A Unified Approach to Singularly Perturbed Quasilinear Schrödinger Equations

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Abstract. In this paper we present a unified approach to investigate existence and concentration of positive solutions for the following class of quasilinear Schrödinger equations,

$$-\varepsilon^2 \Delta u + V(x)u \mp \varepsilon^{2+\gamma} u \Delta u^2 = h(u), \quad x \in \mathbb{R}^N,$$

where $N \geq 3$, $\varepsilon > 0$, $V(x)$ is a positive external potential, h is a real function with subcritical or critical growth. The problem is quite sensitive to the sign changing of the quasilinear term as well as to the presence of the parameter $\gamma > 0$. Nevertheless, by means of perturbation type techniques, we establish the existence of a positive solution $u_{\varepsilon, \gamma}$ concentrating, as $\varepsilon \rightarrow 0$, around minima points of the potential.

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1. Introduction

In this paper, we consider the following class of quasilinear Schrödinger equations,

$$-\varepsilon^2 \Delta u + V(x)u \mp \varepsilon^{2+\gamma} u \Delta u^2 = h(u), \quad x \in \mathbb{R}^N, \quad (1.1)$$

where $N \geq 3$, $\varepsilon, \gamma > 0$, V is the external Schrödinger potential, h is a real function. Solutions of (1.1) are related to standing wave solutions for the time-dependent quasilinear Schrödinger equations

$$i\varepsilon z_t = -\varepsilon^2 \Delta z + W(x)z + \kappa z \Delta |z|^2 - f(|z|^2)z, \quad x \in \mathbb{R}^N, \quad (1.2)$$

where $z : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$, $W : \mathbb{R}^N \rightarrow \mathbb{R}$ is a given potential, $f : \mathbb{R} \rightarrow \mathbb{R}$ is a real function, $\varepsilon > 0$, and κ is a parameter. An equation of type (1.2) appears in

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the literature in the context of plasma physics and the continuum limit of discrete molecular structures. In particular, κ turns out to be small and negative when (1.2) represents the weak nonlocal limit of some general nonlocal models, whereas κ has no prescribed sign in plasma physics, see for instance [6, 7, 20, 26, 28] for more details on the Physics involved.

A well-known standard tool in the investigation of the elliptic side of Schrödinger equations, is the ansatz $z(t, x) = \exp(-iEt/\varepsilon)u(x)$ where $E \in \mathbb{R}$ and u is a real function, by which equation (1.2) turns into the following quasilinear elliptic equation,

$$-\varepsilon^2 \Delta u + V(x)u - \kappa u \Delta u^2 = h(u), \quad x \in \mathbb{R}^N, \quad (1.3)$$

where $V(x) = W(x) - E$ and $h(u) = f(|u|^2)u$.

On the one hand, the study of the existence of ground states (minimal energy solutions) and bound states (finite energy) solutions to (1.3) with $\varepsilon = 1$ has received considerable attention in recent years, see [12, 23, 24, 27, 30, 34] for arbitrary fixed $\kappa > 0$ and [4, 32] for $\kappa < 0$ and $|\kappa|$ small. On the other hand, a lot of work has been done on the existence of semi-classical states for (1.3), namely $\kappa = \varepsilon^2$ and $\varepsilon \rightarrow 0^+$. Assume $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is Hölder continuous and satisfies the following condition (V):

$$0 < V_0 < \inf_{x \in \mathbb{R}^N} V(x)$$

and there is a bounded open set $\mathcal{O}$ such that

$$0 < m := \inf_{x \in \mathcal{O}} V(x) < \min_{x \in \partial \mathcal{O}} V(x).$$

With the assumption (V), the existence of a localized solution concentrating around $\mathcal{M} := \{x \in \mathcal{O} : V(x) = m\}$ has been studied under various conditions on the nonlinearity h . When $N \geq 3$, we refer to [11, 16, 36] for the subcritical case and [18, 33] for the critical case. The case of dimension $N = 2$ has been addressed in [13, 14].

Recently, S. Adachi et al. in [1–3] have studied asymptotic properties of the ground state of the quasilinear Schrödinger equation

$$-\Delta u + \lambda u - \kappa u \Delta u^2 = |u|^{p-1}u, \quad x \in \mathbb{R}^N.$$

As $\kappa \rightarrow 0^+$, they have proved that the ground state u_κ converges to a ground state solution of a limit equation for $2 < p < 2^*$ whence it blows up, in a suitable sense, for $p > 2^*$.

Therefore, it is a legitimate and somewhat interesting question what happens when the two parameters ε and κ in (1.3) tend to zero at different speed rates.

The purpose of this paper is to present a unified approach to study the asymptotic properties of the solutions to (1.3) for $\kappa = \mp \varepsilon^{2+\gamma}$ with $\gamma > 0$ and $\varepsilon \rightarrow 0^+$. Let us stress the fact that both signs for κ , which is assumed to be small in absolute value, turn out to be physically relevant as developed in [31]. We will show that beyond the expected concentration phenomenon, the limit equations for the case $\gamma > 0$ and $\gamma = 0$ turn out to be different when $\kappa = \varepsilon^{2+\gamma}$. Moreover, to the best of our knowledge, no result is known in the case $\kappa = -\varepsilon^{2+\gamma}$.

Observe that (1.1) is the Euler-Lagrange equation associated to the energy functional

$$I_{\varepsilon,\gamma}(u) = \frac{1}{2} \int_{\mathbb{R}^N} [(\varepsilon^2 + \kappa u^2)|\nabla u|^2 + V(x)u^2]dx - \int_{\mathbb{R}^N} H(u)dx, \quad (1.4)$$

where $H(t) = \int_0^t h(s)ds$. From the variational point of view, the first difficulty to handle is to find a proper function space setting where (1.4) is well defined. For $\kappa = \varepsilon^{2+\gamma}$, this difficulty can be overcome by minimization on spheres (mass constraint) or on the Nehari manifold, see [22, 24]. Another important tool for this type of equations is to perform a suitable change of variable [23, 29] which reduces (1.1) to a semilinear elliptic equation for which variational methods are available, see [12]. Unfortunately, the methods which work in the case $\kappa = \varepsilon^{2+\gamma}$ can not be directly adapted to deal with the case $\kappa = -\varepsilon^{2+\gamma}$. Indeed, here the principle part of the corresponding energy functional does change sign, see [4, 32].

As we look for positive solutions, let us assume $h(t) = 0$ for $t \leq 0$ and we require $h \in C(\mathbb{R}^+, \mathbb{R})$ to enjoy the following conditions, which were introduced by Byeon and Jeanjean in [10], see also [9]:

- (h₁) $\lim_{t \rightarrow 0} \frac{h(t)}{t} = 0$;
- (h₂) $\lim_{t \rightarrow +\infty} \frac{h(t)}{t^p} = 0$, $p \in (1, 2^* - 1)$;
- (h₃) there exists $T > 0$ such that $\frac{1}{2}mT^2 < H(T)$.

Our main result is the following:

Theorem 1.1 (Subcritical case). *Let $\gamma > 0$ and assume that (V) and (h₁)–(h₃) hold. Then, for sufficiently small $\varepsilon > 0$, there exists a positive solution $u_{\varepsilon,\gamma}$ of (1.1) satisfying the following:*

- (1) *There exists a local maximum point x_ε of $u_{\varepsilon,\gamma}$ such that $\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon, \mathcal{M}) = 0$ and $u_{\varepsilon,\gamma}(\varepsilon \cdot -\varepsilon x_\varepsilon)$ converges locally uniformly to a positive ground state of*

$$-\Delta u + mu = h(u). \quad (1.5)$$

- (2) *There exist $C, c > 0$ such that*

$$u_{\varepsilon,\gamma}(x) \leq C \exp\left(-\frac{c}{\varepsilon}|x - x_\varepsilon|\right)$$

for all $x \in \mathbb{R}^N$.

Critical case. *Assume in place of (h₂) the following critical growth,*

$$(h_4) \quad \lim_{t \rightarrow +\infty} \frac{h(t)}{t^{2^*-1}} = 1.$$

The above results extend to cover the critical case provided we further assume

$$(h_5) \begin{cases} \lim_{t \rightarrow +\infty} \frac{H(t) - \frac{1}{2^*} t^{2^*}}{t^4} = +\infty, & \text{if } N = 3; \\ \lim_{t \rightarrow +\infty} \frac{H(t) - \frac{1}{2^*} t^{2^*}}{t^2 \ln t} = +\infty, & \text{if } N = 4; \\ \lim_{t \rightarrow +\infty} \frac{H(t) - \frac{1}{2^*} t^{2^*}}{t^2} = +\infty, & \text{if } N \geq 5. \end{cases}$$

In [16], similar results to those of Theorem 1.1 are obtained for the equation (1.3) with $\gamma = 0$, namely $\kappa = \varepsilon^2$ and $p \in (1, 2(2^*) - 1)$ in (h_2) . The limit equation for $\kappa = \varepsilon^2$ being

$$-\Delta u + mu - u\Delta u^2 = h(u).$$

In our case, where $\kappa = \varepsilon^{2+\gamma}$ with $\gamma > 0$, the situation is different and we can not cover the range $p \in (2^* - 1, 2(2^*) - 1)$ as the solution can blow up. Closely related problems were considered in [4], where the authors study the following equation,

$$-\Delta u + V(x)u + \alpha u\Delta u^2 = |u|^{p-1}u, \quad x \in \mathbb{R}^N \quad (1.6)$$

and assuming V is a trapping potential, they have proved the existence of a positive solution for $2 < p < 2^* - 1$ and $\alpha > 0$ small enough. Moreover, in [32] the authors prove that if $V \in C^1(\mathbb{R}^N, [0, +\infty))$ is such that $x \cdot \nabla V(x) + 2V(x) \geq 0$ for all $x \in \mathbb{R}^N$, then the equation (1.6) has only trivial C^2 solutions for any $p \geq 2^* - 1$ and any $\alpha > 0$.

Finally, let us point out that difficulties in (1.3) with $\kappa = -\varepsilon^2$ are due to the fact that it is still an open problem if there exists a solution to equation (1.6) for all $\alpha > 0$.

The outline of the paper is as follows: in Section 2, we modify (1.1) by introducing an auxiliary function. In Section 3, we prove the existence of a positive solution for the modified problem. Then, Sections 4 and 5 are devoted to prove Theorem 1.1 respectively in the subcritical and critical case where we use a truncation approach.

Notation. Without loss of generality, we may assume $0 \in \mathcal{M}$. For any set $\Omega \subset \mathbb{R}^N$ and $\varepsilon, \delta > 0$, we define $\Omega_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Omega\}$ and $\Omega^\delta := \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) \leq \delta\}$. Let

$$E_\varepsilon := \left\{ v \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x) v^2 dx < \infty \right\}$$

the Hilbert space endowed with the norm $\|v\|_\varepsilon := \sqrt{\int_{\mathbb{R}^N} (|\nabla v|^2 + V(\varepsilon x) v^2) dx}$.

We denote by C a positive constants whose exact value may change from line to line without affecting the overall result.

2. An auxiliary problem

Let $w(x) = u(\varepsilon x)$, equation (1.1) is equivalent to

$$-\Delta w + V_\varepsilon(x)w \mp \varepsilon^\gamma w\Delta w^2 = h(w), \quad x \in \mathbb{R}^N, \quad (2.1)$$

where $V_\varepsilon(x) = V(\varepsilon x)$.

Note that (2.1) formally is the Euler-Lagrange equation associated to the energy functional

$$I_{\varepsilon,\gamma}(w) = \frac{1}{2} \int_{\mathbb{R}^N} [(1 \pm \varepsilon^\gamma w^2)|\nabla w|^2 + V_\varepsilon(x)w^2]dx - \int_{\mathbb{R}^N} H(w)dx. \quad (2.2)$$

However, classical variational methods can not be directly applied since $I_{\varepsilon,\gamma}$ is not well defined on the Hilbert space E_ε because of the presence of four homogeneous term $w^2|\nabla w|^2$.

Define an auxiliary function $g_{\varepsilon,\gamma,\pm}(t) : \mathbb{R} \rightarrow \mathbb{R}^+$ as follows:

$$g_{\varepsilon,\gamma,\pm}(t) = \sqrt{1 \pm \varepsilon^\gamma t^2 \eta(t)},$$

where $\eta(t) \in C_0^\infty(\mathbb{R}, [0, 1])$ is a cut-off function such that $\eta(t) = \eta(-t)$, $\eta(t) = 1$ for $0 \leq t < \sqrt[4]{1/\varepsilon^\gamma}$ and $\eta(t) = 0$ for $t \geq 2\sqrt[4]{1/\varepsilon^\gamma}$. Moreover, there exists some $C > 0$ independent of ε and γ such that $-C\sqrt{\eta(t)} \leq \eta'(t)t \leq 0$, for all $t \in \mathbb{R}$.

