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Correction to: Positive Solutions for Slightly Subcritical Elliptic Problems Via Orlicz Spaces

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The authors would like to correct an error in the original paper [1], which was kindly pointed out to us by Prof. Alfonso Castro. In the proof of Proposition 3.1, the two last lines in p. 8, and in the proof of Theorem 1.1, the two last lines in p. 18 were wrong, and so the conclusions. Indeed, let $\Omega \subset \mathbb{R}^N$, $N > 2$, be a bounded, connected open subset, with C^2 boundary $\partial\Omega$, and $\{g_k\}_{k\in\mathbb{N}} \subset L^\infty(\Omega)$ such that $\int_\Omega g_k\psi \to 0$ for any $\psi \in H_0^1(\Omega)$. Density alone is not enough and the above do not imply that $\int_\Omega g_k\psi \to 0$ for any $\psi \in L^2(\Omega)$, it will hold if $\{g_k\}_{k\in\mathbb{N}}$ is uniformly bounded.

We present here a proof based on a different argument. The consideration presented below should modify part of the proof of Proposition 3.1, and part of the proof of Theorem 1.1.

Replace line 3 in page 3, by:

Assume that

(A1) Ω^+, and Ω^- are non-empty sets,

(A2) $\overline{\Omega}^+ \cap \overline{\Omega}^- = \emptyset$.

We add hypothesis **(A1)**–**(A2)** to the statement of Theorem 1.1.

Theorem 1.1. *Assume that $g \in C^1(\mathbb{R})$ satisfies hypothesis* (H). *Let $C_0 > 0$ be defined by* (1.6). *Assume that a changes sign in Ω, that* **(A1)**–**(A2)**, *and that* (1.5) *hold, then there exists a $\Lambda \in \mathbb{R}$,*

$$\lambda_1 < \Lambda \le \min\left\{\lambda_1\big(\,int(\Omega^0)\big),\quad \lambda_1\big(\,int\left(\Omega^+ \cup \Omega^0\right)\big) + C_0 \sup a^+\right\}$$

such that (1.1) *has a classical positive solution if $\lambda < \Lambda$ and there is no positive solutions if $\lambda > \Lambda$.*

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Moreover, there exists a continuum (a closed and connected set) $\mathscr{C}$ of classical positive solutions to (1.1) *emanating from the trivial solution set at the bifurcation point* $(\lambda, u) = (\lambda_1, 0)$ *which is unbounded. Besides,*

(a) *For every,* $\lambda \in \big(\lambda_1, \Lambda\big)$, (1.1) *admits at least two classical ordered positive solutions.*
(b) *Assume* $\Lambda < \lambda_1\big(\, int(\Omega^0)\big)$, *for* $\lambda = \Lambda$ *problem* (1.1) *admits at least one classical positive solution.*
(c) *For every* $\lambda \leq \lambda_1$, *problem* (1.1) *admits at least one classical positive solution.*

We also add hypothesis **(A1)**–**(A2)** to the statement of Proposition 3.1. We present here a more general result, with a new (J_2) condition, more general than usual.

Proposition 3.1. *Assume that* $f \in C^1(\mathbb{R})$ *fulfills hypothesis* (H) *and that* $\lambda < \lambda_1(\, int\Omega^0) < +\infty$. *Assume that* $a \in C^1(\overline{\Omega})$, *and* $\Omega^{\pm}$, Ω^0 *satisfy hypothesis* **(A1)**–**(A2)**.

Then any (PS) *sequence, that is, a sequence* $\{u_n\}$ *satisfying the conditions*

(J_1) $J_\lambda[u_n] \leq C$,
(J_2) $\big|J_\lambda'[u_n]\,\psi\big| \leq \varepsilon_n \,\|u_n\|\,\|\psi\|$, *where* $\varepsilon_n \to 0$ *as* $n \to +\infty$

is a bounded sequence.

Proof. A careful reading of Step 1 show that the conclusion holds under new hypothesis (J_2) assumed here, just changing l. 15–16, p. 7 by the following:

$$\left|-\int_\Omega |\nabla u_n^-|^2 dx - \int_\Omega a(x) f(u_n^+) u_n^- dx\right| = \int_\Omega |\nabla u_n^-|^2 dx \leq \epsilon_n \|u_n\| \|u_n^-\|$$

so $\|v_n^-\| \to 0$ and then $v^- \equiv 0$, and we conclude the proof of the claim.

