An Optimal Bound for Nonlinear Eigenvalues and Torsional Rigidity on Domains with Holes

Francesco Della Pietra and Gianpaolo Piscitelli

Abstract. In this paper we prove an optimal upper bound for the first eigenvalue of a Robin-Neumann boundary value problem for the p -Laplacian operator in domains with convex holes. An analogous estimate is obtained for the corresponding torsional rigidity problem.

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1. Introduction

In this paper we prove an optimal bound for the first eigenvalue and the torsional rigidity of a Robin-Neumann boundary value problem for the p -Laplacian operator in domains with convex holes.

The problem of maximizing or minimizing elliptic functionals with different boundary conditions under various geometric constraints has been extensively studied. Based on the classical isoperimetric inequality, the first evidences are the celebrated Faber-Krahn inequality for the first Dirichlet eigenvalue of the Laplacian, or the Saint-Venant principle for the torsional rigidity with Dirichlet boundary conditions.

Let us consider Ω and D two bounded open sets, with D convex, Ω Lipschitz, connected, and $D \Subset \Omega$. We will denote by $\Sigma = \Omega \setminus \bar{D}$. Here we study the following problem:

$$\lambda(\beta, \Sigma) = \min \left\{ \frac{\int_{\Sigma} |\nabla \varphi|^p dx + \beta \int_{\partial D} |\varphi|^p d\sigma}{\int_{\Sigma} |\varphi|^p dx}, \varphi \in W^{1,p}(\Sigma), \varphi \not\equiv 0 \right\}, \quad (1.1)$$

where β is a positive constant and $1 < p < +\infty$. A minimizer $u \in W^{1,p}(\Sigma)$ of (1.1) satisfies

$$\begin{cases} -\Delta_p u = \lambda(\beta, \Sigma)|u|^{p-2}u & \text{in } \Sigma, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + \beta|u|^{p-2}u = 0 & \text{on } \partial D. \end{cases} \quad (1.2)$$

In this paper we obtain an optimal upper bound on $\lambda(\beta, \Sigma)$, when $\beta > 0$ is fixed and $\Sigma = \Omega \setminus \bar{D}$ varies among all domains such that the volume of Σ and the $(n-1)$ -quermassintegral of D are given (see Section 2.3 for the precise definition). If $n = 2$, the geometrical constraint on D corresponds to fix its perimeter. In particular, we show that $\lambda(\beta, \Sigma)$ is maximized by the spherical shell. The first main result is the following.

Theorem 1.1. *Let $\beta > 0$, and Ω and D be two bounded open sets, with D convex, Ω Lipschitz, connected, and $D \Subset \Omega$. Let be $\Sigma = \Omega \setminus \bar{D}$, $A = A_{R_1, R_2} = B_{R_2} \setminus \bar{B}_{R_1}$, where B_{R_i} is a ball centered at the origin with radius R_i , $i = 1, 2$. Suppose that $|A| = |\Sigma|$, and $W_{n-1}(B_{R_1}) = W_{n-1}(D)$. Then,*

$$\lambda(\beta, \Sigma) \leq \lambda(\beta, A).$$

In the case $p = n = 2$, Theorem 1.1 recovers a result proved by Hersch in [18]. Our result generalizes it to the p -Laplacian operator and to any dimension, giving an answer to a question posed by Henrot in [17, Chapter 3, Open problem 5]. Actually, we are not able to prove or disprove that the first eigenvalue $\lambda(\beta, \Sigma)$ is maximized on the spherical shell when the perimeter of D , and not the W_{n-1} , is fixed. We stress that a related result has been recently proved for an optimal insulating problem (see [14]).

Optimal estimates for eigenvalues with different Robin or Robin-Neumann boundary conditions have been proved by several authors. For example, if $D = \emptyset$, and a Robin condition on $\partial\Omega$ is given, with $\beta > 0$, a Faber-Krahn inequality has been proved in [3, 9] for $p = 2$ and in [5, 8] for $p \in]1, +\infty[$, stating that the first eigenvalue is minimum on the ball of the same volume of Ω . Otherwise, if β is negative the problem is still open; in this direction, in [15] the authors showed that, among planar domains of the same measure, the disk is a maximizer only for small value of the parameter. On the other hand, if the perimeter rather than the volume is fixed, the ball maximizes the first eigenvalue among all open, bounded, convex, smooth enough sets (see [2, 6]).

