in D_w(\mathsf{A})$, then $(p,F)\in D_w(A)$, and hence, by Theorem 3.19 the map $t\mapsto\varphi_t:=M_L[\varphi](t,0)$ is a once continuously differentiable map $\mathbb{R}\to D_w(A)\subset\mathcal{H}_w$. By properties of J stated in [4, Lemma 2.22] we know that $A^kJ:\mathcal{H}_w\to D_w(A^\infty)\subset\mathcal{H}_w$ is locally Lipschitz continuous for any $k\in\mathbb{N}$. By projecting onto field space $\mathcal{F}_w$, cf. Definition 3.25, we obtain that also $\mathsf{A}^k\mathsf{J}:\mathcal{H}_w\to D_w(\mathsf{A}^\infty)\subset\mathcal{F}_w$ is locally Lipschitz continuous. Hence, by the group properties of $(\mathsf{W}_t)_{t\in\mathbb{R}}$, we know that $s\mapsto\mathsf{W}_{-s}\mathsf{A}^k\mathsf{J}(\varphi_s)$ for any $k\in\mathbb{N}$ is continuous. Furthermore, A is closed. This implies the commutation

$$\mathsf{A}^k\int_{\pm T}^{0}\mathrm{d}s\ \mathsf{W}_{-s}\mathsf{J}(\varphi_s)=\int_{\pm T}^{0}\mathrm{d}s\ \mathsf{W}_{-s}\mathsf{A}^k\mathsf{J}(\varphi_s).$$

As this holds for any $k\in\mathbb{N}$, $\int_{\pm T}^0\mathrm{d}s\ \mathsf{W}_{-s}\mathsf{J}(\varphi_s)\in D_w(\mathsf{A}^\infty)$. Furthermore, by Definition 4.8, the term $X^\pm_{\pm T}[p,F]$ is in $D_w(\mathsf{A}^\infty)$ and therefore $\mathsf{W}_{\mp T}X^\pm_{\pm T}[p,F]\in D_w(\mathsf{A}^\infty)$ by the group properties. Hence, the map $S_T^{p,X^\pm}$ is well defined as a map $D_w(A)\to D_w(\mathsf{A}^\infty)$.

(ii) For $F\in D_w(\mathsf{A})$ let $(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}$ denote the charge world-lines $t\mapsto(\mathbf{q}_{i,t},\mathbf{p}_{i,t})_{1\le i\le N}=(\mathsf{Q}+\mathsf{P})\varphi_t$ of $t\mapsto\varphi_t:=M_L[p,F](t,0)$, which by $(p,F)\in D_w(A)$ and Theorem 3.19 are once continuously differentiable. Since the absolute value of the velocity is given by $\|\mathbf{v}(\mathbf{p}_{i,t})\|=\frac{\|\mathbf{p}_{i,t}\|}{\sqrt{m^2+\mathbf{p}_{i,t}^2}}<1$, we conclude that $(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}$ are also time-like and therefore in $\mathcal{T}_\oplus^N$, cf. Definition 2.1. Furthermore, the boundary fields

$X^\pm_{\pm T}[p,F]$ are in $D_w(\mathsf{A}^\infty)\cap\mathcal{F}^N$ and obey the Maxwell constraints by the definition of $\mathcal{A}^n_w$. So we can apply Lemma 3.28 which states for $(X^\pm_{i,\pm T})_{1\le i\le N}:=X^\pm_{\pm T}[p,F]$ that

$$\left(M_{\varrho_i}[X^\pm_{i,\pm T},(\mathbf{q}_i,\mathbf{p}_i)](t,\pm T)\right)_{1\le i\le N}=\mathsf{W}_{t\mp T}X^\pm_{\pm T}[p,F]+\int_{\pm T}^{t}\mathrm{d}s\;\mathsf{W}_{t-s}\mathsf{J}(\varphi_s)\in D_w(\mathsf{A})\cap\mathcal{F}^N. \tag{40}$$

For $t=0$ this proves claim (ii).

(iii) Finally, assume there is an $F\in D_w(\mathsf{A})$ such that $F=S^{p,X^\pm}_T[F]$. By (ii) this implies $F\in D_w(\mathsf{A}^\infty)\cap\mathcal{F}^N$. Let $(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}$ and $t\mapsto\varphi_t$ be defined as in the proof of (ii) which now is infinitely often differentiable as $\mathbb{R}\to\mathcal{H}_w$ since $(p,F)\in D_w(A^\infty)$. We shall show later that the following integral equality holds

$$\varphi_t=(p,0)+\int_0^t\mathrm{d}s\;(\mathtt{Q}+\mathtt{P})J(\varphi_s)+\frac{1}{2}\sum_\pm\left[W_{t\mp T}(0,X^\pm_{\pm T}[p,F])+\int_{\pm T}^{t}\mathrm{d}s\;W_{t-s}\mathtt{F}J(\varphi_s)\right] \tag{41}$$

for all $t\in\mathbb{R}$; note that $t\mapsto\varphi_t:=M_L[p,F](t,0)$ depends also on (p,F). For now, suppose (41) holds. Then the differentiation with respect to time t of the phase space components of $(\mathbf{q}_{i,t},\mathbf{p}_{i,t},\mathbf{E}_{i,t},\mathbf{B}_{i,t})_{1\le i\le N}:=\varphi_t$ yields $\partial_t(\mathtt{Q}+\mathtt{P})\varphi_t=(\mathtt{Q}+\mathtt{P})J(\varphi_t)$, which by definition of J gives

$$\begin{aligned}\partial_t\mathbf{q}_{i,t}&=\mathbf{v}(\mathbf{p}_{i,t}):=\frac{\mathbf{p}_{i,t}}{\sqrt{m^2+\mathbf{p}^2_{i,t}}}\\ \partial_t\mathbf{p}_{i,t}&=\sum_{j\ne i}\int\mathrm{d}^3x\;\varrho_i(\mathbf{x}-\mathbf{q}_{i,t})\left(\mathbf{E}_{j,t}(\mathbf{x})+\mathbf{v}(\mathbf{q}_{i,t})\wedge\mathbf{B}_{j,t}(\mathbf{x})\right).\end{aligned} \tag{42}$$

Furthermore, the field components fulfill

$$\begin{aligned}\mathtt{F}\varphi_t&=\mathtt{F}\frac{1}{2}\sum_\pm\left[W_{t\mp T}(0,X^\pm_{\pm T}[\varphi])+\int_{\pm T}^{t}\mathrm{d}s\;W_{t-s}\mathtt{F}J(\varphi_s)\right]\\ &=\frac{1}{2}\sum_\pm\left[\mathsf{W}_{t\mp T}X^\pm_{\pm T}[p,F]+\int_{\pm T}^{t}\mathrm{d}s\;\mathsf{W}_{t-s}\mathsf{J}(\varphi_s)\right]\end{aligned}$$

where we only used the definition of the projectors, cf. Definition 3.25. Hence, by (40) we know

$$(\mathbf{E}_{i,t},\mathbf{B}_{i,t})=\frac{1}{2}\sum_\pm M_{\varrho_i}[F_i,(\mathbf{q}_i,\mathbf{p}_i)](t,\pm T). \tag{43}$$

Furthermore, we have

$$(\mathbf{q}_{i,t},\mathbf{p}_{i,t})_{1\le i\le N}\big|_{t=0}=p=(\mathbf{q}^0_i,\mathbf{p}^0_i)_{1\le i\le N}. \tag{44}$$

Now, Eqs. (42), (43) and (44) are exactly the conditional WF_ϱ equations (14)–(15) for Newtonian Cauchy data p and boundary fields $X^\pm$. Hence, since in (ii) we proved that $(\mathbf{q}_i,\mathbf{p}_i)_{1\le i\le N}$ are in $\mathcal{T}^N_\ominus$, we conclude that they are also in $\mathcal{T}^{p,X^\pm}_T$, cf. Definition 4.10.

Finally, it is only left to prove that the integral equation (41) holds. By Definition 3.23, φ_t fulfills

$$\varphi_t=W_t(p,F)+\int_0^t\mathrm{d}s\;W_{t-s}J(\varphi_s)$$

for all $t \in \mathbb{R}$. Inserting the fixed point equation $F = S_T^{p,X^\pm}[F]$, that is,

$$F = \frac{1}{2}\sum_{\pm}\left[\mathsf{W}_{\mp T}X^{\pm}_{\pm T}[p,F] + \int_{\pm T}^{t} \mathrm{d}s\ \mathsf{W}_{-s}\mathsf{J}(\varphi_s)\right],$$

we find

$$\varphi_t = (p,0) + \frac{1}{2}\sum_{\pm} W_{t\mp T}\big(0, X^{\pm}_{\pm T}[p,F]\big) + \frac{1}{2}\sum_{\pm} W_t \int_{\pm T}^{0} \mathrm{d}s\ W_{-s}\big(0,\mathsf{J}(\varphi_s)\big) + \int_0^t \mathrm{d}s\ W_{t-s}J(\varphi_s).$$

By the same reasoning as in (i), we may commute W_t with the integral. This together with $J = (\mathtt{Q}+\mathtt{P})J + \mathtt{F}J$ and $(\mathtt{Q}+\mathtt{P})W_t = \mathrm{id}_{\mathcal{P}}$ proves the equality (41) for all $t \in \mathbb{R}$ which concludes the proof. □

Next, we give a simple but physically meaningful element $C \in \widetilde{\mathcal{A}}^n_w \cap \mathcal{A}^{\mathrm{Lip}}_w$ to show that neither $\widetilde{\mathcal{A}}^n_w$ nor $\mathcal{A}^{\mathrm{Lip}}_w$ is empty.

Definition 4.13. (*Coulomb Boundary Field*) Define $C : \mathbb{R} \times D_w(A) \to D_w(\mathsf{A}^\infty)$, $(T,\varphi) \mapsto C_T[\varphi]$ by

$$C_T[\varphi] := \big(\mathbf{E}^C_i(\cdot - \mathbf{q}_{i,T}), 0\big)_{1\le i\le N}$$

where $(\mathbf{q}_{i,T})_{1\le i\le N} := \mathtt{Q}M_L[\varphi](T,0)$ and

$$(\mathbf{E}^C_i, 0) := M_{\varrho_i}[t \mapsto (0,0)](0,-\infty) = \left(\int \mathrm{d}^3z\ \varrho_i(\cdot - \mathbf{z})\frac{\mathbf{z}}{\|\mathbf{z}\|^3}, 0\right). \tag{45}$$

Note that the equality on the right-hand side of (45) holds by Theorem 3.10.