2. Claim: $\lim_{n\to\infty} \int_\Omega a(x) \frac{f(u_n^+)u_n^+}{\|u_n\|^2}\, dx = 1$.

Taking into account that $v \equiv 0$ a.e. in Ω, it follows from (J_2) applied to $\psi = u_n$ that

$$\int_\Omega a(x) \frac{f(u_n^+)u_n^+}{\|u_n\|^2}\, dx = 1 + o(1).$$

3. Claim: $0 \leq \limsup_{n\to\infty} \int_\Omega a^-(x) \frac{f(u_n^+)u_n^+}{\|u_n\|^2}\, dx < +\infty$.

To prove this claim we will make use of hypothesis. It comes from this hypothesis that there exists $0 \leq \psi \in C^1(\Omega)$ such that $\psi = 1$ on Ω^- and $\psi = 0$ on Ω^+. Taking $\phi = u_n \psi$ as a test function in (J_2) we get

$$\int_\Omega \psi |\nabla v_n|^2\, dx + \int_\Omega v_n \nabla v_n \cdot \nabla \psi\, dx + \int_\Omega a^-(x) \frac{f(u_n^+)u_n^+}{\|u_n\|^2}\, dx = \lambda \int_\Omega (v_n^+)^2 \psi\, dx + o(1)$$

from which it comes directly that the sequence $\left\{\int_\Omega a^-(x) \frac{f(u_n^+)u_n^+}{\|u_n\|^2}\, dx\right\}_{n\in\mathbb{N}}$ is bounded and non-negative, so the claim 3 is proved.

In order to achieve a contradiction, we use (J_1) and (J_2). Multiplying by 1/2 the inequality (J_2) applied to u_n and adding (J_1)

$$\int_\Omega a(x) \left[\frac{1}{2} f(u_n^+) u_n^+ - F(u_n^+)\right] dx \leq C + \varepsilon_n \|u_n\|^2.$$

Consequently

$$\limsup_{n\to\infty} \int_\Omega a(x) \left[\frac{\frac{1}{2} f(u_n^+)u_n^+ - F(u_n^+)}{\|u_n\|^2} \right] dx \le 0. \tag{0.1}$$

Since hypothesis $(\mathrm{H})_{g'}$, and using l'Hôpital's rule, we can write

$$\lim_{s\to\infty} \frac{\frac{1}{2} s f(s) - F(s)}{s f(s)} = \frac{1}{2} - \frac{1}{2^*} =: \frac{1}{N} > 0. \tag{0.2}$$

Fixing $\varepsilon > 0$ and using (0.2), there exists C_ε such that

$$\begin{aligned} -C_\varepsilon + \left(\frac{1}{N} - \varepsilon\right) \int_\Omega a^\pm(x)\, f(u_n^+)u_n^+\, dx &\le \int_\Omega a^\pm(x) \left[\frac{1}{2} f(u_n^+)u_n^+ - F(u_n^+)\right] dx \\ &\le C_\varepsilon + \left(\frac{1}{N} + \varepsilon\right) \int_\Omega a^\pm(x)\, f(u_n^+)u_n^+\, dx. \end{aligned}$$

Consequently,

$$\begin{aligned} &\int_\Omega \big(a^+(x) - a^-(x)\big) \left[\frac{1}{2} f(u_n^+)u_n^+ - F(u_n^+)\right] dx \\ &\quad \ge -2C_\varepsilon + \left(\frac{1}{N} - \varepsilon\right) \int_\Omega \big(a^+(x) - a^-(x)\big)\, f(u_n^+)u_n^+\, dx \\ &\qquad - 2\varepsilon \int_\Omega a^-(x)\, f(u_n^+)u_n^+\, dx, \end{aligned}$$

and then

$$\begin{aligned} \left(\frac{1}{N} - \varepsilon\right) \int_\Omega a(x)\, f(u_n^+)u_n^+\, dx \le 2C_\varepsilon &+ \int_\Omega a(x) \left[\frac{1}{2} f(u_n^+)u_n^+ - F(u_n^+)\right] dx \\ &+ 2\varepsilon \int_\Omega a^-(x)\, f(u_n^+)u_n^+\, dx, \end{aligned}$$

so, using (0.1)

$$\lim_{n\to\infty} \int_\Omega a(x)\, \frac{f(u_n^+)u_n^+}{\|u_n\|^2}\, dx \le \frac{2\varepsilon}{\frac{1}{N} - \varepsilon} \limsup_{n\to\infty} \int_\Omega a^-(x) \frac{f(u_n^+)u_n^+}{\|u_n\|^2}\, dx.$$

Letting $\varepsilon \to 0$, we get a contradiction with claim 2 and claim 3. We have just proved that any Palais–Smale sequence (PS) is bounded. □

We observe that Theorem 3.2 still holds under the more general (J_2) condition.