In this framework, the case of Robin eigenvalue problems when $\beta \geq 0$ is not a constant has been considered in [16], and, for example, in [4, 13, 11] where optimization with respect to β is considered.

Coming back to the problem with mixed boundary conditions, we recall that the case of a Neumann condition on ∂D and Robin condition on $\partial\Omega$ has been considered in [22, 20].

Another problem we deal with is the p -torsional rigidity with the same boundary conditions, namely

$$T(\beta, \Sigma) = \max \left\{ \frac{\left(\int_{\Sigma} |\varphi| dx \right)^p}{\int_{\Sigma} |\nabla \varphi|^p dx + \beta \int_{\partial D} |\varphi|^p d\sigma}, \varphi \in W^{1,p}(\Sigma), \varphi \not\equiv 0 \right\}, \quad (1.3)$$

with $\beta > 0$. The unique function $u_{\Sigma} > 0$ such that

$$\begin{cases} -\Delta_p u_{\Sigma} = 1 & \text{in } \Sigma, \\ |\nabla u_{\Sigma}|^{p-2} \frac{\partial u_{\Sigma}}{\partial \nu} = 0 & \text{on } \partial \Omega, \\ |\nabla u_{\Sigma}|^{p-2} \frac{\partial u_{\Sigma}}{\partial \nu} + \beta |u_{\Sigma}|^{p-2} u_{\Sigma} = 0 & \text{on } \partial D, \end{cases} \quad (1.4)$$

is a maximizer of (1.3), and

$$T(\beta, \Sigma) = \left(\int_{\Sigma} u_{\Sigma} dx \right)^{p-1}.$$

Moreover, any maximizer of (1.3) is proportional to u_{Σ} .

We prove that the spherical shell minimizes $T(\beta, \Sigma)$ among all domains such that the volume of Σ and the $(n-1)$ -quermassintegral of D are given.

Theorem 1.2. *Let $\beta > 0$, and Ω and D be two bounded open sets, with D convex, Ω Lipschitz, connected, and $D \Subset \Omega$. Let be $\Sigma = \Omega \setminus \bar{D}$, $A = A_{R_1, R_2} = B_{R_2} \setminus \bar{B}_{R_1}$, where B_{R_i} is a ball centered at the origin with radius R_i , $i = 1, 2$. Suppose that $|A| = |\Sigma|$, and $W_{n-1}(B_{R_1}) = W_{n-1}(D)$. Then,*

$$T(\beta, \Sigma) \geq T(\beta, A).$$

Regarding optimal estimates related to the torsional rigidity when $D = \emptyset$ and a Robin boundary condition is given on $\partial \Omega$, a Saint-Venant inequality can be proved: we refer the reader to [1, 7].

Finally, we recall that in this context, estimates have been obtained also for a more general class of problems, involving the so called Finsler operator. We refer the reader, for example, to [10, 12, 21].

The structure of the paper is the following. In the second section, we prove some properties of the first eigenvalue $\lambda(\beta, \Sigma)$ and of the torsional rigidity $T(\beta, \Sigma)$, as well as we recall some basic tool of convex analysis. In the third section, we prove the main results. We conclude the paper with a section containing the possible future directions.

2. Notation and preliminaries

2.1. Eigenvalue problem

The following result can be proved by a standard argument of Calculus of Variations.

