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An Existence Result for a Class of Magnetic Problems in Exterior Domains

Claudianor O. Alves, Vincenzo Ambrosio, and
César E. Torres Ledesma

Abstract. In this paper we deal with the existence of solutions for the following class of magnetic semilinear Schrödinger equation

$$(P) \qquad \begin{cases} (-i\nabla + A(x))^2 u + u = |u|^{p-2}u, \ \text{ in } \ \Omega, \\ u = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where $N \geq 3$, $\Omega \subset \mathbb{R}^N$ is an exterior domain, $p \in (2, 2^*)$ with $2^* = \frac{2N}{N-2}$, and $A : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous vector potential verifying $A(x) \to 0$ as $|x| \to \infty$.

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1. Introduction

In this paper we investigate the existence of solutions for the following magnetic semilinear Schrödinger equation

$$(P) \qquad \begin{cases} (-i\nabla + A(x))^2 u + u = |u|^{p-2}u \ \text{ in } \ \Omega, \\ u = 0 \ \text{ on } \ \partial\,\Omega, \end{cases}$$

where $N \geq 3$, $p \in (2, 2^*)$, $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent, $\Omega \subset \mathbb{R}^N$ is an exterior domain, i.e. Ω is an unbounded domain with smooth boundary $\partial\Omega \neq \emptyset$ such that $\mathbb{R}^N \backslash \Omega$ is bounded, and $A \in C(\mathbb{R}^N, \mathbb{R}^N)$ satisfies

$$A(x) \to 0 \ \text{ as } \ |x| \to \infty. \tag{A}$$

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During the past years there has been a considerable interest in the existence of solutions for elliptic equations in exterior domains, more precisely, for problems of the type

$$\begin{cases} -\Delta u + u = f(u), & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

where f is a continuous function satisfying some technical conditions. The main difficulty in dealing with (1.1) is the lack of compactness of the Sobolev embedding due to the unboundedness of the domain. In order to overcome this difficulty, in some papers, authors assumed certain type of symmetry on Ω; see for instance [9], [23] and [25].

In [12], Benci and Cerami studied the existence of nontrivial solutions for the problem

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u, & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.2}$$

in an exterior domain Ω without assuming symmetry, with $2 \le p < 2^*$ and $\lambda > 0$. In that article, they proved that (1.2) does not have a ground state solution and this fact yields a series of difficulties. The key idea exploited by the authors was to analyze the behavior of Palais-Smale sequences, obtaining a precise estimate of the energy levels where the Palais-Smale condition fails. The authors proved that if $p = 1 + \frac{8}{N}$ for $N = 5, 6, 7$ or $p < \frac{2(N-1)}{N-2}$ for $N = 3, 4$,

- there exists $\lambda_* > 0$ such that, for every $\lambda \in (0, \lambda_*)$, (1.2) has at least one positive solution,
- for every λ there exists a $\rho = \rho(\lambda)$ such that if $\mathbb{R}^N \backslash \Omega \subset B(x_0, \rho)$, with $x_0 \in \mathbb{R}^N \backslash \Omega$, (1.2) has at least one positive solution.

Later, existence results were obtained for more general problems

$$\begin{cases} -\Delta u + \lambda u = f(x, u) & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.3}$$

where f is a continuous function satisfying

$$\lim_{|x| \to +\infty} f(x, t) = f_\infty(t), \text{ for all } t \in \mathbb{R}.$$

In [9], Bahri and Lions studied (1.3) with $f(x, t) = b(x)|t|^{p-2}t$ where $b(x) \to b > 0$ as $|x| \to +\infty$. Using topological arguments, they showed that (1.3) has a solution when Ω is an arbitrary exterior domain, for all $\lambda > 0$. In the autonomous case, using the technique introduced in [9], Li and Zheng [28] proved that (1.3) possesses at least one positive solution, with f asymptotically linear satisfying some assumptions, in particular, a property of convexity (see also [21]). Whereas in [29], Maia and Pellacci established an existence result without the hypothesis of convexity.

In the above-mentioned papers, a key point to prove the results of existence is the uniqueness, up to a translation, of the positive solution for the "equation at infinity" associated with (1.3) given by

$$-\Delta u + \lambda u = f_\infty(u) \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

However, when the exterior domain is the exterior of a ball, more precisely $\Omega = \mathbb{R}^N \backslash B(0,R)$ for some $R > 0$, it is possible to explore some groups of rotation in order to get multiple solutions without using the uniqueness of solutions of the limit problem when R is large enough, see for example Cao and Noussair [16] and Clapp, Maia and Pellacci [22]. In [2], Alves and de Freitas studied the existence of a positive solution for a class of elliptic problems in exterior domains involving critical growth. Finally, we mention a recent paper due to Alves, Molica Bisci and Torres Ledesma [5], in which a fractional elliptic equation with Dirichlet-type condition set in an exterior domain is considered.

In the last years time-independent magnetic Schrödinger equations in bounded domains or in whole of $\mathbb{R}^N$ have received a special attention. A basic motivation to study these equations stems from the search of standing wave solutions for the time-dependent nonlinear Schrödinger equation of the type

$$ih\frac{\partial \psi}{\partial t} = \left(\frac{h}{i}\nabla - A(z)\right)^2 \psi + U(z)\psi - f(|\psi|^2)\psi,\ z \in \mathcal{D},\, t \in \mathbb{R},$$

where $\mathcal{D} \subset \mathbb{R}^N$, with $N \geq 2$, is a smooth domain, the function ψ takes values in $\mathbb{C}$, h is the Planck constant, i is the imaginary unit and $A : \mathbb{R}^N \to \mathbb{R}^N$ denotes a magnetic potential. For the interested reader in this subject, we cite the papers by Alves, Figueiredo and Furtado [3,4], Ambrosio [6,7], Arioli and Szulkin [8], Barile [10,11], Chabrowski and Szulkin [17], Cingolani [18], Cingolani and Clapp [19], Cingolani, Jeanjean and Secchi [20], Ding and Liu [24], Esteban and Lions [26] and the references therein.

After a careful bibliography review, we did not find any paper concerned with magnetic semilinear Schrödinger equations in exterior domains. Motivated by this fact and the above-mentioned papers, the aim of this paper is to give a first existence result for (P). We emphasize that the main difficulty in dealing with this type of problem is related to the uniqueness, up to a translation, of the solution for the limit problem. Recently, in a very interesting paper due to Bonheure, Nys and Van Schaftingen [14], we found a partial answer for this question when the magnetic field A satisfies some technical conditions; see [14, Theorem 1].

The main result of this paper can be stated as follows:

Theorem 1.1. *Suppose that (A) holds. Then, there exist $\rho_0 > 0$ and $\epsilon > 0$ such that if $\mathbb{R}^N \backslash \Omega \subset B(0,\rho)$, $\rho < \rho_0$ and $\|A\|_\infty \leq \epsilon$, then problem (P) has at least one weak solution.*

The proof of Theorem 1.1 will be done done via variational methods inspired by [2] and [12]. However, with respect to [2,12], a more careful analysis will be needed and some refined estimates will be given. The diamagnetic inequality in [26] will play a fundamental role.

We point out that Theorem 1.1 complements the study in magnetic semilinear Schrödinger equations, in the sense that we obtain an existence result for a magnetic Schrödinger equation in an exterior domain.

The paper is organized as follows. In Sect. 2, we introduce suitable function spaces and collect some useful results concerning the limit problem that we will work with. In Sect. 3, we establish a compactness result in the spirit of [12] for the energy functional associated with problem (P). In Sect. 4, we show some technical estimates that will be used in Sect. 5, where Theorem 1.1 is proved.

Notations: In this paper, we use the following notations:

- For $q \in (2, 2^*)$, we define q' as the conjugate exponent of q, that is, $q' := \frac{q}{q-1}$.
- The usual norm of the Lebesgue spaces $L^t(\Omega)$ for $t \in [1, \infty]$, will be denoted by $|\,.\,|_t$, and the norm of the Sobolev space $H_0^1(\Omega)$, by $\|\,.\,\|$;
- C denotes (possibly different) any positive constant.

2. Preliminary Results and the Limit Problem

In what follows, we denote by $H_A^1(\Omega, \mathbb{C})$ the Hilbert space obtained by the closure of $C_0^\infty(\Omega, \mathbb{C})$ under the scalar product

$$\langle u, v\rangle := \Re\left(\int_\Omega (\nabla_A u(x)\overline{\nabla_A v(x)} + u(x)\overline{v(x)})\, dx\right),$$

where $\Re(w)$ denotes the real part of $w \in \mathbb{C}$, $\overline{w}$ is its complex conjugate, $\nabla_A u := (D_1 u, D_2 u, ..., D_N u)$ where $D_j := -i\partial_j + A_j(x)$, for $j = 1, 2, ..., N$. The norm induced by this inner product is given by

$$\|u\| := \left(\int_\Omega (|\nabla_A u(x)|^2 + |u(x)|^2)\, dx\right)^{\frac{1}{2}}.$$

We also consider the Hilbert space $H_A^1(\mathbb{R}^N, \mathbb{C})$ defined as

$$H_A^1(\mathbb{R}^N, \mathbb{C}) := \{u \in L^2(\mathbb{R}^N, \mathbb{C}) : \nabla_A u \in L^2(\mathbb{R}^N, \mathbb{C}^N)\}$$

endowed with the scalar product

$$\langle u, v\rangle_A := \Re\left(\int_{\mathbb{R}^N} (\nabla_A u(x) \cdot \overline{\nabla_A v(x)} + u(x)\overline{v(x)})dx\right).$$

Then we can define the norm

$$\|u\|_A := \left(\int_{\mathbb{R}^N} \left(|\nabla_A u(x)|^2 + |u(x)|^2\right) dx\right)^{\frac{1}{2}}.$$

By [26, Proposition 2.1-(i)], we know that $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ is dense in $H_A^1(\mathbb{R}^N, \mathbb{C})$. A direct computation shows that $H_A^1(\Omega, \mathbb{C}) \subset H_A^1(\mathbb{R}^N, \mathbb{C})$.