Lemma 4.14. ($\widetilde{\mathcal{A}}^n_w \cap \mathcal{A}^{\mathrm{Lip}}_w$is Non-Empty) *Let $n \in \mathbb{N}$ and $w \in \mathcal{W}$. The map C given in Definition 4.13 is an element of $\widetilde{\mathcal{A}}^n_w \cap \mathcal{A}^{\mathrm{Lip}}_w$.*

Proof. We need to show the properties (i)–(v) given in Definition 4.8. Fix $T > 0$ and $p \in \mathcal{P}$. Let $\varphi \in D_w(A)$ such that $(\mathtt{Q}+\mathtt{P})\varphi = p$ and set $F := \mathtt{F}\varphi$. Furthermore, we define $(\mathbf{q}_{i,T})_{1\le i\le N} := \mathtt{Q}M_L[\varphi](T,0)$. Since $\mathbf{E}^C_i$ is a Liénard–Wiechert field of the constant charge world-line $t \mapsto (\mathbf{q}_{i,T}, 0)$ in $\mathcal{T}^1_{\odot !}$, we can apply Corollary 3.13 to yield the following estimate for any multi-index $\alpha \in \mathbb{N}^3_0$ and $\mathbf{x} \in \mathbb{R}^3$

$$\big\|D^\alpha \mathbf{E}^C_i(\mathbf{x})\big\|_{\mathbb{R}^3} \le \frac{C_1^{(\alpha)}}{1+\|\mathbf{x}\|^2}. \tag{46}$$

which allows to define the finite constants $C_6^{(\alpha)} := \big\|D^\alpha \mathbf{E}^C_i\big\|_{L^2_w}$. Using the properties of the weight $w \in \mathcal{W}$, see (27), we find

$$\begin{aligned}\|C_T[\varphi]\|^2_{\mathcal{F}^n_w} &\le \sum_{k=0}^{n}\|\mathsf{A}^k C_T[\varphi]\|_{\mathcal{F}_w} \le \sum_{k=0}^{n}\sum_{i=1}^{N}\big\|(\nabla\wedge)^k\mathbf{E}^C_i(\cdot - \mathbf{q}_{i,T})\big\|_{L^2_w} \le \sum_{k=0}^{n}\sum_{|\alpha|\le k}\sum_{i=1}^{N}\big\|D^\alpha\mathbf{E}^C_i(\cdot - \mathbf{q}_{i,T})\big\|_{L^2_w}\\ &\le \sum_{k=0}^{n}\sum_{|\alpha|\le k}\sum_{i=1}^{N}\left(1+C_w\|\mathbf{q}_{i,T}\|\right)^{\frac{P_w}{2}}\big\|D^\alpha\mathbf{E}^C\big\|_{L^2_w} \le \sum_{k=0}^{n}\sum_{|\alpha|\le k}\sum_{i=1}^{N}\left(1+C_w\|\mathbf{q}_{i,T}\|\right)^{\frac{P_w}{2}} C_6^{(\alpha)} < \infty.\end{aligned}$$

This implies $C_T[\varphi] \in D_w(\mathsf{A}^\infty) \cap \mathcal{F}^N$ and that $C : \mathbb{R} \times D_w(A) \to D_w(\mathsf{A}^\infty) \cap \mathcal{F}^N$ is well defined. Note that the right-hand side depends only on $\|\mathbf{q}_{i,T}\|$ which is bounded by

$$\|\mathbf{q}_{i,T}\| \le \|\mathtt{Q}p\| + |T| \tag{47}$$

since the maximal velocity is bounded by one, that is, the speed of light. Hence, property (i) holds for

$$C_4^{(n)}(|T|,\|p\|) := \sum_{k=0}^{n}\sum_{|\alpha|\le k}\sum_{i=1}^{N}\left(1+C_w\left(\|\mathtt{Q}p\| + |T|\right)\right)^{\frac{P_w}{2}} C_6^{(\alpha)}.$$

Instead of showing property (ii), we prove the stronger property (v). For this let $\widetilde{\varphi} \in D_w(A)$ such that $(\mathtt{Q}+\mathtt{P})\varphi = (\mathtt{Q}+\mathtt{P})\widetilde{\varphi}$ and set $(\widetilde{\mathbf{q}}_{i,T})_{1\le i\le N} := \mathtt{Q}M_L[\widetilde{\varphi}](T,0)$. Starting with

$$\|C_T[\varphi] - C_T[\widetilde{\varphi}]\|_{\mathcal{F}^1_w} \le \sum_{i=1}^N \sum_{|\alpha|\le 1} \left\| D^\alpha \left(\mathbf{E}^C(\cdot - \mathbf{q}_{i,T}) - \mathbf{E}^C(\cdot - \widetilde{\mathbf{q}}_{i,T}) \right) \right\|_{L^2_w}$$

we compute

$$\begin{aligned}\left\| D^\alpha \left(\mathbf{E}^C(\cdot - \mathbf{q}_{i,T}) - \mathbf{E}^C(\cdot - \widetilde{\mathbf{q}}_{i,T}) \right) \right\|_{L^2_w} &= \left\| \int_0^1 \mathrm{d}\lambda \, (\widetilde{\mathbf{q}}_{i,T} - \mathbf{q}_{i,t}) \cdot \nabla D^\alpha \mathbf{E}^C(\cdot - \widetilde{\mathbf{q}}_{i,T} + \lambda(\widetilde{\mathbf{q}}_{i,T} - \mathbf{q}_{i,t})) \right\|_{L^2_w} \\ &\le \int_0^1 \mathrm{d}\lambda \, \left\| (\mathbf{q}_{i,t} - \widetilde{\mathbf{q}}_{i,T}) \cdot \nabla D^\alpha \mathbf{E}^C(\cdot - \widetilde{\mathbf{q}}_{i,T} + \lambda(\widetilde{\mathbf{q}}_{i,T} - \mathbf{q}_{i,t})) \right\|_{L^2_w}.\end{aligned}$$

Therefore, for all $|\alpha| \le 1$ we get

$$\begin{aligned}&\sum_{|\alpha|\le 1} \left\| D^\alpha \left(\mathbf{E}^C(\cdot - \mathbf{q}_{i,T}) - \mathbf{E}^C(\cdot - \widetilde{\mathbf{q}}_{i,T}) \right) \right\|_{L^2_w} \\ &\quad\le \|\mathbf{q}_{i,T} - \widetilde{\mathbf{q}}_{i,T}\|_{\mathbb{R}^3} \sup_{0\le\lambda\le 1} \sum_{|\beta|\le 2} \left\| D^\beta \mathbf{E}^C(\cdot + \lambda(\mathbf{q}_{i,T} - \widetilde{\mathbf{q}}_{i,T})) \right\|_{L^2_w}.\end{aligned}$$

The estimate (46), $0 \le \lambda \le 1$ and the properties of $w \in \mathcal{W}$ yield

$$\begin{aligned}\left\| D^\beta \mathbf{E}^C(\cdot - \widetilde{\mathbf{q}}_{i,T} + \lambda(\widetilde{\mathbf{q}}_{i,T} - \mathbf{q}_{i,t})) \right\|_{L^2_w} &\le \left(1 + C_w \|\widetilde{\mathbf{q}}_{i,T} - \lambda(\widetilde{\mathbf{q}}_{i,T} - \mathbf{q}_{i,t})\|_{\mathbb{R}^3}\right)^{\frac{P_w}{2}} \left\| D^\beta \mathbf{E}^C \right\|_{L^2_w} \\ &\le (1 + C_w(\|\mathbf{q}_{i,T}\|_{\mathbb{R}^3} + \|\widetilde{\mathbf{q}}_{i,T}\|_{\mathbb{R}^3})^{\frac{P_w}{2}} C_6{}^{(\beta)}\end{aligned}$$

Hence, because of bound (47), property (v) holds for

$$C_5(|T|, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) := N \sum_{|\beta|\le 2} (1 + C_w(\|\mathtt{Q}\varphi\|_{\mathbb{R}^3} + \|\mathtt{Q}\widetilde{\varphi}\|_{\mathbb{R}^3} + 2|T|)^{\frac{P_w}{2}} C_6{}^{(\beta)}$$

(iii) holds by Theorem 3.6. (iv) Let $B_\tau(0) \subset \mathbb{R}^3$ be a ball of radius $\tau > 0$ around the origin. For any $F \in D_w(\mathsf{A})$, we define $(\mathbf{q}_{i,T})_{1\le i\le N} := \mathtt{Q}M_L[\varphi](T,0)$. It holds

$$\begin{aligned}\|C^T[p,F]\|_{\mathcal{F}^n_w(B^c_\tau(0))} &\le \sum_{i=1}^N \sum_{|\alpha|\le n} \left\| D^\alpha \mathbf{E}^C(\cdot - \mathbf{q}_{i,T}) \right\|_{L^2_w(B^c_\tau(0))} \\ &\le \sum_{i=1}^N \sum_{|\alpha|\le n} (1 + C_w\|\mathbf{q}_{i,T}\|)^{\frac{P_w}{2}} \left\| D^\alpha \mathbf{E}^C \right\|_{L^2_w(B^c_\tau(\mathbf{q}_{i,T}))}.\end{aligned}$$

We use again that the maximal velocity is smaller than one, that is, $\|\mathbf{q}_{i,T}\| \le \|\mathbf{q}^0_i\| + T$. Hence, for $\tau > \|\mathbf{q}^0_i\| + T$ define $r(\tau) := \tau - \|\mathbf{q}^0_i\| + T$ such that we can estimate the $L^2_w(B^c_\tau(\mathbf{q}_{i,T}))$ norm by the $L^2_w(B^c_{r(\tau)}(0))$ norm and yield

$$\sup_{F\in D_w(\mathsf{A})} \|C^T[p,F]\|_{\mathcal{F}^n_w(B^c_\tau(0))} \le \sum_{i=1}^N \sum_{|\alpha|\le n} (1 + C_w\|\mathbf{q}_{i,T}\|)^{\frac{P_w}{2}} \left\| D^\alpha \mathbf{E}^C \right\|_{L^2_w(B^c_{r(\tau)}(0))} \xrightarrow[\tau\to\infty]{} 0$$

This concludes the proof. □

Remark 4.15. When looking for global WF_ϱ solutions, in view of (15) and (16), the boundary fields can be seen as a good guess of how the charge world-lines $(\mathbf{q}^0_i, \mathbf{p}_i)_{1\le i\le N}$ continue outside of the time interval $[-T,T]$. Without much modification of Lemma 4.14 one can also treat the Liénard–Wiechert fields of a charge world-line which starts at $\mathbf{q}_{i,T}$ and has constant momentum $\mathbf{p}_{i,T}$ using the notation $(\mathbf{q}_{i,T}, \mathbf{p}_{i,T})_{1\le i\le N} := (\mathtt{Q}+\mathtt{P})M_L[\varphi](T,0)$ (the result is the Lorentz boosted Coulomb field). Such boundary fields are also in $\widetilde{\mathcal{A}}_w^n \cap \mathcal{A}_w^{\mathrm{Lip}}$ since the derivative $\partial_s \mathbf{p}_{i,s}$ for $s \in [-T,T]$ can be expressed by J which is locally Lipschitz continuous by Bauer et al. [4, Lemma 2.22] and the ML-SI$_\varrho$ dynamics are well controllable on the interval $[-T,T]$; see (ii) of Theorem 3.19.