Proposition 2.1. *Let $\beta > 0$, D, Ω Lipschitz domains with $D \Subset \Omega$, and $\Sigma = \Omega \setminus \bar{D}$.*

- *There exists a positive minimizer $u \in W^{1,p}(\Sigma)$ of (1.1). Moreover, u is a solution of (1.2).*
- *The eigenvalue $\lambda(\beta, \Sigma)$ is simple, that is all the associated eigenfunctions are scalar multiple of each other.*

A trivial upper bound for $\lambda(\beta, \Sigma)$ is given, by choosing constant test functions, by

$$\lambda(\beta, \Sigma) \leq \beta \frac{P(D)}{|\Sigma|}.$$

Another simple upper bound for $\lambda(\beta, \Sigma)$ is given by

$$\lambda(\beta, \Sigma) \leq \Lambda(\Sigma),$$

where

$$\Lambda(\Sigma) = \min \left\{ \frac{\int_{\Sigma} |\nabla \varphi|^p dx}{\int_{\Sigma} |\varphi|^p dx}, \varphi \in W^{1,p}(\Sigma), \varphi|_{\partial D} = 0 \right\},$$

that is the first eigenvalue of the corresponding Dirichlet-Neumann problem.

In the radial case, we have that the eigenfunctions are radially symmetric and strictly increasing in the radial variable, as stated in the following

Proposition 2.2. *Let be $\beta > 0$, $R_2 > R_1 > 0$ and let v be a positive minimizer of problem (1.1) on the spherical shell $\Sigma = A_{R_1, R_2}$. Then v is radially symmetric, in the sense that $v(x) = \psi(|x|)$. Moreover, $\psi'(r) > 0$.*

Proof. The radial symmetry of the minimizers follows from the rotational invariance of problem (1.1) and the simplicity of $\lambda(\beta, A_{R_1, R_2})$.

Now, let ψ be a positive radial eigenfunction. We observe that ψ solves

$$\begin{cases} -\frac{1}{r^{n-1}} (|\psi'(r)|^{p-2} \psi'(r) r^{n-1})' = \lambda(\beta, A_{R_1, R_2}) \psi^{p-1}(r) & \text{if } r \in (R_1, R_2), \\ \psi'(R_2) = 0, \\ |\psi'(R_1)|^{p-2} \psi'(R_1) + \beta \psi^{p-1}(R_1) = 0. \end{cases}$$

For every $r \in (R_1, R_2)$ it holds that

$$-\frac{1}{r^{n-1}} (|\psi'(r)|^{p-2} \psi'(r) r^{n-1})' = \lambda(\beta, A_{R_1, R_2}) \psi^{p-1}(r) > 0,$$

and, as a consequence,

$$(|\psi'(r)|^{p-2} \psi'(r) r^{n-1})' < 0.$$

Taking into account the boundary conditions $\psi'(R_2) = 0$, it follows by integrating that $\psi'(r) > 0$, and this concludes the proof. $\square$

The following proposition is a complementary result which states some properties of the first eigenvalue $\lambda(\beta, \Sigma)$, related to its dependence on the parameter β .

Proposition 2.3. *Let $\beta > 0$, D, Ω Lipschitz domains with $D \Subset \Omega$, and $\Sigma = \Omega \setminus \bar{D}$. The following properties hold.*

1. *The first eigenvalue $\lambda(\beta, \Sigma)$ is differentiable and nondecreasing in $\beta > 0$. Moreover,*

$$\frac{d}{d\beta} \lambda(\beta, \Sigma) = \frac{\int_{\partial D} |u|^p d\sigma}{\int_{\Sigma} |u|^p dx},$$

where u is a minimizer of (1.1).

2. *It holds that*

$$\lim_{\beta \rightarrow +\infty} \lambda(\beta, \Sigma) = \Lambda(\Sigma). \quad (2.1)$$

3. *The first eigenvalue $\lambda(\beta, \Sigma)$ is concave with respect to β .*

Proof. Let us denote

$$Q(\beta, \varphi) = \frac{\int_{\Sigma} |\nabla \varphi|^p dx + \beta \int_{\partial D} |\varphi|^p d\sigma}{\int_{\Sigma} |\varphi|^p dx}, \quad \varphi \in W^{1,p}(\Sigma).$$

Let $u_\beta, u_{\beta+h}$ be two positive minimizers of (1.1) associated to β and $\beta + h$, respectively. Moreover, suppose $\int_{\Sigma} u_\beta^p dx = \int_{\Sigma} u_{\beta+h}^p dx = 1$. Then,

$$\lambda(\beta + h, \Sigma) - \lambda(\beta, \Sigma) \geq \lambda(\beta + h, \Sigma) - Q(\beta, u_{\beta+h}) = h \int_{\partial D} u_{\beta+h}^p d\sigma.$$