Remark 2.1. The cut-off function $\eta(t)$ can be constructed as follows. Let

$$\zeta(t) = \frac{1}{1 + e^{\frac{1}{t} - \frac{1}{1-t}}}$$

for $0 < t < 1$, $\zeta(t) = 1$ for $t \geq 1$ and $\zeta(t) = 0$ for $t \leq 0$. For $t \in \mathbb{R}^+$ let $\rho(t) = \zeta^2 \left[2 - \sqrt[4]{\varepsilon^\gamma t} \right]$. Then, the function $\eta(t) = \rho(t)$ for $t \geq 0$ and $\eta(t) = \rho(-t)$ for $t < 0$, enjoys the required properties.

Let

$$G_{\varepsilon,\gamma,\pm}(t) = \int_0^t g_{\varepsilon,\gamma,\pm}(s)ds,$$

then $G_{\varepsilon,\gamma,\pm}(t)$ is an odd and smooth function as well as the inverse function $G_{\varepsilon,\gamma,\pm}^{-1}(t)$. For any fixed $\gamma > 0$ and small $\varepsilon > 0$, next we establish a few properties of $G_{\varepsilon,\gamma,\pm}^{-1}(t)$.

Lemma 2.2. *The following properties hold:*

- (i) $\lim_{t \rightarrow 0} \frac{G_{\varepsilon,\gamma,\pm}^{-1}(t)}{t} = 1$;
- (ii) $\lim_{t \rightarrow +\infty} \frac{G_{\varepsilon,\gamma,\pm}^{-1}(t)}{t} = 1$;
- (iii) $\frac{1}{\sqrt{1+4\varepsilon^{\frac{\gamma}{2}}}}|t| \leq |G_{\varepsilon,\gamma,+}^{-1}(t)| \leq |t|$ and $|t| \leq |G_{\varepsilon,\gamma,-}^{-1}(t)| \leq \frac{1}{\sqrt{1-4\varepsilon^{\frac{\gamma}{2}}}}|t|$ for all $t \in \mathbb{R}$;
- (iv) there exists $C > 0$ such that $|G_{\varepsilon,\gamma,\pm}^{-1}(t) - t| \leq C\sqrt{\varepsilon^\gamma}|t|$;
- (v) $|G_{\varepsilon,\gamma,\pm}^{-1}(t)|^2$ is convex in t for small $\varepsilon > 0$.

Proof. (i) and (ii) are consequences of L'Hôpital's rule:

$$\lim_{t \rightarrow 0(+\infty)} \frac{G_{\varepsilon,\gamma,\pm}^{-1}(t)}{t} = \lim_{t \rightarrow 0(+\infty)} \frac{1}{g_{\varepsilon,\gamma,\pm}(G_{\varepsilon,\gamma,\pm}^{-1}(t))} = 1.$$

By definition of $g_{\varepsilon,\gamma,\pm}(t)$ and $G_{\varepsilon,\gamma,\pm}(t)$, (iii) follows.

Let us prove (iv): for $t > 0$ we have

$$|G_{\varepsilon,\gamma,\pm}(t) - t| \leq \int_0^t |g_{\varepsilon,\gamma,\pm}(s) - 1|ds \leq \varepsilon^\gamma \int_0^t s^2 \eta(s)ds \leq 4\sqrt{\varepsilon^\gamma}t. \quad (2.3)$$

From (iii), we have $|G_{\varepsilon,\gamma,\pm}^{-1}(t) - t| \leq 4\sqrt{\varepsilon\gamma}|G_{\varepsilon,\gamma,\pm}^{-1}(t)| \leq C\sqrt{\varepsilon\gamma}|t|$.

Finally let $T_{\pm}(t) := |G_{\varepsilon,\gamma,\pm}^{-1}(t)|^2$. Then, for small $\varepsilon > 0$, by (iii) we have

$$\begin{aligned} \frac{1}{2}g^4(G_{\varepsilon,\gamma,+}^{-1}(t))T_+''(G_{\varepsilon,\gamma,+}^{-1}(t)) &= 1 - \varepsilon^\gamma|G_{\varepsilon,\gamma,+}^{-1}(t)|^2\eta(G_{\varepsilon,\gamma,+}^{-1}(t)) \\ &\quad - \varepsilon^\gamma|G_{\varepsilon,\gamma,+}^{-1}(t)|^2\eta'(G_{\varepsilon,\gamma,+}^{-1}(t))G_{\varepsilon,\gamma,+}^{-1}(t) > 0 \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}g^4(G_{\varepsilon,\gamma,-}^{-1}(t))T_-''(G_{\varepsilon,\gamma,-}^{-1}(t)) &= 1 + \varepsilon^\gamma|G_{\varepsilon,\gamma,-}^{-1}(t)|^2\eta(G_{\varepsilon,\gamma,-}^{-1}(t)) \\ &\quad + \varepsilon^\gamma|G_{\varepsilon,\gamma,-}^{-1}(t)|^2\eta'(G_{\varepsilon,\gamma,-}^{-1}(t))G_{\varepsilon,\gamma,-}^{-1}(t) > 0, \end{aligned}$$

which imply (v). $\square$

Next we consider the following modified quasilinear Schrödinger equation,

$$- \operatorname{div}(g_{\varepsilon,\gamma,\pm}^2(w)\nabla w) + g_{\varepsilon,\gamma,\pm}(w)g'_{\varepsilon,\gamma,\pm}(w)|\nabla w|^2 + V_\varepsilon(x)w = h(w). \quad (2.4)$$

Direct calculations show that (2.4) turns into (2.1) when $g_{\varepsilon,\gamma,\pm}(t) = \sqrt{1 \pm \varepsilon\gamma t^2}$ and the energy functional related to (2.4) is given by

$$\bar{I}_{\varepsilon,\gamma,\pm}(w) = \frac{1}{2} \int_{\mathbb{R}^N} [g_{\varepsilon,\gamma,\pm}^2(w)|\nabla w|^2 + V_\varepsilon(x)w^2] dx - \int_{\mathbb{R}^N} H(w) dx \quad (2.5)$$

which is well defined on E_ε , though it is still non-smooth. As in [29], we exploit the change of variable $w = G_{\varepsilon,\gamma,\pm}^{-1}(v)$ to replace the functional $\bar{I}_{\varepsilon,\gamma,\pm}$ by the following smooth functional:

$$P_{\varepsilon,\gamma,\pm}(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V_\varepsilon(x)|G_{\varepsilon,\gamma,\pm}^{-1}(v)|^2) dx - \int_{\mathbb{R}^N} H(G_{\varepsilon,\gamma,\pm}^{-1}(v)) dx. \quad (2.6)$$

Indeed, under conditions (h_1) – (h_3) and (i)–(ii) of Lemma 2.2, it is standard to check that $P_{\varepsilon,\gamma,\pm} \in C^1(E_\varepsilon, \mathbb{R})$.

The Euler-Lagrange equation associated to $P_{\varepsilon,\gamma,\pm}$ is

$$-\Delta v + V_\varepsilon(x) \frac{G_{\varepsilon,\gamma,\pm}^{-1}(v)}{g_{\varepsilon,\gamma,\pm}(G_{\varepsilon,\gamma,\pm}^{-1}(v))} = \frac{h(G_{\varepsilon,\gamma,\pm}^{-1}(v))}{g_{\varepsilon,\gamma,\pm}(G_{\varepsilon,\gamma,\pm}^{-1}(v))}, \quad x \in \mathbb{R}^N. \quad (2.7)$$

Now notice that if $v \in E_\varepsilon$ is a critical point of $P_{\varepsilon,\gamma,\pm}(v)$, then it satisfies for all $\phi \in E_\varepsilon$,

$$\int_{\mathbb{R}^N} \left[\nabla v \nabla \phi + V_\varepsilon(x) \frac{G_{\varepsilon,\gamma,\pm}^{-1}(v)}{g_{\varepsilon,\gamma,\pm}(G_{\varepsilon,\gamma,\pm}^{-1}(v))} \phi \right] dx - \int_{\mathbb{R}^N} \frac{h(G_{\varepsilon,\gamma,\pm}^{-1}(v))}{g_{\varepsilon,\gamma,\pm}(G_{\varepsilon,\gamma,\pm}^{-1}(v))} \phi dx = 0.$$

Let $w = G_{\varepsilon,\gamma,\pm}^{-1}(v) \in E_\varepsilon$, $\varphi \in C_0^\infty(\mathbb{R}^N)$, then $\phi := \varphi g_{\varepsilon,\gamma,\pm}(G_{\varepsilon,\gamma,\pm}^{-1}(v)) \in E_\varepsilon$ and

$$\int_{\mathbb{R}^N} [\nabla w \nabla \varphi + g_{\varepsilon,\gamma,\pm}(w)g'_{\varepsilon,\gamma,\pm}(w)|\nabla w|^2\varphi + V_\varepsilon(x)w\varphi] dx - \int_{\mathbb{R}^N} h(w)\varphi dx = 0,$$

which implies that w is a weak solution of (2.4).

Define the functional

$$\Gamma_{\varepsilon,\gamma,\pm}(v) = P_{\varepsilon,\gamma,\pm}(v) + Q_\varepsilon(v), \quad (2.8)$$

where

$$Q_\varepsilon(v) = \left(\int_{\mathbb{R}^N} \chi_\varepsilon v^2 dx - 1 \right)_+^2 \quad (2.9)$$

with $\chi_\varepsilon(x) = 0$ for $x \in \mathcal{O}_\varepsilon$ and $\chi_\varepsilon(x) = \varepsilon^{-1}$ for $x \notin \mathcal{O}_\varepsilon$. The functional Q_ε will have a penalization effect which forces concentration phenomena to occur inside $\mathcal{O}$. This type of penalization was first introduced in [8]. Clearly, $\Gamma_{\varepsilon,\gamma,\pm} \in C^1(E_\varepsilon, \mathbb{R})$.

3. Existence of solutions for the modified problem

Let us recall some well known results about the limit equation (1.5) under conditions (h_1) – (h_3) . The energy functional is defined by

$$L_m(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + mv^2) dx - \int_{\mathbb{R}^N} H(v) dx.$$

In [5], Berestycki and Lions proved that there exists a ground state (least energy solution) of (1.5) and each solution v of (1.5) satisfies Pohozaev's identity

$$\frac{N-2}{2N} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} m \int_{\mathbb{R}^N} v^2 dx - \int_{\mathbb{R}^N} H(v) dx = 0. \quad (3.1)$$

Let S_m be the set of ground states U of (1.5) satisfying $U(0) = \max_{x \in \mathbb{R}^N} U(x)$. Then, S_m is compact and

$$|U(x)| + |\nabla U(x)| \leq C \exp(-c|x|), \quad x \in \mathbb{R}^N, \quad (3.2)$$

for some $C, c > 0$ independent of U , see Proposition 1 in [10]. In [19], Jeanjean and Tanaka proved that the ground state has a mountain pass characterization, i.e.,

$$L_m(U) = C_m = \inf_{\eta \in \Phi} \max_{t \in [0,1]} L_m(\eta(t)),$$

where $\Phi = \{\eta \in C([0,1], H^1(\mathbb{R}^N)) : \eta(0) = 0, L_m(\eta(1)) < 0\}$.

Now set $U_t(x) = U(\frac{x}{t})$ for $t > 0$. By (3.1), for any $t > 0$, we get

$$L_m(U_t) = \left(\frac{t^{N-2}}{2} - \frac{t^N}{2^*} \right) \int_{\mathbb{R}^N} |\nabla U|^2 dx.$$

So, there exists $t_0 > 1$ such that $L_m(U_t) < -2$ for all $t \geq t_0$.

Choose a positive number $\beta < \frac{\text{dist}(\mathcal{M}, \mathbb{R}^N \setminus \mathcal{O})}{100}$ and a cut-off function $\varphi(x) \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $\varphi(x) = 1$ for $|x| \leq \beta$ and $\varphi(x) = 0$ for $|x| \geq 2\beta$. We define $\varphi_\varepsilon(x) := \varphi(\varepsilon x)$ and $U_\varepsilon^y(x) := \varphi_\varepsilon(x - \frac{y}{\varepsilon}) U(x - \frac{y}{\varepsilon})$ for each $y \in \mathcal{M}^\beta$, $U \in S_m$. For sufficiently small $\varepsilon > 0$, we will find a solution near the set $X_\varepsilon := \{U_\varepsilon^y(x) : y \in \mathcal{M}^\beta, U \in S_m\}$.

Let $W_{\varepsilon,t}(x) = \varphi_\varepsilon(x) U_t(x)$. Note that for fixed $x \in \mathbb{R}^N$, $W_{\varepsilon,t}(x) \rightarrow 0$ as $t \rightarrow 0$. Thus, we set $W_{\varepsilon,0}(x) = 0$.

Lemma 3.1. $\lim_{\varepsilon \rightarrow 0} \max_{t \in (0, t_0]} |\Gamma_{\varepsilon,\gamma,\pm}(W_{\varepsilon,t}) - L_m(U_t)| = 0$.