As proved in [26, Sect. 2], for any $u \in H_A^1(\mathbb{R}^N, \mathbb{C})$, there holds the *diamagnetic inequality*, namely

$$|\nabla |u|| \leq |\Re\left(\nabla_A u \,\mathrm{sign}(u)\right)| \leq |\nabla u_A|, \text{ a.e. in } \mathbb{R}^N, \tag{2.1}$$

where

$$\mathrm{sign}(u) = \begin{cases} \frac{\overline{u(x)}}{|u(x)|} & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0. \end{cases}$$

Hence, if $u \in H_A^1(\mathbb{R}^N, \mathbb{C})$, then $|u| \in H^1(\mathbb{R}^N, \mathbb{R})$. Furthermore, as a consequence of the diamagnetic inequality, we have that the embedding

$$H_A^1(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^s(\mathbb{R}^N, \mathbb{C}) \tag{2.2}$$

is continuous for any $s \in [2, 2^*]$.

2.1. Limit Problem

In this subsection, we consider the scalar limit problem associated with (P), namely

$$(P_0) \qquad \begin{cases} -\Delta u + u = |u|^{p-2}u \ \text{ in } \ \mathbb{R}^N, \\ u \in H_0^1(\mathbb{R}^N, \mathbb{C}), \end{cases}$$

where $p \in (2, 2^*)$. The reader is invited to see that $u = u_1 + \imath u_2$, with u_1 and u_2 real valued, is a solution of (P_0) if and only if u_1 and u_2 solve the following elliptic system

$$(S_0) \qquad \begin{cases} -\Delta u_1 + u_1 = \left(\sqrt{|u_1|^2 + |u_2|^2}\right)^{p-2} u_1, \ \text{ in } \mathbb{R}^N, \\ \\ -\Delta u_2 + u_2 = \left(\sqrt{|u_1|^2 + |u_2|^2}\right)^{p-2} u_2, \ \text{ in } \mathbb{R}^N, \end{cases}$$

which is a system of the gradient type.

Note that, the solutions of (P_0) are critical points of the functional

$$\begin{aligned} &I_0 : H^1(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R} \\ &\qquad u \to I_0(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u(x)|^2 + |u(x)|^2)dx - \frac{1}{p}\int_{\mathbb{R}^N} |u(x)|^p dx. \end{aligned} \tag{2.3}$$

If c_0 denotes the mountain pass level of I_0 and $\mathcal{N}_0$ is the Nehari manifold defined as

$$\mathcal{N}_0 := \{u \in H_0^1(\mathbb{R}^N, \mathbb{C})\backslash\{0\} : I_0'(u)u = 0\},$$

it is well-known (see [30]) that

$$c_0 = c_0^* := \inf_{u \in \mathcal{N}_0} I_0(u),$$

from where it follows that c_0 is the least energy of (P_0). We recall that $u \in H_0^1(\mathbb{R}^N, \mathbb{C})$ is a least energy solution of (P_0) if $I_0(u) = c_0^*$ and $I_0'(u) = 0$, and c_0^* is called the least energy of (P_0).

Lemma 2.1. *The following fact holds: u is a least energy solution of (P_0) if, and only if, $v(x) := |u(x)| \in H^1(\mathbb{R}^N)$ is a least energy solution of*

$$(P_\infty) \qquad -\Delta v + v = |v|^{p-2}v, \ \textit{in } \mathbb{R}^N, \ v > 0.$$

Moreover, (P_0) and (P_∞) have the same least energy.

Proof. The proof can be done as in [24, Lemma 2.5]. □

Lemma 2.2. *The following facts hold:*

(1) $c_0 > 0$ *is the least energy of* (P_∞);
(2) $\mathcal{N}_0 \neq \emptyset$;

(3) c_0 *is attained, and the set*

$$\mathcal{R}_0 := \{u \in \mathcal{N}_0 : I_0(u) = c_0, \ \ u(0) = \|u\|_\infty\}$$

is compact in $H_0^1(\mathbb{R}^N, \mathbb{C})$*;*

(4) *There exists* $C, c > 0$ *such that*

$$|u(x)| \leq Ce^{-c|x|} \ \ \forall x \in \mathbb{R}^N, u \in \mathcal{R}_0.$$

Proof. See [24, Lemma 2.6]. □

Let $I_\infty : H^1(\mathbb{R}^N, \mathbb{R}) \to \mathbb{R}$ be the energy functional given by

$$I_\infty(w) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla w(x)|^2 + |w(x)|^2)dx - \frac{1}{p}\int_{\mathbb{R}^N} |w(x)|^p dx.$$

Note that I_∞ is defined on $H^1(\mathbb{R}^N, \mathbb{R})$ while I_0 is defined on $H^1(\mathbb{R}^N, \mathbb{C})$. If c_∞ denotes the mountain pass level of I_∞ and $\mathcal{N}_\infty$ is the Nehari manifold given by

$$\mathcal{N}_\infty := \{w \in H^1(\mathbb{R}^N, \mathbb{R})\backslash\{0\} : I'_\infty(w)w = 0\},$$

then (see [30])

$$c_\infty = c_\infty^* := \inf_{w \in \mathcal{N}_\infty} I_\infty(w).$$

Let $\varphi \in H^1(\mathbb{R}^N, \mathbb{R})$ be a positive ground state solution of (P_∞), that is,

$$I_\infty(\varphi) = c_\infty^* = c_\infty \quad \text{and} \quad I'_\infty(\varphi) = 0. \tag{2.4}$$

The function φ can be chosen radial and decreasing with respect to $|x|$; see [13].

An immediate consequence of Lemmas 2.1 and 2.2, we have the equality $c_0 = c_\infty$, and so, φ is also a ground state solution for (P_0).

Before concluding this section, we state an important result that is a particular case of a result due to Bonheure, Nys and Van Schaftingen [14, Theorem 1], which will be crucial in our approach.

Theorem 2.3. *There is* $\kappa > 0$ *such that if* $w \in H_0^1(\mathbb{R}^N, \mathbb{C})$ *is a critical point of* I_0 *with* $I_0(w) \leq c_0 + \kappa$*, then there are* $a \in \mathbb{R}^N$ *and* $\theta \in \mathbb{R}$ *such that* $w(x) = e^{i\theta}\varphi(x - a)$*, for all* $x \in \mathbb{R}^N$*. Hence,* $I_0(w) = c_0$*.*

3. A Compactness Result for Energy Functional

In this section, we study some compactness property of the energy functional $I_A : H_A^1(\Omega, \mathbb{C}) \to \mathbb{R}$ associated with (P) given by

$$I_A(u) := \frac{1}{2}\int_\Omega (|\nabla_A u(x)|^2 + |u(x)|^2)dx - \frac{1}{p}\int_\Omega |u(x)|^p dx.$$

In the sequel, we denote by c_A the mountain pass level of I that satisfies the equality below

$$c_A = \inf_{u \in \mathcal{N}_A} I_A(u), \tag{3.1}$$

where $\mathcal{N}_A$ is the Nehari manifold of I_A given by

$$\mathcal{N}_A := \{u \in H_A^1(\Omega, \mathbb{C})\backslash\{0\} : I'_A(u)u = 0\}.$$

Theorem 3.1. *The equality $c_0 = c_A$ holds true. Hence, there is no $u \in H^1_A(\Omega, \mathbb{C})$ such that*

$$I_A(u) = c_A \quad \textit{and} \quad I'_A(u) = 0,$$

and so, problem (P) has no ground state solution.

Proof. By using the diamagnetic inequality (2.1),

$$c_0 \le c_A. \tag{3.2}$$

Recalling that φ satisfies (2.4) and that $c_0 = c_\infty$, we have that

$$I_0(\varphi) = c_0 \quad \text{and} \quad I'_0(\varphi) = 0.$$

Let $(y_n) \subset \Omega$ be a sequence such that $|y_n| \to +\infty$, and ρ be the smallest positive number satisfying

$$\mathbb{R}^N \backslash \Omega \subset B(0, \rho) = \{x \in \mathbb{R}^N : |x| < \rho\}.$$

Furthermore, let us define $\zeta \in C^\infty(\mathbb{R}^N, [0,1])$ by

$$\zeta(x) := \xi\left(\frac{|x|}{\rho}\right),$$

where $\xi : [0, +\infty) \to [0,1]$ is a smooth non-decreasing function such that

$$\xi(t) = 0 \ \ \forall t \le 1 \quad \text{and} \quad \xi(t) = 1, \ \ \forall t \ge 2.$$

Now, we consider the sequence

$$\psi_n(x) := \zeta(x)\varphi(x - y_n),$$

and fix $t_n > 0$ such that $t_n \psi_n \in \mathcal{N}_A$. By making the change of variable $z = x - y_n$, we deduce

$$\|\zeta\varphi(\cdot - y_n) - \varphi(\cdot - y_n)\|_{L^p(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |(\zeta(x + y_n) - 1)\varphi(x)|^p dx\right)^{1/p}.$$

Since $|y_n| \to +\infty$ as $n \to +\infty$, it is easy to check that

$$|(\zeta(x + y_n) - 1)\varphi(x)|^p \to 0 \quad \text{a.e. } x \in \mathbb{R}^N.$$

As

$$|(\zeta(x + y_n) - 1)\varphi(x)|^p \le |\zeta(x + y_n) - 1|^p |\varphi(x)|^p \le 2^p |\varphi(x)|^p \in L^1(\mathbb{R}^N),$$

the dominated convergence theorem ensures that

$$\int_{\mathbb{R}^N} |\zeta(x + y_n)\varphi(x) - \varphi(x)|^p dx \to 0 \ \text{ as } \ n \to +\infty,$$

or equivalently,

$$\zeta(\cdot + y_n)\varphi \to \varphi \ \text{ in } \ L^p(\mathbb{R}^N). \tag{3.3}$$