We collect the needed estimates and properties of $S_T^{p,X^\pm}$ in the following three lemmas.

Lemma 4.16. ($\mathcal{F}_w^n$Estimates) *For $n \in \mathbb{N}_0$ the following is true:*

(i) For all $t \in \mathbb{R}$ and $F \in D_w(\mathsf{A}^n)$ it holds that $\|\mathsf{W}_t F\|_{\mathcal{F}_w^n} \le e^{\gamma|t|}\|F\|_{\mathcal{F}_w^n}$.

(ii) For all $\varphi \in \mathcal{H}_w$ there is a $C_7{}^{(n)} \in$ `Bounds` such that

$$\|\mathsf{J}(\varphi)\|_{\mathcal{F}_w^n} \le C_7{}^{(n)}(\|\mathsf{Q}\varphi\|_{\mathcal{H}_w}).$$

(iii) For all $\varphi, \widetilde{\varphi} \in \mathcal{H}_w$ there is a $C_8{}^{(n)} \in$ `Bounds` such that

$$\|\mathsf{J}(\varphi) - \mathsf{J}(\widetilde{\varphi})\|_{\mathcal{F}_w^n} \le C_8{}^{(n)}(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w})\|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}.$$

Proof. (i) As shown in [4, Lemma 2.19], A on $D_w(A)$ generates a γ-contractive group $(W_t)_{t\in\mathbb{R}}$; cf. Definition 3.21. This property is inherited from A on $D_w(\mathsf{A})$ which generates the group $(\mathsf{W}_t)_{t\in\mathbb{R}}$. Hence, A and W_t commute for any $t \in \mathbb{R}$ which implies for all $F \in D_w(\mathsf{A}^n)$ that

$$\|\mathsf{W}_t F\|_{\mathcal{F}_w^n}^2 = \sum_{k=0}^n \|\mathsf{A}^k \mathsf{W}_t F\|_{\mathcal{F}_w}^2 = \sum_{k=0}^n \|\mathsf{W}_t \mathsf{A}^k F\|_{\mathcal{F}_w}^2 \le e^{\gamma|t|} \sum_{k=0}^n \|\mathsf{A}^k F\|_{\mathcal{F}_w}^2 = e^{\gamma|t|}\|F\|_{\mathcal{F}_w^n}.$$

For (ii) let $(\mathbf{q}_i, \mathbf{p}_i, \mathbf{E}_i, \mathbf{B}_i)_{1\le i\le N} = \varphi \in \mathcal{H}_w$. Using then the definition of J, cf. Definitions 3.18 and 3.25, we find

$$\|\mathsf{J}(\varphi)\|_{\mathcal{F}_w^n} \le \sum_{i=1}^N \sum_{k=0}^n \|(\nabla\wedge)^k \mathbf{v}(\mathbf{p}_i)\varrho_i(\cdot - \mathbf{q}_i)\|_{L_w^2}.$$

By applying the triangular inequality, one finds a constant C_9 such that

$$\|(\nabla\wedge)^k \mathbf{v}(\mathbf{p}_i)\varrho_i(\cdot - \mathbf{q}_i)\|_{L_w^2} \le (C_9)^n \sum_{|\alpha|\le n} \|\mathbf{v}(\mathbf{p}_i) D^\alpha \varrho_i(\cdot - \mathbf{q}_i)\|_{L_w^2} \le (C_9)^n \sum_{|\alpha|\le n} \|D^\alpha \varrho_i(\cdot - \mathbf{q}_i)\|_{L_w^2}$$

whereas in the last step, we used the fact that the maximal velocity is smaller than one. Using the properties of the weight function $w \in \mathcal{W}$, cf. Definition 3.14, we conclude

$$\|D^\alpha \varrho_i(\cdot - \mathbf{q}_i)\|_{L_w^2} \le (1 + C_w\|\mathbf{q}_i\|)^{\frac{P_w}{2}} \|D^\alpha \varrho_i\|_{L_w^2}.$$

Collecting these estimates, we yield that claim (ii) holds for

$$C_7{}^{(n)}(\|\mathsf{Q}\varphi\|_{\mathcal{H}_w}) := (C_9)^n \sum_{i=1}^N (1 + C_w\|\mathbf{q}_i\|)^{\frac{P_w}{2}} \sum_{|\alpha|\le n} \|D^\alpha \varrho_i\|.$$

Claim (iii) is shown by repetitively applying estimate of [4, Lemma 2.22] on the right-hand side of

$$\|\mathsf{J}(\varphi) - \mathsf{J}(\widetilde{\varphi})\|_{\mathcal{F}_w^n} \le \sum_{k=0}^n \|A^k[J(\varphi) - J(\widetilde{\varphi})]\|_{\mathcal{H}_w}$$

which yields a constant $C_8{}^{(n)} := \sum_{k=0}^n C_{10}{}^{(k)}(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w})$ where $C_{10} \in$ `Bounds` is given in the proof of [4, Lemma 2.22]. This concludes the proof. □

Lemma 4.17. (Properties of $S_T^{p,X^\pm}$) *Let $0 < T < \infty$, $p \in \mathcal{P}$ and $X^\pm \in \mathcal{A}_w^n$ for $n \in \mathbb{N}$. Then it holds:*

(i) There is a $C_{11} \in$ `Bounds` such that for all $F \in \mathcal{F}_w^1$ we have

$$\|S_T^{p,X^\pm}[p,F]\|_{\mathcal{F}_w^n} \le C_{11}{}^{(n)}(T, \|p\|).$$

(ii) $F \mapsto S_T^{p,X^\pm}[F]$ as $\mathcal{F}_w^1 \to \mathcal{F}_w^1$ is continuous.

If $X^\pm \in \mathcal{A}_w^{\mathrm{Lip}}$*, it is also true that:*

(iii) There is a $C_{12} \in \texttt{Bounds}$ *such that for all* $F, \widetilde{F} \in \mathcal{F}_w^1$ *we have*

$$\|S_T^{p,X^\pm}[F] - S_T^{p,X^\pm}[\widetilde{F}]\|_{\mathcal{F}_w^1} \leq T C_{12}(T, \|p\|, \|F\|_{\mathcal{F}_w}, \|\widetilde{F}\|_{\mathcal{F}_w}) \|F - \widetilde{F}\|_{\mathcal{F}_w}.$$

Proof. Fix a finite $T > 0$, $p \in \mathcal{P}$, $X^\pm \in \mathcal{A}_w^n$ for $n \in \mathbb{N}$. Before we prove the claims we preliminarily recall the relevant estimates of the ML-SI$_\varrho$ dynamics. Throughout the proof and for any $F, \widetilde{F} \in \mathcal{F}_w^n$, we use the notation

$$D_w(A^n) \ni \varphi \equiv (p, F), \qquad D_w(A^n) \ni \widetilde{\varphi} \equiv (p, \widetilde{F}),$$

and furthermore,

$$\varphi_t := M_L[\varphi](t, 0), \qquad \widetilde{\varphi}_t := M_L[\widetilde{\varphi}](t, 0),$$

for any $t \in \mathbb{R}$. Recall the estimates given in (ii) of Theorem 3.19 which gives the following T dependent upper bounds on these ML-SI$_\varrho$ solutions:

$$\sup_{t\in[-T,T]} \|\varphi_t - \widetilde{\varphi}_t\|_{\mathcal{H}_w} \leq C_{??}(T, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}, \tag{48}$$

$$\sup_{t\in[-T,T]} \|\varphi_t\|_{\mathcal{H}_w} \leq C_2(T, \|\varphi\|_{\mathcal{H}_w}, 0) \|\varphi\|_{\mathcal{H}_w} \text{and} \sup_{t\in[-T,T]} \|\widetilde{\varphi}_t\|_{\mathcal{H}_w} \leq C_2(T, \|\widetilde{\varphi}\|_{\mathcal{H}_w}, 0) \|\widetilde{\varphi}\|_{\mathcal{H}_w}. \tag{49}$$

To prove claim (i) we estimate

$$\|S_T^{p,X^\pm}[F]\|_{\mathcal{F}_w^n} \leq \left\|\tfrac{1}{2}\textstyle\sum_\pm \mathsf{W}_{\mp T} X^\pm_{\pm T}[p, F]\right\|_{\mathcal{F}_w^n} + \left\|\tfrac{1}{2}\textstyle\sum_\pm \int_{\pm T}^0 \mathrm{d}s\, \mathsf{W}_{-s} \mathsf{J}(\varphi_s)\right\|_{\mathcal{F}_w^n} =: \boxed{1} + \boxed{2},$$

cf. Definition 4.11. By the estimate given in (i) of Lemma 4.16 and the property (i) of Definition 4.8 we find

$$\boxed{1} \leq \frac{1}{2}\sum_\pm \|\mathsf{W}_{\mp T} X^\pm_{\pm T}[p, F]\|_{\mathcal{F}_w^n} \leq e^{\gamma T} \|X^\pm_{\pm T}[p, F]\|_{\mathcal{F}_w^n} \leq e^{\gamma T} C_4{}^{(n)}(T, \|\varphi\|_{\mathcal{H}_w}).$$

Furthermore, using in addition the estimates (i)–(ii) of Lemma 4.16, we get a bound for the other term by

$$\boxed{2} \leq T e^{\gamma T} \sup_{s\in[-T,T]} \|\mathsf{J}(\varphi_s)\|_{\mathcal{F}_w^n} \leq T e^{\gamma T} \sup_{s\in[-T,T]} C_7(\|\mathsf{Q}\varphi_s\|_{\mathcal{H}_w}) \leq T e^{\gamma T} C_7(\|p\| + T)$$

whereas the last step is implied by the fact that the maximal velocity is below one. These estimates prove claim (i) for

$$C_{11}{}^{(n)}(T, \|\phi\|_{\mathcal{H}_w^n}) := e^{\gamma T} \left(C_4{}^{(n)}(T, \|p\|) + T C_7(\|p\| + T)\right).$$

