



Equivariant Fusion Subcategories

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Abstract

We provide a parameterization of all fusion subcategories of the equivariantization by a group action on a fusion category. As applications, we classify the Hopf subalgebras of a family of semisimple Hopf algebras of Kac-Paljutkin type and recover Naidu-Nikshych-Witherspoon classification of the fusion subcategories of the representation category of a twisted quantum double of a finite group.

1 Introduction

Fusion categories generalize the representation categories of finite-dimensional semi-simple (quasi) Hopf algebras [6]. A fundamental construction in the theory of fusion categories is equivariantization by a group action, [5]. Given a categorical action of a finite group G on a fusion category \mathcal{C} , the equivariantization \mathcal{C}^G is a new fusion category, consisting of categorical “fixed points” under the G action. Equivariantization, and its reciprocal construction called de-equivariantization, have been extensively studied and used in recent years to classify and characterize families of fusion categories, see [4, 7]. The goal of this note is to identify the lattice of fusion subcategories of \mathcal{C}^G in terms of the lattice of fusion subcategory of \mathcal{C} and cohomological data arising from the G -action (Theorem 4.8).

If a fusion category \mathcal{C} is equivalent to the category of (right) comodules of a finite-dimensional semisimple Hopf algebra H , then the lattice of fusion subcategories of \mathcal{C} is isomorphic to the lattice of Hopf subalgebras of H , see [14, Theorem 6]. Many interesting Hopf algebras (in particular Hopf algebras of Kac-Paljutkin type and

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quantum groups at roots of unity) contain a central Hopf subalgebra \mathbb{C}^G , and this implies the existence of a central exact sequence of Hopf algebras of the form

$$\mathbb{k} \rightarrow \mathbb{k}^G \rightarrow H \rightarrow Q \rightarrow \mathbb{k}.$$

In this case, the category of H -comodules is a G -equivariantization of the category of Q -comodules, [1]. Thus, we can apply our results to obtain a parameterization of the Hopf subalgebras of central extensions of semisimple Hopf algebras, particularly to Hopf algebras of Kac-Paljutkin type associated to semidirect products of groups.

It is well known how to parameterize the simple objects of the equivariantization and compute their fusion rules in terms of the data of the G action (for example, [3]). Fusion subcategories of a fusion category \mathcal{D} correspond to fusion sub-rules of the Grothendieck ring $K_0(\mathcal{D})$. Hence, in principle, having the fusion rules of \mathcal{C}^G , the computation of fusion subcategories of \mathcal{C}^G reduces to the computation of fusion sub-rules of $K_0(\mathcal{C}^G)$. However, even the computation of $K_0(\mathcal{C}^G)$ and its multiplication could be computationally demanding. Therefore, it is desirable to obtain a parameterization of the fusion subcategories of an equivariantization directly in terms of the data of the group action. The main result of this paper, Theorem 4.8, is a parameterization of all fusion subcategories of the equivariantization \mathcal{C}^G in terms of the lattice of fusion subcategories of \mathcal{C} stables by the G -action and cohomological data arising from the G -action.

After posting an initial version of this paper to the arXiv, Dmitri Nikshych brought to our attention that our Theorem 4.8 was previously obtained by Alex Levin and presented at conferences in 2017. Levin's work has not yet been made publicly available. We thank Dmitri for bringing this to our attention.

The paper is organized as follows. In Sects. 2 and 3, we recall the basic definitions and results concerning equivariantization by a categorical group action and de-equivariantization of a fusion category containing a central inclusion of $\text{Rep}(G)$. In Sect. 4, we prove our main result, which provides a parameterization of equivariant fusion subcategories of a fusion category. In Sect. 5, we take a closer look at the parameterization in the case of group actions on pointed fusion categories, providing an obstruction theory for G -invariant trivializations. Finally, in Sect. 6, we apply this to parameterize Hopf subalgebras of a class of Kac-Paljutkin type studied in [15] and [10].

2 Preliminaries

A fusion category is a finite semi-simple rigid tensor category. In this paper, we assume our field is algebraically closed and characteristic 0. We refer the reader to the comprehensive reference [5] for categorical background. In this section, we will recall some basic notions and set up notation for categorical actions of groups on fusion categories and equivariantization.

Let $\underline{\text{Aut}}(\mathcal{C})$ denote the 2-group of tensor autoequivalences of a fusion category and tensor natural isomorphisms with tensor product given by the composition of tensor functors. A *categorical action* of a finite group G on \mathcal{C} is a strong monoidal functor

$\underline{G} \rightarrow \underline{\text{Aut}}(\mathcal{C})$, where \underline{G} is the discrete monoidal category where the objects are the elements of G and the tensor product is defined by the product of G . We denote the assignment $g \mapsto g_*$, and the monoidal natural isomorphisms $\mu_{g,h} : g_* \circ h_* \rightarrow (gh)_*$. We denote the component of a natural isomorphism with a superscript, i.e., the x component of $\mu_{g,h}$ is denoted $\mu_{g,h}^x$ for $x \in \mathcal{C}$.

Given a categorical action of G on \mathcal{C} , recall that the equivariantization \mathcal{C}^G is a new fusion category whose objects are *equivariant objects*, given by pairs (x, ρ) , where $x \in \mathcal{C}$ and $\rho = \{\rho_g : g_*(x) \cong x\}_{g \in G}$ is a family of isomorphisms satisfying $\mu_{g,h}^x \rho_g g_*(\rho_h) = \rho_{gh}$ for all $g, h \in G$. Morphisms between equivariant objects (x, ρ) and (y, δ) are morphisms $f \in \mathcal{C}(x, y)$ such that for all $g \in G$, $f \rho_g = \delta_g g_*(f)$.

This forms a new monoidal category called the equivariantization, denoted \mathcal{C}^G . The product $(x, \rho) \otimes (y, \delta)$ is defined by

$$(x \otimes y, (\rho \otimes \delta) \circ \beta^{x,y}),$$

where $\beta_g^{x,y} : g_*(x \otimes y) \cong g_*(x) \otimes g_*(y)$ is the tensorator of the monoidal equivalence g_* .

An important point of note is that since the unit object of \mathcal{C} is simple, there is a canonical copy of $\text{Rep}(G) \leq \mathcal{C}^G$ embedded as a full subcategory, consisting of equivariant objects which are direct sums of the unit object $\mathbb{1}$. Furthermore, this subcategory naturally lifts to the center $\mathcal{Z}(\mathcal{C}^G)$; see [5, Proposition 8.23.1]. Fusion categories \mathcal{C} equipped with a fully faithful monoidal functor $F : \text{Rep}(G) \rightarrow \mathcal{C}$ that factors through a braided monoidal functor $\text{Rep}(G) \rightarrow \mathcal{Z}(\mathcal{C})$ are called fusion categories *over* $\text{Rep}(G)$.