On the other hand,

$$\lambda(\beta + h, \Sigma) - \lambda(\beta, \Sigma) \leq Q(\beta + h, u_\beta) - \lambda(\beta, \Sigma) = h \int_{\partial D} u_\beta^p d\sigma.$$

Finally,

$$\int_{\partial D} u_{\beta+h}^p d\sigma \leq \frac{\lambda(\beta + h, \Sigma) - \lambda(\beta, \Sigma)}{h} \leq \int_{\partial D} u_\beta^p d\sigma.$$

Then, being $\{u_{\beta+h}\}_h$ bounded in $W^{1,p}(\Omega)$, there exists a subsequence, still denoted by $u_{\beta+h}$ such that $u_{\beta+h} \rightarrow u_\beta$ strongly in L^p and almost everywhere and $\nabla u_{\beta+h} \rightharpoonup \nabla u_\beta$ weakly in $L^p(\Omega)$. As a consequence, by the compactness of the trace operator (see for example [19, Cor. 18.4]), $u_{\beta+h}$ converges strongly to u_β in $L^p(\partial D)$.

The monotonicity of $\lambda(\beta, \Sigma)$ with respect to β is obvious.

Now we show the continuity of λ with respect to β . Define

$$Q(\beta, \varphi) = \frac{\int_{\Sigma} |\nabla \varphi|^p dx + \beta \int_{\partial D} |\varphi|^p d\sigma}{\int_{\Sigma} |\varphi|^p dx}, \quad \varphi \in W^{1,p}(\Sigma).$$

Let φ_k be a minimizing sequence for $\lambda(\beta, \Sigma)$, normalized with $\|\varphi_k\|_{L^p(\Sigma)} = 1$. Then $\int_{\partial D} \varphi_k d\sigma \leq C = C(\Sigma, \beta)$. Hence

$$\lambda(\beta, \Sigma) \leq \lambda(\beta + \varepsilon, \Sigma) \leq Q(\beta + \varepsilon, \varphi_k) \leq Q(\beta, \varphi_k) + \varepsilon C.$$

Passing to the limit as $k \rightarrow +\infty$, we get

$$0 \leq \lambda(\beta + \varepsilon, \Sigma) - \lambda(\beta, \Sigma) \leq \varepsilon C.$$

Let u_β be a minimizer for $\lambda(\beta, \Sigma)$, with $\int_\Sigma |u_\beta|^p dx = 1$. Then being

$$\lambda(\beta, \Sigma) = \int_\Sigma |\nabla u_\beta|^p dx + \beta \int_{\partial D} |u_\beta|^p d\sigma \leq \lambda(\Sigma),$$

if $\beta_j \rightarrow +\infty$ as $j \rightarrow +\infty$, there exists $u_\infty \in W^{1,p}(\Sigma)$ such that, up to a subsequence,

$$u_{\beta_j} \rightharpoonup u_\infty \text{ in } W^{1,p}(\Sigma), \quad u_{\beta_j} \rightarrow u_\infty \equiv 0 \text{ in } L^p(\partial D), \quad u_{\beta_j} \rightarrow u_\infty \text{ in } L^p(\Sigma).$$

Hence by lower semicontinuity,

$$\lambda(\Sigma) \leq \int_\Sigma |\nabla u_\infty|^p dx \leq \liminf_j \lambda(\beta_j, \Sigma) \leq \lambda(\Sigma),$$

that gives (2.1).