Next we prove claim (ii). Therefore, we consider

$$\|S_T^{p,X^\pm}[F] - S_T^{p,X^\pm}[\widetilde{F}]\|_{\mathcal{F}_w^n} \leq e^{\gamma T} \|X^\pm_{\pm T}[\varphi] - X^\pm_{\pm T}[\widetilde{\varphi}]\|_{\mathcal{F}_w^n} + T e^{\gamma T} \sup\nolimits_{s\in[-T,T]} \|\mathsf{J}(\varphi_s) - \mathsf{J}(\widetilde{\varphi}_s)\|_{\mathcal{F}_w^n}$$
$$=: \boxed{3} + \boxed{4}$$

where we have already applied (i) of Lemma 4.16. Next we apply (iii) of Lemma 4.16 to the term $\boxed{4}$ and yield

$$\boxed{4} \leq T e^{\gamma T} \sup_{s\in[-T,T]} C_8{}^{(n)}(\|\varphi_s\|_{\mathcal{H}_w}, \|\widetilde{\varphi}_s\|_{\mathcal{H}_w}) \|\varphi_s - \widetilde{\varphi}_s\|_{\mathcal{H}_w}.$$

Finally, by the ML-SI$_\varrho$ estimates (48) and (49), we have

$$\boxed{4} \leq T C_{13}(T, \|p\|, \|F\|_{\mathcal{F}_w^n}, \|\widetilde{F}\|_{\mathcal{F}_w^n}) \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w} \tag{50}$$

for

$$C_{13}(T,\|p\|,\|F\|_{\mathcal{F}_w^n},\|\widetilde{F}\|_{\mathcal{F}_w^n}) := e^{\gamma T}C_8{}^{(n)}\Big(C_2(T,\|\varphi\|_{\mathcal{H}_w},0)\|\varphi\|_{\mathcal{H}_w},C_2(T,0,\|\widetilde{\varphi}\|_{\mathcal{H}_w})\|\varphi\|_{\mathcal{H}_w}\Big)$$
$$\times C_2(T,\|\varphi\|_{\mathcal{H}_w},\|\widetilde{\varphi}\|_{\mathcal{H}_w}).$$

By this estimate and (ii) of Definition 4.8, the limit $\widetilde{F}\to F$ in $\mathcal{F}_w^1$ implies $S_T^{p,X^\pm}[\widetilde{F}]\to S_T^{p,X^\pm}[F]$ in $\mathcal{F}_w^1$ since here $\|\varphi-\widetilde{\varphi}\|_{\mathcal{H}_w}=\|F-\widetilde{F}\|_{\mathcal{F}_w}$. Hence, the claim (ii) is true.

(iii) Let now $X^\pm\in\mathcal{A}_w^{\mathrm{Lip}}$. By (v) of Definition 4.8 the term $\boxed{3}$ then behaves according to

$$\boxed{3}\le TC_5{}^{(n)}(|T|,\|\varphi\|_{\mathcal{H}_w},\|\widetilde{\varphi}\|_{\mathcal{H}_w})\,\|\varphi-\widetilde{\varphi}\|_{\mathcal{H}_w}$$

Together with the estimate (50) this proves claim (ii) for

$$C_{12}{}^{(n)}(T,\|p\|,\|F\|_{\mathcal{F}_w},\|\widetilde{F}\|_{\mathcal{F}_w}) := C_5{}^{(n)}(|T|,\|\varphi\|_{\mathcal{H}_w},\|\widetilde{\varphi}\|_{\mathcal{H}_w})+C_{13}(T,\|p\|,\|F\|_{\mathcal{F}_w^n},\|\widetilde{F}\|_{\mathcal{F}_w^n})$$

since in our case $\|\varphi-\widetilde{\varphi}\|_{\mathcal{H}_w}=\|F-\widetilde{F}\|_{\mathcal{F}_w}$. □

Remark 4.18. Let $p\in\mathcal{P}$, $X^\pm\in A_w^{\mathrm{Lip}}$. Then claim (iii) of Lemma 4.17 has an immediate consequence: For sufficiently small T the mapping $S_T^{p,X^\pm}$ has a unique fixed point, which follows by Banach's fixed point theorem. Consider therefore $X^\pm\in\mathcal{A}_w^{\mathrm{Lip}}\subset\mathcal{A}_w^1$, then (i) of Lemma 4.17 states

$$\|S_T^{p,X^\pm}[p,F]\|_{\mathcal{F}_w^1}\le C_{11}{}^{(1)}(T,\|p\|)=:r.$$

Hence, the map $S_T^{p,X^\pm}$ restricted to the ball $B_r(0)\subset\mathcal{F}_w^1$ with radius r around the origin is a nonlinear self-mapping. Claim (iii) of Lemma 4.17 states for all $T>0$ and $F,\widetilde{F}\in B_r(0)\subset D_w(\mathsf{A})$ that

$$\|S_T^{p,X^\pm}[F]-S_T^{p,X^\pm}[\widetilde{F}]\|_{\mathcal{F}_w^1}\le TC_{12}(T,\|p\|,\|F\|_{\mathcal{F}_w},\|\widetilde{F}\|_{\mathcal{F}_w})\|F-\widetilde{F}\|_{\mathcal{F}_w}$$
$$\le TC_{12}(T,\|p\|,r,r)\|F-\widetilde{F}\|_{\mathcal{F}_w}.$$

where we have also used that $C_{12}\in\texttt{Bounds}$ is a continuous and strictly increasing function of its arguments. Hence, for T sufficiently small we have $TC_{12}(T,\|p\|,r,r)<1$ such that $S_T^{p,X^\pm}$ is a contraction on $B_r(0)\subset\mathcal{F}_w^1$. However, for larger T, we loose control on the uniqueness of the fixed point. This highlights an interesting aspect of dynamical systems, for example, for the ML-SI$_\varrho$ dynamics, it means that solutions are still uniquely characterized not only by Newtonian Data and fields $(p,F)\in D_w(A)$ at time $t=0$ but also by specifying Newtonian Cauchy data $p\in\mathcal{P}$ at time $t=0$ and fields F at a different time $t=T$. The maximal T will in general be inverse proportional to the Lipschitz constant of the vector field.

We come to the proof of Theorem 2.6 where we shall use the following criterion for precompactness of sequences in L_w^2.

Lemma 4.19. (Criterion for Precompactness) *Let $(\mathbf{F}_n)_{n\in\mathbb{N}}$ be a sequence in $L_w^2(\mathbb{R}^3,\mathbb{R}^3)$ such that*

(i) The sequence $(\mathbf{F}_n)_{n\in\mathbb{N}}$ is uniformly bounded in $H_w^\triangle$, defined in (29).
(ii) $\lim_{\tau\to\infty}\sup_{n\in\mathbb{N}}\|\mathbf{F}_n\|_{L_w^2(B_\tau^c(0))}=0$.

Then the sequence $(F_n)_{n\in\mathbb{N}}$ is precompact, that is, it contains a convergent subsequence.

Proof. The proof of this claim can be seen as a special case of [16, Chapter 8,Proof of Theorem 8.6, p.208] and can be found in [5]. □

Of course, one solely needs control on the gradient. However, the Laplace turns out to be more convenient for our later application of this lemma.

Proof of Theorem 2.6 (Existence of Conditional WF$_\varrho$ Solutions). Fix $p \in \mathcal{P}$. Given a finite $T > 0$, $p \in \mathcal{P}$ and $X^\pm \in \widetilde{\mathcal{A}}^3_w$ claim (i) of Lemma 4.12 states for all $F \in \mathcal{F}^1_w$

$$\|S^{p,X^\pm}_T[p,F]\|_{\mathcal{F}^1_w} \le \|S^{p,X^\pm}_T[p,F]\|_{\mathcal{F}^3_w} \le C_{11}{}^{(3)}(T,\|p\|) =: r. \tag{51}$$

Let K be the closed convex hull of $M := \{S^{p,X^\pm}_T[F] \mid F \in \mathcal{F}^1_w\} \subset B_r(0) \subset \mathcal{F}^1_w$. By (ii) of Lemma 4.12, we know that the map $S^{p,X^\pm}_T : K \to K$ is continuous as a map $\mathcal{F}^1_w \to \mathcal{F}^1_w$. Note that if M were compact so would be K and we could infer by Schauder's fixed point theorem the existence of a fixed point.

It remains to verify that M is compact. Therefore, let $(G_m)_{m\in\mathbb{N}}$ be a sequence in M. With the help of Lemma 4.19, we shall show now that it contains an $\mathcal{F}^1_w$ convergent subsequence. By definition, there is a sequence $(F_m)_{m\in\mathbb{N}}$ in $B_r(0) \subset \mathcal{F}^1_w$ such that $G_m := S^{p,X^\pm}_T[F_m]$, $m \in \mathbb{N}$; note that G_m is an element of $D_w(\mathsf{A}^\infty)$ and therefore also of $\mathcal{F}^n_w$ for any $n \in \mathbb{N}$. We define for $m \in \mathbb{N}$ the electric and magnetic fields

$$(\mathbf{E}^{(m)}_i, \mathbf{B}^{(m)}_i)_{1\le i\le N} := S^{p,X^\pm}_T[F_m].$$

Recall the definition of the norm of $\mathcal{F}^n_w$, cf. Definition 4.6, for some $(\mathbf{E}_i,\mathbf{B}_i)_{1\le i\le N} = F \in \mathcal{F}^n_w$ and $n \in \mathbb{N}$

$$\|F\|^2_{\mathcal{F}^n_w} = \sum_{k=0}^n \|\mathsf{A}^k F\|^2_{\mathcal{F}_w} = \sum_{k=0}^n \sum_{i=1}^N \left(\|(\nabla\wedge)^k \mathbf{E}_i\|^2_{L^2_w} + \|(\nabla\wedge)^k \mathbf{B}_i\|^2_{L^2_w} \right). \tag{52}$$

Therefore, since A on $D_w(\mathsf{A})$ is closed, $(G_m)_{m\in\mathbb{N}}$ has an $\mathcal{F}^1_w$ convergent subsequence if and only if all the sequences $((\nabla\wedge)^k \mathbf{E}^{(m)}_i)_{m\in\mathbb{N}}$, $((\nabla\wedge)^k \mathbf{B}^{(m)}_i)_{m\in\mathbb{N}}$ for $k = 0, 1$ and $1 \le i \le N$ have a common convergent subsequence in L^2_w.