Recalling that $\zeta(x)=0$ for $x\in\mathbb{R}^N\backslash\Omega$, by the previous discussion,

$$\begin{aligned}\|\zeta\varphi(\cdot-y_n)\|_{L^p(\Omega)} &= \left(\int_\Omega |\zeta(x)\varphi(x-y_n)|^p dx\right)^{1/p}\\ &= \left(\int_{\mathbb{R}^N} |\zeta(x+y_n)\varphi(x)|^p dx\right)^{1/p} = \left(\int_{\mathbb{R}^N} |\varphi(x)|^p dx\right)^{1/p} + o_n(1).\end{aligned}$$

On the other hand, for each $j\in\{1,..,N\}$,

$$\begin{aligned}&\int_{\mathbb{R}^N} |D_j(\zeta(x)\varphi(x-y_n))|^2dx = \int_{\mathbb{R}^N} |(-i\partial_j+A_j(x))(\zeta(x)\varphi(x-y_n))|^2dx\\ &= \int_{\mathbb{R}^N}\left(|(\partial_j(\zeta(x+y_n)\varphi(x))|^2+|A_j(x+y_n)\zeta(x+y_n)\varphi(x)|^2\right)dx\\ &= \int_{\mathbb{R}^N}\left(|\zeta(x+y_n)\partial_j\varphi(x)+\varphi(x)\partial_j\zeta(x+y_n)|^2+|A_j(x+y_n)\zeta(x+y_n)\varphi(x)|^2\right)dx.\end{aligned}$$

Since $\zeta(x+y_n)\to 1$, $A(x+y_n)\to 0$ as $n\to+\infty$ and $\partial_j\zeta(x+y_n)\to 0$, the dominated convergence theorem implies that

$$\int_{\mathbb{R}^N} |D_j(\zeta(x)\varphi(x-y_n))|^2dx = \int_{\mathbb{R}^N} |\partial_j\varphi(x)|^2dx + o_n(1),$$

and so,

$$\int_{\mathbb{R}^N} |\nabla_A(\zeta(x)\varphi(x-y_n))|^2dx = \int_{\mathbb{R}^N} |\nabla\varphi(x)|^2dx + o_n(1). \tag{3.4}$$

By the previous analysis together with the fact that $\varphi\in\mathcal{N}_0$, using translation invariance it is not difficult to prove that $t_n\to 1$. Thus, by definition of c_A, (3.3) and (3.4), we get

$$c_A\le I_A(t_n\psi_n)=c_0+o_n(1),$$

that leads to

$$c_A\le c_0. \tag{3.5}$$

From (3.2) and (3.5),

$$c_0=c_A. \tag{3.6}$$

Now, suppose by contradiction that there is $v_0\in H^1_A(\Omega,\mathbb{C})$ such that

$$I_A(v_0)=c_A \quad \text{and} \quad I'_A(v_0)=0.$$

By (2.1), (3.6), and recalling that $c_0=c_\infty$, we deduce that the function $w=|v_0|\in H^1_0(\mathbb{R}^N,\mathbb{R})$ is a ground state solution of (P_∞), that is,

$$-\Delta w+w=|w|^{p-2}w, \quad \text{in} \quad \mathbb{R}^N.$$

Since $w\ge 0$ in $\mathbb{R}^N$ and $w\ne 0$, the strong maximum principle ensures that $w(x)>0$ for all $x\in\mathbb{R}^N$, which is impossible because $v_0=0$ in $\mathbb{R}^N\backslash\Omega$. $\square$

3.1. A Compactness Lemma

In this section, we prove a compactness result involving the energy functional I_A associated with (P). In order to do this, we need to consider the energy functional $I_0 : H_0^1(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ associated with (P_0) defined as in (2.3). With the above notations, we are able to prove the following compactness result.

Lemma 3.2. *Let $(u_n) \subset H_A^1(\Omega, \mathbb{C})$ be a sequence such that*

$$I_A(u_n) \to c \quad \textit{and} \quad I_A'(u_n) \to 0 \ \textit{ as } \ n \to +\infty. \tag{3.7}$$

Then, up to a subsequence, there exists a weak solution $u^0 \in H_A^1(\Omega, \mathbb{C})$ of (P) such that

$$u_n \to u^0 \quad \textit{in} \quad H^1(\mathbb{R}^N)$$

or there are k functions $(u_n^j) \subset H_0^1(\mathbb{R}^N, \mathbb{C})$, $1 \leq j \leq k$ such that

$$u_n^0 = u_n \rightharpoonup u^0 \ \textit{ in } \ H_A^1(\Omega, \mathbb{C}),$$
$$u_n^j \rightharpoonup u^j \ \textit{ in } \ H_0^1(\mathbb{R}^N, \mathbb{C}) \ \textit{ for } \ 1 \leq j \leq k,$$

where u^j are nontrivial weak solutions of (P_0), for every $1 \leq j \leq k$. Furthermore

$$\|u_n\|^2 \to \|u_0\|^2 + \sum_{j=1}^{k} \|u^j\|_0^2$$

and

$$I_A(u_n) \to I_A(u^0) + \sum_{j=1}^{k} I_0(u^j).$$

Proof. We proceed by steps.

Step 1. The sequence (u_n) is bounded in $H_A^1(\Omega, \mathbb{C})$.
By (3.7),

$$\begin{aligned}\langle I_A'(u_n), \psi \rangle &= \Re \int_\Omega (\nabla_A u_n(x) \cdot \overline{\nabla_A \psi(x)} + u_n(x)\overline{\psi(x)} - |u_n(x)|^{p-2} u_n(x) \overline{\psi(x)}) dx \\ &= o(1)\end{aligned} \tag{3.8}$$

for any $\psi \in H_A^1(\Omega, \mathbb{C})$ and

$$\frac{1}{2} \int_\Omega \left(|\nabla_A u_n(x)|^2 + |u_n(x)|^2 \right) dx - \frac{1}{p} \int_\Omega |u_n(x)|^p dx \to c. \tag{3.9}$$

Choosing $\psi = u_n$ in (3.8), we obtain

$$\|u_n\|^2 - \int_\Omega |u_n(x)|^p dx = o_n(1) \tag{3.10}$$

which combined with (3.9) and (3.10) gives

$$\left(\frac{1}{2} - \frac{1}{p} \right) \|u_n\|^2 = c + o_n(1). \tag{3.11}$$

Therefore, (u_n) is bounded in $H_A^1(\Omega, \mathbb{C})$.

Consequently, up to a subsequence, there exists $u^0 \in H^1_A(\Omega,\mathbb{C})$ such that

$$\begin{aligned} &u_n \rightharpoonup u^0 \ \text{ in } \ H^1_A(\Omega,\mathbb{C}),\\ &u_n \to u^0 \ \text{ in } \ L^p_{loc}(\Omega,\mathbb{C}) \ \text{ for } \ p\in[2,2^*),\\ &u_n(x) \to u^0(x) \ \text{ a.e. in } \ \Omega. \end{aligned} \tag{3.12}$$

We claim that u^0 is a weak solution of (P). In fact, for an arbitrary function $\psi \in H^1_A(\Omega,\mathbb{C})$, the limit $I'_A(u_n)\to 0$ in $(H^1_A(\Omega,\mathbb{C}))^*$ yields

$$\langle I'_A(u_n),\psi\rangle \to 0 \ \text{ as } \ n\to\infty,$$

that is,

$$\lim_{n\to\infty} \Re\left(\int_\Omega (\nabla_A u_n(x)\overline{\nabla_A\psi(x)} + u_n(x)\overline{\psi(x)})dx - \int_\Omega |u_n(x)|^{p-2}u_n(x)\overline{\psi(x)}dx\right) = 0. \tag{3.13}$$

Since $u_n \rightharpoonup u^0$ in $H^1_A(\Omega,\mathbb{C})$, it follows that

$$\begin{aligned} &\lim_{n\to\infty} \Re\int_\Omega (\nabla_A u_n(x)\overline{\nabla_A\psi(x)} + u_n(x)\overline{\psi(x)})dx\\ &\quad = \Re\int_\Omega (\nabla_A u^0(x)\overline{\nabla_A\psi(x)} + u^0(x)\overline{\psi(x)})dx. \end{aligned} \tag{3.14}$$

From the boundedness of (u_n) in $H^1_A(\Omega,\mathbb{C})$ and Sobolev embedding, we know that $(|u_n|^{p-2}u_n)$ is a bounded sequence in $L^{\frac{p}{p-1}}(\Omega,\mathbb{C})$. Moreover, by (3.12), we see that

$$|u_n|^{p-2}u_n \to |u^0|^{p-2}u^0 \ \text{ a.e. in } \ \Omega.$$

Consequently, by [27, Lemma 4.8], $|u^0|^{p-2}u^0$ is the weak limit of the sequence $(|u_n|^{p-2}u_n)$ in $L^{\frac{p}{p-1}}(\Omega,\mathbb{C})$. Hence,

$$\lim_{n\to\infty} \Re\int_\Omega |u_n(x)|^{p-2}u_n(x)\overline{\psi(x)}dx = \Re\int_\Omega |u^0(x)|^{p-2}u^0(x)\overline{\psi(x)}dx. \tag{3.15}$$

From (3.13)–(3.15),

$$\Re\left(\int_\Omega (\nabla_A u^0(x)\overline{\nabla_A\psi(x)} + u^0(x)\overline{\psi(x)})dx - \int_\Omega |u^0(x)|^{p-2}u^0(x)\overline{\psi(x)}dx\right) = 0,$$

which means that $I'_A(u^0)=0$, and so, u^0 is a weak solution of (P).