We finally prove that $\lambda(\beta, \Sigma)$ is concave in β . Indeed, for fixed $\beta_0 \in \mathbb{R}$, we have to show that

$$\lambda(\beta, \Sigma) \leq \lambda(\beta_0, \Sigma) + \frac{d}{d\beta} \lambda(\beta_0, \Omega) (\beta - \beta_0), \quad (2.2)$$

for every $\beta > 0$. Let u_0 the eigenfunction associated to $\lambda(\beta_0, \Sigma)$ and normalized such that $\int_\Sigma u_0^p dx = 1$. Hence, we have

$$\lambda(\beta, \Sigma) \leq \int_\Sigma |Du_0|^p dx + \beta \int_{\partial D} |u_0|^p d\mathcal{H}^{n-1}. \quad (2.3)$$

Now, summing and subtracting to the right hand side of (2.3), $\beta_0 \int_{\partial D} |u_0|^p d\mathcal{H}^{n-1}$, taking into account that we have just showed that

$$\frac{d}{d\beta} \lambda(\beta_0, \Omega) = \int_{\partial D} |u_0|^p d\mathcal{H}^{n-1},$$

we obtain (2.2). □

2.2. The p -torsional rigidity

Similarly to the case of the first eigenvalue, the following results hold.

Proposition 2.4. *Let $\beta > 0$. There exists a unique positive maximizer $u_\Sigma \in W^{1,p}(\Sigma)$ of (1.3) which solves (1.4), and any other maximizer of (1.4) is a scalar multiple of u_Σ .*

Proposition 2.5. *Let be $\beta > 0$, $R_2 > R_1 > 0$ and let w be a positive maximizer of problem (1.3) on the spherical shell $\Sigma = A_{R_1, R_2}$. Then w is radially symmetric, $w(x) = \Psi(|x|)$, with $\Psi'(r) > 0$.*

2.3. Quermassintegrals and the Aleksandrov-Fenchel inequalities

Here we list some basic properties of convex analysis which will be useful in the following. For an extended discussion on the subject we refer the reader to [23].

Let K be a bounded convex open set, and $B_1 = \{x: |x| < 1\}$. The outer parallel body of K at distance $\rho > 0$ is the Minkowski sum

$$K + \rho B_1 = \{x + \rho y \in \mathbb{R}^n \mid x \in K, y \in B_1\}.$$

The Steiner formulas assert that

$$|K + \rho B_1| = \sum_{i=0}^n \binom{n}{i} W_i(K) \rho^i,$$

and

$$P(K + \rho B_1) = n \sum_{i=0}^{n-1} \binom{n-1}{i} W_{i+1}(K) \rho^i.$$

The coefficients $W_i(K)$ are known as the quermassintegrals of K . In particular, it holds that

$$W_0(K) = |K|, \quad nW_1(K) = P(K), \quad W_n(K) = \omega_n.$$

The Aleksandrov-Fenchel inequalities state that

$$\left(\frac{W_j(K)}{\omega_n} \right)^{\frac{1}{n-j}} \geq \left(\frac{W_i(K)}{\omega_n} \right)^{\frac{1}{n-i}}, \quad 0 \leq i < j \leq n-1, \quad (2.4)$$

where the inequality is replaced by an equality if and only if K is a ball.

In what follows, we use the Aleksandrov-Fenchel inequalities for particular values of i and j . When $i = 0$ and $j = 1$, we have the classical isoperimetric inequality:

$$P(K) \geq n\omega_n^{\frac{1}{n}} |K|^{1-\frac{1}{n}}.$$

Let us denote by K^* a ball such that $W_{n-1}(K) = W_{n-1}(K^*)$. Then by Aleksandrov-Fenchel inequalities (2.4), for $0 \leq i < n-1$

$$\left(\frac{W_i(K)}{\omega_n} \right)^{\frac{1}{n-i}} \leq \frac{W_{n-1}(K)}{\omega_n} = \left(\frac{W_i(K^*)}{\omega_n} \right)^{\frac{1}{n-i}},$$

hence

$$W_i(K) \leq W_i(K^*), \quad 0 \leq i < n-1.$$

3. Proof of main results

In this section we prove the main results (Theorem 1.1 and Theorem 1.2).

Proof of Theorem 1.1. Let $v(x) = \psi(|x|)$ be a positive radial solution of problem (1.1) on A , and denote by $v_m = \psi(R_1)$ and $v_M = \psi(R_2)$ be the minimum and maximum of v , respectively.