Both G -fusion categories and fusion categories over $\text{Rep}(G)$ naturally form 2-categories (see [4, Section 4]). Equivariantization extends to a 2-functor from G -fusion categories to fusion categories over $\text{Rep}(G)$. This 2-functor has an explicit inverse called the *de-equivariantization functor*.

Given a fusion category over $\text{Rep}(G)$, then identifying $\text{Rep}(G)$ with its image under F , the étale algebra $\text{Fun}(G) \in \text{Rep}(G)$ can be viewed as a central algebra in \mathcal{C} . Thus, the category of right $\text{Fun}(G)$ modules $\mathcal{C}_{\text{Fun}(G)}$ is a fusion category called the *de-equivariantization* of \mathcal{C} , which we also denote by \mathcal{C}_G . The free module functor $\mathcal{C} \rightarrow \mathcal{C}_{\text{Fun}(G)} = \mathcal{C}_G$ is monoidal. Furthermore, one can equip the de-equivariantization with a natural categorical action of G , and in fact this extends to a 2-functor from the 2-category of fusion categories over $\text{Rep}(G)$ to the 2-category of G -fusion categories. The fundamental theorem of G -actions on fusion categories states that equivariantization and de-equivariantization are mutually inverse 2-functors [4, Theorem 4.4].

3 Sequential Equivariantization

Let G be a group acting on a monoidal category \mathcal{C} and let $H \leq G$ be a normal subgroup. First pick a section $\iota : G/H \rightarrow G$ of the canonical projection $\pi : G \rightarrow G/H$ and define $\alpha : G/H \times G/H \rightarrow H$ by $\alpha(a, b) := \iota(a)\iota(b)\iota(ab)^{-1}$ for all $a, b \in G/H$.

Using the bijection

$$\begin{aligned} H \times G/H &\rightarrow G \\ (h, a) &\mapsto h\iota(a). \end{aligned}$$

the multiplication on G transports to

$$(h, a)(k, b) = (h\iota(a)k\iota(a)^{-1}\alpha(a, b), ab).$$

Given $(x, \rho) \in \mathcal{C}^H$, and $a \in G/H$, we define $a_*(x, \rho) := (x', \rho')$ where

$$x' := \iota(a)_*(x)$$

and ρ'_h is defined via the composition

$$h_*(\iota(a)_*(x)) \xrightarrow{\text{can}} \iota(a)_*((\iota(a)^{-1}h\iota(a))_*(x)) \xrightarrow{\iota(a)_*(\rho_h)} \iota(a)_*(x).$$

It is straightforward to check for each $a \in G/H$ this is a monoidal functor, using the tensorator for $\iota(a)_*$.

To obtain a categorical action, we need monoidal natural isomorphisms $a_* \circ b_* \rightarrow (ab)_*$. Recall that $\iota(a)\iota(b) = \iota(ab)(\iota(ab)^{-1}\alpha(a, b)\iota(ab))$, and thus we define $v_{a,b}^{(x,\rho)}$ to be the morphism making the following diagram commute

$$\begin{array}{ccc} \iota(a)_*(\iota(b)_*(x)) & \xrightarrow{\text{can}} & \iota(ab)_*((\iota(ab)^{-1}\alpha(a, b)\iota(ab))_*(x)) \\ & \searrow v_{a,b}^{(x,\rho)} & \downarrow \iota(ab)_*(\rho_{\iota(ab)^{-1}\iota(a)\iota(b)}) \\ & & \iota(ab)_*(x) \end{array}$$

which is an equivariant morphism. $v_{a,b} : a_* \circ b_* \rightarrow (ab)_*$ assembles into a monoidal natural isomorphism. This gives us a categorical action $\underline{G/H} \rightarrow \underline{\text{Aut}}(\mathcal{C}^H)$.

It is not difficult to see there is a canonical monoidal equivalence $(\mathcal{C}^H)^{G/H} \cong \mathcal{C}^G$ (for example [4, Equation 75]). One way to see this is using the strictification result of [8], which implies it suffices to consider the case where the G action on \mathcal{C} is strict. Here it is very easy to build this equivalence by simply rearranging the data of equivariant objects.

4 The Parametrization

In this section, we parameterize *fusion subcategories* of an equivariantization of a fusion category by a group action. If \mathcal{C} is a fusion category, by a fusion subcategory, we mean a full, replete monoidal subcategory (which itself is necessarily a fusion category). These correspond bijectively to unital-based subrings of the fusion ring of \mathcal{C} .

We define the trivial categorical action Tr of a group G on a monoidal category \mathcal{C} as the action that assigns each group element to the identity monoidal functor $\text{Id}_{\mathcal{C}}$, with all structure maps being the identities as well. If we have a categorical action $\alpha = (g_*, \mu_{g,h})_{g,h \in G} : \underline{G} \rightarrow \underline{\text{Aut}}(\mathcal{C})$, a *trivialization* is a monoidal natural isomorphism $\alpha \cong \text{Tr}$. Unpacking this, a trivialization consists of

- For each $g \in G$, a monoidal natural isomorphism $\eta_g : g_* \cong \text{Id}_{\mathcal{C}}$.
- These $\{\eta_g\}_{g \in G}$ must satisfy the equation

$$\eta_g^x \alpha_g(\eta_h^x) = \eta_{gh}^x \mu_{g,h}^x. \quad (4.1)$$

Recall if \mathcal{E} is a full, replete subcategory of \mathcal{C} , we say \mathcal{E} is *invariant* under the G -action if α_g restricts to an autoequivalence of \mathcal{E} .

Definition 4.1 Let $\mathcal{E} \leq \mathcal{C}$ be a G -invariant fusion subcategory, and let $H \leq G$ be a normal subgroup. We say a trivialization η of $\alpha|_H$ on the category \mathcal{E} is G -equivariant if the following diagram commutes:

$$\begin{array}{ccc} h_*(g_*(x)) & \xrightarrow{\text{can}} & g_*((g^{-1}hg)_*(x)) \\ & \searrow \eta_h^{g_*(x)} & \downarrow g_*(\eta_{g^{-1}hg}^x(x)) \\ & & g_*(x) \end{array} \quad (4.2)$$

where **can** is the canonical morphism arising from coherence.