To show that this is the case we first provide the bounds needed for condition (i) of Lemma 4.19. Estimate (51) implies that

$$\sum_{k=0}^3 \sum_{i=1}^N \left(\|(\nabla\wedge)^k \mathbf{E}^{(m)}_i\|^2_{L^2_w} + \|(\nabla\wedge)^k \mathbf{B}^{(m)}_i\|^2_{L^2_w} \right) = \|G_m\|^2_{\mathcal{F}^3_w} \le r^2 \tag{53}$$

for all $m \in \mathbb{N}$. Furthermore, by (ii) of Lemma 4.12 the fields $(\mathbf{E}^{(m)}_i, \mathbf{B}^{(m)}_i)_{1\le i\le N}$ are the fields of a Maxwell solution at time zero, and hence, by Theorem 3.6 fulfill the Maxwell constraints for $(\mathbf{q}^0_i, \mathbf{p}^0_i)_{1\le i\le N} := p$ which read

$$\nabla \cdot \mathbf{E}^{(m)}_i = 4\pi\varrho_i(\cdot - \mathbf{q}^0_i), \quad \nabla \cdot \mathbf{B}^{(m)}_i = 0.$$

Also by Theorem 3.6, G_m is in $\mathcal{F}^N$ so that for every $k \in \mathbb{N}_0$

$$(\nabla\wedge)^{k+2}\mathbf{E}^{(m)}_i = 4\pi(\nabla\wedge)^k \nabla\varrho_i(\cdot - \mathbf{q}^0_i) - \triangle(\nabla\wedge)^k \mathbf{E}^{(m)}_i, \;\; (\nabla\wedge)^{k+2}\mathbf{B}^{(m)}_i = -\triangle(\nabla\wedge)^k \mathbf{B}^{(m)}_i.$$

Estimate (53) implies for all $m \in \mathbb{N}$ that

$$\begin{aligned}
&\sum_{k=0}^1 \sum_{i=1}^N \left(\|\triangle(\nabla\wedge)^k \mathbf{E}^{(m)}_i\|^2_{L^2_w} + \|\triangle(\nabla\wedge)^k \mathbf{B}^{(m)}_i\|^2_{L^2_w} \right) \\
&\quad \le 2\sum_{k=0}^1 \sum_{i=1}^N \left(\|(\nabla\wedge)^{k+2} \mathbf{E}^{(m)}_i\|^2_{L^2_w} + \|(\nabla\wedge)^{k+2} \mathbf{B}^{(m)}_i\|^2_{L^2_w} \right) + 2\sum_{i=1}^N \|4\pi\nabla\varrho_i(\cdot - \mathbf{q}^0_i)\|_{L^2_w} \\
&\quad \le 2r^2 + 8\pi \sum_{i=1}^N \left(1 + C_w \left\|\mathbf{q}^0_i\right\|\right)^{P_w} \|\nabla\varrho_i\|^2_{L^2_w}
\end{aligned}$$

where we made use of the properties of the weight $w \in \mathcal{W}$. Note that the right-hand side does not depend on m. Therefore, all the sequences $((\nabla\wedge)^k \mathbf{E}^{(m)}_i)_{m\in\mathbb{N}}$, $(\triangle(\nabla\wedge)^k \mathbf{E}^{(m)}_i)_{m\in\mathbb{N}}$, $((\nabla\wedge)^k \mathbf{B}^{(m)}_i)_{m\in\mathbb{N}}$, $(\triangle(\nabla\wedge)^k \mathbf{B}^{(m)}_i)_{m\in\mathbb{N}}$ for $k = 0, 1$ and $1 \le i \le N$ are uniformly bounded in L^2_w. This ensures condition (i) of Lemma 4.19.

Second, we need to show that all the sequences $((\nabla\wedge)^k\mathbf{E}_i^{(m)})_{m\in\mathbb{N}}$, $((\nabla\wedge)^k\mathbf{B}_i^{(m)})_{m\in\mathbb{N}}$ for $k=0,1$ and $1\le i\le N$ decay uniformly at infinity to meet condition (ii) of Lemma 4.19. Define $(\mathbf{E}_{i,\pm T}^{(m),\pm},\mathbf{B}_{i,\pm T}^{(m),\pm})_{1\le i\le N}:=X_{\pm T}^{\pm}[p,F_m]$ for $m\in\mathbb{N}$ and denote the ith charge world-line $t\mapsto(\mathbf{q}_{i,t}^{(m)},\mathbf{p}_{i,t}^{(m)}):=(\mathtt{Q}+\mathtt{P})M_L[p,F_m](t,0)$ by $(\mathbf{q}_i^{(m)},\mathbf{p}_i^{(m)})$, $1\le i\le N$. Using Lemma 4.12(ii) and afterward Theorem 3.6, we can write the fields as

$$\begin{aligned}\begin{pmatrix}\mathbf{E}_i^{(m)}\\ \mathbf{B}_i^{(m)}\end{pmatrix}&=\frac{1}{2}\sum_{\pm}M_{\varrho_i}[(\mathbf{E}_{i,\pm T}^{\pm},\mathbf{E}_{i,\pm T}^{\pm}),(\mathbf{q}_i^{(m)},\mathbf{p}_i^{(m)})](0,\pm T)\\ &=\frac{1}{2}\sum_{\pm}\Bigg[\begin{pmatrix}\partial_t & \nabla\wedge\\ -\nabla\wedge & \partial_t\end{pmatrix}K_{t\mp T}*\begin{pmatrix}\mathbf{E}_{i,\pm T}^{(m),\pm}\\ \mathbf{B}_{i,\pm T}^{(m),\pm}\end{pmatrix}+K_{t\mp T}*\begin{pmatrix}-4\pi\mathbf{j}_{i,\pm T}^{(m)}\\ 0\end{pmatrix}\\ &\quad+4\pi\int_{\pm T}^{t}\mathrm{d}s\,K_{t-s}*\begin{pmatrix}-\nabla & -\partial_s\\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\rho_{i,s}^{(m)}\\ \mathbf{j}_{i,s}^{(m)}\end{pmatrix}\Bigg]_{t=0}=:\boxed{5}+\boxed{6}+\boxed{7}\end{aligned}$$

where $\rho_{i,t}^{(m)}:=\varrho_i(\cdot-\mathbf{q}_{i,t}^{(m)})$ and $\mathbf{j}_{i,t}^{(m)}:=\mathbf{v}(\mathbf{p}_{i,t}^{(m)})\rho_{i,t}$ for all $t\in\mathbb{R}$.

We shall show that there is a $\tau^*>0$ such that for all $m\in\mathbb{N}$ the terms $\boxed{6}$ and $\boxed{7}$ are pointwise zero on $B_{\tau^*}^c(0)\subset\mathbb{R}^3$. Recalling the computation rules for K_t from Lemma 3.4, we calculate for term $\boxed{6}$

$$\|4\pi[K_{\mp T}*\mathbf{j}_{i,\pm T}^{(m)}](\mathbf{x})\|_{\mathbb{R}^3}\le 4\pi T\fint_{\partial B_T(\mathbf{x})}\mathrm{d}\sigma(y)\;|\varrho_i(\mathbf{y}-\mathbf{q}_{\pm T}^{(m)})|.$$

The right-hand side is zero for all $\mathbf{x}\in\mathbb{R}^3$ such that $\partial B_T(\mathbf{x})\cap\operatorname{supp}\varrho_i(\cdot-\mathbf{q}_{\pm T})=\emptyset$. Because the charge distributions have compact support, there is a $R>0$ such that $\operatorname{supp}\varrho_i\subseteq B_R(0)$ for all $1\le i\le N$. Now for any $1\le i\le N$ and $m\in\mathbb{N}$ we have

$$\operatorname{supp}\varrho_i(\cdot-\mathbf{q}_{i,\pm T}^{(m)})\subseteq B_R(\mathbf{q}_{i,\pm T}^{(m)})\subseteq B_{R+T}(\mathbf{q}_i^0)$$

since the supremum of the velocities of the charge $\sup_{t\in[-T,T],m\in\mathbb{N}}\|\mathbf{v}(\mathbf{p}_{i,t}^{(m)})\|$ is less than one. Hence, $\partial B_T(\mathbf{x})\cap B_{R+T}(\mathbf{q}_i^0)=\emptyset$ for all $\mathbf{x}\in B_\tau^c(0)$ with $\tau>\|p\|+R+2T$.

Considering $\boxed{7}$ we have

$$\left\|4\pi\int_{\pm T}^{0}\mathrm{d}s\left[K_{-s}*\begin{pmatrix}-\nabla & -\partial_s\\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\rho_{i,s}^{(m)}\\ \mathbf{j}_{i,s}^{(m)}\end{pmatrix}\right](\mathbf{x})\right\|_{\mathbb{R}^6}\le 4\pi\int_{\pm T}^{0}\mathrm{d}s\;s\fint_{\partial B_{|s|}(\mathbf{x})}\mathrm{d}\sigma(y)\;\|\mathbf{G}(\mathbf{y}-\mathbf{q}_s^{(m)})\|_{\mathbb{R}^6}\tag{54}$$

where we used the abbreviation

$$\mathbf{G}(\mathbf{y}-\mathbf{q}_s^{(m)}):=\begin{pmatrix}-\nabla & -\partial_s\\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\rho_{i,s}^{(m)}\\ \mathbf{j}_{i,s}^{(m)}\end{pmatrix}(\mathbf{y})$$

and the computation rules for K_t given in Lemma 3.4. As $\operatorname{supp}\mathbf{G}\subseteq\operatorname{supp}\varrho_i\subseteq B_R(0)$, the right-hand side of (54) is zero for all $\mathbf{x}\in\mathbb{R}$ such that

$$\bigcup_{s\in[-T,T]}\left[\partial B_{|s|}(\mathbf{x})\cap B_R(\mathbf{q}_{i,s}^{(m)})\right]=\emptyset.$$

Now the left-hand side above is a subset of

$$\cup_{s\in[-T,T]}\partial B_{|s|}(\mathbf{x})\bigcap\cup_{s\in[-T,T]}B_R(\mathbf{q}_{i,s}^{(m)})\subseteq B_T(\mathbf{x})\cap B_{R+T}(\mathbf{q}_i^0)$$

which equals the empty set for all $\mathbf{x}\in B_\tau^c(0)$ with $\tau>\|p\|+R+2T$.