For $x \in \Sigma$, let us denote by $d(x)$ the distance of x from D ,

$$d(x) = \inf\{|x - y|, y \in D\},$$

and consider as test function

$$w(x) := \begin{cases} G(d(x)) & \text{if } d(x) < R_2 - R_1, \\ v_M & \text{if } d(x) \geq R_2 - R_1, \end{cases}$$

where G is defined as

$$G^{-1}(t) = \int_{v_m}^t \frac{1}{g(\tau)} d\tau, \quad v_m < t < v_M,$$

with $g(t) = |Dv|_{v=t}$. We observe that $v(x) = G(|x| - R_1)$ and w satisfies the following properties:

$$\begin{aligned} w &\in W^{1,p}(\Sigma) \cap C(\bar{\Sigma}), \\ |Dw|_{w=t} &= |Dv|_{v=t}, \\ w_m &:= \min_{\bar{\Sigma}} w = v_m = G(0), \\ w_M &:= \max_{\bar{\Sigma}} w = v_M. \end{aligned}$$

Since $R_2 - R_1 = G^{-1}(v_M)$, to prove that the maximum value of w is v_M , we need to verify that $G^{-1}(v_M) = \int_{v_m}^{v_M} \frac{1}{g(\tau)} d\tau$. Indeed since $v(x) = \psi(|x|)$, then $\int_{v_m}^{v_M} \frac{1}{\psi'(v^{-1}(s))} ds = \psi^{-1}(v_M) - \psi^{-1}(v_m) = R_2 - R_1$.

Let us denote

$$E_t = \bar{D} \cup \{x \in \Sigma : w < t\}, \quad F_t = \bar{B}_{R_1} \cup \{x \in A : v < t\}.$$

Observe that

$$E_t \subseteq \{x \in \mathbb{R}^n : d(x) < G^{-1}(t)\} := \tilde{E}_t, \quad F_t = \{x \in \mathbb{R}^n : |x| < R_1 + G^{-1}(t)\}.$$

By the Steiner formula and the Aleksandrov-Fenchel inequalities, we get, as $\rho = G^{-1}(t)$, that

$$\begin{aligned} \mathcal{H}^{n-1}(\partial E_t \cap \Sigma) &\leq P(\tilde{E}_t) = P(D + \rho B_1) = n \sum_{k=0}^{n-1} \binom{n-1}{k} W_{k+1}(D) \rho^k \\ &\leq n \sum_{k=0}^{n-1} \binom{n-1}{k} W_{k+1}(B_{R_1}) \rho^k = P(B_{R_1} + \rho B_1) = P(F_t). \end{aligned} \quad (3.1)$$

Using now the coarea formula and (3.1):

$$\begin{aligned} \int_{\Omega} |\nabla w|^p dx &= \int_{w_m}^{w_M} g(t)^{p-1} \mathcal{H}^{n-1}(\partial E_t \cap \Sigma) dt \\ &\leq \int_{w_m}^{w_M} g(t)^{p-1} P(\tilde{E}_t) dt \leq \int_{v_m}^{v_M} g(t)^{p-1} P(F_t) dt = \int_A |\nabla v|^p dx. \end{aligned} \quad (3.2)$$

Since, by construction, $w(x) = w_m = v_m$ on ∂D , then

$$\int_{\partial D} w^p d\mathcal{H}^{n-1} = w_m^p P(D) \leq v_m^p P(B_{R_1}) = \int_{\partial B_{R_1}} v^p d\mathcal{H}^{n-1}. \quad (3.3)$$

Now, we define $\mu(t) = |E_t \cap \Sigma|$ and $\eta(t) = |F_t|$ and using again coarea formula, we obtain, for $v_m \leq t < v_M$,

$$\begin{aligned} \mu'(t) &= \int_{\{w=t\} \cap \Sigma} \frac{1}{|\nabla w(x)|} d\mathcal{H}^{n-1} = \frac{\mathcal{H}^{n-1}(\partial E_t \cap \Sigma)}{g(t)} \leq \frac{P(\tilde{E}_t)}{g(t)} \\ &\leq \frac{P(F_t)}{g(t)} = \int_{\{v=t\}} \frac{1}{|\nabla v(x)|} d\mathcal{H}^{n-1} = \eta'(t). \end{aligned}$$