Writing **can** out explicitly gives the equation:

$$g(\eta_{g^{-1}hg}^x \mu_{g^{-1},hg}^x \mu_{h,g}^x)(\mu_{g,g^{-1}}^{h(g(x))})^{-1} = \eta_h^{g_*(x)}.$$

Definition 4.2 Given a group G acting on a monoidal category \mathcal{C} , a *lifting* of a fusion subcategory $\mathcal{E} \leq \mathcal{C}$ is a choice, for every $x \in \mathcal{E}$ of equivariant structure (x, ρ^x) such that

- (1) $(x, \rho^x) \otimes_{\mathcal{C}G} (y, \rho^y) = (x \otimes y, \rho^{x \otimes y})$
- (2) $\text{Hom}_{\mathcal{C}G}((x, \rho^x), (y, \rho^y)) = \text{Hom}_{\mathcal{C}}(x, y)$.

We denote the image of \mathcal{E} in \mathcal{C}^G by $\mathcal{E}^\rho \leq \mathcal{C}^G$.

Lemma 4.3 Let G be a group acting on a monoidal category \mathcal{C} and let $\mathcal{E} \leq \mathcal{C}$ be a fusion subcategory. A family of natural isomorphisms

$$\{\eta_g^x : g_*(x) \rightarrow x : x \in \mathcal{E}, g \in G\}$$

defines a lifting of \mathcal{E} if and only if it defines a trivialization of the G -action on \mathcal{E} .

Proof Given a lifting of \mathcal{E} , we can define a trivialization of the G action on \mathcal{E} as follows. Set $\eta_g^x : g_*(x) \cong x$ by $\eta_g^x := \rho_g^x$. Equation (4.1) is (4.2) in the definition of lifting and Eq. (4.2) is equivalent to the naturality of η_g for every g .

Conversely, given a trivialization of G on \mathcal{E} , then (x, η_g^x) defines an equivariant structure. Since the η_g are natural, we see

$$\mathrm{Hom}_{\mathcal{C}^G}((x, \rho^x), (y, \rho^y)) = \mathrm{Hom}_{\mathcal{C}}(x, y).$$

□

Lemma 4.4 *Given a G action on \mathcal{C} , $\mathcal{E} \leq \mathcal{C}$ be a fusion subcategory, and a lifting ρ of \mathcal{E} to \mathcal{C}^H for some normal subgroup $H \leq G$, then the corresponding trivialization of the H action on \mathcal{E} is G -equivariant if and only if $\mathcal{E}^\rho \leq \mathcal{C}^H$ is invariant under the G -action.*

Proof Let ρ be a lifting of \mathcal{E} to \mathcal{C}^H , and let $\{\eta_h\}_{h \in H}$ be the trivialization defined by $\eta_h^x := \rho_h^x$. If $g \in G$ then $g_*(x, \rho) = (x', \rho')$ where $x' = g_*(x)$ and $\rho_h'^{g_*(x)}$ is defined by the composition

$$h_*(g_*(x)) \xrightarrow{\text{can}} g_*((g^{-1}hg)_*(x)) \xrightarrow{g_*(\rho_h^x)} g_*(x).$$

Hence, $\mathcal{E}^\rho \leq \mathcal{C}^H$ is G -invariant if and only if $\rho' = \rho^{g_*(x)}$, and that is exactly the commutativity of diagram Eq. (4.2). □

Definition 4.5 [12, Definition 3.1]

Let \mathcal{C} be a fusion category, let $\mathcal{E} \subseteq \mathcal{C}$ be a fusion subcategory, and let A be a separable algebra in \mathcal{C} . We will assume that $\mathrm{Hom}_{\mathcal{C}}(A, \mathbb{1}) \cong \mathbb{k}$, that is, that A is a connected algebra. We say that \mathcal{E} is transversal to A if

$$\mathrm{Hom}_{\mathcal{C}}(X, A) = \mathrm{Hom}_{\mathcal{C}}(X, \mathbb{1})$$

for all $X \in \mathcal{E}$.

It is easy to see that transversality is equivalent to requiring

$$\mathrm{Hom}_{\mathcal{C}}(x, A) \cong \mathrm{Hom}_{\mathcal{C}}(x, \mathbb{1})$$

for all $x \in \mathcal{E}$. We denote by \mathcal{C}_A the semi-simple category of right A modules in \mathcal{C} . We have the following lemma:

Lemma 4.6 *Let \mathcal{C} be fusion, $\mathcal{E} \leq \mathcal{C}$ a fusion subcategory, and $A \in \mathcal{C}$ a connected separable algebra. Then, \mathcal{E} is transversal to A if and only if the free module functor $F_A : \mathcal{E} \rightarrow \mathcal{C}_A$, $F_A(x) := x \otimes A$, is fully faithful.*

Proof Let $x, y \in \mathcal{E}$. Suppose \mathcal{E} is transversal to A . Recall that the left adjoint of F_A is simply the forgetful functor, i.e., the functor that forgets the right A -module structure on an object. Then, $\mathrm{Hom}_{\mathcal{C}_A}(F_A(x), F_A(y)) \cong \mathrm{Hom}_{\mathcal{C}}(x, F_A(y)) \cong \mathrm{Hom}_{\mathcal{C}}(y^* \otimes x, A) \cong \mathrm{Hom}_{\mathcal{C}}(y^* \otimes x, \mathbb{1}) \cong \mathrm{Hom}_{\mathcal{C}}(x, y)$. Thus, the inclusion

$$F_A : \mathrm{Hom}_{\mathcal{C}}(x, y) \rightarrow \mathrm{Hom}_{\mathcal{C}_A}(F_A(x), F_A(y))$$

is an isomorphism, hence F_A is fully faithful.