Hence, setting $\tau^* := \|p\| + R + 2T$ we conclude that for all $\tau > \tau^*$, the terms $\boxed{6}$ and $\boxed{7}$ and all their derivatives are zero on $B^c_\tau(0) \subset \mathbb{R}^3$. That means in order to show that all the sequences $((\nabla\wedge)^k \mathbf{E}_i^{(m)})_{m\in\mathbb{N}}$, $((\nabla\wedge)^k \mathbf{B}_i^{(m)})_{m\in\mathbb{N}}$ for $k = 0, 1$ and $1 \le i \le N$ decay uniformly at spatial infinity, it suffices to show

$$\lim_{\tau\to\infty} \sup_{m\in\mathbb{N}} \sum_{k=0}^{1} \sum_{i=1}^{N} \left(\|(\nabla\wedge)^k \mathbf{e}_i^{(m)}\|_{L^2_w(B^c_\tau(0))} + \|(\nabla\wedge)^k \mathbf{b}_i^{(m)}\|_{L^2_w(B^c_\tau(0))} \right) = 0. \tag{55}$$

for

$$\begin{pmatrix} \mathbf{e}_i^{(m)} \\ \mathbf{b}_i^{(m)} \end{pmatrix} := \boxed{5} = \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t\mp T} * \begin{pmatrix} \mathbf{E}_{i,\pm T}^{(m),\pm} \\ \mathbf{B}_{i,\pm T}^{(m),\pm} \end{pmatrix} \Bigg|_{t=0}$$

for $1 \le i \le N$. Let $\mathbf{F} \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}^3)$ and $\tau > 0$. By computation rules for K_t given in Lemma 3.4 we get

$$\begin{aligned}
\|\nabla \wedge K_{\mp T} * \mathbf{F}\|_{L^2_w(B^c_{\tau+T}(0))} &= \|K_{\mp T} * \nabla \wedge \mathbf{F}\|_{L^2_w(B^c_{\tau+T}(0))} \le \left\| T \fint_{\partial B_T(0)} d\sigma(y)\, \nabla \wedge \mathbf{F}(\cdot - \mathbf{y}) \right\|_{L^2_w(B^c_{\tau+T}(0))} \\
&\le T \fint_{\partial B_T(0)} d\sigma(y)\, \|\nabla \wedge \mathbf{F}(\cdot - \mathbf{y})\|_{L^2_w(B^c_{\tau+T}(0))} \le T \sup_{\mathbf{y}\in\partial B_T(0)} \|\nabla \wedge \mathbf{F}(\cdot - \mathbf{y})\|_{L^2_w(B^c_{\tau+T}(0))} \\
&\le T \sup_{\mathbf{y}\in\partial B_T(0)} (1 + C_w\|\mathbf{y}\|)^{\frac{P_w}{2}} \|\nabla \wedge \mathbf{F}(\cdot - \mathbf{y})\|_{L^2_w(B^c_{\tau+T}(0))} \le T(1 + C_w T)^{\frac{P_w}{2}} \|\nabla \wedge \mathbf{F}\|_{L^2_w(B^c_\tau(0))},
\end{aligned}$$

and

$$\begin{aligned}
\|\partial_t K_{t\mp T}|_{t=0} * \mathbf{F}\|_{L^2_w(B^c_{\tau+T}(0))} &= \left\| \fint_{\partial B_T(0)} d\sigma(y)\, \mathbf{F}(\cdot - \mathbf{y}) + \frac{T^2}{3} \fint_{B_T(0)} d^3y\, \triangle \mathbf{F}(\cdot - \mathbf{y}) \right\|_{L^2_w(B^c_{\tau+T}(0))} \\
&\le \fint_{\partial B_T(0)} d\sigma(y)\, \|\mathbf{F}(\cdot - \mathbf{y})\|_{L^2_w(B^c_{\tau+T}(0))} + \frac{T^2}{3} \fint_{B_T(0)} d^3y\, \|\triangle \mathbf{F}(\cdot - \mathbf{y})\|_{L^2_w(B^c_{\tau+T}(0))} \\
&\le (1 + C_w T)^{\frac{P_w}{2}} \|\mathbf{F}\|_{L^2_w(B^c_\tau(0))} + \frac{T^2}{3}(1 + C_w T)^{\frac{P_w}{2}} \|\triangle \mathbf{F}\|_{L^2_w(B^c_\tau(0))}.
\end{aligned}$$

Substituting $\mathbf{F}$ with $(\nabla\wedge)^k \mathbf{E}_{i,\pm T}^{(m),\pm}$ and $(\nabla\wedge)^k \mathbf{B}_{i,\pm T}^{(m),\pm}$ for $k = 0, 1$ and $1 \le i \le N$ in the two estimates above yields

$$\begin{aligned}
&\sum_{k=0}^{1} \sum_{i=1}^{N} \left(\|(\nabla\wedge)^k \mathbf{e}_i^{(m)}\|_{L^2_w(B^c_{\tau+T}(0))} + \|(\nabla\wedge)^k \mathbf{b}_i^{(m)}\|_{L^2_w(B^c_{\tau+T}(0))} \right) \\
&\quad \le (1 + C_w T)^{\frac{P_w}{2}} \Bigg(\|(\nabla\wedge)^k \mathbf{E}_{i,\pm T}^{(m),\pm}\|_{L^2_w(B^c_\tau(0))} + \|(\nabla\wedge)^k \mathbf{B}_{i,\pm T}^{(m),\pm}\|_{L^2_w(B^c_\tau(0))} \\
&\qquad + \frac{T^2}{3} \left(\|(\nabla\wedge)^k \triangle \mathbf{E}_{i,\pm T}^{(m),\pm}\|_{L^2_w(B^c_\tau(0))} + \|(\nabla\wedge)^k \triangle \mathbf{B}_{i,\pm T}^{(m),\pm}\|_{L^2_w(B^c_\tau(0))} \right) \\
&\qquad + T \left(\|(\nabla\wedge)^{k+1} \mathbf{E}_{i,\pm T}^{(m),\pm}\|_{L^2_w(B^c_\tau(0))} + \|(\nabla\wedge)^{k+1} \mathbf{B}_{i,\pm T}^{(m),\pm}\|_{L^2_w(B^c_\tau(0))} \right) \Bigg).
\end{aligned} \tag{56}$$

Now the boundary fields $X^\pm$ lie in $\widetilde{\mathcal{A}}^3_w$ which means that the fields $\mathbf{E}_{i,\pm T}^{(m),\pm}$ and $\mathbf{B}_{i,\pm T}^{(m),\pm}$ for $1 \le i \le N$ fulfill the Maxwell constraints so that

$$\|(\nabla\wedge)^k \triangle \mathbf{E}_{i,\pm T}^{(m),\pm}\|_{L^2_w(B^c_\tau(0))} = \|(\nabla\wedge)^{k+2} \mathbf{E}_{i,\pm T}^{(m),\pm}\|_{L^2_w(B^c_\tau(0))} + 4\pi \|(\nabla\wedge)^k \nabla \varrho_i(\cdot - \mathbf{q}_{i,\pm T}^{(m)}\|_{L^2_w(B^c_\tau(0))}$$

and

$$\|(\nabla\wedge)^k \triangle \mathbf{B}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))} = \|(\nabla\wedge)^{k+2}\mathbf{B}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))}.$$

Applying (iv) of Definition 4.8 yields

$$\lim_{\tau\to\infty}\sup_{m\in\mathbb{N}}\sum_{j=0}^{3}\sum_{i=1}^{N}\left(\left\|(\nabla\wedge)^j\mathbf{E}^{(m),\pm}_{i,\pm T}\right\|^2_{L^2_w(B^c_\tau(0))}+\|(\nabla\wedge)^j\mathbf{B}^{(m),\pm}_{i,\pm T}\|^2_{L^2_w(B^c_\tau(0))}\right)=\lim_{\tau\to\infty}\sup_{m\in\mathbb{N}}\|\chi^\pm_{\pm T}[p,F_m]\|^2_{\mathcal{F}^n_w(B^c_\tau(0)}=0$$

because $F_m \in B_r(0) \subset \mathcal{F}^1_w$ for all $m \in \mathbb{N}$. Hence, (55) holds which, as we have shown, implies the uniform decay at spatial infinity of all the sequences $((\nabla\wedge)^k\mathbf{E}^{(m)}_i)_{m\in\mathbb{N}}$, $((\nabla\wedge)^k\mathbf{B}^{(m)}_i)_{m\in\mathbb{N}}$ for $k = 0, 1$ and $1 \le i \le N$. This ensures condition (ii) of Lemma 4.19.

Using the abbreviations $\mathbf{E}^{(m,k)}_i := (\nabla\wedge)^k\mathbf{E}^{(m)}_i$ and $\mathbf{B}^{(m,k)}_i := (\nabla\wedge)^k\mathbf{B}^{(m)}_i$ for $1 \le i \le N$, $k = 0, 1$, and $m \in \mathbb{N}$, we summarize: The sequences $(\mathbf{E}^{(m,k)}_i)_{m\in\mathbb{N}}$, $(\mathbf{B}^{(m,k)}_i)_{m\in\mathbb{N}}$, $(\triangle\mathbf{E}^{(m,k)}_i)_{m\in\mathbb{N}}$ and $(\triangle\mathbf{B}^{(m,k)}_i)_{m\in\mathbb{N}}$ are all uniformly bounded in L^2_w and decay uniformly at spatial infinity.

Successively application of Lemma 4.19 produces the common $\mathcal{F}^1_w$ convergent subsequence: Fix $1 \le i \le N$. Let $(\mathbf{E}^{(m^0_l,0)}_i)_{l\in\mathbb{N}}$ be the L^2_w convergent subsequence of $(\mathbf{E}^{(m,0)}_i)_{m\in\mathbb{N}}$ and $(\mathbf{E}^{(m^1_l,1)}_i)_{l\in\mathbb{N}}$ the L^2_w convergent subsequence of $(\mathbf{E}^{(m^0_l,1)}_i)_{l\in\mathbb{N}}$. In the same way, we proceed with the other indices $1 \le i \le N$ and the magnetic fields, every time choosing a further subsequence of the previous one. Let us denote the final subsequence by $(m_l)_{l\in\mathbb{N}} \subset \mathbb{N}$. Then, we have constructed sequences $(G_{m_l})_{l\in\mathbb{N}}$ as well as $(\mathsf{A}G_{m_l})_{l\in\mathbb{N}}$ which are convergent in $\mathcal{F}^0_w$ and implies the convergence in $\mathcal{F}^1_w$. As $(G_m)_{m\in\mathbb{N}}$ was arbitrary, we conclude that every sequence in M has an $\mathcal{F}^1_w$ convergent subsequence and therefore M is compact which had to be shown. This concludes the proof. □

Having established the existence of a fixed point F for any finite $T > 0$, $p \in \mathcal{P}$ and $(X^\pm_{i,\pm T})_{1\le i\le N} = X^\pm \in \widetilde{\mathcal{A}}^3_w$, claim (iii) of Theorem 4.12 states that the charge world-lines $t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1\le i\le N} := (\mathsf{Q}+\mathsf{P})M_L[p, F](t, 0)$ are in $\mathcal{T}^{p,X^\pm}_T$, that is, they are time-like charge world-lines that solve the conditional WF$_\varrho$ equations (14)–(15) for all times $t \in \mathbb{R}$. As discussed in the introduction, it is interesting to verify that among those solutions we see truly advanced and delayed interactions between the charges. This is the content of Theorem 2.7 which we prove next. We introduce:

Definition 4.20. (*Partial* WF$_\varrho$ *solutions*) For $p \in \mathcal{P}$ we define $\mathcal{T}^L_{\mathrm{WF}}$ to be the set of time-like charge world-lines $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N} \in \mathcal{T}^N_\oplus$ which solve the WF$_\varrho$ equations (5)–(6) for times $t \in [-L, L]$ and have initial conditions $(\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1\le i\le N}|_{t=0} = p$. We shall call every element of $\mathcal{T}^L_{\mathrm{WF}}$ a partial WF$_\varrho$ solution on $[-L, L]$ for initial value p.