Now, let Ψ^1_n be the function defined as

$$\Psi^1_n(x) := \begin{cases} (u_n-u^0)(x), & x\in\Omega\\ 0, & x\in\mathbb{R}^N\backslash\Omega. \end{cases}$$

With the above notations, we are able to prove the following steps:

Step 2.

$$I_0(\Psi^1_n) = I_A(\Psi^1_n) + o_n(1) = I_A(u_n) - I_A(u^0) + o_n(1). \tag{3.16}$$

Note that by the Brezis-Lieb lemma [15],

$$\int_\Omega |u_n(x)|^p dx = \int_\Omega |u^0(x)|^p dx + \int_{\mathbb{R}^N} |\Psi^1_n(x)|^p dx + o_n(1). \tag{3.17}$$

Moreover,

$$\begin{aligned}\|\Psi_n^1\|_0^2 &= \int_{\mathbb{R}^N} |\nabla_0 \Psi_n^1(x)|^2 dx = \int_{\mathbb{R}^N} |-i\nabla\Psi_n^1(x)|^2 dx \\ &= \int_{\mathbb{R}^N} |-i\nabla\Psi_n^1(x) + (A(x)-A(x))\Psi_n^1(x)|^2 dx \\ &= \int_{\mathbb{R}^N} |\nabla_A \Psi_n^1(x) - A(x)\Psi_n^1(x)|^2 dx.\end{aligned}$$

Using condition (A), it is easy to prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |A(x)\Psi_n^1(x)|^2 dx = 0.$$

Therefore,

$$\int_{\mathbb{R}^N} |\nabla_A\Psi_n^1(x) - A(x)\Psi_n^1(x)|^2 dx = \int_{\mathbb{R}^N} |\nabla_A\Psi_n^1(x)|^2 dx + o_n(1) = \|\Psi_n^1\|_A^2 + o_n(1).$$

Consequently,

$$\|\Psi_n^1\|_0^2 = \|\Psi_n^1\|_A^2 + o_n(1) = \|u_n\|_A^2 - \|u^0\|_A^2 + o_n(1). \tag{3.18}$$

Now, (3.16) follows from (3.17) and (3.18).

Step 3.

$$I_0'(\Psi_n^1) = I_A'(\Psi_n^1) + o_n(1) = I_A'(u_n) - I_A'(u^0) + o_n(1) = o_n(1). \tag{3.19}$$

For each $v \in H_A^1(\Omega, \mathbb{C}) \subset H_0^1(\mathbb{R}^N, \mathbb{C})$ with $\|v\| \le 1$, one has

$$I_A'(\Psi_n^1)v = \Re\left(\int_\Omega (\nabla_A\Psi_n^1(x)\cdot\overline{\nabla_A v(x)} + \Psi_n^1(x)\overline{v(x)})dx - \int_\Omega |\Psi_n^1(x)|^{p-2}\Psi_n^1(x)\overline{v(x)}dx\right)$$

and

$$I_0'(\Psi_n^1)v = \Re\left(\int_{\mathbb{R}^N} (\nabla_0\Psi_n^1(x)\cdot\overline{\nabla_0 v(x)} + \Psi_n^1(x)\overline{v(x)})dx - \int_{\mathbb{R}^N} |\Psi_n^1(x)|^{p-2}\Psi_n^1(x)\overline{v(x)}dx\right).$$

Then, as in the previous step,

$$|\langle I_A'(\Psi_n^1) - I_0'(\Psi_n^1), v\rangle| = \left|\Re\left(\int_\Omega (\nabla_A\Psi_n^1(x)\cdot\overline{\nabla_A v(x)} - \nabla_0\Psi_n^1(x)\cdot\overline{\nabla_0 v(x)})dx\right)\right| = o_n(1)$$

for every $v \in H_A^1(\Omega, \mathbb{C})$ with $\|v\| \le 1$. Consequently,

$$I_0'(\Psi_n^1) = I_A'(\Psi_n^1) + o_n(1), \quad \text{in} \quad H_A^1(\Omega, \mathbb{C})'. \tag{3.20}$$

Now, we are going to show that

$$I_A'(\Psi_n^1) = I_A'(u_n) - I_A'(u^0) + o_n(1) = o_n(1). \tag{3.21}$$

Note that

$$
\begin{aligned}
&\langle I'_A(\Psi^1_n) - I'_A(u_n) + I'_A(u^0), v\rangle \\
&\quad = \Re\left(\int_\Omega \nabla_A \Psi^1_n(x)\overline{\nabla_A v(x)} + \Psi^1_n(x)\overline{v(x)}dx - \int_\Omega |\Psi^1_n(x)|^{p-2}\Psi^1_n(x)\overline{v(x)}dx\right) \\
&\qquad - \Re\left(\int_\Omega \nabla_A u_n(x)\overline{\nabla_A v(x)} + u_n(x)\overline{v(x)}dx - \int_\Omega |u_n(x)|^{p-2}u_n(x)\overline{v(x)}dx\right) \\
&\qquad + \Re\left(\int_\Omega \nabla_A u^0(x)\overline{\nabla_A v(x)} + u^0(x)\overline{v(x)}dx - \int_\Omega |u^0(x)|^{p-2}u^0(x)\overline{v(x)}dx\right).
\end{aligned}
$$

As $\Psi^1_n = u_n - u^0$ in Ω, by Hölder inequality,

$$
\begin{aligned}
&|\langle I'_A(\Psi^1_n) - I'_A(u_n) + I'_A(u^0), v\rangle| \\
&\quad = \left|\Re\int_\Omega \left(|u_n(x)|^{p-2}u_n(x) - |u^0(x)|^{p-2}u^0(x) - |\Psi^1_n(x)|^{p-2}(x)\Psi^1_n(x)\right)\overline{v}dx\right| \\
&\quad \le \int_\Omega ||u_n(x)|^{p-2}u_n(x) - |u^0(x)|^{p-2}u^0(x) - |\Psi^1_n(x)|^{p-2}\Psi^1_n(x)||v(x)|dx \\
&\quad \le C\left(\int_\Omega ||u_n(x)|^{p-2}u_n(x) - |u^0(x)|^{p-2}u^0(x) - |\Psi^1_n(x)|^{p-2}\Psi^1_n(x)|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}.
\end{aligned}
$$

Recalling that by [1, Lemma 3] there holds

$$
\left(\int_\Omega ||u_n(x)|^{p-2}u_n(x) - |u^0(x)|^{p-2}u^0(x) - |\Psi^1_n(x)|^{p-2}\Psi^1_n(x)|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}} = o_n(1),
$$

it follows that

$$
I'_A(\Psi^1_n) = I'_A(u_n) - I'_A(u^0) + o_n(1).
$$

As u^0 is critical point of I_A, we have $I'_A(u^0) = 0$ and thus (3.19) holds.

If $\Psi^1_n \to 0$ in $H^1_A(\Omega, \mathbb{C})$ the statements of the main result are verified. Thus, we can suppose that

$$
\Psi^1_n \not\to 0 \ \text{ in } \ H^1_A(\Omega, \mathbb{C}). \tag{3.22}
$$

By using the fact that

$$
I_0(\Psi^1_n) = \frac{1}{2}\|\Psi^1_n\|^2_0 - \frac{1}{p}\int_{\mathbb{R}^N} |\Psi^1_n(x)|^p dx,
$$

and $I'_0(\Psi^1_n) = o_n(1)$, we have

$$
I'_0(\Psi^1_n)\Psi^1_n = \|\Psi^1_n\|^2_0 - \int_{\mathbb{R}^N} |\Psi^1_n(x)|^p dx = o_n(1). \tag{3.23}
$$

Therefore,

$$
I_0(\psi^1_n) = \left(\frac{1}{2} - \frac{1}{p}\right)\|\Psi^1_n\|^2_0 + o_n(1).
$$

By (3.22), there is $\alpha > 0$ such that

$$
I_0(\Psi^1_n) \ge \alpha > 0. \tag{3.24}
$$

Now, let us decompose $\mathbb{R}^N$ into N-dimensional unit hypercubes Q_i whose vertices have integer coordinates and put

$$d_n := \max_i \|\Psi_n^1\|_{L^p(Q_i)}. \tag{3.25}$$

Arguing as in [12, Lemma 3.1], there is $\gamma > 0$ such that

$$d_n \geq \gamma > 0. \tag{3.26}$$

Denote by (y_n^1) the center of a hypercube Q_i in which $\|\Psi_n^1\|_{L^p(Q_i)} = d_n$. We claim that (y_n^1) is unbounded sequence in $\mathbb{R}^N$. Arguing by contradiction, let us suppose that (y_n^1) is bounded in $\mathbb{R}^N$. Then, there is $R > 0$ such that

$$\int_{B(0,R)} |\Psi_n^1(x)|^p dx \geq \int_{Q_i(y_n^1)} |\Psi_n^1(x)|^p dx = d_n^p > \gamma^p > 0. \tag{3.27}$$

On the other hand, since $\Psi_n^1 \rightharpoonup 0$ in $H_0^1(\mathbb{R}^N, \mathbb{C})$, the local compactness of the Sobolev embedding gives

$$\int_{B(0,R)} |\Psi_n^1(x)|^p dx \to 0, \text{ as } n \to +\infty,$$

against (3.27). Therefore, the sequence (y_n^1) is unbounded. Since

$$\|\Psi_n^1(\cdot + y_n^1)\|_0 = \|\Psi_n^1\|_0 \quad \forall n \in \mathbb{N},$$

we deduce that $(\Psi_n^1(\cdot + y_n^1))$ is a bounded sequence in $H_0^1(\mathbb{R}^N, \mathbb{C})$. Then, there is $u^1 \in H_0^1(\mathbb{R}^N, \mathbb{C})$ such that

$$\Psi_n^1(\cdot + y_n^1) \rightharpoonup u^1 \text{ in } H_0^1(\mathbb{R}^N, \mathbb{C})$$

and

$$\Psi_n^1(\cdot + y_n^1) \to u^1 \text{ in } L^p_{loc}(\mathbb{R}^N, \mathbb{C}).$$

Step 4. u^1 is a nontrivial weak solution of (P_0).