Conversely, if F_A is fully faithful on \mathcal{E} , then

$$\mathrm{Hom}_{\mathcal{C}}(x, \mathbb{1}) \cong \mathrm{Hom}_{\mathcal{C}_A}(F_A(x), F_A(\mathbb{1})) \cong \mathrm{Hom}_{\mathcal{C}}(x, A),$$

so \mathcal{E} is transversal to A . □

Lemma 4.7 *Let \mathcal{C} be a fusion category with G action. There is a bijection between the following:*

- (1) *G -invariant fusion subcategories $\mathcal{D} \leq \mathcal{C}$ with trivializations of the G action on \mathcal{D} .*
- (2) *Fusion subcategories $\mathcal{E} \leq \mathcal{C}^G$ such that $F_{\mathrm{Fun}(G)}$ restricted to \mathcal{E} is fully faithful.*
- (3) *Fusion subcategories of \mathcal{C}^G which are transversal to the algebra $\mathrm{Fun}(G) \in \mathrm{Rep}(G) \leq \mathcal{C}^G$.*

Proof First, we establish a bijection between items (1) and (2). Given $\mathcal{D} \leq \mathcal{C}$ and a trivialization η , this uniquely defines a lifting ρ by Lemma 4.3. Then, we consider the full subcategory $\mathcal{E} := \mathcal{D}^\rho \leq \mathcal{C}^G$, spanned by objects $\{(x, \rho^x)\}_{x \in \mathcal{D}}$. By the definition of lifting, this is a full monoidal category of \mathcal{C}^G , such that the forgetful functor is fully faithful.

In the other direction, suppose we have a $\mathcal{E} \leq \mathcal{C}^G$ such that the forgetful functor is fully faithful. Define \mathcal{D} to be the image of \mathcal{E} under the forgetful functor. Since the forgetful functor is fully faithful, we have

$$\mathrm{Hom}_{\mathcal{E}}((x, \eta^x), (y, \lambda^y)) = \mathrm{Hom}_{\mathcal{C}}(x, y),$$

in particular, that implies that if $(x, \eta^x), (x, \lambda^x) \in \mathcal{E}$, then $\eta^x = \lambda^x$. This follows since $1_x \in \mathrm{Hom}_{\mathcal{C}}(x, x) = \mathrm{Hom}_{\mathcal{E}}((x, \eta^x), (x, \lambda^x))$. Since \mathcal{E} is monoidal, the uniqueness of η^x for all $x \in \mathcal{D}$ implies Eq. (4.1). Now for each object $x \in \mathcal{D}$, by definition, there exists a unique pre-image $(x, \eta^x) \in \mathcal{C}^G$, which is isomorphic to an object in \mathcal{E} , and thus by repleteness is in \mathcal{E} . The isomorphisms $\{\eta^x\}_{x \in \mathcal{D}}$ define a lifting of \mathcal{D} .

The equivalence of items (2) and (3) follows from Lemma 4.6. □

Theorem 4.8 *Let \mathcal{C} be a fusion category with G action. Fusion subcategories of \mathcal{C}^G are parameterized by triples (\mathcal{E}, H, η) where*

- (1) *$\mathcal{E} \leq \mathcal{C}$ is a G -invariant fusion subcategory.*
- (2) *$H \leq G$ is a normal subgroup.*
- (3) *η is a G -equivariant trivialization of H on \mathcal{E} .*

Proof As explained in Sect. 3, a G/H -action on \mathcal{C}^H can be defined given a normal subgroup $H \leq G$ and a section $\iota : G/H \rightarrow G$. This action satisfies $(\mathcal{C}^H)^{G/H} \cong \mathcal{C}^G$. It is worth noting that any pair of sections $\iota, \iota' : G/H \rightarrow G$ are related by a map $\omega : G/H \rightarrow H$ that ensures the associated (non-abelian) 2-cocycles are cohomologous, and the actions associated with ι and ι' are canonically equivalent. Thus, the sequential equivariantization $(\mathcal{C}^H)^{G/H} \cong \mathcal{C}^G$, up to a canonical equivalence, is independent of the choice of ι or the associated 2-cocycle.

Given the above data, let ρ be the lift of \mathcal{E} to \mathcal{C}^H defined by the trivialization (Lemma 4.3). The fusion subcategory $\mathcal{E}^\rho \leq \mathcal{C}^H$ is G/H invariant by Lemma 4.4. Define \mathcal{D} to be the replete image of $(\mathcal{E}^\rho)^{G/H}$ under the canonical equivalence $(\mathcal{C}^H)^{G/H} \cong \mathcal{C}^G$. This gives a subcategory of \mathcal{C}^G from our data.

In the other direction, let $\mathcal{D} \leq \mathcal{C}^G$ be a fusion subcategory. Let $\mathcal{E} \leq \mathcal{C}$ be the full fusion subcategory generated by the image of \mathcal{D} under the forgetful functor $F_G : \mathcal{C}^G \rightarrow \mathcal{C}$. Clearly \mathcal{E} is G -invariant.

Let H be defined by $\text{Rep}(G) \cap \mathcal{D} \cong \text{Rep}(G/H)$. To get the trivialization of the H action on \mathcal{E} note that we have the following diagram of monoidal functors commutes up to canonical natural isomorphism:

$$\begin{array}{ccc} \mathcal{C}^G & & \\ \downarrow F_G & \searrow F_{G/H} & \\ & \mathcal{C}^H & \\ & \swarrow F_H & \\ & \mathcal{C} & \end{array} \quad (4.3)$$

Thus, if we define $\mathcal{E}' = F_{G/H}(\mathcal{D}) \leq \mathcal{C}^H$, then we have $F_H(\mathcal{E}') = F_G(\mathcal{D}) = \mathcal{E}$. Alternatively, using the equivalence $(\mathcal{C}^G)^{G/H} \cong \mathcal{C}^H$, then we see $\mathcal{E}' := (\mathcal{D})_{G/H} \leq \mathcal{C}^H$.

We claim that the separable, connected algebra $\text{Fun}(H) \in (\mathcal{C}^G)_{G/H}$ is transversal to \mathcal{E}' , or in other words, $\text{Fun}(H) \cap \mathcal{E}' \cong \mathbb{1}$. Let $F_{G/H} : \mathcal{C} \rightarrow \mathcal{C}_{\text{Fun}(G/H)}^G \cong \mathcal{C}^H$ be the forgetful functor, which is equivalent as a monoidal functor to $F_{\text{Fun}(G/H)}$. This functor is *normal* by [2, Corollary 5.4]. Thus, by [2, Corollary 4.3], for any two simple objects x, y in \mathcal{C}^G , $F_{G/H}(x)$ and $F_{G/H}(y)$ are either isomorphic or disjoint. Since $\text{Rep}(G)$ and \mathcal{D} are full, replete subcategories of \mathcal{C}^G this implies

$$F_{G/H}(\text{Rep}(G) \cap \mathcal{D}) = F_{G/H}(\text{Rep}(G)) \cap F_{G/H}(\mathcal{D}).$$

But $\text{Rep}(G) \cap \mathcal{D} \cong \text{Rep}(G/H)$, so the left hand side reduces to Vec . The right hand side is $\text{Rep}(H) \cap \mathcal{E}'$. Since $\text{Fun}(H) \in \text{Rep}(H)$ it must be transversal to \mathcal{E}' .