Proof. Since $0 \in \mathcal{M}$, we get $\text{supp}(W_{\varepsilon,t}(x)) \subset \{x \in \mathbb{R}^N : |\varepsilon x| \leq 2\beta\} \subset (\mathcal{M}^{2\beta})_\varepsilon \subset \mathcal{O}_\varepsilon$. Thus, $Q_\varepsilon(W_{\varepsilon,t}(x)) = 0$. For $t \in (0, t_0]$, we have

$$\begin{aligned} \Gamma_{\varepsilon,\gamma,\pm}(W_{\varepsilon,t}) - L_m(U_t) &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla W_{\varepsilon,t}|^2 - |\nabla U_t|^2) dx \\ &+ \frac{1}{2} \int_{\mathbb{R}^N} (V_\varepsilon |G_{\varepsilon,\gamma,\pm}^{-1}(W_{\varepsilon,t})|^2 - m U_t^2) dx + \int_{\mathbb{R}^N} [H(U_t) - H(G_{\varepsilon,\gamma,\pm}^{-1}(W_{\varepsilon,t}))] dx. \end{aligned} \quad (3.3)$$

Next we prove that the three integrals on the right-hand side of (3.3) tend to zero as $\varepsilon \rightarrow 0$. By (3.2) we have, as $\varepsilon \rightarrow 0$,

$$\int_{\mathbb{R}^N} ||\nabla W_{\varepsilon,t}|^2 - |\nabla U_t|^2| dx \leq C \int_{\mathbb{R}^N} [C\varepsilon + (1 - \varphi_\varepsilon^2)] e^{-c|x|/t_0} dx \rightarrow 0. \quad (3.4)$$

By Lemma 2.2-(iv) we have

$$\int_{\mathbb{R}^N} ||G_{\varepsilon,\gamma,\pm}^{-1}(U_t)|^2 - |U_t|^2| dx \leq C\sqrt{\varepsilon^\gamma} \int_{\mathbb{R}^N} U^2 dx \leq C\sqrt{\varepsilon^\gamma}. \quad (3.5)$$

Thus, by (3.5) we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |V_\varepsilon |G_{\varepsilon,\gamma,\pm}^{-1}(W_{\varepsilon,t})|^2 - m |U_t|^2| dx \\ &\leq C \int_{\mathbb{R}^N} |V_\varepsilon - m| |U_t|^2 dx + Cm \int_{\mathbb{R}^N} |\xi_{\varepsilon,t}| |(\varphi_\varepsilon - 1) U_t| dx + C\sqrt{\varepsilon^\gamma} \\ &\leq C \int_{\mathbb{R}^N} |V_\varepsilon - m| e^{-2c|x|/t_0} dx + C \int_{\mathbb{R}^N} |\varphi_\varepsilon - 1| e^{-c|x|/t_0} dx + C\sqrt{\varepsilon^\gamma} \rightarrow 0, \end{aligned} \quad (3.6)$$

as $\varepsilon \rightarrow 0$, where $\xi_{\varepsilon,t} = \tau W_{\varepsilon,t} + (1 - \tau)U_t$, $0 < \tau < 1$, $|\xi_{\varepsilon,t}| \leq C$.

By (h_1) – (h_2) we have

$$\int_{\mathbb{R}^N} |H(U_t) - H(G_{\varepsilon,\gamma,\pm}^{-1}(U_t))| dx \leq C \int_{\mathbb{R}^N} (|U| + |U|^p) |U - G_{\varepsilon,\gamma,\pm}^{-1}(U)| dx \leq C\sqrt{\varepsilon^\gamma}. \quad (3.7)$$

From (3.7) we get

$$\begin{aligned} &\int_{\mathbb{R}^N} |H(U_t) - H(G_{\varepsilon,\gamma,\pm}^{-1}(W_{\varepsilon,t}))| dx \\ &\leq \int_{\mathbb{R}^N} |H(G_{\varepsilon,\gamma,\pm}^{-1}(U_t)) - H(G_{\varepsilon,\gamma,\pm}^{-1}(W_{\varepsilon,t}))| dx + \int_{\mathbb{R}^N} |H(U_t) - H(G_{\varepsilon,\gamma,\pm}^{-1}(U_t))| dx \\ &\leq \int_{\mathbb{R}^N} \frac{h(G_{\varepsilon,\gamma,\pm}^{-1}(\eta))}{g_{\varepsilon,\gamma,\pm}(G_{\varepsilon,\gamma,\pm}^{-1}(\eta))} |U_t - W_{\varepsilon,t}| dx + C\sqrt{\varepsilon^\gamma} \\ &\leq C \int_{\mathbb{R}^N} [U_t + W_{\varepsilon,t} + U_t^p + W_{\varepsilon,t}^p] |U_t - W_{\varepsilon,t}| dx + C\sqrt{\varepsilon^\gamma} \\ &\leq C \int_{\mathbb{R}^N} |\varphi_\varepsilon - 1| \exp(-c|x|/t_0) dx + C\sqrt{\varepsilon^\gamma} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0. \end{aligned} \quad (3.8)$$

Combining (3.3)–(3.8) we obtain the result. $\square$

From Lemma 3.1, there exists $\varepsilon_0 > 0$, such that for $\varepsilon \in (0, \varepsilon_0)$,

$$\Gamma_{\varepsilon, \gamma, \pm}(W_{\varepsilon, t_0}) \leq L_m(U_{t_0}) + 1 < -1.$$

Fix $\varepsilon \in (0, \varepsilon_0)$ and define the minimax level

$$C_{\varepsilon, \gamma, \pm} = \inf_{\eta_\varepsilon \in \Phi_\varepsilon} \max_{s \in [0, 1]} \Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon(s)),$$

where $\Phi_\varepsilon = \{\eta_\varepsilon \in C([0, 1], E_\varepsilon) : \eta_\varepsilon(0) = 0, \eta_\varepsilon(1) = W_{\varepsilon, t_0}\}$.

Lemma 3.2. $\lim_{\varepsilon \rightarrow 0} C_{\varepsilon, \gamma, \pm} = C_m$.

Proof. Let $\eta_\varepsilon(s) = W_{\varepsilon, st_0}$, $s \in [0, 1]$, then $\eta_\varepsilon(s) \in \Phi_\varepsilon$. Since $t_0 > 1$, from Lemma 3.1 we have

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} C_{\varepsilon, \gamma, \pm} &\leq \limsup_{\varepsilon \rightarrow 0} \max_{s \in [0, 1]} \Gamma_{\varepsilon, \gamma, \pm}(W_{\varepsilon, st_0}) \\ &= \limsup_{\varepsilon \rightarrow 0} \max_{t \in [0, t_0]} \Gamma_{\varepsilon, \gamma, \pm}(W_{\varepsilon, t}) \\ &\leq \max_{t \in [0, t_0]} L_m(U_t) = C_m. \end{aligned}$$

The result follows if we prove that $\liminf_{\varepsilon \rightarrow 0} C_{\varepsilon, \gamma, \pm} \geq C_m$. By the definition of $C_{\varepsilon, \gamma, \pm}$, for any $\tilde{\varepsilon} > 0$, there exists $\tilde{\eta}_\varepsilon \in \Phi_\varepsilon$ such that

$$\max_{s \in [0, 1]} \Gamma_{\varepsilon, \gamma, \pm}(\tilde{\eta}_\varepsilon(s)) < C_{\varepsilon, \gamma, \pm} + \tilde{\varepsilon}. \quad (3.9)$$

Since $P_{\varepsilon, \gamma, \pm}(\tilde{\eta}_\varepsilon(0)) = 0$ and $P_{\varepsilon, \gamma, \pm}(\tilde{\eta}_\varepsilon(1)) \leq \Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon(1)) = \Gamma_{\varepsilon, \gamma, +}(W_{\varepsilon, t_0}) < -1$, there exists $s_0 \in (0, 1)$ such that

$$P_{\varepsilon, \gamma, \pm}(\tilde{\eta}_\varepsilon(s_0)) = -1 \quad \text{and} \quad P_{\varepsilon, \gamma, \pm}(\tilde{\eta}_\varepsilon(s)) \geq -1, \quad s \in [0, s_0].$$

Then we have

$$Q_\varepsilon(\tilde{\eta}_\varepsilon(s)) \leq \Gamma_{\varepsilon, \gamma, \pm}(\tilde{\eta}_\varepsilon(s)) + 1 < C_{\varepsilon, \gamma, \pm} + \tilde{\varepsilon} + 1, \quad s \in [0, s_0].$$

By Lemma 2.2-(iii) we also have

$$\begin{aligned} \int_{\mathbb{R}^N \setminus \mathcal{O}_\varepsilon} |G_{\varepsilon, \gamma, \pm}^{-1}(\tilde{\eta}_\varepsilon(s))|^2 dx &\leq C \int_{\mathbb{R}^N \setminus \mathcal{O}_\varepsilon} |\tilde{\eta}_\varepsilon(s)|^2 dx \\ &\leq C\varepsilon \left[\sqrt{Q_\varepsilon(\tilde{\eta}_\varepsilon(s))} + 1 \right] \\ &\leq C\varepsilon \left[\sqrt{C_{\varepsilon, \gamma, \pm} + \tilde{\varepsilon} + 1} + 1 \right], \quad s \in [0, s_0]. \end{aligned}$$

Let us estimate $P_{\varepsilon, \gamma, +}$ and $P_{\varepsilon, \gamma, -}$ separately:

(i) Since $g_{\varepsilon, \gamma, +}^2(t) \geq 1$, we get

$$\begin{aligned} &P_{\varepsilon, \gamma, +}(\tilde{\eta}_\varepsilon(s)) \\ &\geq L_m(G_{\varepsilon, \gamma, +}^{-1}(\tilde{\eta}_\varepsilon(s))) - \frac{1}{2}m \int_{\mathbb{R}^N \setminus \mathcal{O}_\varepsilon} |G_{\varepsilon, \gamma, +}^{-1}(\tilde{\eta}_\varepsilon(s))|^2 dx \\ &\geq L_m(G_{\varepsilon, \gamma, +}^{-1}(\tilde{\eta}_\varepsilon(s))) - \frac{1}{2}m\varepsilon C \left[\sqrt{C_{\varepsilon, \gamma, +} + \tilde{\varepsilon} + 1} + 1 \right], \quad s \in [0, s_0], \end{aligned} \quad (3.10)$$

which yields

$$L_m(G_{\varepsilon,\gamma,+}^{-1}(\tilde{\eta}_\varepsilon(s_0))) \leq \frac{1}{2}m\varepsilon C \left[\sqrt{C_{\varepsilon,\gamma,+} + \tilde{\varepsilon} + 1 + 1} \right] - 1 < 0,$$

for small $\varepsilon > 0$.

Thus, $G_{\varepsilon,\gamma,+}^{-1}(\tilde{\eta}_\varepsilon(ts_0)) \in \Phi$ and $\max_{t \in [0,1]} L_m(G_{\varepsilon,\gamma,+}^{-1}(\tilde{\eta}_\varepsilon(ts_0))) \geq C_m$. So, by (3.9) and (3.10), we get

$$\begin{aligned} C_{\varepsilon,\gamma,+} + \tilde{\varepsilon} &> \max_{s \in [0,s_0]} \Gamma_{\varepsilon,\gamma,+}(\tilde{\eta}_\varepsilon(s)) \\ &\geq C_m - \frac{1}{2}m\varepsilon C \left[\sqrt{C_{\varepsilon,\gamma,+} + \tilde{\varepsilon} + 1 + 1} \right], \end{aligned}$$

which yields $\liminf_{\varepsilon \rightarrow 0} C_{\varepsilon,\gamma,+} \geq C_m$ since $\tilde{\varepsilon}$ is arbitrary.

(ii) For simplicity, let $u_s = G_{\varepsilon,\gamma,-}^{-1}(\tilde{\eta}_\varepsilon(s))$, $\omega_s(x) = u_s(\sqrt{1 - 4\sqrt{\varepsilon\gamma}}x)$, to have

$$\begin{aligned} &P_{\varepsilon,\gamma,-}(\tilde{\eta}_\varepsilon(s)) \\ &\geq \frac{1}{2} \int_{\mathbb{R}^N} (g_{\varepsilon,\gamma,-}^2(u_s) |\nabla u_s|^2 + m u_s^2) dx - \int_{\mathbb{R}^N} H(u_s) dx - \frac{1}{2} m \int_{\mathbb{R}^N \setminus \mathcal{O}_\varepsilon} u_s^2 dx \\ &\geq (1 - 4\sqrt{\varepsilon\gamma}) \left[\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_s|^2 + \frac{1}{1 - 4\sqrt{\varepsilon\gamma}} m u_s^2) dx - \frac{1}{1 - 4\sqrt{\varepsilon\gamma}} \int_{\mathbb{R}^N} H(u_s) dx \right] \\ &\quad - \frac{1}{2} m \int_{\mathbb{R}^N \setminus \mathcal{O}_\varepsilon} u_s^2 dx \tag{3.11} \\ &\geq (1 - 4\sqrt{\varepsilon\gamma})^{-\frac{N}{2}} \left[\int_{\mathbb{R}^N} \frac{1}{2} (|\nabla \omega_s|^2 + m \omega_s^2) dx - \int_{\mathbb{R}^N} H(\omega_s) dx \right] \\ &\quad - \frac{1}{2} m \int_{\mathbb{R}^N \setminus \mathcal{O}_\varepsilon} u_s^2 dx \\ &\geq (1 - 4\sqrt{\varepsilon\gamma})^{-\frac{N}{2}} L_m(\omega_s) - \frac{1}{2} m \varepsilon C \left(\sqrt{C_{\varepsilon,\gamma,-} + \tilde{\varepsilon} + 1 + 1} \right), \quad s \in [0, s_0]. \end{aligned}$$

Hence

$$L_m(\omega_{s_0}) \leq (1 - 4\sqrt{\varepsilon\gamma})^{\frac{N}{2}} \left[\frac{1}{2} m \varepsilon C \left(\sqrt{C_{\varepsilon,\gamma,-} + \tilde{\varepsilon} + 1 + 1} \right) - 1 \right] < 0,$$

which implies $\omega_{ts_0}(x) \in \Phi$ and $\max_{t \in [0,1]} L_m(\omega_{ts_0}) \geq C_m$. So, by (3.9) and (3.11), we get

$$\begin{aligned} C_{\varepsilon,\gamma,-} + \tilde{\varepsilon} &> \max_{s \in [0,s_0]} \Gamma_{\varepsilon,\gamma,-}(\tilde{\eta}_\varepsilon(s)) \\ &\geq (1 - 4\sqrt{\varepsilon\gamma})^{-\frac{N}{2}} C_m - \frac{1}{2} m \varepsilon C \left[\sqrt{C_{\varepsilon,\gamma,-} + \tilde{\varepsilon} + 1 + 1} \right], \end{aligned}$$

which yields $\liminf_{\varepsilon \rightarrow 0} C_{\varepsilon,\gamma,-} \geq C_m$.