First, by (3.27), we derive that $u^1 \neq 0$, and by a straightforward computation

$$I_0'(\Psi_n^1(\cdot + y_n^1))\varphi = o_n(1), \;\; \forall \varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{C}).$$

Thus, passing to the limit as $n \to +\infty$, we find

$$\begin{aligned}\Re &\left(\int_{\mathbb{R}^N} (\nabla_0 u^1(x) \cdot \overline{\nabla_d \varphi(x)} + u^1(x)\overline{\varphi(x)}) dx \right) \\ &= \Re \left(\int_{\mathbb{R}^N} |u^1(x)|^{p-2} u^1(x) \overline{\varphi(x)} dx \right), \;\; \forall \varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{C}).\end{aligned}$$

Now, the density of $C_0^\infty(\mathbb{R}^N, \mathbb{C})$ in $H_0^1(\mathbb{R}^N, \mathbb{C})$ ensures that

$$\begin{aligned}\Re &\left(\int_{\mathbb{R}^N} (\nabla_0 u^1(x) \cdot \overline{\nabla_d w(x)} + u^1(x)\overline{w(x)}) dx \right) \\ &= \Re \left(\int_{\mathbb{R}^N} |u^1(x)|^{p-2} u^1(x) \overline{w(x)} dx \right), \;\; \forall w \in H_0^1(\mathbb{R}^N, \mathbb{C}),\end{aligned}$$

i.e., the function u^1 is a nontrivial weak solution of problem (P_0).

We can repeat this process obtaining the sequences

$$\Psi_n^j(x) = \Psi_n^{j-1}(x + y_n^{j-1}) - u^{j-1}(x), \;\; j \geq 2,$$

with

$$|y_n^j| \to +\infty, \quad \text{as} \quad n \to +\infty$$

and

$$\Psi_n^{j-1}(x + y_n^{j-1}) \rightharpoonup u^{j-1} \quad \text{in} \quad H_0^1(\mathbb{R}^N, \mathbb{C}), \tag{3.28}$$

where each function u^j is a nontrivial weak solution of problem (P_0).

Now, an inductive argument ensures that

$$\|\Psi_n^j\|_0^2 = \|\Psi_n^{j-1}\|_0^2 - \|u^{j-1}\|_0^2 + o_n(1) = \|u_n\|_A^2 - \|u^0\|_A^2 - \sum_{i=1}^{j-1} \|u^i\|_0^2 + o_n(1) \tag{3.29}$$

and

$$I_0(\Psi_n^j) = I_0(\Psi_n^{j-1}) - I_0(u^{j-1}) + o_n(1) = I_A(u_n) - I_A(u^0) - \sum_{i=1}^{j-1} I_0(u^i) + o_n(1). \tag{3.30}$$

Since u^j is a nontrivial solution of (P_0), it follows that

$$I_0(u^j) \geq c_0, \tag{3.31}$$

for every $1 \leq j \leq k$. Now, the rest of the proof follows as in [12, Lemma 3.1]. □

Corollary 3.3. *Let (u_n) be as in Lemma 3.2 with $c < c_0$. Then (u_n) admits a strongly convergent subsequence. Hence, the functional I_A verifies the $(PS)_c$ condition, for every $c \in (0, c_0)$.*

Proof. The argument is standard. However, we give the details for the reader's convenience. Thanks to our hypotheses, one has

$$I_A(u_n) \to c \quad \text{and} \quad I_A'(u_n) \to 0 \quad \text{as} \quad n \to +\infty,$$

with $c < c_0$. Without loss of generality, we can suppose that (u_n) is bounded in $H_A^1(\Omega, \mathbb{C})$. Then, up to some subsequence, there exists $u^0 \in H_A^1(\Omega, \mathbb{C})$ such that

$$u_n \rightharpoonup u^0 \quad \text{in} \quad H_A^1(\Omega, \mathbb{C}).$$

If $u \not\to u^0$ in $H_A^1(\Omega, \mathbb{C})$, by Lemma 3.2 we must have $k \geq 1$. Hence,

$$I_A(u_n) \to c \geq c_0,$$

which contradicts $c < c_0$. Thereby,

$$u_n \rightharpoonup u^0 \quad \text{and} \quad \|u_n\|^2 \to \|u^0\|^2,$$

and this implies that $u_n \to u^0$ in $H_A^1(\Omega, \mathbb{C})$. □

Corollary 3.4. *Assume that there exists (u_n) for which all the assumptions of Lemma 3.2 hold. If*

$$c_0 < c < c_0 + \kappa_1, \tag{3.32}$$

where $\kappa_1 < \min\{\kappa, c_0\}$, then (u_n) admits a strongly convergent subsequence. Hence, the energy functional I_A satisfies the $(PS)_c$ condition, for every $c \in (c_0, c_0 + \kappa_1)$, where κ is given by Theorem 2.3.

Proof. Assume by contradiction that $u_n \not\to u^0$ in $H^1_A(\Omega, \mathbb{C})$. By Lemma 3.2, it follows that $k \geq 1$. As $I_0(u^j) \geq c_0$, we must have $k = 1$, and so,

$$I_A(u_n) \to I_A(u^0) + I_0(u^1).$$

We claim that $u^0 \neq 0$, otherwise

$$I_0(u^1) = I_A(u_n) + o_n(1) = c + o_n(1)$$

and thus u^1 is a critical point of I_0 with $I_0(u^1) < c_0 + \kappa_1$. Hence, by Theorem 2.3, we must have $I_0(u^1) = c_0$, which is impossible because $c > c_0$. From this, $u^0 \neq 0$ and

$$I_A(u^0) \geq c_A = c_0.$$

Hence, the limit equality

$$I_A(u_n) = I_A(u^0) + I_0(u^1) + o_n(1),$$

yields that

$$\lim_{n \to +\infty} I_A(u_n) = I_A(u^0) + I_0(u^1) \geq 2c_0,$$

which gives an absurd. Thus, we must have $u_n \to u^0$ in $H^1_A(\Omega, \mathbb{C})$. This shows the desired result. □

4. Technical Estimates

The main goal this section is to establish some technical estimates that we will use in the proof of Theorem 1.1.

We start by introducing the following operator

$$\begin{aligned} \Phi_\rho : \mathbb{R}^N &\to H^1_0(\mathbb{R}^N, \mathbb{C}) \\ y &\to \Phi_\rho(y) := \phi_{y,\rho}, \end{aligned}$$

where

$$\phi_{y,\rho}(x) := \zeta(x)\varphi(x-y) = \xi\left(\frac{|x|}{\rho}\right)\varphi(x-y),$$

φ is the positive ground state of (P_∞) satisfying (2.4) and ζ, ξ are given as in the proof of Theorem 3.1. A direct computation ensures that the functions $\phi_{y,\rho}$ belong to $H^1_A(\Omega, \mathbb{C})$ and $L^p(\Omega, \mathbb{C})$, respectively. From now on, we take $t_{y,\rho} > 0$ such that $\psi_\rho(y) = t_{y,\rho}\phi_{y,\rho} \in \mathcal{N}_A$.

Lemma 4.1. *The following relations hold:*

(i) $\limsup_{\rho \to 0} I_A(\psi_\rho(y)) \leq c_0 + M\Gamma\|\varphi\|^2_{L^2(\Omega)}$ *uniformly in* y;

(ii) $I_A(\psi_\rho(y)) \to c_0$ *as* $|y| \to +\infty$, *for every* ρ,

where $M := \sum_{j=1}^N \|A_j\|^2_\infty$ *and* $\Gamma > 0$ *is a constant independent of* y.

Proof. (i) Note that

$$\begin{aligned}\int_{\mathbb{R}^N} |\phi_{y,\rho}(x) - \varphi(x-y)|^2 dx &= \int_{\mathbb{R}^N} \left|\left(\xi\left(\frac{|x|}{\rho}\right) - 1\right)\varphi(x-y)\right|^2 dx \\ &= \int_{\mathbb{R}^N} \left|\left(\xi\left(\frac{|x|}{\rho}\right) - 1\right)\varphi(x-y)\right|^2 dx \\ &= \int_{B(0,2\rho)} \left|\left(\xi\left(\frac{|x|}{\rho}\right) - 1\right)\varphi(x-y)\right|^2 dx \\ &\le C_1\varphi^2(0)|B(0,2\rho)| \to 0 \ \text{ as } \ \rho \to 0,\end{aligned}$$

from where it follows that

$$\phi_{y,\rho} \to \varphi(\cdot - y) \quad \text{in} \quad L^2(\mathbb{R}^N) \quad \text{as } \ \rho \to 0, \ \text{ uniformly in } \ y \in \mathbb{R}^N.$$

Consequently,

$$\|\phi_{y,\rho}\|^2_{L^2(\Omega)} \to \|\varphi(\cdot - y)\|^2_{L^2(\mathbb{R}^N)} = \|\varphi\|^2_{L^2(\mathbb{R}^N)} \ \text{ as } \ \rho \to 0, \ \text{ uniformly in } \ y \in \mathbb{R}^N. \tag{4.1}$$

In the same way, we can show that

$$\phi_{y,\rho} \to \varphi(\cdot - y) \quad \text{in} \quad L^p(\mathbb{R}^N) \quad \text{as } \ \rho \to 0, \ \text{ uniformly in } \ y \in \mathbb{R}^N$$

and

$$\|\phi_{y,\rho}\|^p_{L^p(\Omega)} \to \|\varphi(\cdot - y)\|^p_{L^p(\mathbb{R}^N)} = \|\varphi\|^p_{L^p(\mathbb{R}^N)} \ \text{ as } \ \rho \to 0, \ \text{ uniformly in } \ y \in \mathbb{R}^N. \tag{4.2}$$

On the other hand, note that for each $j \in \{1, .., N\}$,

$$\begin{aligned}\int_\Omega |D_j\phi_{y,\rho}(x)|^2 dx &= \int_{\mathbb{R}^N} |-i\partial_j\phi_{y,\rho}(x) + A_j(x)\phi_{y,\rho}(x)|^2 dx \\ &= \int_{\mathbb{R}^N} \left(|\partial_j\phi_{y,\rho}(x)|^2 + |A_j(x)\phi_{y,\rho}(x)|^2\right) dx \\ &\le \int_{\mathbb{R}^N} |\partial_j\phi_{y,\rho}(x)|^2\, dx + \|A_j\|^2_\infty \|\varphi\|^2_{L^2(\Omega)}.\end{aligned}$$

We claim that

$$\int_{\mathbb{R}^N} |\partial_j\phi_{y,\rho}(x)|^2\, dx \to \int_{\mathbb{R}^N} |\partial_j\varphi(x)|^2\, dx \ \text{ as } \ \rho \to 0. \tag{4.3}$$

Indeed, note that

$$\int_{\mathbb{R}^N} \left|\varphi(x-y)\partial_j\xi\left(\frac{|x|}{\rho}\right)\right|^2 dx \le \|\varphi\|^2_\infty |B(0,\rho)|,$$

which yields

$$\lim_{\rho\to 0} \int_{\mathbb{R}^N} \left|\varphi(x-y)\partial_j\xi\left(\frac{|x|}{\rho}\right)\right|^2 dx = 0, \quad \text{uniformly in } \ y \in \mathbb{R}^N.$$

Since

$$\lim_{\rho\to 0} \int_{\mathbb{R}^N} \left|\xi\left(\frac{|x|}{\rho}\right)\partial_j\varphi(x-y)\right|^2 dx = \int_{\mathbb{R}^N} |\partial_j\varphi(x-y)|^2\, dx, \quad \text{uniformly in } \ y \in \mathbb{R}^N,$$

we can deduce that (4.3) holds.