Proof of Theorem 1.1. (b) Step 2. Existence of a classical positive solution for $\lambda = \Lambda$. We prove the existence of a solution for $\lambda = \Lambda$. First notice that for any $\lambda \in (\lambda_1, \Lambda)$,

$$\lim_{t\to 0^+} J_\lambda[t\varphi_1]/t^2 = (\lambda_1 - \lambda) \int_\Omega \varphi_1^2 dx < 0.$$

Consequently there exists $t_0 > 0$ such that $J_\lambda[t\varphi_1] \le 0$ for all $0 < t \le t_0$. Besides, we recall that $t\varphi_1$ is a subsolution of problem (1.1) for all $t > 0$ small enough. It follows that for each $\lambda \in (\lambda_1, \Lambda)$, problem $(1.1)_\lambda$ admits a positive weak solution $\overline{u}_\lambda$ minimising J_λ in some interval of the form $[t\varphi_1, \overline{u}]$ with $t < t_0$ and $\overline{u}$ a positive solution of (1.1) for a $\mu \in (\lambda, \Lambda)$. Hence $J_\lambda[\overline{u}_\lambda] \le J_\lambda[t\varphi_1]$ and

$$J_\Lambda[\overline{u}_\lambda] = J_\lambda[\overline{u}_\lambda] + \frac{1}{2}(\lambda - \Lambda)\int_\Omega \overline{u}_\lambda^2 dx \le J_\lambda[t\varphi_1] \le 0.$$

Moreover

$$J'_\Lambda[\overline{u}_\lambda]\psi = J'_\lambda[\overline{u}_\lambda]\psi + (\lambda - \Lambda)\int_\Omega \overline{u}_\lambda \psi dx = (\lambda - \Lambda)\int_\Omega \overline{u}_\lambda \psi dx$$

so, for any sequence $\lambda_n \to \Lambda$, the sequence $(\overline{u}_{\lambda_n})_n$ satisfies (J_1) and (J_2) for the functional J_Λ. Since $\Lambda < \lambda_1\big(int\,(\Omega^0)\big)$, by Proposition 3.1 the sequence $(\overline{u}_{\lambda_n})_n$ is bounded. Hence, using Theorem 3.2 of [1], there exists a function $\overline{u}_\Lambda$ such that, up to a subsequence, $\overline{u}_{\lambda_n} \to u_\Lambda$ strongly in $H_0^1(\Omega)$. The possibility of $u_\Lambda = 0$ is ruled out by considering the sequence $v_n = \frac{\overline{u}_{\lambda_n}}{\|\overline{u}_{\lambda_n}\|}$ which will converge weakly to some $0 \le v \not\equiv 0$ since $1 = \Lambda\|v\|^2$, and satisfying, weakly in $H_0^1(\Omega)$, $-\Delta v = \Lambda v$, contradicting that $\Lambda > \lambda_1$. We conclude that $\overline{u}_\Lambda$ is a positive solution of (1.1) for $\lambda = \Lambda$.

(c) Case $\lambda = \lambda_1$. A solution can be obtained by a Mountain pass as follows. As above, any sequence of solutions $\{u\}_n$ converging to 0 in $H_0^1(\Omega)$ satisfies, up to a sub-sequence, $\frac{u_n}{\|u_n\|} \rightharpoonup c\varphi_1$ for some $c > 0$. Then, using $(\mathrm{H})_0$ and l'Hôpital rule we have,

$$\lim_{\|u_n\|\to 0} \frac{J_{\lambda_1}[u_n]}{\|u_n\|^p} = -L_1 c^p \int_\Omega a(x)\varphi_1^p dx > 0$$

so 0 is a strict local minimum of J_{λ_1}. Besides, by choosing suitably $u_0 \in C_0^1(\Omega^+)$, $\lim_{t\to+\infty} J_{\lambda_1}[tu_0] = -\infty$. Since $\lambda_1 < \lambda_1\big(\mathrm{int}\,(\Omega^0)\big)$, J_{λ_1} satisfies the (PS) condition and we then inferee the existence of a nontrivial solution of (1.1) for $\lambda = \lambda_1$. By the strong maximum principle, this solution is > 0. □

Reference

[1] Cuesta, M., Pardo, R.: Positive solutions for slightly subcritical elliptic problems via Orlicz spaces. Milan J. Math. **90**(1), 229–255 (2022)

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