Combine (i) and (ii) to have $\liminf_{\varepsilon \rightarrow 0} C_{\varepsilon,\gamma,\pm} \geq C_m$ which completes the proof. $\square$

Remark 3.3. Let $D_{\varepsilon,\gamma,\pm} = \max_{s \in [0,1]} \Gamma_{\varepsilon,\gamma,\pm}(w_{\varepsilon,st_0})$; from Lemma 3.2 one has $\lim_{\varepsilon \rightarrow 0} D_{\varepsilon,\gamma,\pm} = C_m$.

Next we borrow some ideas from Byeon and Jeanjean [10] in order to prove the existence of critical points for the modified problem. This method has been used to deal with the case $\gamma = 0$ in [16]. Here the situation is quite different as the limit equation is different from the case $\gamma = 0$ and we set the problem in a Sobolev space setting instead of the Orlicz space setting used in [16].

We set the problem in the space $E_\varepsilon^R := H_0^1(B_{R/\varepsilon}(0))$ endowed with the norm

$$\|v\|_{\varepsilon,R} = \left[\int_{B_{R/\varepsilon}(0)} (|\nabla v|^2 + V_\varepsilon(x)v^2) dx \right]^{\frac{1}{2}}.$$

Note that any $v \in E_\varepsilon^R$ can be regarded as an element in E_ε by defining $v = 0$ on $\mathbb{R}^N \setminus B_{R/\varepsilon}(0)$, so that we may assume the two norms $\|\cdot\|_\varepsilon$ and $\|\cdot\|_{\varepsilon,R}$ coincide.

Define the level set

$$\Gamma_{\varepsilon,\gamma,\pm}^c := \{u \in E_\varepsilon^R : \Gamma_{\varepsilon,\gamma,\pm} \leq c\}$$

and

$$A^d := \{u \in E_\varepsilon^R : \inf_{v \in A} \|u - v\|_{\varepsilon,R} < d\}, \quad d > 0.$$

Let $v_n \in X_{\varepsilon_n}^d \cap E_{\varepsilon_n}^{R_n}$ with $\varepsilon_n \rightarrow 0$ and $R_n \rightarrow +\infty$ such that

$$\lim_{n \rightarrow \infty} \Gamma_{\varepsilon_n,\gamma,\pm}(v_n) \leq C_m \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\Gamma'_{\varepsilon_n,\gamma,\pm}(v_n)\|_{(E_{\varepsilon_n}^{R_n})'} = 0.$$

From the definition of $X_{\varepsilon_n}^d$, we can find $\{U_n\} \subset S_m$ and a sequence of points $\{y_n\} \subset \mathcal{M}^\beta$ such that

$$\left\| v_n - \varphi_{\varepsilon_n} \left(\cdot - \frac{y_n}{\varepsilon_n} \right) U_n \left(\cdot - \frac{y_n}{\varepsilon_n} \right) \right\|_{\varepsilon_n, R_n} \leq d. \quad (3.12)$$

Since S_m and $\mathcal{M}^\beta$ are compact, there exist $U \in S_m$ and $y_0 \in \mathcal{M}^\beta$ such that $U_n \rightarrow U$ in $H^1(\mathbb{R}^N)$, $U_n(x) \rightarrow U(x)$ a.e. in $\mathbb{R}^N$ and $y_n \rightarrow y_0$. It is easy to check that for $d \in (0, d_0)$ with $d_0 > 0$ sufficiently small, we have

$$\left\| v_n - \varphi_{\varepsilon_n} \left(\cdot - \frac{y_n}{\varepsilon_n} \right) U \left(\cdot - \frac{y_n}{\varepsilon_n} \right) \right\|_{\varepsilon_n, R_n} \leq 2d, \quad (3.13)$$

provided n is large enough and in particular $\{v_n\}$ stays bounded in E_ε .

Lemma 3.4.

$$\lim_{n \rightarrow \infty} \sup_{z \in \{z \in \mathbb{R}^N : \beta/2 \leq |\varepsilon_n z - y_n| \leq 3\beta\}} \int_{B_R(z)} v_n^2 dx = 0, \quad \forall R > 0.$$

Proof. Suppose by contradiction that there exist $R > 0$ and a sequence $\{z_n\} \subset \{z \in \mathbb{R}^N : \frac{1}{2}\beta \leq |\varepsilon_n z - y_n| \leq 3\beta\}$ such that

$$\lim_{n \rightarrow \infty} \int_{B_R(z_n)} v_n^2 dx > 0. \quad (3.14)$$

Let $\varepsilon_n z_n \rightarrow z_0 \in \{z \in \mathbb{R}^N : \frac{1}{2}\beta \leq |z - y_0| \leq 3\beta\}$, as $n \rightarrow \infty$. Set $\tilde{v}_n := v_n(\cdot + z_n)$ and, up to a subsequence, we may assume $\tilde{v}_n \rightharpoonup \tilde{v}$ in $H^1(\mathbb{R}^N)$ and $\tilde{v}_n \rightarrow \tilde{v}$ in $L^p_{loc}(\mathbb{R}^N)$, $p \in [2, 2^*)$. Then, (3.14) yields

$$\int_{B_R(0)} |\tilde{v}|^2 dx = \lim_{n \rightarrow \infty} \int_{B_R(0)} \tilde{v}_n^2 dx > 0, \quad (3.15)$$

which implies $\tilde{v} \not\equiv 0$.

From $\lim_{n \rightarrow \infty} \|\Gamma'_{\varepsilon_n, \gamma, \pm}(v_n)\|_{(E_{\varepsilon_n}^{R_n})'} = 0$, we have $\langle \Gamma'_{\varepsilon_n, \gamma, \pm}(v_n), \phi \rangle = o_n(1)\|\phi\|$ for any $\phi \in C_0^\infty(\mathbb{R}^N)$. Hence

$$\begin{aligned} & \int_{\mathbb{R}^N} \nabla \tilde{v}_n \nabla \phi dx \\ & + \int_{\mathbb{R}^N} V_{\varepsilon_n}(x + z_n) \frac{G_{\varepsilon_n, \gamma, \pm}^{-1}(\tilde{v}_n)}{g_{\varepsilon_n, \gamma, \pm}(G_{\varepsilon_n, \gamma, \pm}^{-1}(\tilde{v}_n))} \phi - \frac{h(G_{\varepsilon_n, \gamma, \pm}^{-1}(\tilde{v}_n))}{g_{\varepsilon_n, \gamma, \pm}(G_{\varepsilon_n, \gamma, \pm}^{-1}(\tilde{v}_n))} \phi dx \\ & - 4 \left(\int_{\mathbb{R}^N} \chi_{\varepsilon_n} v_n^2 dx - 1 \right) \int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x + z_n) \tilde{v}_n \phi dx = o_n(1)\|\phi\|. \end{aligned} \quad (3.16)$$

Clearly,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x + z_n) \tilde{v}_n \phi dx = 0.$$

Now let us prove

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V_{\varepsilon_n}(x + z_n) \frac{G_{\varepsilon_n, \gamma, \pm}^{-1}(\tilde{v}_n)}{g_{\varepsilon_n, \gamma, \pm}(G_{\varepsilon_n, \gamma, \pm}^{-1}(\tilde{v}_n))} \phi dx = V(z_0) \int_{\mathbb{R}^N} \tilde{v} \phi dx \quad (3.17)$$

and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{h(G_{\varepsilon_n, \gamma, \pm}^{-1}(\tilde{v}_n))}{g_{\varepsilon_n, \gamma, \pm}(G_{\varepsilon_n, \gamma, \pm}^{-1}(\tilde{v}_n))} \phi dx = \int_{\mathbb{R}^N} h(\tilde{v}) \phi dx. \quad (3.18)$$

Indeed, by Lemma 2.2-(iv), one has $G_{\varepsilon_n, \gamma, \pm}^{-1}(\tilde{v}_n) - \tilde{v}_n \rightarrow 0$, a.e. in $\mathbb{R}^N$. Since $\tilde{v}_n \rightarrow \tilde{v}$, a.e. in $\mathbb{R}^N$, we get $G_{\varepsilon_n, \gamma, \pm}^{-1}(\tilde{v}_n) \rightarrow \tilde{v}$, a.e. in $\mathbb{R}^N$. Moreover, $g_{\varepsilon_n, \gamma, \pm}(G_{\varepsilon_n, \gamma, \pm}^{-1}(\tilde{v}_n)) \rightarrow 1$, a.e. in $\mathbb{R}^N$. Thus, (3.17) and (3.18) are consequences of the Lebesgue dominated convergence theorem.

Combine (3.16)–(3.18) to obtain

$$\int_{\mathbb{R}^N} [\nabla \tilde{v} \nabla \phi + V(z_0) \tilde{v} \phi - h(\tilde{v}) \phi] dx = 0, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N),$$

so that $\tilde{v}$ turns out to be a nontrivial solution to the following equation,

$$-\Delta \tilde{v} + V(z_0) \tilde{v} = h(\tilde{v}), \quad x \in \mathbb{R}^N.$$

By the maximum principle $\tilde{v} > 0$. From $V(z_0) \geq m$, we have $C_{V(z_0)} \geq C_m$.

Next choose $R > 0$ sufficiently large and apply Pohozaev's identity to get, on the one hand,

$$2 \lim_{n \rightarrow \infty} \int_{B_R(z_n)} |\nabla v_n|^2 dx \geq \int_{\mathbb{R}^N} |\nabla \tilde{v}|^2 dx = NL_{V(z_0)}(\tilde{v}) \geq NC_{V(z_0)} \geq NC_m,$$

on the other hand, for large n , it follows from (3.13) that

$$\begin{aligned} \int_{B_R(z_n)} |\nabla v_n|^2 dx &\leq 4d^2 + 2\varepsilon_n^2 \int_{B_R(z_n)} |\nabla \varphi_{\varepsilon_n}(x - \frac{y_n}{\varepsilon_n}) U(x - \frac{y_n}{\varepsilon_n})|^2 dx \\ &\quad + 2 \int_{B_R(z_n)} |\varphi_{\varepsilon_n}(x - \frac{y_n}{\varepsilon_n}) \nabla U(x - \frac{y_n}{\varepsilon_n})|^2 dx \\ &\leq d + C\varepsilon_n^2 + C \int_{B_R(0)} \exp(-c|x + z_n - \frac{y_n}{\varepsilon_n}|) dx. \end{aligned}$$

Note that $\lim_{n \rightarrow \infty} |z_n - \frac{y_n}{\varepsilon_n}| = +\infty$ since $\frac{1}{2\varepsilon_n}\beta \leq |z_n - \frac{y_n}{\varepsilon_n}| \leq \frac{3\beta}{\varepsilon_n}$. Thus, for n large, we get

$$\frac{1}{2}NC_m \leq \lim_{n \rightarrow \infty} \int_{B_R(z_n)} |\nabla v_n|^2 dx \leq 3d,$$

which is a contradiction for small $d > 0$. This completes the proof of Lemma 3.4. $\square$

Next choose $\eta \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \eta \leq 1$, $\eta(z) = 1$ if $z \in \{z \in \mathbb{R}^N : \beta \leq |z| \leq 2\beta\}$, $\eta(z) = 0$ if $z \in \mathbb{R}^N \setminus \{z \in \mathbb{R}^N : \frac{1}{2}\beta \leq |z| \leq 3\beta\}$. Setting $\eta_n(z) = \eta(\varepsilon_n z - y_n)v_n$, clearly, η_n is bounded in $H^1(\mathbb{R}^N)$. Thus, from Lemma 3.7, we have

$$\lim_{n \rightarrow \infty} \sup_{z \in \mathbb{R}^N} \int_{\mathbb{R}^N} |\eta_n|^2 dx = 0. \quad (3.19)$$

This fact, together with Lions' lemma [21], gives $\eta_n \rightarrow 0$ in $L^{p+1}(\mathbb{R}^N)$. So that we obtain

$$\begin{aligned} &\lim_{n \rightarrow \infty} \int_{\{x \in \mathbb{R}^N : \beta \leq |\varepsilon_n x - y_n| \leq 2\beta\}} |v_n|^{p+1} dx \\ &= \lim_{n \rightarrow \infty} \int_{\{x \in \mathbb{R}^N : \beta \leq |\varepsilon_n x - y_n| \leq 2\beta\}} |\eta_n|^{p+1} dx \\ &\leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\eta_n|^{p+1} dx = 0. \end{aligned} \quad (3.20)$$

Set $v_{n,1} = \varphi_{\varepsilon_n}(\cdot - \frac{y_n}{\varepsilon_n})v_n$ and $v_{n,2} = v_n - v_{n,1}$. Then, we can prove the following lemma.