As $\psi_\rho(y) = t_{y,\rho}\phi_{y,\rho} \in \mathcal{N}_A$, it follows that

$$t_{y,\rho}\|\phi_{y,\rho}\|_A^2 = t_{y,\rho}^{p-1}\int_\Omega |\phi_{y,\rho}(x)|^p dx.$$

This combined with diamagnetic inequality (2.1) leads to

$$t_{y,\rho}\|\phi_{y,\rho}\|_{H^1(\mathbb{R}^N)}^2 \leq t_{y,\rho}^{p-1}\int_{\mathbb{R}^N} |\phi_{y,\rho}(x)|^p dx.$$

Now, recalling that

$$\|\phi_{y,\rho}\|_{H^1(\mathbb{R}^N)}^2 \to \|\varphi\|_{H^1(\mathbb{R}^N)}^2 \quad \text{and} \quad \|\phi_{y,\rho}\|_{L^p(\mathbb{R}^N)}^2 \to \|\varphi\|_{L^p(\mathbb{R}^N)}^2$$

as $\rho \to 0$, uniformly in y, we can infer that

$$\liminf_{\rho\to 0} t_{y,\rho} \geq 1 \ \text{ uniformly in } \ y \in \mathbb{R}^N.$$

On the other hand, we also know that the limits below

$$\limsup\nolimits_{\rho\to 0} \|\phi_{y,\rho}\|_A^2 \leq \|\varphi\|_{H^1(\mathbb{R}^N)}^2 + M\|\varphi\|_{L^2(\Omega)}^2 \quad \text{and}$$
$$\limsup\nolimits_{\rho\to 0} \textstyle\int_\Omega |\phi_{y,\rho}(x)|^p dx = \int_{\mathbb{R}^N} |\varphi(x)|^p dx,$$

are uniform in $y \in \mathbb{R}^N$. These facts imply that there exists $C > 0$ that depends on M, which is independent of $y \in \mathbb{R}^N$, such that

$$\limsup_{\rho\to 0} t_{y,\rho} \leq C \ \text{ uniformly in } \ y \in \mathbb{R}^N.$$

Indeed,

$$\limsup_{\rho\to 0} t_{y,\rho} = \limsup_{\rho\to 0}\left(\frac{\|\phi_{y,\rho}\|_A^2}{\|\phi_{y,\rho}\|_{L^p(\mathbb{R}^N)}^p}\right)^{\frac{1}{p-2}} \leq \left(\frac{\|\varphi\|_{H^1(\mathbb{R}^N)}^2 + M\|\varphi\|_{L^2(\Omega)}^2}{\|\varphi\|_{L^p(\mathbb{R}^N)}^p}\right)^{\frac{1}{p-2}} =: C.$$

Therefore, using the fact that φ satisfies (2.4) and that $c_0 = c_\infty$, we obtain

$$\limsup_{\rho\to 0} I_A(\psi_\rho(y)) = \limsup_{\rho\to 0} I_A(t_{y,\rho}\phi_{y,\rho}) \leq C^2 I_\infty(\varphi) + M\Gamma\|\varphi\|_{L^2(\Omega)}^2 = c_0 + M\Gamma\|\varphi\|_{L^2(\Omega)}^2,$$

where $\Gamma := \frac{C^2}{2}$. We point out that, from the above calculations, Γ is bounded when $M \to 0$. This information will be useful in the next section.
(ii) For each fixed ρ, let us consider an arbitrary sequence $(y_n) \subset \mathbb{R}^N$ with $|y_n| \to \infty$ as $n \to +\infty$ and let $t_{y_n,\rho} > 0$ such that $t_{y_n,\rho}\phi_{y_n,\rho} \in \mathcal{N}_A$. As in the proof of Theorem 3.1,

$$\|\psi_{y_n,\rho}\|_A^2 \to \|\varphi(\cdot - y_n)\|_0^2, \quad \int_\Omega |\phi_{y_n,\rho}(x)|^p dx \to \int_{\mathbb{R}^N} |\varphi(x-y_n)|^p dx \quad \text{and} \quad t_{y_n,\rho} \to 1.$$

From this,

$$I_A(\psi_\rho(y_n)) = I_A(t_{y_n,\rho}\phi_{y_n,\rho}) = I_0(\varphi) + o_n(1) = I_\infty(\varphi) + o_n(1) = c_\infty + o_n(1) = c_0 + o_n(1).$$

□

In light of the previous lemma, we can prove the corollary below.

Corollary 4.2. *There is $\rho_0 > 0$ and $\epsilon > 0$ such that if $\|A\|_\infty < \epsilon$, then*

$$\sup_{y\in\mathbb{R}^N} I_A(\psi_\rho(y)) < c_0 + \kappa_1, \quad \forall \rho \leq \rho_0. \tag{4.4}$$

Proof. By Lemma 4.1—Part (i), one has

$$\limsup_{\rho\to 0} I_A(\psi_\rho(y)) \leq c_0 + M\Gamma\|\varphi\|^2_{L^2(\mathbb{R}^N)},$$

where $M = \sum_{j=1}^N \|A_j\|^2_\infty$, for every $y \in \mathbb{R}^N$. So, there is $\rho_0 > 0$ small enough such that

$$\sup_{y\in\mathbb{R}^N} I_A(\psi_\rho(y)) \leq c_0 + 2M\Gamma\|\varphi\|^2_{L^2(\mathbb{R}^N)}, \quad \forall \rho < \rho_0.$$

Fixing $\epsilon > 0$ such that if $\|A\|_\infty < \epsilon$ then $2M\Gamma\|\varphi\|^2_{L^2(\Omega)} < \kappa_1$, we must have that

$$\sup_{y\in\mathbb{R}^N} I_A(\psi_\rho(y)) \leq c_0 + 2M\Gamma\|\varphi\|^2_{L^2(\Omega)} < c_0 + \kappa_1.$$

This ends the proof of the corollary. □

Hereafter, let us fix $\rho \in (0, \rho_0)$, such that

$$\mathbb{R}^N\backslash\Omega \subset B(0, \rho).$$

Furthermore, we consider the barycenter function given by

$$\tau : H^1_0(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}^N$$
$$u \to \tau(u) := \int_{\mathbb{R}^N} |u(x)|^2\chi(|x|)x\,dx,$$

where $\chi \in C(\mathbb{R}^+, \mathbb{R})$ is a non-increasing real function such that

$$\chi(t) = \begin{cases} 1, & t \in (0, R] \\ \frac{R}{t}, & t > R, \end{cases}$$

for some $R > 0$ such that $\mathbb{R}^N\backslash\Omega \subset B(0, R)$. By definition of χ,

$$\chi(|x|)|x| \leq R, \ \ \forall x \in \mathbb{R}^N. \tag{4.5}$$

Set

$$\mathcal{T}_0 := \{u \in \mathcal{N}_A : \tau(u) = 0\} \subset \mathcal{N}_A \subset H^1_A(\Omega, \mathbb{C}).$$

Lemma 4.3. *If*

$$\lambda_0 := \inf_{u\in\mathcal{T}_0} I_A(u),$$

then

$$c_0 < \lambda_0 \tag{4.6}$$

and there exists $R_0 > 0$, with $R_0 > \rho$ such that:

(i) *If $y \in \mathbb{R}^N$ with $|y| \geq R_0$, then*

$$I_A(\psi_\rho(y)) \in \Big(c_0, \frac{c_0 + \lambda_0}{2}\Big).$$

(ii) *If* $y \in \mathbb{R}^N$ *with* $|y| = R_0$, *then*

$$\langle \tau(\psi_\rho(y)), y \rangle > 0.$$

Proof. Since $\mathcal{T}_0 \subset \mathcal{N}_A$ and

$$c_0 = c_A = \inf_{u \in \mathcal{N}_A} I_A(u),$$

we have

$$c_0 \le \lambda_0.$$

Now we are going to show that $c_0 \neq \lambda_0$. Suppose by contradiction that $c_0 = \lambda_0$. Then, there exists a minimizing sequence $(u_n) \subset \mathcal{T}_0 \subset H^1_A(\Omega, \mathbb{C})$ such that

$$I_A(u_n) \to c_0 \quad \text{and} \quad \tau(u_n) = 0 \;\; \forall n \in \mathbb{N}.$$

By the Ekeland variational principle [30, Theorem 2.4], we can find a sequence $(w_n) \subset \mathcal{N}_A$ such that

$$I_A(w_n) \to c_0, \quad \nabla_{\mathcal{N}_A} I_A(w_n) \to 0 \quad \text{and} \quad \|w_n - u_n\|_A \to 0.$$