By Lemma 4.7, this implies F_H induces an equivalence from \mathcal{E}' to \mathcal{E} , and such an equivalence uniquely defines a lifting $\mathcal{E} \rightarrow \mathcal{C}^H$ which is G/H invariant (since \mathcal{E}' is G/H invariant by construction). This gives a G -equivariant trivialization η by Lemma 4.3.

It is easy to see that these two constructions are mutually inverse. Indeed, starting with a full subcategory $\mathcal{D} \leq \mathcal{C}^G$, apply our construction to obtain a normal subgroup $H \leq G$, a full subcategory $\mathcal{E} \leq \mathcal{C}$ and a trivialization for H on \mathcal{E} . By construction $\mathcal{E}' = F_{G/H}(\mathcal{D})$ is the lift of \mathcal{E} corresponding to the trivialization of H on \mathcal{E} Eq. (4.7), and $(F_{G/H}(\mathcal{D}))^{G/H} \cong \mathcal{D}$.

Starting from the data (\mathcal{E}, H, η) , then \mathcal{D} is obtained by using η to lift \mathcal{E} to an $\mathcal{E}' \leq \mathcal{C}^H$ Eq. (4.7) and then equivariantizing, so that $\mathcal{D} = (\mathcal{E}')^{G/H}$. We see that applying our construction in the reverse direction gives $F_{G/H}(\mathcal{D}) = F_{G/H}((\mathcal{E}')^{G/H}) = \mathcal{E}'$,

and thus the associated trivialization on \mathcal{E} will agree with the original (again by Lemma 4.7). \square

From the construction, we have the following corollary

Corollary 4.9 *Let $\mathcal{D}(\mathcal{E}, H, \eta) \leq \mathcal{C}^G$ denote the subcategory constructed in the previous theorem. Then,*

- (1) $FPdim(\mathcal{D}(\mathcal{E}, H, \eta)) = [G : H]FPdim(\mathcal{E})$.
- (2) $\mathcal{D}(\mathcal{E}, H, \eta) \leq \mathcal{D}(\mathcal{F}, K, \nu)$ if and only if $\mathcal{E} \leq \mathcal{F}$, $K \leq H$ and $\eta|_K = \nu$ on \mathcal{E} .

5 Pointed Fusion Categories

Let K be a finite group and $\omega \in Z^3(K, \mathbb{k}^\times)$. The data of a categorical action of the group G on $\text{Vec}(K, \omega)$ can be described by the following:

- (1) A homomorphism $G \rightarrow \text{Aut}(K)$, $g \mapsto g_*$
- (2) A collection of scalars $\beta_g^{k,l} \in \mathbb{k}^\times$ for $k, l \in K$, $g \in G$ and a collection of scalars $\mu_{g,h}^k \in \mathbb{k}^\times$ for $k \in K$ and $g, h \in G$, satisfying the following equations:

$$\frac{\omega_{k,l,m}}{\omega_{g_*(k),g_*(l),g_*(m)}} = \frac{\beta_g^{k,l} \beta_g^{kl,m}}{\beta_g^{l,m} \beta_g^{k,lm}} \quad (5.1)$$

$$\mu_{g,h}^k \mu_{f,gh}^k = \mu_{f,g}^{h(k)} \mu_{fg,h}^k \quad (5.2)$$

$$\frac{\beta_{gh}^{k,l}}{\beta_{g_*(k),h_*(l)}^{k,l}} = \frac{\mu_{g,h}^k \mu_{g,h}^l}{\mu_{g,h}^{kl}} \quad (5.3)$$

To realize this data, pick a skeletal model of $\text{Vec}(K, \omega)$ with the associator given by ω explicitly. Then, extend the action of g_* on K linearly to a functor. By semi-simplicity, it suffices to define natural transformations of functors on simple objects. The tensorator $g_*(k \otimes l) \cong g_*(k) \otimes g_*(l)$ is given by $\beta_g^{k,l}$, and Eq. (5.1) above is exactly the required compatibility with the associator. We define the natural isomorphisms $gh_*(k) \cong g_* \circ h_*(k)$ by $\mu_{g,h}^k$. Equation (5.3) ensures these are monoidal natural isomorphisms, and Eq. (5.2) guarantees these assemble into a categorical action.

Now, our goal is to apply our general results to parameterize subcategories of the equivariantization in terms of the above data. The only remaining task is to interpret the G -equivariant trivializations in terms of this data.

Let $L \leq K$ be a G -invariant subgroup, and $H \leq G$ a normal subgroup such that $h_*|_L$ is trivial. First, we can unpack the definition of a trivialization of the H action on L .

It consists of a family $\eta_h^k \in \mathbb{k}^\times$ for $k \in L$, $h \in H$ satisfying for all $g, h \in H$

$$\eta_g^{lk} = \eta_g^l \eta_g^k \beta_g^{l,k} \quad (5.4)$$

$$\eta_g^l \eta_h^l = \eta_{gh}^l \mu_{g,h}^l \quad (5.5)$$

Viewing $\eta : L \times H \rightarrow \mathbb{k}$, we see that it is almost a bicharacter. In particular, it is a character in each variable up to β and μ respectively. Thus, we will call $\eta : L \times H \rightarrow \mathbb{k}$ satisfying the above equations a (β, μ) -bicharacter.

Now, we can unpack the definition of a G -equivariant trivialization to obtain

$$\eta_h^{g_*(k)} = \eta_{g^{-1}hg}^k \left(\frac{\mu_{g^{-1},hg}^k \mu_{h,g}^k}{\mu_{g,g^{-1}}^{gh_*(k)}} \right) \quad (5.6)$$

or equivalently

$$\eta_{ghg^{-1}}^{g_*(k)} = \eta_h^k \left(\frac{\mu_{g^{-1},gh}^k \mu_{ghg^{-1},g}^k}{\mu_{g,g^{-1}}^{g(k)}} \right) \quad (5.7)$$

where we have used in the superscript on the denominator that H is normal in G , and since H acts trivially on L , $g(ghg^{-1})_*(k) = g(k)$ for all $k \in L$.