Lemma 3.5. $\Gamma_{\varepsilon_n, \gamma, \pm}(v_n) \geq \Gamma_{\varepsilon_n, \gamma, \pm}(v_{n,1}) + \Gamma_{\varepsilon_n, \gamma, \pm}(v_{n,2}) + o_n(1)$.

Proof. Note that $\text{supp}(v_{n,1}) \subset \{z \in \mathbb{R}^N : |\varepsilon_n z - y_n| \leq 2\beta\} \subset \mathcal{M}_\varepsilon^{3\beta} \subset \mathcal{O}_\varepsilon$, thus, $Q_{\varepsilon_n}(v_{n,1}) = 0$ and $Q_{\varepsilon_n}(v_{n,2}) = Q_{\varepsilon_n}(v_n)$. On the other hand, we note that $\text{supp}(v_{n,2}) \subset \{z \in \mathbb{R}^N : |\varepsilon_n z - y_n| \geq \beta\}$. Therefore, by the convexity of $|G_{\varepsilon_n, \gamma, \pm}^{-1}(t)|^2$

and since $G_{\varepsilon_n, \gamma, \pm}^{-1}(0) = 0$, for large n , we get

$$\begin{aligned}
 & \Gamma_{\varepsilon_n, \gamma, \pm}(v_{n,1}) + \Gamma_{\varepsilon_n, \gamma, \pm}(v_{n,2}) \\
 &= \Gamma_{\varepsilon_n, \gamma, \pm}(v_n) + \int_{\mathbb{R}^N} \varphi_{\varepsilon_n}(x - \frac{y_n}{\varepsilon_n}) \left[\varphi_{\varepsilon_n}(x - \frac{y_n}{\varepsilon_n}) - 1 \right] |\nabla v_n|^2 dx \\
 & \quad + \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon_n x) [|G_{\varepsilon_n, \gamma, \pm}^{-1}(v_{n,1})|^2 + |G_{\varepsilon_n, \gamma, \pm}^{-1}(v_{n,2})|^2 - |G_{\varepsilon_n, \gamma, \pm}^{-1}(v_n)|^2] dx \\
 & \quad + \int_{\mathbb{R}^N} [H(G_{\varepsilon_n, \gamma, \pm}^{-1}(v_n)) - H(G_{\varepsilon_n, \gamma, \pm}^{-1}(v_{n,1})) - H(G_{\varepsilon_n, \gamma, \pm}^{-1}(v_{n,2}))] dx + o_n(1) \\
 &\leq \Gamma_{\varepsilon_n, \gamma, \pm}(v_n) + \int_{\mathbb{R}^N} [H(G_{\varepsilon_n, \gamma, \pm}^{-1}(v_n)) \\
 & \quad - H(G_{\varepsilon_n, \gamma, \pm}^{-1}(v_{n,1})) - H(G_{\varepsilon_n, \gamma, \pm}^{-1}(v_{n,2}))] dx + o_n(1) .
 \end{aligned} \tag{3.21}$$

By (h_1) – (h_2) , given any $\sigma > 0$, there exists some $C_\sigma > 0$ such that

$$\begin{aligned}
 & \int_{\mathbb{R}^N} [H(G_{\varepsilon_n, \gamma, \pm}^{-1}(v_n)) - H(G_{\varepsilon_n, \gamma, \pm}^{-1}(v_{n,1})) - H(G_{\varepsilon_n, \gamma, \pm}^{-1}(v_{n,2}))] dx \\
 & \leq \sigma \int_{\mathbb{R}^N} v_n^2 dx + C_\sigma \int_{\{x \in \mathbb{R}^N : \beta \leq |\varepsilon_n x - y_n| \leq 2\beta\}} |v_n|^{p+1} dx .
 \end{aligned} \tag{3.22}$$

Since σ is arbitrary, by (3.21), (3.22) and (3.20) we obtain the result. $\square$

Lemma 3.6. $\Gamma_{\varepsilon_n, \gamma, \pm}(v_{n,2}) > 0$.

Proof. From (3.13), we get

$$\begin{aligned}
 & \int_{\{x \in \mathbb{R}^N : |\varepsilon_n x - y_n| \geq 2\beta\}} |\nabla v_{n,2}|^2 dx = \int_{\{x \in \mathbb{R}^N : |\varepsilon_n x - y_n| \geq 2\beta\}} |\nabla v_n|^2 dx \\
 &= \int_{\{x \in \mathbb{R}^N : |\varepsilon_n x - y_n| \geq 2\beta\}} \left| \nabla \left[v_n - \varphi_{\varepsilon_n}(\cdot - \frac{y_n}{\varepsilon_n}) U(\cdot - \frac{y_n}{\varepsilon_n}) \right] \right|^2 dx \\
 &\leq \left\| v_n - \varphi_{\varepsilon_n}(\cdot - \frac{y_n}{\varepsilon_n}) U(\cdot - \frac{y_n}{\varepsilon_n}) \right\|_{\varepsilon_n}^2 \leq d^2 .
 \end{aligned} \tag{3.23}$$

Similarly, by (3.13), we get

$$\begin{aligned}
 & \int_{\{x \in \mathbb{R}^N : \beta \leq |\varepsilon_n x - y_n| \leq 2\beta\}} |\nabla v_{n,2}|^2 dx \\
 & \leq C \int_{\{x \in \mathbb{R}^N : \beta \leq |\varepsilon_n x - y_n| \leq 2\beta\}} (|\nabla v_n|^2 + v_n^2) dx \\
 & \leq Cd^2 + \left\| \varphi_{\varepsilon_n}(\cdot - \frac{y_n}{\varepsilon_n}) U(\cdot - \frac{y_n}{\varepsilon_n}) \right\|_{\varepsilon_n}^2 \Big|_{\{x \in \mathbb{R}^N : \beta \leq |\varepsilon_n x - y_n| \leq 2\beta\}} \\
 & \leq Cd^2 + o_n(1) .
 \end{aligned} \tag{3.24}$$

Consequently, as n is large enough, we have $\|\nabla v_{n,2}\|_2 \leq Cd$. For small $d > 0$, using (h_1) – (h_2) again, we get

$$\begin{aligned}
 & \Gamma_{\varepsilon_n, \gamma, \pm}(v_{n,2}) \\
 & \geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_{n,2}|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} V_{\varepsilon_n}(x) |G_{\varepsilon_n, \gamma, \pm}^{-1}(v_{n,2})|^2 dx - C \int_{\mathbb{R}^N} |v_{n,2}|^{2^*} dx \\
 & \geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_{n,2}|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} V_{\varepsilon_n}(x) |G_{\varepsilon_n, \gamma, \pm}^{-1}(v_{n,2})|^2 dx - C \left(\int_{\mathbb{R}^N} |\nabla v_{n,2}|^2 dx \right)^{\frac{2^*}{2}} \\
 & \geq \frac{1}{4} \int_{\mathbb{R}^N} |\nabla v_{n,2}|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} V_{\varepsilon_n}(x) |G_{\varepsilon_n, \gamma, \pm}^{-1}(v_{n,2})|^2 dx > 0.
 \end{aligned} \tag{3.25}$$

This completes the proof of Lemma 3.6. $\square$

Lemma 3.7. *For $d > 0$ sufficiently small, there exist a sequence $\{z_n\} \subset \mathbb{R}^N$, $y_0 \in \mathcal{M}$, and $U_0 \in S_m$ satisfying, up to a subsequence, the following:*

$$\lim_{n \rightarrow \infty} |\varepsilon_n z_n - y_0| = 0, \quad \lim_{n \rightarrow \infty} \|v_n - \varphi_{\varepsilon_n}(\cdot - z_n) U_0(\cdot - z_n)\|_{\varepsilon_n, R_n} = 0.$$

Proof. Let $w_n := v_{n,1}(\cdot + \frac{y_n}{\varepsilon_n})$. Then, by (3.13), we get $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Thus, up to a subsequence if necessary, we may assume $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N)$, $w_n \rightarrow w$ in $L_{loc}^p(\mathbb{R}^N)$, $p \in [2, 2^*)$ and $w_n \rightarrow w$ a.e. in $\mathbb{R}^N$. From (3.13), for given $R > 0$, when n large, we get

$$\begin{aligned}
 4d^2 & \geq \int_{\{x \in \mathbb{R}^N : |\varepsilon_n x - y_n| \leq \beta\}} V_{\varepsilon_n}(x) \left| v_{n,1} - \varphi_{\varepsilon_n}\left(x - \frac{y_n}{\varepsilon_n}\right) U\left(x - \frac{y_n}{\varepsilon_n}\right) \right|^2 dx \\
 & = \int_{\{x \in \mathbb{R}^N : |\varepsilon_n x| \leq \beta\}} V(\varepsilon_n x + y_n) |w_n - U(x)|^2 dx \\
 & \geq m \int_{B_R(0)} |w_n - U(x)|^2 dx.
 \end{aligned}$$

Thus, we have

$$\begin{aligned}
 \int_{B_R(0)} w^2 dx & = \lim_{n \rightarrow \infty} \int_{B_R(0)} w_n^2 dx \\
 & \geq \int_{B_R(0)} |U(x)|^2 dx - \lim_{n \rightarrow \infty} \int_{B_R(0)} |w_n - U(x)|^2 dx \\
 & \geq C - \frac{4}{m} d^2,
 \end{aligned}$$

which yields $w \neq 0$.

Now let $\phi \in C_0^\infty(\mathbb{R}^N)$ and note that $w_n(x) = v_{n,1}(x + \frac{y_n}{\varepsilon_n}) = v_n(x + \frac{y_n}{\varepsilon_n})$ for $x \in \text{supp}(\phi)$ and large n . Moreover, $\text{supp}(w_n(x)) \subset \{x \in \mathbb{R}^N : |\varepsilon_n x| \leq 2\beta\} \subset \mathcal{O}$. Thus, from $\langle \Gamma'_{\varepsilon_n, \gamma, \pm}(v_n), \phi \rangle = o_n(1) \|\phi\|$ and analogous to the proof of (3.17) and (3.18), w is a nontrivial solution of equation

$$-\Delta w + V(y_0)w = h(w), \quad x \in \mathbb{R}^N. \tag{3.26}$$

By the maximum principle, $w > 0$.

Now we *claim*:

$$\lim_{n \rightarrow \infty} \sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |w_n - w|^2 dx = 0. \quad (3.27)$$

In fact, if by contradiction (3.27) does not occur, then there exists a sequence $\{z_n\} \subset \mathbb{R}^N$ with $|z_n| \rightarrow +\infty$ such that

$$\lim_{n \rightarrow \infty} \int_{B_1(z_n)} |w_n - w|^2 dx > 0.$$

Thus, we have

$$\lim_{n \rightarrow \infty} \int_{B_1(z_n)} |w|^2 dx = 0, \quad \lim_{n \rightarrow \infty} \int_{B_1(z_n)} |w_n|^2 dx > 0.$$

We have $|\varepsilon_n z_n - y_n| \leq \frac{1}{2}\beta$. In fact, if $|\varepsilon_n z_n - y_n| \geq \frac{1}{2}\beta$, by (3.14), we have

$$0 < \lim_{n \rightarrow \infty} \int_{B_1(z_n)} |w_n|^2 dx \leq \lim_{n \rightarrow \infty} \sup_{z \in \{z \in \mathbb{R}^N : \frac{1}{2}\beta \leq |\varepsilon_n z - y_n| \leq 3\beta\}} \int_{B_1(z)} |v_n|^2 dx = 0,$$

which is impossible. Thus, up to a subsequence, we may assume that $\varepsilon_n z_n \rightarrow z_0 \in \{z \in \mathbb{R}^N : |z| \leq 2\beta\}$. Assume that $v_{n,1}(\cdot + z_n + \frac{y_n}{\varepsilon_n}) \rightharpoonup v_1$ in $H^1(\mathbb{R}^N)$. Analogously to the proof of (3.26), we have

$$-\Delta v_1 + V(z_0 + y_0)v_1 = h(v_1), \quad x \in \mathbb{R}^N.$$

By the maximum principle, $v_1 > 0$.