A well-known computation shows that (u_n) and (w_n) are bounded. Moreover, there exists a sequence $(\lambda_n) \subset \mathbb{R}$ such that

$$I'_A(w_n) - \lambda_n J'_A(w_n) = o_n(1), \tag{4.7}$$

where $J_A(u) = \langle I'_A(u), u \rangle$. Using standard arguments, we have that $\lambda_n \to 0$, and then (4.7) yields

$$I'_A(w_n) \to 0.$$

Assuming that $w_n \rightharpoonup w^0$ in $H^1_A(\Omega, \mathbb{C})$, we have that $I'_A(w^0) = 0$, and so, w^0 is a solution for (P). Hence, by Theorem 3.1, we cannot have $w_n \to w^0$ in $H^1_A(\Omega, \mathbb{C})$, because this convergence would imply in $I_A(w^0) = c_0 = c_A$. Thereby, by Lemma 3.2,

$$c_0 = \lim_{n \to +\infty} I_A(w_n) = I_A(w^0) + \sum_{j=1}^{k} I_0(u^j) \ge I_A(w^0) + k c_0.$$

As $I_A(w^0) \ge 0$, then $k = 1$, $w^0 = 0$ and u^1 is a nontrivial solution of (P_0) with $I_0(u^1) = c_0$. From Theorem 2.3, there are $\theta > 0$ and $a \in \mathbb{R}^N$ such that

$$u^1(x) = e^{i\theta} \varphi(x - a), \quad \forall x \in \mathbb{R}^N. \tag{4.8}$$

Since $w_n \rightharpoonup w^0 = 0$, we get

$$\Psi_n(x + y^1_n) = w_n(x + y^1_n) \rightharpoonup u^1(x)$$

and

$$\|\Psi_n(\cdot + y^1_n)\|^2_A = \|w_n(\cdot + y^1_n)\|^2_A \to \|u^1\|^2_0,$$

where (y^1_n) must be a sequence satisfying $|y^1_n| \to \infty$. Therefore,

$$w_n(\cdot + y^1_n) \to u^1 \;\text{ in }\; H^1_A(\mathbb{R}^N, \mathbb{C}).$$

Setting

$$u = u^1, \;\; y_n = y^1_n \quad \text{and} \quad v_n(x + y^1_n) = w_n(x + y^1_n) - u^1(x),$$

we have

$$v_n(x) = w_n(x) - u(x - y_n) \quad \text{and} \quad \|v_n\|_A^2 = \|w_n(\cdot + y_n) - u\|_A^2.$$

Therefore, the strong convergence of $(w_n(\cdot + y_n))$ to u^1 yields $v_n \to 0$ in $H_0^1(\mathbb{R}^N, \mathbb{C})$.

Next, we consider the following sets

$$(\mathbb{R}^N)_n^+ := \{x \in \mathbb{R}^N : \langle x, y_n + a\rangle > 0\} \quad \text{and} \quad (\mathbb{R}^N)_n^- = \mathbb{R}^N \backslash (\mathbb{R}^N)_n^+,$$

where the vector a is given in (4.8). Using the fact that $|y_n| \to +\infty$ as $n \to +\infty$, we claim that there is a ball

$$B(y_n + a, r_*) = \{x \in \mathbb{R}^N : |x - y_n - a| < r_*\} \subset (\mathbb{R}^N)_n^+$$

such that

$$\varphi(x - y_n - a) \geq \frac{1}{2}\varphi(0) > 0, \ \ \forall x \in B(y_n + a, r_*). \tag{4.9}$$

It is easy to see that (4.9) holds for $r^* > 0$ small enough, because φ is positive, radial, strictly decreasing with respect to $|x|$ and

$$\varphi(0) = \max_{z \in \mathbb{R}^N} \varphi(z).$$

On the other hand, for each $r_* > 0$ fixed, there is n_0 such that

$$\langle x, y_n + a\rangle > \frac{|x|^2 + |y_n + a|^2 - r_*^2}{2} \geq \frac{|y_n + a|^2 - r_*^2}{2} > 0, \ \ \forall n \geq n_0, \ \ \forall x \in B(y_n + a, r_*),$$

showing that

$$B(y_n + a, r_*) \subset (\mathbb{R}^N)_n^+, \ \ \text{for } n \text{ large enough}.$$

Hence, for n large enough,

$$|\varphi(x - y_n - a)|^2, \chi(|x|), \langle x, y_n\rangle > 0 \quad \forall x \in (\mathbb{R}^N)_n^+, \qquad B(y_n + a, r_*) \subset (\mathbb{R}^N)_n^*,$$

and $|x| > R$ for every $x \in B(y_n + a, r_*)$. Using this information, we find

$$\begin{aligned}\int_{(\mathbb{R}^N)_n^+} |u(x - y_n)|^2 \chi(|x|)\langle x, y_n + a\rangle dx &= \int_{(\mathbb{R}^N)_n^+} |\varphi(x - y_n - a)|^2 \chi(|x|)\langle x, y_n + a\rangle dx \\ &\geq \frac{|\varphi(0)|^2}{4} \int_{B(y_n + a, r_*)} \chi(|x|)\langle x, y_n + a\rangle dx \\ &= \frac{|\varphi(0)|^2}{8} R|B(y_n + a, r_*)||y_n + a| > 0.\end{aligned}$$

Recalling that for each $x \in (\mathbb{R}^N)_n^-$,

$$|x - y_n - a| \geq |x|,$$

and using again the fact that φ is radial with relation to the origin and decreasing, it follows that

$$|u(x - y_n)|^2 \chi(|x|)|x| = |\varphi(x - y_n - a)|^2 \chi(|x|)|x| \leq R|\varphi(|x|)|^2 \in L^1(\mathbb{R}^N).$$

This fact, combined with the limit

$$\varphi(\cdot - y_n - a) \to 0 \ \ \text{as} \ \ |y_n| \to +\infty$$

ensures that

$$\int_{(\mathbb{R}^N)_n^-} |u(x-y_n)|^2\chi(|x|)|x|dx = o_n(1). \tag{4.10}$$

Therefore, by the Cauchy-Schwarz inequality and (4.10),

$$\begin{aligned}\langle \tau(u(x-y_n)), \frac{y_n+a}{|y_n+a|}\rangle &= \int_{(\mathbb{R}^N)_n^+} |\varphi(x-y_n-a)|^2\chi(|x|)\langle x, \frac{y_n+a}{|y_n+a|}\rangle dx \\ &+ \int_{(\mathbb{R}^N)_n^-} |\varphi(x-y_n-a)|^2\chi(|x|)\langle x, \frac{y_n+a}{|y_n+a|}\rangle dx \\ &\geq \frac{R|\varphi(0)|^2}{8}|B(y_n+a, r_*)| - \int_{(\mathbb{R}^N)_n^-} |\varphi(x-y_n-a)|^2\chi(|x|)|x|dx \\ &\geq \frac{R|\varphi(0)|^2}{8}|B(y_n+a, r_*)| - o_n(1) > 0.\end{aligned} \tag{4.11}$$

Now, using the fact that $w_n \to u^1$ in $H_0^1(\mathbb{R}^N, \mathbb{C})$ together with the limit $\tau(w_n) = o_n(1)$, we find that

$$\tau(u(x-y_n)) = o_n(1), \tag{4.12}$$

which contradicts (4.11), and so,

$$c_0 < \lambda_0.$$

Now we are ready to prove the assertion (i) of Lemma 4.3. As $\psi_\rho(y) = t_{y,\rho}\phi_{y,\rho} \in \mathcal{N}_0$, by Theorem 3.1,

$$I_A(\psi_\rho(y)) > c_A = c_0, \ \ \forall y \in \mathbb{R}^N.$$

By Lemma 4.1-part (ii), for each ρ fixed

$$I_A(\psi_\rho(y)) \to c_0 \ \text{ as } \ |y| \to \infty. \tag{4.13}$$

Thereby, for a given $\epsilon_1 \in (0, \frac{\lambda_0 - c_0}{2})$, there is $R_0 > 0$ such that

$$\Big| I_A(\psi_\rho(y)) - c_0 \Big| < \epsilon_1 \ \text{ whenever } \ |y| \geq R_0.$$

From this

$$I_A(\psi_\rho(y)) \in \left(c_0, \frac{\lambda_0 + c_0}{2}\right), \ \ \forall y \in \mathbb{R}^N \ \text{ such that } \ |y| \geq R_0.$$

Finally, let us show assertion (ii) of Lemma 4.3, by definition of $\psi_\rho(y)$ and arguing as above with $|y|$ large enough, we derive

$$\begin{aligned}
\langle \tau(\psi_\rho(y)), \frac{y}{|y|}\rangle &= t_{y,\rho}^2 \int_{(\mathbb{R}^N)_n^+} |\phi_{y,\rho}(x)|^2 \chi(|x|)\langle x, \frac{y}{|y|}\rangle dx \\
&\quad + t_{y,\rho}^2 \int_{(\mathbb{R}^N)_n^-} |\phi_{y,\rho}(x)|^2 \chi(|x|)\langle x, \frac{y}{|y|}\rangle dx \\
&\geq t_{y,\rho}^2 \int_{B(y,r_*)} |\phi_{y,\rho}(x)|^2 \chi(|x|)\langle x, \frac{y}{|y|}\rangle dx \\
&\quad + t_{y,\rho}^2 \int_{(\mathbb{R}^N)_n^-} |\phi_{y,\rho}(x)|^2 \chi(|x|)\langle x, \frac{y}{|y|}\rangle dx \\
&\geq t_{y,\rho}^2 \frac{|\varphi(0)|^2}{4} \int_{B(y,r_*)} \left|\xi\left(\frac{|x|}{\rho}\right)\right|^2 \chi(|x|)\langle x, \frac{y}{|y|}\rangle dx - o(1).
\end{aligned}$$