In order to show that a conditional WF$_\varrho$ solution $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N} \in \mathcal{T}^{p,X^\pm}_T$ is also a partial WF$_\varrho$ solution, we have to regard the difference between the WF fields produce by them:

$$\begin{aligned} &M_{\varrho_i}[X^\pm_{i,\pm T}, (\mathbf{q}_i, \mathbf{p}_i)](t, \pm T) - M_{\varrho_i}[\mathbf{q}_i, \mathbf{p}_i](t, \pm\infty) \\ &\quad= \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t\mp T} * X^\pm_{i,\pm T} + K_{t\mp T} * \begin{pmatrix} -4\pi\mathbf{v}(\mathbf{p}_{i,\pm T})\varrho_i(\cdot - \mathbf{q}_{i,\pm T}) \\ 0 \end{pmatrix} \\ &\quad\quad -4\pi\int_{\pm\infty}^{\pm T} \mathrm{d}s\, K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix}\begin{pmatrix} \varrho_i(\cdot - \mathbf{q}_{i,s}) \\ \mathbf{v}(\mathbf{p}_{i,s})\varrho_i(\cdot - \mathbf{q}_{i,s}) \end{pmatrix}. \end{aligned} \tag{57}$$

The equality holds by Definition 3.9, Theorem 3.6 and (23) in Theorem 3.10. Let $R > 0$ be the smallest radius such that supp $\varrho_i \subseteq B_R(0)$ for all $1 \le i \le N$. Whenever there is an $L > 0$ such that this difference is zero at least for times $t \in [-L, L]$ and in tubes of radius R around the charge world-lines $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N}$ of a conditional WF$_\varrho$ solution, these world-lines form already a partial WF$_\varrho$ solution in $\mathcal{T}^L_{\mathrm{WF}}$.

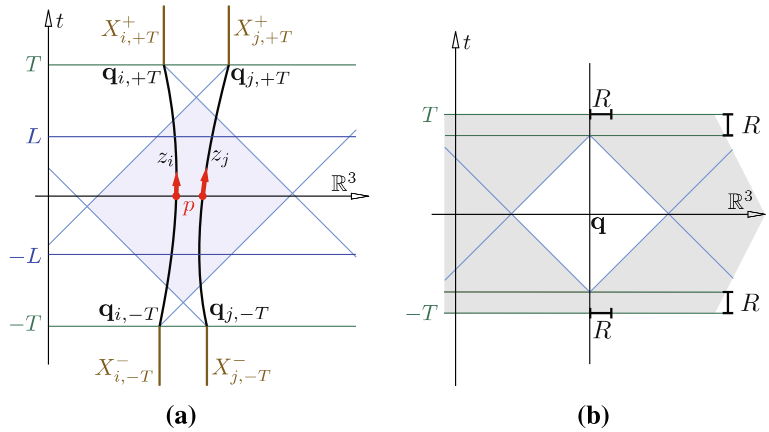

FIG. 3. **a** Choosing Liénard–Wiechert fields for $X^{\pm}_{i,\pm T}$, $1 \leq i \leq N$, the difference (57) between the conditional WF$_\varrho$ and partial WF$_\varrho$ solution vanishes inside the *shaded* (*sheared diamond shaped*) space-time region, which is given by the intersection of the forward and backward light-cones of $\mathbf{q}_{k,-T}$ and $\mathbf{q}_{k,+T}$ for $1 \leq k \leq N$, respectively. **b** The *nonshaded* region visualizes the set of space-time points $(t, \mathbf{x})$ such that $t \in (-T + R, T - R)$ and $\mathbf{x} \in B_{|t \mp T| - R}(\mathbf{q})$ which is used in Lemma 4.21.

Suppose that the boundary fields $X^{\pm}_{i,\pm T}$ are given by the advanced and retarded Liénard–Wiechert fields of asymptotes which continue the conditional WF$_\varrho$ solution $(\mathbf{q}_i, \mathbf{p}_i)_{1\leq i\leq N}$ for times $|t| > T$ into the future and past, respectively. Only by looking at the geometry of the interaction, see Fig. 3a, one may expect that the difference (57) is zero in the intersection of all forward and backward light-cones of the space-time points $(-T, \mathbf{q}_{k,-T})$ and $(+T, \mathbf{q}_{k,+T})$ for all $1 \leq k \leq N$, respectively. However, this intersection might be empty, either if T is chosen too small compared to the maximal distance of the charges at time zero or if the charges approach the speed of light sufficiently fast; see Fig. 2b for an extreme case. As discussed, the properties of known WF$_\varrho$ solutions suggest that the latter case will never occur as their velocities are expected to be uniformly bounded away from the speed of light. In the particular case of the Coulomb boundary fields C, cf. Definition 4.13, we shall now show that, even without having such a uniform velocity estimate, for fixed $T > 0$, there is always a suitable choice of Newtonian Cauchy data $p \in \mathcal{P}$ and nontrivial charge densities $\varrho_i \in \mathcal{C}^\infty_c$ such that partial WF$_\varrho$ solutions exist.

Observe that the difference (57) is given by the free Maxwell time evolution of the boundary fields which at time $\pm T$ carry divergences at $\mathbf{q}_{k,\pm T}$ due to the Maxwell constraints. We shall now exploit the remarkable feature of the free Maxwell time evolution that justifies the discussed geometric picture in Fig. 3a: Any initial field with a nonvanishing divergence will be evolved by the free Maxwell time evolution in a way that the forward and backward light-cones of the support of the divergence are cleared to zero. The next Lemma proves this explicitly in the case of the Coulomb field. Using Lorentz boosts, the presented proof can easily be generalized to Coulomb fields of a moving charge with constant velocity, and with a bit more work it can be generalized further to Liénard–Wiechert fields of any strictly time-like charge world-line.

Lemma 4.21. (Shadows of the Boundary Fields and *WF*$_\varrho$ fields) *Let* $\mathbf{q}, \mathbf{v} \in \mathbb{R}^3$, $\varrho \in \mathcal{C}^\infty_c(\mathbb{R}^3, \mathbb{R})$ *such that* $\operatorname{supp} \varrho \subseteq B_R(0)$ *for some finite* $R > 0$, *and let* $t \mapsto (\mathbf{q}_t, \mathbf{p}_t) \in \mathcal{T}^N_\oplus$ *such that* $\mathbf{q}_t|_{t=0} = \mathbf{q}$. *Furthermore,*$\mathbf{E}^C$ *be the Coulomb field of a charge at rest at the origin*

$$\mathbf{E}^C := \int \mathrm{d}^3 z \; \varrho(\cdot - \mathbf{z}) \frac{\mathbf{z}}{\|\mathbf{z}\|^3}$$

Then for $T > R$ *the expressions*

$$\left[\begin{pmatrix}\partial_t & \nabla\wedge\\ -\nabla\wedge & \partial_t\end{pmatrix} K_{t\mp T} * \begin{pmatrix}\mathbf{E}^C(\cdot-\mathbf{q})\\ 0\end{pmatrix} + K_{t\mp T} * \begin{pmatrix}-4\pi\mathbf{v}\varrho(\cdot-\mathbf{q})\\ 0\end{pmatrix}\right](\mathbf{x}) \tag{58}$$

and

$$\int_{\pm\infty}^{\pm T} \mathrm{d}s\; K_{t-s} * \begin{pmatrix}-\nabla & -\partial_s\\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\varrho(\cdot-\mathbf{q}_s)\\ \mathbf{v}(\mathbf{p}_s)\varrho(\cdot-\mathbf{q}_s)\end{pmatrix}(\mathbf{x}) \tag{59}$$

equal zero for $t\in(-T+R,T-R)$ and $\mathbf{x}\in B_{|t\mp T|-R}(\mathbf{q})$; see Fig. 3b.

Proof. Let $t\in(-T+R,T-R)$. With regard to the second term in (58), we compute

$$\begin{aligned}\|-4\pi\mathbf{v}\left[K_{t\mp T}*\varrho(\cdot-\mathbf{q})\right](\mathbf{x})\| &= 4\pi\|\mathbf{v}\|\left|(t\mp T)\fint_{\partial B_{|t\mp T|}(0)} \mathrm{d}\sigma(y)\;\varrho(\mathbf{x}-\mathbf{y}-\mathbf{q})\right|\\ &\le 4\pi\|\mathbf{v}\||t\mp T|\sup|\varrho|\fint_{\partial B_{|t\mp T|}(\mathbf{q})}\mathrm{d}\sigma(y)\;\mathbb{1}_{B_R(\mathbf{x})}(\mathbf{y})\end{aligned}$$

where we used Definition 3.3 for $K_{t\mp T}$. Now $\mathbf{x}\in B_{|t\mp T|-R}(\mathbf{q})$ implies $\partial B_{|t\mp T|}(\mathbf{q})\cap B_R(\mathbf{x})=\emptyset$, and hence, that the term above is zero.