Thus, for large R , we get

$$\begin{aligned} \frac{1}{2}NC_m &\leq \int_{B_R(0)} |\nabla v_{n,1}(x + z_n + \frac{y_n}{\varepsilon_n})|^2 dx \\ &= \int_{B_R(z_n + \frac{y_n}{\varepsilon_n})} |\nabla v_{n,1}(x)|^2 dx \\ &\leq C\varepsilon_n^2 + C \int_{B_R(z_n + \frac{y_n}{\varepsilon_n})} |\nabla v_n|^2 dx \\ &\leq C\varepsilon_n^2 + Cd + C \int_{B_R(0)} e^{-c|x+z_n|} dx. \end{aligned}$$

We get a contradiction for large n and small d since $|z_n| \rightarrow +\infty$.

Therefore, (3.27) holds. By Lions' Lemma, we have $w_n \rightarrow w$ in $L^{p+1}(\mathbb{R}^N)$. Then, for given $\sigma > 0$, there exists $C_\sigma > 0$ such that

$$\begin{aligned} &\int_{\mathbb{R}^N} |H(G_{\varepsilon_n, \gamma, \pm}^{-1}(w_n)) - H(G_{\varepsilon_n, \gamma, \pm}^{-1}(w))| dx \\ &= \int_{\mathbb{R}^N} \left| \frac{h(G_{\varepsilon_n, \gamma, \pm}^{-1}(\varpi))}{g_{\varepsilon_n, \gamma, \pm}(G_{\varepsilon_n, \gamma, \pm}^{-1}(\varpi))} \right| |w_n - w| dx \\ &\leq \sigma \|w_n - w\|_2^2 + C_\sigma \|w_n - w\|_{p+1}^{p+1}, \end{aligned}$$

where $\varpi(x) = tw_n(x) + (1-t)w(x)$, $t \in (0, 1)$. Thus, by the Lebesgue dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} H(G_{\varepsilon_n, \gamma, \pm}^{-1}(w_n)) dx = \int_{\mathbb{R}^N} H(w) dx.$$

By Fatou's Lemma, Lemmas 3.5, 3.6 and (3.25), up to a subsequence, we have

$$\begin{aligned} C_m &\geq \lim_{n \rightarrow \infty} \Gamma_{\varepsilon_n, \gamma, \pm}(v_n) \\ &\geq \lim_{n \rightarrow \infty} \Gamma_{\varepsilon_n, \gamma, \pm}(v_{n,1}) + \lim_{n \rightarrow \infty} \Gamma_{\varepsilon_n, \gamma, \pm}(v_{n,2}) \\ &\geq \lim_{n \rightarrow \infty} \Gamma_{\varepsilon_n, \gamma, \pm}(v_{n,1}) \\ &= \lim_{n \rightarrow \infty} \left[\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w_n|^2 + V(\varepsilon_n x + y_n) |G_{\varepsilon_n, \gamma, \pm}^{-1}(w_n)|^2) dx \right. \\ &\quad \left. - \int_{\mathbb{R}^N} H(G_{\varepsilon_n, \gamma, \pm}^{-1}(w_n)) dx \right] \\ &\geq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w|^2 + V(y_0) w^2) dx - \int_{\mathbb{R}^N} H(w) dx \\ &\geq C_{V(y_0)} \\ &\geq C_m. \end{aligned} \tag{3.28}$$

Hence, we get $\lim_{n \rightarrow \infty} \Gamma_{\varepsilon_n, \gamma, \pm}(v_{n,1}) = C_m$. Moreover, we get $V(y_0) = m$ and as a consequence we see that w is a ground state to (1.5). Thus, there exists some $z \in \mathbb{R}^N$ such that $U_0 := w(\cdot + z) \in S_m$. By (3.28), we have

$$\begin{aligned} &\lim_{n \rightarrow \infty} \left[\int_{\mathbb{R}^N} |\nabla w_n|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) |G_{\varepsilon_n, \gamma, \pm}^{-1}(w_n)|^2 dx \right] \\ &= \left[\int_{\mathbb{R}^N} |\nabla w|^2 dx + m \int_{\mathbb{R}^N} w^2 dx \right]. \end{aligned}$$

This together Fatou's Lemma again give

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla w_n|^2 dx = \int_{\mathbb{R}^N} |\nabla w|^2 dx, \tag{3.29}$$

and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) |G_{\varepsilon_n, \gamma, \pm}^{-1}(w_n)|^2 dx = m \int_{\mathbb{R}^N} w^2 dx. \tag{3.30}$$

Moreover, for any set $A \subset \mathbb{R}^N$, we get

$$\lim_{n \rightarrow \infty} \int_A V(\varepsilon_n x + y_n) |G_{\varepsilon_n, \gamma, \pm}^{-1}(w_n)|^2 dx = m \int_A w^2 dx. \tag{3.31}$$

By (3.29), we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left\| \nabla \left[v_{n,1} - \varphi_{\varepsilon_n} \left(\cdot - z - \frac{y_n}{\varepsilon_n} \right) w \left(\cdot - \frac{y_n}{\varepsilon_n} \right) \right] \right\|_2 \\ &= \lim_{n \rightarrow \infty} \left\| \nabla [w_n - \varphi_{\varepsilon_n}(\cdot - z)w] \right\|_2 \\ &\leq \lim_{n \rightarrow \infty} \left\| \nabla (w_n - w) \right\|_2 + \lim_{n \rightarrow \infty} \left\| \nabla [(1 - \varphi_{\varepsilon_n}(\cdot - z))w] \right\|_2 = 0. \end{aligned}$$

According to (3.30) and (3.31), buying the line of Proposition 2.1-(3) in [24], for any $\sigma > 0$, we can take a bounded set $A_\sigma \subset \mathbb{R}^N$ and $n_0 \in \mathbb{N}$ such that $|G_{\varepsilon_n, \gamma, \pm}^{-1}(t)|^2$ is convex and for $n > n_0$,

$$\int_{\mathbb{R}^N \setminus A_\sigma} V(\varepsilon_n x + y_n) [|G_{\varepsilon_n, \gamma, \pm}^{-1}(\varphi_{\varepsilon_n}(x - z)w)|^2 + |G_{\varepsilon_n, \gamma, \pm}^{-1}(w_n)|^2] dx \leq \frac{1}{2}\sigma.$$

Thus, we have

$$\int_{\mathbb{R}^N \setminus A_\sigma} V(\varepsilon_n x + y_n) |G_{\varepsilon_n, \gamma, \pm}^{-1}(w_n - \varphi_{\varepsilon_n}(x - z)w)|^2 dx \leq C\sigma, \quad n > n_0.$$

Thus, for small $\delta > 0$, we get

$$\begin{aligned} & \int_{\mathbb{R}^N \setminus A_\sigma} V(\varepsilon_n x + y_n) [|G_{\varepsilon_n, \gamma, \pm}^{-1}(w_n - \varphi_{\varepsilon_n}(x - z)w)|^2 \\ & \quad - |w_n - \varphi_{\varepsilon_n}(x - z)w|^2] dx \\ & \leq \int_{\{x \in \mathbb{R}^N \setminus A_\sigma : 0 < |(w_n - w)(x)| \leq \delta\}} V(\varepsilon_n x + y_n) |w_n - \varphi_{\varepsilon_n}(x - z)w|^2 \\ & \quad \cdot \left| \frac{|G_{\varepsilon_n, \gamma, \pm}^{-1}(w_n - \varphi_{\varepsilon_n}(x - z)w)|^2}{|w_n - \varphi_{\varepsilon_n}(x - z)w|^2} - 1 \right| dx \\ & \quad + C \int_{\{x \in \mathbb{R}^N \setminus A_\sigma : \delta \leq |(w_n - \varphi_{\varepsilon_n}(x - z)w)(x)|\}} |w_n - \varphi_{\varepsilon_n}(x - z)w|^{p+1} dx \\ & \leq C\sigma \|w_n - \varphi_{\varepsilon_n}(x - z)w\|_2^2 + C \|w_n - \varphi_{\varepsilon_n}(x - z)w\|_{p+1}^{p+1}. \end{aligned} \tag{3.32}$$

On the other hand, since $w_n \rightarrow w$ in $L_{loc}^2(\mathbb{R}^N)$, we have

$$\lim_{n \rightarrow \infty} \int_{A_\sigma} V(\varepsilon_n x + y_n) |w_n - \varphi_{\varepsilon_n}(x - z)w|^2 dx = 0. \tag{3.33}$$

Combining (3.32) and (3.33), we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(\varepsilon_n x) \left[v_{n,1} - \varphi_{\varepsilon_n} \left(x - z - \frac{y_n}{\varepsilon_n} \right) w \left(x - \frac{y_n}{\varepsilon_n} \right) \right]^2 dx = 0.$$

Let $z_n = z + \frac{y_n}{\varepsilon_n}$, then

$$\|v_{n,1} - \varphi_{\varepsilon_n}(\cdot - z_n)U_0(\cdot - z_n)\|_{\varepsilon_n, R_n} \rightarrow 0.$$

Finally, using (3.28) again,

$$0 = \lim_{n \rightarrow \infty} \Gamma_{\varepsilon_n, \gamma, \pm}(v_{n,2}) \geq \frac{1}{4} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [|\nabla v_{n,2}|^2 + V(\varepsilon_n x) |G_{\varepsilon_n, \gamma, \pm}^{-1}(v_{n,2})|^2] dx,$$

which yields $\lim_{n \rightarrow \infty} \|v_{n,2}\|_{\varepsilon_n, R_n} = 0$ by Lemma 2.2-(iii). The result follows. $\square$

Let $d \in (0, d_0)$ such that Lemma 3.7 holds.

Lemma 3.8. *For any $d \in (0, d_0)$, there exist positive constants δ_d , R_d and ε_d such that*

$$\|\Gamma'_{\varepsilon, \gamma, \pm}(v)\|_{(E_\varepsilon^R)'} \geq \delta_d$$

for any $v \in E_\varepsilon^R \cap \Gamma_{\varepsilon, \gamma, \pm}^{D_{\varepsilon, \gamma, \pm}} \cap (X_\varepsilon^{d_0} \setminus X_\varepsilon^d)$, $R \geq R_d$, and $\varepsilon \in (0, \varepsilon_d)$.

Proof. By contradiction, we assume that for some $d \in (0, d_0)$, there exists $\varepsilon_n < \frac{1}{n}$, $R_n > n$ and $v_n \in E_{\varepsilon_n}^{R_n} \cap \Gamma_{\varepsilon_n, \gamma, \pm}^{D_{\varepsilon_n, \gamma, \pm}} \cap (X_{\varepsilon_n}^{d_0} \setminus X_{\varepsilon_n}^d)$ such that $\|\Gamma'_{\varepsilon_n, \gamma, \pm}(v_n)\|_{(E_{\varepsilon_n}^{R_n})'} < \frac{1}{n}$. By Lemma 3.7, there exist a sequence $\{z_n\} \subset \mathbb{R}^N$, $y_0 \in \mathcal{M}$ and $U_0 \in S_m$ satisfying

$$\lim_{n \rightarrow \infty} |\varepsilon_n z_n - y_0| = 0, \quad \lim_{n \rightarrow \infty} \|v_n - \varphi_{\varepsilon_n}(\cdot - z_n)U_0(\cdot - z_n)\|_{\varepsilon_n, R_n} = 0$$

up to a subsequence. Thus, for large n , $\varepsilon_n z_n \in \mathcal{M}^\beta$, $\varphi_{\varepsilon_n}(\cdot - z_n)U_0(\cdot - z_n) \in X_{\varepsilon_n}$ and $v_n \in X_{\varepsilon_n}^d$, which contradicts the fact $v_n \in X_{\varepsilon_n}^{d_0} \setminus X_{\varepsilon_n}^d$. $\square$

Lemma 3.9. *For any given $\nu > 0$, there exist small positive constants ε_1 and d such that $\Gamma_{\varepsilon, \gamma, \pm}(v) > C_m - \nu$ for any $v \in X_\varepsilon^d$ and $\varepsilon \in (0, \varepsilon_1)$.*

Proof. See [15]. $\square$

Lemma 3.10. *For sufficiently small $\varepsilon > 0$ and large R , there exists a sequence $\{v_{\varepsilon, n}^R\} \subset E_\varepsilon^R \cap \Gamma_{\varepsilon, \gamma, \pm}^{D_{\varepsilon, \gamma, \pm}} \cap X_\varepsilon^{d_0}$ such that $\|\Gamma'_{\varepsilon, \gamma, \pm}(v_{\varepsilon, n}^R)\|_{(E_\varepsilon^R)'} \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. The proof follows [15, 16]. However, for convenience the reader, we give a detailed proof. By contradiction, for small $\varepsilon > 0$ and $R > R_0$, there exists $C(\varepsilon, R) > 0$ such that

$$\|\Gamma'_{\varepsilon, \gamma, \pm}(v)\|_{(E_\varepsilon^R)'} \geq C(\varepsilon, R), \quad v \in E_\varepsilon^R \cap \Gamma_{\varepsilon, \gamma, \pm}^{D_{\varepsilon, \gamma, \pm}} \cap X_\varepsilon^{d_0}.$$