As $t_{y,\rho} \to 1$ as $|y| \to +\infty$, we have for $|y| = R_0$ large,

$$\left\langle \tau(\psi_\rho(y)), \frac{y}{|y|} \right\rangle > 0. \tag{4.14}$$

$\square$

5. Proof of Theorem 1.1

In the sequel, we consider the sets

$$\begin{aligned}
&\Sigma := \{\psi_\rho(y) : |y| \leq R_0\} \subset H_A^1(\Omega, \mathbb{C}), \\
&\mathcal{H} := \left\{h \in C(\mathcal{N}_A, \mathcal{N}_A) : h(u) = u, \ \ \forall u \in \mathcal{N}_A \ \text{ with } \ I_A(u) < \frac{\lambda_0 + c_0}{2}\right\}
\end{aligned}$$

and

$$\Upsilon := \{B \in \mathcal{N}_A : \ \ B = h(\Sigma), \ \ h \in \mathcal{H}\}.$$

Lemma 5.1. *If $B \in \Upsilon$, then $B \cap \mathcal{T}_0 \neq \emptyset$.*

Proof. We are going to show that, for every $B \in \Upsilon$, there exists $u \in B$ such that $\tau(u) = 0$. Equivalently, we prove that: for every $h \in \mathcal{H}$, there exists $\tilde{y} \in \mathbb{R}^N$ with $|\tilde{y}| \leq R_0$ such that

$$(\tau \circ h \circ \psi_\rho)(\tilde{y}) = 0. \tag{5.1}$$

For any $h \in \mathcal{H}$, we set the functions

$$\mathcal{J} := \tau \circ h \circ \psi_\rho : \mathbb{R}^N \to \mathbb{R}^N$$

and $\mathcal{F} : [0,1] \times \overline{B}(0, R_0) \to \mathbb{R}^N$ given by

$$\mathcal{F}(t, z) := t\mathcal{J}(z) + (1-t)z.$$

We claim that $0 \notin \mathcal{F}(t, \partial B(0, R_0))$. Indeed, for $|y| = R_0$, by Lemma 4.3—Part (i) we have

$$I_A(\psi_\rho(y)) < \frac{\lambda_0 + c_0}{2}.$$

Hence,

$$\mathcal{F}(t, y) = t(\tau \circ \psi_\rho)(y) + (1 - t)y$$

and

$$\langle \mathcal{F}(t, y), y\rangle = t\langle \tau(\psi_\rho(y)), y\rangle + (1 - t)\langle y, y\rangle. \tag{5.2}$$

Now

- If $t = 0$, then $\langle \mathcal{F}(0, y), y\rangle = |y|^2 = R_0^2 > 0$;
- If $t = 1$, then by Lemma 4.3—Part (ii) we have $\langle \mathcal{F}(1, y), y\rangle = \langle \tau(\psi_\rho(y)), y\rangle > 0$;
- If $t \in (0, 1)$, then $\langle \mathcal{F}(t, y), y\rangle > 0$, since the terms $t, 1 - t, \langle \tau(\psi_\rho(y)), y\rangle$ and $|y|^2$ are positives.

Then, by using the homotopy-invariance of the Brouwer degree, one has

$$d(\mathcal{F}(t, \cdot), B(0, R_0), 0) = \text{constant}, \quad \forall t \in [0, 1].$$

Recalling that

$$d(\mathcal{J}, B(0, R_0), 0) = 1 \neq 0,$$

there exists $\tilde{y} \in B(0, R_0)$ such that $\mathcal{J}(\tilde{y}) = 0$, that is,

$$\mathcal{J}(\tilde{y}) = (\tau \circ h \circ \psi_\rho)(\tilde{y}) = 0.$$

This completes the proof of Lemma 5.1. $\square$

Now, let us define

$$c := \inf_{B \in \Upsilon} \sup_{u \in B} I_A(u), \tag{5.3}$$

$$\mathcal{K}_c := \{u \in \mathcal{N}_A : I_A(u) = c \ \text{ and } \ \nabla I_A\Big|_{\mathcal{N}_A}(u) = 0\}$$

and

$$L_\gamma := \{u \in \mathcal{N}_A : I_A(u) \leq \gamma\},$$

for every $\gamma \in \mathbb{R}$.

Proof of Theorem 1.1. We choose $\rho = \rho_0$, where ρ_0 is given in Corollary 4.2. We claim that the constant c defined in (5.3) is a critical value for I_A, that is, $\mathcal{K}_c \neq \emptyset$. We start our analysis by noting that

$$c_0 < c < c_0 + \kappa_1. \tag{5.4}$$

First of all, we recall that by Lemma 5.1, $B \cap \mathcal{T}_0 \neq \emptyset$ for every $B \in \Upsilon$. Then, for each $B \in \Upsilon$, there is $\tilde{u} \in B \cap \mathcal{T}_0$ such that

$$\inf_{u \in \mathcal{T}_0} I_A(u) \leq \inf_{u \in B \cap \mathcal{T}_0} I_A(u) \leq I_A(\tilde{u}) \leq \sup_{u \in B \cap \mathcal{T}_0} I_A(u) \leq \sup_{u \in B} I_A(u). \tag{5.5}$$

By Lemma 4.3 and (5.5),

$$c_0 < \lambda_0 = \inf_{u\in\mathcal{I}_0} I_A(u) \le \sup_{u\in B} I_A(u), \quad \forall B \in \Upsilon.$$

Thus

$$c_0 < \lambda_0 \le \inf_{B\in\Upsilon} \sup_{u\in B} I_A(u) = c. \tag{5.6}$$

Since

$$c \le \sup_{u\in B} I_A(u), \ \ \forall B \in \Upsilon, \tag{5.7}$$

it follows that

$$c \le \sup_{|y|\le R_0} I_A(h(\psi_\rho(y))), \ \ \forall h \in \mathcal{H}.$$

Now, taking $h \equiv I$, we find

$$c \le \sup_{|y|\le R_0} I_A(\psi_\rho(z)) \le \sup_{y\in\mathbb{R}^N} I_A(\psi_\rho(y)).$$

The last inequality, together with Corollary 4.2 and (5.6) leads to

$$c_0 < c < c_0 + \kappa_1, \tag{5.8}$$

which proves (5.4).

Suppose by contradiction that $\mathcal{K}_c = \emptyset$. Recall that

$$\frac{\lambda_0 + c_0}{2} \le \frac{c + c_0}{2} < c < c_0 + \kappa_1.$$

By Corollary 3.4 and the deformation lemma [30], there is a continuous map

$$\eta : [0,1] \times \mathcal{N}_A \to \mathcal{N}_A$$

and a positive number ε_0 such that

(a) $L_{c+\varepsilon_0} \backslash L_{c-\varepsilon_0} \subset\subset L_{c_0+\kappa_1} \backslash L_{\frac{\lambda_0+c_0}{2}}$,
(b) $\eta(t,u) = u, \ \ \forall u \in L_{c-\varepsilon_0} \cup \{\mathcal{N}_A \backslash L_{c+\varepsilon_0}\}$ and $\forall t \in [0,1]$,
(c) $\eta(1, L_{c+\frac{\varepsilon_0}{2}}) \subset L_{c-\frac{\varepsilon_0}{2}}$.

Fix $\tilde{B} \in \Upsilon$ such that

$$c \le \sup_{u\in\tilde{B}} I_A(u) < c + \frac{\varepsilon_0}{2}.$$

Since

$$I_A(u) < c + \frac{\varepsilon_0}{2}, \ \ \forall u \in \tilde{B},$$

it follows that

$$\tilde{B} \subset L_{c+\frac{\varepsilon_0}{2}}.$$

Now, by (c), one has

$$I_A(u) < c - \frac{\varepsilon_0}{2}, \ \ \forall u \in \eta(1, \tilde{B}),$$

that is,

$$\sup_{u\in\eta(1,\tilde{B})} I_A(u) < c - \frac{\varepsilon_0}{2}. \tag{5.9}$$

On the other hand, we notice that $\eta(1,\cdot) \in C(\mathcal{N}_A, \mathcal{N}_A)$. Moreover, since $\tilde{B} \in \Upsilon$, there exists $h \in \mathcal{H}$ such that $\tilde{B} = h(\Sigma)$. Consequently,

$$\tilde{h} = \eta(1,\cdot) \circ h \in C(\mathcal{N}_A, \mathcal{N}_A).$$

Since $h \in \mathcal{H}$, it follows that

$$h(u) = u, \ \ \forall u \in \mathcal{N}_A \ \text{ with } \ I_A(u) < \frac{\lambda_0 + c_0}{2}$$

and

$$\tilde{h}(u) = \eta(1,u) \ \ \forall u \in \mathcal{N}_A \ \text{ with } \ I_A(u) < \frac{\lambda_0 + c_0}{2}.$$

Taking into account that

$$\frac{\lambda_0 + c_0}{2} < c - \epsilon_0,$$

by item (b), we easily have

$$\tilde{h}(u) = \eta(1,u) = u, \ \ \forall u \in \mathcal{N}_A \ \text{ with } \ I_A(u) < \frac{\lambda_0 + c_0}{2} < c - \epsilon_0.$$

Then $\tilde{h} \in \mathcal{H}$. Moreover

$$\eta(1, \tilde{A}) \in \Gamma,$$

owing to $\eta(1,\tilde{B}) = \tilde{h}(\Sigma)$. Therefore, exploiting the definition of c, we have

$$c \leq \sup_{u\in\eta(1,\tilde{B})} I_A(u),$$

which contradicts (5.9). Thereby, $\mathcal{K}_c \neq \emptyset$ and c is a critical value of I_A on $\mathcal{N}_A$, namely there is at least one nontrivial weak solution of (P). Hence, Theorem 1.1 is proved.

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Claudianor O. Alves
Unidade Acadêmica de Matemática
Universidade Federal de Campina Grande
Campina Grande PB 58429-970
Brazil
e-mail: coalves@mat.ufcg.edu.br

Vincenzo Ambrosio
Dipartimento di Ingegneria Industriale e Scienze Matematiche
Università Politecnica delle Marche
Via Brecce Bianche, 12
60131 Ancona
Italy
e-mail: v.ambrosio@staff.univpm.it

César E. Torres Ledesma
Departamento de Matemáticas
Universidad Nacional de Trujillo
Av. Juan Pablo II s/n.
Trujillo
Peru
e-mail: ctl_576@yahoo.es

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