With regard to the first term, we note that the only nonzero contribution is $\partial_t K_{t\mp T}*\mathbf{E}^C_i$ since $\nabla\wedge\mathbf{E}^C=0$. Lemma 3.4 and in particular equation (19) give

$$\left[\partial_t K_{t\mp T}*\mathbf{E}^C(\cdot-\mathbf{q})\right](\mathbf{x}) = \fint_{\partial B_{|t\mp T|}(0)}\mathrm{d}\sigma(y)\;\mathbf{E}^C(\mathbf{x}-\mathbf{y}-\mathbf{q}) + (t\mp T)\partial_t\fint_{\partial B_{|t\mp T|}(0)}\mathrm{d}\sigma(y)\;\mathbf{E}^C(\mathbf{x}-\mathbf{y}-\mathbf{q}) \tag{60}$$

$$= \fint_{\partial B_{|t\mp T|}(0)}\mathrm{d}\sigma(y)\;\mathbf{E}^C(\mathbf{x}-\mathbf{y}-\mathbf{q}) + \frac{(t\mp T)^2}{3}\fint_{B_{|t\mp T|}(0)}\mathrm{d}^3y\;\triangle\mathbf{E}^C(\mathbf{x}-\mathbf{y}) =: \boxed{8}+\boxed{9}. \tag{61}$$

Using Lebesgue's theorem, we start with

$$\begin{aligned}\boxed{8} &= \mathbf{E}^C(\mathbf{x}-\mathbf{q}) + \int_0^{|t\mp T|}\mathrm{d}s\;\partial_s\fint_{\partial B_s(0)}\mathrm{d}\sigma(y)\;\mathbf{E}^C(\mathbf{x}-\mathbf{y}-\mathbf{q})\\ &= \mathbf{E}^C(\mathbf{x}-\mathbf{q}) + \int_0^{|t\mp T|} dr\;\frac{r}{3}\fint_{B_r(0)}\mathrm{d}^3y\;\triangle\mathbf{E}^C(\mathbf{x}-\mathbf{y}-\mathbf{q}).\end{aligned}$$

Furthermore, we know that $0=(\nabla\wedge)^2\mathbf{E}^C=\nabla(\nabla\cdot\mathbf{E}^C)-\triangle\mathbf{E}^C$ and $\nabla\cdot\mathbf{E}^C=4\pi\varrho$. So we continue the computation with

$$\begin{aligned}\boxed{8} &= \mathbf{E}^C(\mathbf{x}-\mathbf{q}) + \int_0^{|t\mp T|} dr\;\frac{r}{3}\fint_{B_r(0)}\mathrm{d}^3y\;4\pi\nabla\varrho(\mathbf{x}-\mathbf{y}-\mathbf{q})\\ &= \mathbf{E}^C(\mathbf{x}-\mathbf{q}) - \int_0^{|t\mp T|} dr\;\frac{1}{r^2}\int_{\partial B_r(0)}\mathrm{d}\sigma(y)\;\frac{\mathbf{y}}{r}\varrho(\mathbf{x}-\mathbf{y}-\mathbf{q})\end{aligned}$$

where we have used (60) to evaluate the derivative and in addition used Stoke's Theorem. Note that the minus sign in the last line is due to the fact that ∇ acts on $\mathbf{x}$ and not on $\mathbf{y}$. Inserting the definition of the Coulomb field $\mathbf{E}^C$ we finally get

$$\boxed{8} = \int_{B^c_{|t\mp T|}(0)} \mathrm{d}^3 y \; \varrho(\mathbf{x}-\mathbf{y}-\mathbf{q}) \frac{\mathbf{y}}{\|\mathbf{y}\|^3}.$$

This integral is zero if, for example, $B^c_{|t\mp T|}(\mathbf{q}) \cap B_R(\mathbf{x}) = \emptyset$ and this is the case for $\mathbf{x} \in B_{|t\mp T|-R}(\mathbf{q})$. So it remains to show that $\boxed{9}$ also vanishes. Therefore, using $\triangle \mathbf{E}^C = 4\pi \nabla \varrho$ as before, we get

$$\boxed{9} = -\int_{\partial B_{|t\mp T|}(0)} \mathrm{d}\sigma(y) \; \frac{\mathbf{y}}{(t\mp T)^2} \varrho(\mathbf{x}-\mathbf{y}-\mathbf{q}).$$

This expression is zero, for example, when $\partial B_{|t\mp T|}(\mathbf{q}) \cap B_R(\mathbf{x}) = \emptyset$ which is true for $\mathbf{x} \in B_{|t\mp T|-R}(\mathbf{q})$. Hence, we have shown that for $t \in (-T+R, T-R)$ and $\mathbf{x} \in B_{|t\mp T|}(\mathbf{q})$ the term (58) is zero.

Looking at the support of the integrand and the integration domain in term (59), we find that for all $t \in (-T+R, T-R)$ it is zero for all $\mathbf{x} \in \mathbb{R}^3$ such that

$$\bigcup_{|s|>T} \left(\partial B_{|t-s|}(\mathbf{x}) \cap B_R(\mathbf{q}_s)\right) = \emptyset. \tag{62}$$

Hence, for $t \in (-T+R, T-R)$ and $\mathbf{x} \in B_{|t\mp T|}(\mathbf{q})$, the term (59) is also zero which concludes the proof. $\square$

It remains to get a bound on the velocities of the charge world-lines within $[-T, T]$.

Lemma 4.22. (Uniform Velocity Bound) *For finite a, b there is a continuous and strictly increasing map $v^{a,b} : \mathbb{R}^+ \to [0, 1)$, $T \mapsto v^{a,b}_T$ such that*

$$\sup \left\{ \|\mathbf{v}(\mathbf{p}_{i,t})\|_{\mathbb{R}^3} \;\middle|\; t \in [-T,T], \|p\| \le a, F \in \operatorname{Range} S^{p,C}_T, \|\varrho_i\|_{L^2_w} + \|w^{-1/2}\varrho_i\|_{L^2} \le b, 1 \le i \le N \right\}$$
$$\le v^{a,b}_T < 1.$$

for $(\mathbf{p}_{i,t})_{1\le i\le N} := \mathsf{P}M_L[p, F](t, 0)$ for all $t \in \mathbb{R}$.

Proof. Recall the estimate (32) from the ML-SI$_\varrho$ existence and uniqueness Theorem 3.19 which gives the following T dependent upper bounds on these ML-SI$_\varrho$ solutions for all $\varphi \in D_w(A)$:

$$\sup_{t\in[-T,T]} \|M_L[\varphi](t, 0)\|_{\mathcal{H}_w} \le C_3 \left(T, \|\varrho_i\|_{L^2_w}, \|w^{-1/2}\varrho_i\|_{L^2}; 1 \le i \le N\right) \; \|\varphi\|_{\mathcal{H}_w}. \tag{63}$$

Note further that by Lemma 4.17 since $C \in \mathcal{A}^1_w$, there is a ${C_{11}}^{(1)} \in \mathtt{Bounds}$ such that fields $F \in \operatorname{Range} S^{p,C}_T \in D_w(\mathsf{A}^\infty)$ fulfill

$$\|F\|_{\mathcal{F}_w} \le {C_{11}}^{(1)}(T, \|p\|) \le {C_{11}}^{(1)}(T, a).$$

Therefore, setting $c := a + {C_{11}}^{(1)}(T, a)$, we estimate the maximal momentum of the charges by

$$\sup \left\{ \|\mathbf{p}_{i,t}\|_{\mathbb{R}^3} \;\middle|\; t \in [-T,T], \|p\| \le a, F \in \operatorname{Range} S^{p,C}_T, \|\varrho_i\|_{L^2_w} + \|w^{-1/2}\varrho_i\|_{L^2} \le b, 1 \le i \le N \right\}$$
$$\le \sup \left\{ \|\mathbf{p}_{i,t}\|_{\mathbb{R}^3} \;\middle|\; t \in [-T,T], \varphi \in D_w(A), \|\varphi\|_{\mathcal{H}_w} \le c, \|\varrho_i\|_{L^2_w} + \|w^{-1/2}\varrho_i\|_{L^2} \le b, 1 \le i \le N \right\}$$
$$\le C_3\,(T, b, b,)\, c =: p^{a,b}_T < \infty.$$

Now, since C_2 as well as ${C_{11}}^{(1)}$ are in `Bounds`, the map $T \mapsto p_T^{a,b}$ as $\mathbb{R}^+ \to \mathbb{R}^+$ is continuous and strictly increasing. We conclude that claim is fulfilled for the choice

$$v_T^{a,b} := \frac{p_T^{a,b}}{\sqrt{m^2 + (p_T^{a,b})^2}} < 1.$$

□

With this we can prove our last result, that is, Theorem 2.7.

Proof of Theorem 2.7 (True WF$_\varrho$ Interaction). Let F^* be a fixed point $F^* = S_T^{p,C}[F^*]$ which exists by Theorem 2.6. Define the charge world-lines $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N}$ by $t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1\le i\le N} := (\mathtt{Q}+\mathtt{P})M_L[p, F^*](t, 0)$. By the fixed point properties of F^* we know that $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N} \in \mathcal{T}_T^{p,C}$ and therefore solve the conditional WF$_\varrho$ equations (14)–(15) for Newtonian Cauchy data p and boundary fields C. In order to show that the charge world-lines $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N}$ are also in $\mathcal{T}_{\mathrm{WF}}^L$ for the given L we need to show that the difference (57), which equals

$$\begin{aligned}
&M_{\varrho_i}[X_{i,\pm T}^{\pm}, (\mathbf{q}_i, \mathbf{p}_i)](t, \pm T) - M_{\varrho_i}[\mathbf{q}_i, \mathbf{p}_i](t, \pm\infty) \\
&= \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t\mp T} * X_{i,\pm T}^{\pm} + K_{t\mp T} * \begin{pmatrix} -4\pi \mathbf{v}(\mathbf{p}_{i,\pm T})\varrho_i(\cdot - \mathbf{q}_{i,\pm T}) \\ 0 \end{pmatrix} \\
&\quad -4\pi \int_{\pm\infty}^{\pm T} \mathrm{d}s\, K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \varrho_i(\cdot - \mathbf{q}_{i,s}) \\ \mathbf{v}(\mathbf{p}_{i,s})\varrho_i(\cdot - \mathbf{q}_{i,s}) \end{pmatrix},
\end{aligned}$$

is zero for times $t \in [-L, L]$ at least for all points $\mathbf{x}$ in a tube around the position of the $j \neq i$ charge world-lines. Lemma 4.21 states that this expression is zero for all $t \in (-T+R, T-R)$ and $\mathbf{x} \in B_{|t\mp T|-R}(\mathbf{q}_{i,\pm T})$. So it is sufficient to show that the charge world-lines spend the time interval $[-L, L]$ in this particular space-time region. Clearly, the position $\mathbf{q}_i^0$ at time zero is in $B_{T-R}(\mathbf{q}_{i,\pm T})$. Hence, we estimate the earliest exit time L of this space-time region of a charge world-line j, that is, the time when the jth charge world-line leaves the region $B_{|L\mp T|-R}(\mathbf{q}_{i,\pm T})$. By Lemma 4.22, the charges can in the worst case move apart from each other with velocity $v_T^{a,b}$ during the time interval $[-T, T]$. Putting the origin at $\mathbf{q}_i^0$, we can compute the exit time L by

$$-v_T^{a,b} T = \|\mathbf{q}_j^0 - \mathbf{q}_i^0\| + 2R + v_T^{a,b} L - (T - L)$$

This gives the lower bound $L := \frac{(1-v_T^{a,b})T - \triangle q_{max} - 2R}{1+v_T^{a,b}} > 0$. Since $v_T^{a,b} < 1$ we have $(1 - v_T^{a,b})T > 0$. Hence, for sufficiently small R and $\triangle q_{max}$, it is true that $L > 0$ which concludes the proof. □

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