On the other hand, by Lemma 3.8, there exists $\delta > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ and $R > R_0$ such that

$$\|\Gamma'_{\varepsilon, \gamma, \pm}(v)\|_{(E_\varepsilon^R)'} \geq \delta, \quad v \in E_\varepsilon^R \cap \Gamma_{\varepsilon, \gamma, \pm}^{D_{\varepsilon, \gamma, \pm}} \cap (X_\varepsilon^{d_0} \setminus X_\varepsilon^{d_1}).$$

Thus, there exists a pseudo-gradient vector field Υ_ε^R on neighborhood $N_\varepsilon^R \subset E_\varepsilon^R$ of $E_\varepsilon^R \cap \Gamma_{\varepsilon, \gamma, \pm}^{D_{\varepsilon, \gamma, \pm}} \cap (X_\varepsilon^{d_0} \setminus X_\varepsilon^{d_1})$. Let $\tilde{N}_\varepsilon^R \subset N_\varepsilon^R$ such that

$$\|\Gamma'_{\varepsilon, \gamma, \pm}(v)\|_{(E_\varepsilon^R)'} \geq \frac{1}{2}C(\varepsilon, R), \quad v \in \tilde{N}_\varepsilon^R.$$

We choose two positive Lipschitz continuous functions ζ_ε^R and ξ satisfying

$$\zeta_\varepsilon^R(v) = 1, \quad \text{if } v \in \Gamma_{\varepsilon, \gamma, \pm}^{D_{\varepsilon, \gamma, \pm}} \cap X_\varepsilon^{d_0} \quad \text{and} \quad \zeta_\varepsilon^R(v) = 0, \quad \text{if } v \in E_\varepsilon^R \setminus \tilde{N}_\varepsilon^R,$$

$0 \leq \zeta_\varepsilon^R \leq 1$, and

$$\xi \leq 1, \quad \xi(a) = 1, \quad \text{if } |a - C_m| \leq \frac{1}{2}C_m \quad \text{and} \quad \xi(a) = 0, \quad \text{if } |a - C_m| \geq C_m.$$

Now, we define

$$\Psi_\varepsilon^R = \begin{cases} -\zeta_\varepsilon^R(v)\xi(\Gamma_{\varepsilon, \gamma, \pm}(v))\Upsilon_\varepsilon^R, & \text{if } v \in N_\varepsilon^R, \\ 0, & \text{if } v \notin E_\varepsilon \setminus N_\varepsilon^R. \end{cases} \quad (3.34)$$

Then the initial value problem

$$\begin{cases} \frac{d}{dt} F_\varepsilon^R(v, t) = \Psi_\varepsilon^R(F_\varepsilon^R(v, t)), \\ F_\varepsilon^R(v, 0) = v \end{cases} \quad (3.35)$$

yields a unique global solution $F_\varepsilon^R : E_\varepsilon^R \times \mathbb{R} \rightarrow E_\varepsilon^R$. For the properties of F_ε^R , we refer to [16]. Let $\eta_\varepsilon(s) = W_{\varepsilon, st_0} = \varphi_\varepsilon U_{st_0}$, $s \in [0, 1]$ as before. Then, for the small $d_1 > 0$, there exists some $\mu > 0$ such that if $|st_0 - 1| \leq \mu$, we have

$$\|\eta_\varepsilon(s) - \varphi_\varepsilon U\|_\varepsilon = \|\varphi_\varepsilon(U_{st_0} - U)\|_\varepsilon \leq C\|U_{st_0} - U\|_\varepsilon \leq d_1,$$

which implies that $\eta_\varepsilon(s) \in X_\varepsilon^{d_1}$ since $0 \in \mathcal{M}$. On the other hand, if $|st_0 - 1| \geq \mu$, since $t = 1$ is the unique maximum point of $L_m(U_t)$ and $\max_{t \geq 0} L_m(U_t) = L_m(U_1) = C_m$, there exists $\rho > 0$ such that $L_m(U_{st_0}) < C_m - 2\rho$ for $|st_0 - 1| \geq \mu$. From Lemma 3.1, there exists $\varepsilon_1 > 0$ such that

$$\max_{s \in (0, 1]} |\Gamma_{\varepsilon, \gamma, \pm}(W_{\varepsilon, st_0}) - L_m(U_{st_0})| < \rho, \quad \varepsilon \in (0, \varepsilon_1).$$

So, for $|st_0 - 1| \geq \mu$, we have

$$\Gamma_{\varepsilon, \gamma, \pm}(W_{\varepsilon, st_0}) \leq |\Gamma_{\varepsilon, \gamma, \pm}(W_{\varepsilon, st_0}) - L_m(U_{st_0})| + L_m(U_{st_0}) < C_m - \rho, \quad \varepsilon \in (0, \varepsilon_1).$$

Define $\eta_\varepsilon^R(s, t) := F_\varepsilon^R(\eta_\varepsilon(s), t)$, $(s, t) \in [0, 1] \times \mathbb{R}$. Since $\Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon(0))$, $\Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon(1)) \notin (0, 2C_m)$, we get $\eta_\varepsilon^R(s, t) \in \Phi_\varepsilon$ for any $t > 0$.

If $|st_0 - 1| \geq \mu$, we have

$$\Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon^R(s, t)) \leq \Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon(s)) < C_m - \rho,$$

which is a contradiction.

If $|st_0 - 1| \leq \mu$, we get $\eta_\varepsilon(s) \in X_\varepsilon^{d_1}$. In this case, one has (a) $\eta_\varepsilon^R(s, t) \in X_\varepsilon^{d_0}$ for all $t > 0$ or (b) there exists some $t_s > 0$ such that $\eta_\varepsilon^R(s, t) \notin X_\varepsilon^{d_0}$. If (a) holds, we have

$$\begin{aligned} \Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon^R(s, t)) &= \Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon(s)) + \int_0^t \frac{d}{dt} \Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon^R(s, \tau)) d\tau \\ &\leq D_{\varepsilon, \gamma, \pm} - \min\{\delta^2, C(\varepsilon, R)^2\}t. \end{aligned}$$

Thus, $\lim_{t \rightarrow +\infty} \Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon^R(s, t)) = -\infty$, which contradicts to Lemma 3.9. So, (b) holds.

For any fixed s with $|st_0 - 1| \leq \mu$, we find $t_s^1, t_s^2 > 0$ such that $\eta_\varepsilon^R(s, t) \in X_\varepsilon^{d_0} \setminus \tilde{X}_\varepsilon^{d_1}$ for $t \in [t_s^1, t_s^2] \subset (0, t_s)$ for $|t_s^1 - t_s^2| > \sigma$ for some $\sigma > 0$ dependent of d_0 and d_1 . Thus,

$$\begin{aligned} \Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon^R(s, t_{s_0})) &\leq \Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon(s)) + \int_{t_s^1}^{t_s^2} \frac{d}{dt} \Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon^R(s, \tau)) d\tau \\ &\leq D_{\varepsilon, \gamma, \pm} - \delta^2(t_s^2 - t_s^1) \\ &\leq C_m - \frac{1}{2}\delta^2\sigma, \quad t \in [t_s^1, t_s^2], \quad \text{if } |st_0 - 1| \leq \mu. \end{aligned}$$

Therefore, since $[0, 1]$ is compact, using infinite covering theorem, for all $s \in [0, 1]$, we can find t_ε^R such that $\Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon^R(s, t_\varepsilon^R)) < C_m - \frac{1}{2}\delta^2d_1$. On the other hand, we

note that $\eta_\varepsilon^R(s, t_\varepsilon^R) \in \Phi_\varepsilon$. Thus,

$$C_{\varepsilon, \gamma, \pm} \leq \max_{s \in [0, 1]} \Gamma_{\varepsilon, \gamma, \pm}(\eta_\varepsilon^R(s, t_\varepsilon^R)) < C_m - \rho,$$

which is a contradiction. $\square$

Lemma 3.11. *For sufficiently small $\varepsilon > 0$, there exists a critical point $v_\varepsilon \in X_\varepsilon^{d_0} \cap \Gamma_{\varepsilon, \gamma, \pm}^{D_{\varepsilon, \gamma, \pm}}$ of $\Gamma_{\varepsilon, \gamma, \pm}$.*

Proof. By Lemma 8, there exists ε_0 and $R_0 > 0$ such that there exists a sequence $\{v_{\varepsilon, n}^R\} \subset E_\varepsilon^R \cap \Gamma_{\varepsilon, \gamma, \pm}^{D_{\varepsilon, \gamma, \pm}} \cap X_\varepsilon^{d_0}$, such that $\|\Gamma'_{\varepsilon, \gamma, \pm}(v_{\varepsilon, n}^R)\|_{(E_\varepsilon^R)'} \rightarrow 0$ as $n \rightarrow \infty$ for $\varepsilon \in (0, \varepsilon_0)$, and $R \in (R_0, +\infty)$. Clearly, $\{v_{\varepsilon, n}^R\}$ is bounded in $H_0^1(B_{\frac{R}{\varepsilon}}(0))$ since $v_{\varepsilon, n}^R \in X_\varepsilon^{d_0}$. By defining $v_{\varepsilon, n}^R = 0$ on $\mathbb{R}^N \setminus B_{\frac{R}{\varepsilon}}(0)$, $v_{\varepsilon, n}^R$ can be regarded as an element in E_ε . Up to a subsequence if necessary, we may assume $v_{\varepsilon, n}^R \rightharpoonup v_\varepsilon^R$ in $H_0^1(B_{\frac{R}{\varepsilon}}(0))$, $v_{\varepsilon, n}^R \rightarrow v_\varepsilon^R$ in $L^p(B_{\frac{R}{\varepsilon}}(0))$, $p \in [1, 2^*)$ and $v_{\varepsilon, n}^R \rightarrow v_\varepsilon^R$ a.e. in $\mathbb{R}^N$. Thus v_ε^R is a solution of

$$-\Delta v = \frac{h(G_{\varepsilon, \gamma, \pm}^{-1}(v)) - V(\varepsilon x)G_{\varepsilon, \gamma, \pm}^{-1}(v)}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}^{-1}(v))} - 4 \left(\int_{B_{\frac{R}{\varepsilon}}(0)} \chi_\varepsilon |v|^2 dx - 1 \right) \chi_\varepsilon v, \quad x \in B_{\frac{R}{\varepsilon}}(0). \quad (3.36)$$

From (3.36) we have $v_{\varepsilon, n}^R \rightarrow v_\varepsilon^R$ in $H_0^1(B_{\frac{R}{\varepsilon}}(0))$ and $v_\varepsilon^R \in X_\varepsilon^{d_0} \cap \Gamma_{\varepsilon, \gamma, \pm}^{D_{\varepsilon, \gamma, \pm}}$. By the maximum principle, $v_\varepsilon^R > 0$. In consequence of (h_1) – (h_3) and $|G_{\varepsilon, \gamma, \pm}^{-1}(t)| \leq \lambda_2 |t|$ for all $t \in \mathbb{R}$, any positive solutions of (3.36) satisfies

$$-\Delta v \leq C v^p, \quad x \in B_{\frac{R}{\varepsilon}}(0),$$

where $C > 0$ depends only on h . In particular, $-\Delta v_\varepsilon^R \leq C(v_\varepsilon^R)^p$, $x \in B_{\frac{R}{\varepsilon}}(0)$. By applying the standard Moser iteration argument (see [17]), $\{v_\varepsilon^R\}$ is bounded in $L_{loc}^p(\mathbb{R}^N)$ uniformly on $R \geq R_0$ and $\varepsilon \in (0, \varepsilon_0)$ for any $p < \infty$ and is bounded in L^∞ . Thus, since $\|v_\varepsilon^R\|_\varepsilon$ and $\{\Gamma_{\varepsilon, \gamma, \pm}(v_\varepsilon^R)\}$ are bounded, we get $\{Q_\varepsilon(v_\varepsilon^R)\}$ is uniformly bounded on $R \geq R_0$ and $\varepsilon \in (0, \varepsilon_0)$. So, we have

$$\int_{\mathbb{R}^N \setminus B_{\frac{R_0}{\varepsilon}}(0)} |v_\varepsilon^R|^2 dx \leq \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} |v_\varepsilon^R|^2 dx = \varepsilon \int_{\mathbb{R}^N} \chi_\varepsilon |v_\varepsilon^R|^2 dx \leq \varepsilon C$$

for any $R \geq R_0$ and $\varepsilon \in (0, \varepsilon_0)$. Thus, for $|x| \geq \frac{R_0}{\varepsilon}$ and $R \geq R_0$, we have

$$h(G_{\varepsilon, \gamma, \pm}^{-1}(v_\varepsilon^R)) \leq \frac{V_0}{2} G_{\varepsilon, \gamma, \pm}^{-1}(v_\varepsilon^R).$$