This inequality is trivially true also if $0 < t < v_m$. Since $\mu(0) = \eta(0) = 0$, by integrating from 0 to $t < v_M$, we have:

$$\mu(t) \leq \eta(t).$$

On the other hand, we have

$$\begin{aligned} \int_{\Sigma} w^p dx &= \int_0^{v_M} p t^{p-1} (|\Sigma| - \mu(t)) dt \\ &\geq \int_0^{v_M} p t^{p-1} (|A| - \eta(t)) dt = \int_A v^p dx. \end{aligned} \quad (3.4)$$

Using (3.2), (3.3) and (3.4), we achieve

$$\begin{aligned} \lambda(\beta, \Sigma) &\leq \frac{\int_{\Sigma} |\nabla w|^p dx + \beta \int_{\partial D} w^p d\mathcal{H}^{n-1}}{\int_{\Sigma} w^p dx} \\ &\leq \frac{\int_A |\nabla v|^p dx + \beta \int_{\partial B_{R_1}} v^p d\mathcal{H}^{n-1}}{\int_A v^p dx} = \lambda(\beta, A), \end{aligned}$$

and this concludes the proof. $\square$

Proof of Theorem 1.2. As regards the case of the torsional rigidity, the proof follows line by line the proof of Theorem 1.1 making use of the stress function v_A instead of the first eigenfunction on the spherical shell. $\square$

Remark 3.1. We stress that with the same argument it is possible to prove that if

$$\lambda_q(\beta, \Sigma) = \min \left\{ \frac{\int_{\Sigma} |\nabla \varphi|^p dx + \beta \int_{\partial D} |\varphi|^p d\sigma}{\left(\int_{\Sigma} |\varphi|^q dx \right)^{\frac{p}{q}}}, \varphi \in W^{1,p}(\Sigma), \varphi \not\equiv 0 \right\},$$

with $1 \leq q \leq p$ and $1 < p < +\infty$, it holds that

$$\lambda_q(\beta, \Sigma) \leq \lambda_q(\beta, A)$$

where $A = B_{R_2} \setminus \bar{B}_{R_1}$ is the spherical shell with $|A| = |\Sigma|$ and $W_{n-1}(B_{R_1}) = W_{n-1}(D)$.

4. Conclusions

In conclusion, we have considered the first eigenvalue and the torsional rigidity of the p -Laplacian operator, defined in a bounded Lipschitz domain of $\mathbb{R}^n$ with a convex hole, where a Robin boundary condition on the hole and a Neumann condition on the other parts of the boundary is given. We have proved that the first eigenvalue of the spherical shell is maximal among the domains with fixed volume and fixed $(n-1)$ -quermassintegral of the hole, while the torsional rigidity is minimal. It is still an open problem to understand what happens if instead of W_{n-1} the perimeter or the other k -th quermassintegrals of the hole, are fixed. Further developments of the present work could go in this direction.

Another possible direction of investigation is to understand if this kind of results may be generalized to different kind of operators, as the Finsler Laplacian or, more generally, the Finsler p -Laplacian (see for example [10, 12] and the references therein contained). In this case, is reasonable that quermassintegrals with respect to the Finsler metric will be involved, and the candidate as optimal set will be the spherical shell with respect to the polar norm of the considered Finsler metric.

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Francesco Della Pietra

Dipartimento di Matematica e Applicazioni “R. Caccioppoli”

Università degli studi di Napoli Federico II

Via Cintia, Monte S. Angelo, 80126 Napoli, Italia.

e-mail: f.dellapietra@unina.it

Gianpaolo Piscitelli

Dipartimento di Ingegneria Elettrica e dell’Informazione “M. Scarano”

Università degli Studi di Cassino e del Lazio Meridionale

Via G. Di Biasio n. 43, 03043 Cassino (FR), Italy.

e-mail: gianpaolo.piscitelli@unicas.it

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