Note that we can assume $\mu_{h,g}^k$ is *normalized* so that $\mu_{h,g}^k = 1$ for all k if h or g is 1. Then, the defining condition for μ implies $\mu_{g^{-1},g}^k = \mu_{g,g^{-1}}^{g_*(k)}$. Furthermore, the defining equation, together with normalization and the fact that h acts trivially on L implies the equation

$$\mu_{g,h}^k \mu_{g^{-1},gh}^k = \mu_{g^{-1},g}^k.$$

Solving for $\mu_{g^{-1},gh}^k$, replacing $\mu_{g^{-1},g}^k$ with $\mu_{g,g^{-1}}^{g_*(k)}$, and substituting into Eq. (5.7) gives

$$\eta_{ghg^{-1}}^{g_*(k)} = \eta_h^k \left(\frac{\mu_{ghg^{-1},g}^k}{\mu_{g,h}^k} \right). \quad (5.8)$$

Definition 5.1 A (β, μ) bicharacter satisfying Eqs. (5.7) or (5.8) is called a G -equivariant (β, μ) bicharacter.

Corollary 5.2 Consider a categorical action of G on $\text{Vec}(K, \omega)$, given by a homomorphism $G \rightarrow \text{Aut}(K)$, together with (β, μ) as above. Then, fusion subcategories of $\text{Vec}(K, \omega)^G$ are parameterized by triples (L, H, η) where $L \leq K$ is a G -invariant subgroup, H is a normal subgroup of G fixing every element of L , and η is a G -invariant (β, μ) bicharacter.

5.1 $\mathcal{Z}(\text{Vec}(G, \omega))$

Now, we compare our parametrizations with the results of [13]. To state their results, we need to introduce some notation. Let $\omega \in Z^3(G, \mathbb{k}^\times)$. Following [13, Section 5.1]), for any $a, x, y \in G$ define the quantities

$$\eta_a(x, y) := \frac{\omega(x, y, a)\omega(x, y, a^{-1}x^{-1}, x, y)}{\omega(x, y, a^{-1}, y)}$$

$$v_a(x, y) := \frac{\omega(axa^{-1}, aya^{-1}, a)\omega(a, x, y)}{\omega(axa^{-1}, a, y)}$$

Setting $\mu_{g,h}^k := \eta_k(g, h)$ and $\beta_g^{h,k} = v_g(h, k)$, and $g_*(h) := ghg^{-1}$, then it's direct calculations show this data satisfies Eqs. (5.1), (5.2), (5.3) (see [13, Equation 21]) and thus provides a G action on $\text{Vec}(G, \omega)$ via conjugation. It is well known that $\text{Vec}(G, \omega)^G \cong \mathcal{Z}(\text{Vec}(G, \omega))$.

Thus, according to Corollary 5.2, since the action of G on G is by conjugation, the subcategories of $\mathcal{Z}(\text{Vec}(G, \omega))$ are parameterized by triples (L, H, η) , where L, H are commuting normal subgroups of G and η is a G -invariant (β, μ) -bicharacter.

The parameterization in [13] is given by triples (L, H, B) , where L, H are commuting normal subgroups of G , and $B : L \times H \rightarrow \mathbb{k}^\times$ is a G -invariant ω -bicharacter ([13, Definition 5.4]). Using [13, Equation 19], we see that B is an ω -bicharacter if and only if B^{-1} is a (β, μ) -bicharacter in our sense.

5.2 Obstruction Theory

In Corollary 5.2, we parameterized the subcategories of the equivariantization of a pointed fusion category in terms of algebraic data, which can be described as solutions to families of equations with a cohomological flavor. It is desirable to have an obstruction theory that allows one to determine whether such data exists or not, and to be able to count the solutions if they do exist. In this section, we present some results in this direction.

Let G be a finite group acting on $\text{Vec}(K, \omega)$ with associated data $\beta_g^{k,l}$ and $\mu_{g,h}^k$. Let $L \leq K$ be a G -invariant subgroup and $H \leq G$ a normal subgroup such that H acts trivially on L . We will analyze the cohomological conditions for the existence of a G -equivariant (β, μ) -bicharacter $\eta : L \times H \rightarrow \mathbb{k}^\times$.

5.2.1 Existence of (β, μ) Bicharacters.

Since H acts trivially on L , it follows from Eq. 5.1 that

$$(\beta_h)|_{L \times L} \in Z^2(L, \mathbb{k}^\times),$$

for all $h \in H$. It follows from Eq. 5.4 that a *first necessary condition* for the existence of a (β, μ) bicharacter is that the cohomology class of $(\beta_h)|_{L \times L}$ is trivial for all $h \in H$. If the cohomology class $(\beta_h)|_{L \times L}$ is trivial for all $h \in H$, there exist $\tau : L \times H \rightarrow \mathbb{k}^\times$ such that

$$\tau_{l_1 l_2}^h = \tau_{l_1}^h \tau_{l_2}^h \beta_h^{l_1, l_2}$$

holds for all $h \in H, l_1, l_2 \in L$.

The function

$$\begin{aligned}\tilde{\mu} : H \times H \times L &\rightarrow \mathbb{k}^\times \\ (h_1, h_2, l) &\mapsto \mu_{h_1, h_2}^l \frac{\tau_l^{h_1 h_2}}{\tau_l^{h_1} \tau_l^{h_2}},\end{aligned}$$

defines an element in $\tilde{\mu} \in Z^2(H, \text{Hom}(L, \mathbb{k}^\times))$. In fact,

$$\begin{aligned}\tilde{\mu}_{h_1, h_2}^{l_1} \tilde{\mu}_{h_1, h_2}^{l_2} &= \mu_{h_1, h_2}^{l_1} \mu_{h_1, h_2}^{l_2} \frac{\tau_{l_1}^{h_1 h_2}}{\tau_{l_1}^{h_1} \tau_{l_1}^{h_2}} \frac{\tau_{l_2}^{h_1 h_2}}{\tau_{l_2}^{h_1} \tau_{l_2}^{h_2}} \\ &= \mu_{h_1, h_2}^{l_1} \mu_{h_1, h_2}^{l_2} \beta_{h_1}^{l_1, l_2} \beta_{h_2}^{l_1, l_2} \frac{\tau_{l_1}^{h_1 h_2} \tau_{l_2}^{h_1 h_2}}{\tau_{l_1 l_2}^{h_1} \tau_{l_1 l_2}^{h_2}} \\ &= \mu_{h_1, h_2}^{l_1 l_2} \beta_{h_1 h_2}^{l_1, l_2} \frac{\tau_{l_1}^{h_1 h_2} \tau_{l_2}^{h_1 h_2}}{\tau_{l_1 l_2}^{h_1} \tau_{l_1 l_2}^{h_2}} \\ &= \mu_{h_1, h_2}^{l_1 l_2} \frac{\tau_{l_1 l_2}^{h_1 h_2}}{\tau_{l_1 l_2}^{h_1} \tau_{l_1 l_2}^{h_2}} \\ &= \tilde{\mu}_{h_1, h_2}^{l_1 l_2},\end{aligned}$$