Similarly to Proposition 3 in [8], we get

$$\lim_{A \rightarrow +\infty} \int_{\mathbb{R}^N \setminus B_A(0)} [|\nabla v_\varepsilon^R|^2 + V_\varepsilon(x)|G_\varepsilon^{-1}(v_\varepsilon^R)|^2] dx = 0 \quad (3.37)$$

uniformly on $R \geq R_0$. Let $v_k = v_\varepsilon^{R_k}$ and $R_k \rightarrow +\infty$ as $k \rightarrow \infty$. Then $\{v_k\}$ is bounded in E_ε and we may assume that $v_k \rightharpoonup v_\varepsilon$ in $H^1(\mathbb{R}^N)$, $v_k \rightarrow v_\varepsilon$ a.e. in $\mathbb{R}^N$. By iterating the above argument, we get $\|v_\varepsilon\|_\infty \leq C$. Since v_k satisfies (3.36) and

using (3.37), we get $\|v_k - v_\varepsilon\|_\varepsilon \rightarrow 0$ as $k \rightarrow \infty$. Thus, $v_\varepsilon \in X_\varepsilon^{d_0} \cap \Gamma_{\varepsilon, \gamma, \pm}^{D_{\varepsilon, \gamma, \pm}}$. Moreover, $\Gamma'_{\varepsilon, \gamma, \pm}(v_\varepsilon) = 0$. $\square$

4. Proof of Theorem 1.1: the subcritical case

By Lemma 3.11, for small $\varepsilon > 0$, there exists a nonnegative solution v_ε to

$$-\Delta v + V_\varepsilon(x) \frac{G_{\varepsilon, \gamma, \pm}^{-1}(v)}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}^{-1}(v))} = \frac{h(G_{\varepsilon, \gamma, \pm}^{-1}(v))}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}^{-1}(v))} - 4 \left(\int_{\mathbb{R}^N} \chi_\varepsilon v^2 dx - 1 \right)_+ \chi_\varepsilon v, \quad (4.1)$$

in $\mathbb{R}^N$. By the maximum principle we have $v_\varepsilon > 0$. Since $v_\varepsilon \in X_\varepsilon^{d_0}$, by Moser's iteration and Lemma 2.2, $\{v_\varepsilon\}$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$ for small $\varepsilon > 0$. Moreover, there exists $\alpha > 0$ such that $\|v_\varepsilon\|_\infty \geq \alpha$, see e.g. [35]. Thus, by choosing $\varepsilon > 0$ sufficiently small, we get $g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}^{-1}(v_\varepsilon)) = 1 \pm \varepsilon^\gamma G_{\varepsilon, \gamma, \pm}^{-1}(v_\varepsilon)^2$.

As in the proof of Lemma 3.7, there exist a sequence $\{z_\varepsilon\} \subset \mathbb{R}^N$ such that $\varepsilon z_\varepsilon \in \mathcal{M}^{2\beta}$ and for any sequence $\varepsilon_n \rightarrow 0$ there exist $y_0 \in \mathcal{M}$ and $U_0 \in S_m$ satisfying

$$\lim_{n \rightarrow \infty} |\varepsilon_n z_{\varepsilon_n} - y_0| = 0, \quad \lim_{n \rightarrow \infty} \|v_{\varepsilon_n} - \varphi_{\varepsilon_n}(\cdot - z_{\varepsilon_n}) U_0(\cdot - z_{\varepsilon_n})\|_{\varepsilon_n} = 0$$

up to a subsequence. Thus, we get

$$\lim_{n \rightarrow \infty} \|v_{\varepsilon_n}(\cdot + z_{\varepsilon_n}) - U_0\|_{H^1(\mathbb{R}^N)} = 0,$$

which implies that for given $\sigma > 0$, there exist $R > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{\varepsilon \in (0, \varepsilon_0)} \int_{\mathbb{R}^N \setminus B_R(0)} v_{\varepsilon_n}^2(x + z_{\varepsilon_n}) dx \leq \sigma.$$

Setting $w_\varepsilon = v_\varepsilon(\cdot + z_\varepsilon)$, we have $-\Delta w_\varepsilon \leq M w_\varepsilon$. Hence, from [17], there exists a constant $C_0 = C_0(N, C)$ such that

$$\sup_{B_1(y)} w_\varepsilon \leq C_0 \|w_\varepsilon\|_{L^2(B_2(y))} \quad \text{for all } y \in \mathbb{R}^N.$$

Since $\|w_\varepsilon\|_{L^2(B_2(y))} \rightarrow 0$, as $|y| \rightarrow \infty$, we conclude that $w_\varepsilon(x) \rightarrow 0$, as $|x| \rightarrow \infty$. Let y_ε be a maximum point of $w_\varepsilon(x)$, then $\{y_\varepsilon\}$ is bounded. By the comparison principle we have

$$w_\varepsilon(x) \leq C \exp(-c|x|) \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

Hence, $x_\varepsilon := y_\varepsilon + z_\varepsilon$ is a maximum point of v_ε and

$$v_\varepsilon(x) = w_\varepsilon(x - x_\varepsilon) \leq C \exp(-c|x - x_\varepsilon|).$$

Therefore, $Q_\varepsilon(v_\varepsilon) = 0$ for small $\varepsilon > 0$ and v_ε is a positive critical point of $P_{\varepsilon, \gamma, \pm}$. Hence, $w_{\varepsilon, \gamma} = G_{\varepsilon, \gamma, \pm}^{-1}(v_\varepsilon)$ is a positive solution of (2.1). Let $y_{\varepsilon_n} \rightarrow y_0$ and $x = y_0 + x_0$, then $\|w_{\varepsilon_n}(\cdot + x_{\varepsilon_n}) - U(\cdot + y_0)\|_{H^1} \rightarrow 0$. Furthermore,

$$\|w_{\varepsilon_n, \gamma}(\cdot + x_{\varepsilon_n}) - U(\cdot + y_0)\|_{H^1}^2 \leq C \|v_{\varepsilon_n}(\cdot + x_{\varepsilon_n}) - U(\cdot + y_0)\|_{H^1}^2 \rightarrow 0.$$

The results follow by letting $u_{\varepsilon, \gamma}(x) = w_{\varepsilon, \gamma}(\frac{x}{\varepsilon})$.

5. Proof of Theorem 1.1: the critical case

In this Section, we deal with the existence and concentration of solutions to (1.1). We use a truncation approach, see also [37]. In the following, we first show the existence and qualitative properties of ground state solutions to the associated limit problem (1.5).

In [25], the authors proved that, if (h_1) and (h_4) – (h_5) hold, then there exists a ground state solution of (1.5) and the least energy E_m is strictly less than $S^{N/2}/N$, where S is the Sobolev best constant, namely

$$S = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}$$

Moreover, let

$$\Upsilon(u) = \int_{\mathbb{R}^N} \left(H(u) - \frac{m}{2} u^2 \right) dx, u \in H^1(\mathbb{R}^N)$$

and

$$M := \inf \{ T(u) : \Upsilon(u) = 1, u \in H^1(\mathbb{R}^N) \}. \quad (5.1)$$

Similarly to [38], we have

$$E_m = \frac{2}{N} M^{\frac{N}{2}} \left(\frac{N-2}{N} \right)^{\frac{N-2}{2}}.$$

and then

$$0 < M < \frac{1}{2} (2^*)^{\frac{N-2}{N}} S.$$

Let S_m be the set of ground state solutions U of (1.5) satisfying $U(0) = \max_{x \in \mathbb{R}^N} U(x)$. Then $S_m \neq \emptyset$ and analogously to [9], the following result holds true:

Proposition 5.1.

- (1) S_m is compact in $H^1(\mathbb{R}^N)$;
- (2) $0 < \inf \{ \|U\|_\infty : U \in S_m \} \leq \sup \{ \|U\|_\infty : U \in S_m \} < \infty$.

By Proposition 5.1 there exists $\kappa > 0$ such that

$$\sup_{U \in S_m} \|U\|_\infty < \kappa. \quad (5.2)$$

In the following, for any fixed $k > \max_{t \in [0, \kappa]} h(t)$, we consider the truncated problem

$$-\Delta u + mu = h_k(u), \quad u \in H^1(\mathbb{R}^N), \quad (5.3)$$

where

$$h_k(t) := \min\{h(t), k\}, \quad t \in \mathbb{R}.$$

Let S_m^k be the set of positive ground state solutions U of (5.3) satisfying $U(0) = \max_{x \in \mathbb{R}^N} U(x)$. Then, similarly to [39], we have

Lemma 5.2.

$$S_m^k = S_m \text{ if } k > \max_{t \in [0, \kappa]} h(t).$$

Now we are in the position to prove the critical part of Theorem 1.1 by means of a truncation approach. Precisely, let us fix $k > \max_{t \in [0, \kappa]} f(t)$ and consider the truncated problem

$$-\Delta w + V_\varepsilon(x)w \mp \varepsilon^\gamma w \Delta w^2 = h_k(w), \quad x \in \mathbb{R}^N, \quad (5.4)$$

Similarly to Section 2, by the change of variables $w = G_{\varepsilon, \gamma, \pm}^{-1}(v)$, v satisfies

$$-\Delta v + V_\varepsilon(x) \frac{G_{\varepsilon, \gamma, \pm}^{-1}(v)}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}^{-1}(v))} = \frac{h_k(G_{\varepsilon, \gamma, \pm}^{-1}(v))}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}^{-1}(v))}, \quad x \in \mathbb{R}^N. \quad (5.5)$$

In view of Lemma 5.2, $S_m^k = S_m$. For some fixed $\beta > 0$ small and a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ for $|x| \leq \beta$ and $\varphi(x) = 0$ for $|x| \geq 2\beta$. Let $\varphi_\varepsilon(y) = \varphi(\varepsilon y)$, $y \in \mathbb{R}^N$ and for each $x \in \mathcal{M}^\beta$ and $U \in S_m$, similarly as above, we define a set of approximating solutions by

$$X_\varepsilon = \{U_\varepsilon^x(y) \mid x \in \mathcal{M}^\beta, U \in S_m\},$$

where

$$U_\varepsilon^x(y) = \varphi_\varepsilon\left(y - \frac{x}{\varepsilon}\right) U\left(y - \frac{x}{\varepsilon}\right).$$

Moreover, it follows from Theorem 1.1 that for some small $\varepsilon_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$, (5.5) admits a positive solution v_ε satisfying the following property: there exist $U \in S_m$ and a maximum point $x_\varepsilon \in \mathbb{R}^N$ of v_ε , such that $\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon, \mathcal{M}) = 0$ and $v_\varepsilon(\varepsilon \cdot + x_\varepsilon) \rightarrow U(\cdot + z_0)$ as $\varepsilon \rightarrow 0$ in $H^1(\mathbb{R}^N)$ for some $z_0 \in \mathbb{R}^N$. Clearly, $w_\varepsilon(\cdot) = v_\varepsilon(\varepsilon \cdot + x_\varepsilon)$ satisfies

$$-\Delta w_\varepsilon + V_\varepsilon\left(x + \frac{x_\varepsilon}{\varepsilon}\right) \frac{G_{\varepsilon, \gamma, \pm}^{-1}(w_\varepsilon)}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}^{-1}(w_\varepsilon))} = \frac{h_k(G_{\varepsilon, \gamma, \pm}^{-1}(w_\varepsilon))}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}^{-1}(w_\varepsilon))}, \quad x \in \mathbb{R}^N.$$

Without loss of generality, we can choose $\varepsilon_0 > 0$ small enough such that $\frac{1}{2} \leq g_{\varepsilon, \gamma, \pm}(t) \leq \frac{3}{2}$ uniformly for $t \in \mathbb{R}$ and $\varepsilon < \varepsilon_0$. Thanks to the fact that $h_k(t) \leq k$ for all $t \in \mathbb{R}$ and Lemma 2.2, in the weak sense, w_ε satisfies $-\Delta w_\varepsilon + \frac{2}{3}V_0 w_\varepsilon \leq 2k$, $x \in \mathbb{R}^N$. Then it follows from standard regularity estimates (see [17]) that $w_\varepsilon(\cdot) \rightarrow U(\cdot + z_0)$ in $C_{loc}^{1, \alpha}(\mathbb{R}^N)$ as $\varepsilon \rightarrow 0$. In particular, $w_\varepsilon(0) \rightarrow U(z_0)$ as $\varepsilon \rightarrow 0$. Then it follows from (5.2) that $\|v_\varepsilon\|_\infty = w_\varepsilon(0) < \kappa$ holds for sufficiently small $\varepsilon > 0$. Recalling that $0 \leq G_{\varepsilon, \gamma, \pm}^{-1}(t) \leq t$ for $t \geq 0$ and sufficiently small $\varepsilon > 0$, we know $h_k(G_{\varepsilon, \gamma, \pm}^{-1}(v_\varepsilon(x))) \equiv h(G_{\varepsilon, \gamma, \pm}^{-1}(v_\varepsilon(x)))$, $x \in \mathbb{R}^N$ for sufficiently small $\varepsilon > 0$. Therefore, v_ε turns out to be a positive solution of the original problem

$$-\Delta v + V_\varepsilon(x) \frac{G_{\varepsilon, \gamma, \pm}^{-1}(v)}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}^{-1}(v))} = \frac{h(G_{\varepsilon, \gamma, \pm}^{-1}(v))}{g_{\varepsilon, \gamma, \pm}(G_{\varepsilon, \gamma, \pm}^{-1}(v))}, \quad x \in \mathbb{R}^N.$$

The proof of Theorem 1.1 is now complete. $\square$

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