for all $l_1, l_2 \in L, h_1, h_2 \in H$. It is easy to see that the cohomology class of $\tilde{\mu} \in Z^2(H, \text{Hom}(L, \mathbb{k}^\times))$ does not depend of the choice of τ . Hence, it follows from Eq. 5.5 that a *second necessary condition* for the existence of a (β, μ) -bicharacter is the triviality of the cohomology of $\tilde{\mu}$. Moreover, the triviality of the cohomology of $\tilde{\mu}$ is a sufficient condition for the existence of a (β, μ) -bicharacter. In fact, if $\chi \in \text{Hom}(L, \mathbb{k}^\times)$ is such that $\tilde{\mu}(h_1, h_2) = \frac{\chi_{h_1 h_2}}{\chi_{h_1} \chi_{h_2}}$, then $\eta_h^l = \tau_h^l \chi_h^l$ is a (β, μ) -bicharacter.

5.2.2 Existence of G -invariant (β, μ) -bicharacters

It is easy to see that the set of all (β, μ) -characters is a torsor over the abelian group of all bicharacters $H \times L \rightarrow \mathbb{k}^\times$. Let $\eta : H \times L \rightarrow \mathbb{k}^\times$ be a (β, μ) -bicharacter. Define the function

$$\begin{aligned}\Omega(\eta, \mu) : G \times H \times L &\rightarrow \mathbb{k}^\times \\ (g, h, l) &\mapsto \left(\frac{\eta_h^l}{\eta_{ghg^{-1}}^{g*(l)}} \right) \left(\frac{\mu_{ghg^{-1}, g}^l}{\mu_{g, h}^l} \right).\end{aligned}\tag{5.9}$$

Proposition 5.3 *The following conditions are necessary and sufficient for the existence of a G -invariant (β, μ) -bicharacter.*

- (1) $\Omega(\eta, \mu)$ defines an element in $Z^1(G, \text{Hom}(L, \text{Hom}(H, \mathbb{k}^\times)))$ whose cohomology class does not depend on the choice of η . Here, the action of G on $\text{Hom}(L, \text{Hom}(H, \mathbb{k}^\times))$ is induced by the action by conjugation of G on the normal subgroup H and the action $*$ of G on L .
- (2) The cohomology class of $\Omega(\eta, \mu)$ vanishes.

Proof Let $\tau : H \times L \rightarrow \mathbb{k}^\times$ be a G -invariant (β, μ) -bicharacter. There is a bicharacter $\chi : L \times H \rightarrow \mathbb{k}^\times$ such that $\eta = \tau \chi$. Hence,

$$\Omega(\eta, \mu)_{g,h,l} = \frac{\chi_h^l}{\chi_{ghg^{-1}}^{g_*(l)}},$$

and

$$\begin{aligned} \Omega(\eta, \mu)_{g_1, h, l} {}^{g_1} (\Omega(\eta, \mu)_{g_2, h, l}) &= \frac{\chi_h^l}{\chi_h^{(g_1)_*(l)}} {}^{g_1} \left(\frac{\chi_h^l}{\chi_h^{(g_2)_*(l)}} \right) \\ &= \frac{\chi_h^l}{\chi_h^{(g_1 g_2)_*(l)}} \\ &= \Omega(\eta, \mu)_{g_1 g_2, h, l}, \end{aligned}$$

so $\Omega(\eta, \mu) \in Z^1(G, \text{Hom}(K, \text{Hom}(H, \mathbb{k}^\times)))$ and the cohomology does not depend on η .

It follows by the definition of $\Omega(\eta, \mu)$ that it vanishes if η is G -invariant. Conversely, if $\Omega(\eta, \mu) \in Z^1(G, \text{Hom}(K, \text{Hom}(H, \mathbb{k}^\times)))$ and there exists a bicharacter χ such that $\Omega(\eta, \mu)_{g,h,l} = \frac{\chi_h^l}{\chi_h^{g_*(l)}}$ then $\tau = \frac{\eta}{\chi}$ is a G -invariant \square

6 The Lattice of Hopf Subalgebras of a Hopf Algebra of Kac-Paljutkin Type

6.1 Hopf Algebra of Kac-Paljutkin Type

Let G be a finite group acting on a finite group K and (β, μ) a data defining an action of G on $\text{Vec}(K)$.

Let us denote $\mathbb{k}^G := \text{Maps}(G, \mathbb{k})$, let $\delta_g \in \mathbb{k}^G$ be the function that assigns 1 to g and 0 otherwise, and let $\delta_{g,h}$ be the Dirac's delta associated to the pair $g, h \in G$, namely $\delta_{g,h}$ is 1 whenever $g = h$ and 0 otherwise.

The vector space $\mathbb{k}^G \#_\mu^\beta \mathbb{k}K$ with basis $\{\delta_g \# x \mid g \in G, x \in K\}$ is a Hopf algebra with product

$$(\delta_g \# x)(\delta_h \# y) = \delta_{g,h} \beta_g^{x,y} \delta_g \# xy,$$

coproduct,

$$\Delta(\delta_g \# x) = \sum_{a,b \in G: ab=g} \mu_{a,b}^x \delta_a \# b \cdot x \otimes \delta_b \# x,$$

counit, unit and antipode

$$\varepsilon(\delta_g \# x) = \delta_{g,e}, \quad 1 \# e, \quad S(\delta_g \# x) = \frac{1}{\beta_{(gx)^{-1}}^{gx, g^{-1}} \mu_{g^{-1}, g}^x} \delta_{g^{-1}} \# (gx)^{-1},$$

for all $g, h \in G, x, y \in K$. See [11] for more details.

Corollary 6.1 *Let $\mathbb{k}^G \#_{\mu}^{\beta} \mathbb{k}K$ be a semisimple Hopf algebra of Kac-Paljutkin type, then the lattice of Hopf subalgebras is in correspondence with the lattice of triples (L, H, η) where*

- (1) $L \leq K$ is a G -invariant subgroup.
- (2) $H \leq G$ is a normal subgroup.
- (3) $\eta : H \times L \rightarrow \mathbb{k}^{\times}$ is a G -invariant (β, μ) -bicharacter.

Proof It follows from Corollary 5.2 that the triples are in correspondence with fusion subcategories of $\text{Vec}(K)^G$. Now, the fusion category of right $\mathbb{k}^G \#_{\mu}^{\beta} \mathbb{k}K$ -comodules is tensor isomorphic to the fusion category $\text{Vec}(K)^G$, (see [9, Theorem 7.1]) hence by [14, Theorem 6] they correspond to Hopf subalgebras of $\mathbb{k}^G \#_{\mu}^{\beta} \mathbb{k}K$. \square

6.2 Concrete Example

Let $G = C_2 = \langle \sigma \rangle$ a cyclic group of order two and $K = \mathbb{Z}/n \times \mathbb{Z}/n$. Consider the action of G on K given by $\sigma(a, b) = (b, a)$. Since we can consider only normalized maps $\beta : C_2 \times K \times K \rightarrow \mathbb{k}^{\times}$ $\mu : C_2 \times C_2 \times K \rightarrow \mathbb{k}^{\times}$, hence the pair (β, μ) is complete determined by elements $\beta \in Z^2(K, \mathbb{k}^{\times})$ and $\mu \in C^1(K, \mathbb{k}^{\times})$ such that

$$\delta_K(\mu) = \beta(\sigma \beta), \quad \mu(\sigma \mu) = 1. \quad (6.1)$$

Up to equivalences, the pairs (β, μ) are classified by \mathbb{G}_n the group n -th root of unit, see [10, Theorem 2.1]. In fact, given $q \in \mathbb{G}_n$, we can define

$$\beta_q(\vec{a}, \vec{b}) = q^{a_1 b_2}, \quad \mu_q(\vec{a}) = q^{-a_1 a_2}, \quad (6.2)$$

where $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$.

We now can divide the possible Hopf subalgebras of $\mathbb{k}^{C_2} \#_{\mu_q}^{\beta_q} \mathbb{k}(\mathbb{Z}/n \times \mathbb{Z}/n)$ in two cases depending if $H \leq C_2$ is trivial or not.

6.2.1 When $H = C_2$

In this case, the subgroup L is just a subgroup of $\Delta(\mathbb{Z}/n) = \{(a, a) : a \in \mathbb{Z}/n\}$. Let m be a divisor of n and $x = n/m$. Hence, $L = \langle (x, x) \rangle$ and

$$\beta_q(a(x, x), b(x, x)) = q^{x^2 ab}.$$

The 1-cochain

$$\tau_q(l(x, x)) = q^{-x^2 \frac{l(l+1)}{2}}$$

satisfies $\delta_K(\tau_q) = \beta_q$ for all $l(x, x) \in L$.

In order to compute the obstruction, let us consider

$$\begin{aligned} \tilde{\mu}_q(l(x, x)) &= \frac{\mu_q(l(x, x))}{\tau_q(l(x, x))^2} \\ &= q^{-l^2 x^2} q^{x^2 l(l+1)} \\ &= q^{x^2 l}. \end{aligned}$$

Then, the obstruction vanishes if and only if there is $\zeta \in \mathbb{G}_m$ such that

$$q^{x^2} = \zeta^2.$$

If $q^{x^2} = \zeta^2$, the associated (β, μ) -bicharacter is

$$\eta_\zeta(l(x, x)) = \tau_q(l(x, x))\zeta^l = q^{-x^2 \frac{l(l+1)}{2}} \zeta^l.$$

Moreover every η_ζ is invariant.

6.2.2 When H is Trivial

In general, if H is trivial the fusion subcategories associated are given just by pairs (L, η) where $L \leq K$ is a G -invariant subgroup and $\eta \in \widehat{G}$ is G -invariant linear character.

Using Goursat's lemma it is straightforward to see that C_2 -invariant subgroups of $\mathbb{Z}/n \times \mathbb{Z}/n$ corresponds to triple (B, N, f) where $N \subset B \subset \mathbb{Z}/n$ is a tower of groups and $f : B/N \rightarrow B/N$ is an automorphism of order two. The subgroup associated with the triple (B, N, f) is

$$H_{(B, N, f)} = \{(a, b) : f(aN) = bN\}.$$

As a conclusion of the Sections 6.2.1 and 6.2.2, we obtain the following result.

Theorem 6.2 *The Hopf subalgebras of $\mathbb{k}^{C_2 \#_{\mu_q}^{\beta_q} \mathbb{k}(\mathbb{Z}/n \times \mathbb{Z}/n)}$ come in two types, and correspond to*

Type 1: *pairs (m, ζ) where m is divisor of n and ζ is a root of unity of order $\frac{n}{m}$ such that $q^{(\frac{n}{m})^2} = \zeta^2$.*

Type 2: *four-tuples (B, N, f, η) where $N \subset B \subset \mathbb{Z}/n$ is a tower of groups, $f : B/N \rightarrow B/N$ is an automorphism of order two, and $\eta : H_{(B, N, f)} \rightarrow \mathbb{k}^\times$ is a character such that $\eta(a, b) = \eta(b, a)$ for all $(a, b) \in H_{(B, N, f)}$.*

□

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Declarations

Conflict of Interest The authors declare no competing interests.

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References

1. Angiono, I., Galindo, C., Pereira, M.: De-equivariantization of Hopf algebras. *Algebr. Represent. Theory*, **17**(1), 161–180 (2014)
2. Bruguières, A., Burciu, S.: On normal tensor functors and coset decompositions for fusion categories. *Appl. Categ. Structures* **23**(4), 591–608 (2015)
3. Burciu, S., Natale, S.: Fusion rules of equivariantizations of fusion categories. *J. Math. Phys.* **54**(1), 013511, 21, (2013)
4. Drinfeld, V., Gelaki, S., Nikshych, D., Ostrik, V.: On braided fusion categories. I. *Selecta Math. (N.S.)* **16**(1), 1–119 (2010)
5. Etingof, P., Gelaki, S., Nikshych, D., Ostrik, V.: *Tensor categories*. Mathematical Surveys and Monographs, vol. 205. American Mathematical Society, Providence, RI (2015)
6. Etingof, P., Nikshych, D., Ostrik, V.: On fusion categories. *Ann. of Math. (2)*, **162**(2), 581–642 (2005)
7. Etingof, P., Nikshych, D., Ostrik, V.: Weakly group-theoretical and solvable fusion categories. *Adv. Math.* **226**(1), 176–205 (2011)
8. Galindo, C.: Coherence for monoidal G -categories and braided G -crossed categories. *J. Algebra* **487**, 118–137 (2017)
9. Galindo, C., Gutiérrez, I., Uribe, B.: Multiplicative structures on the twisted equivariant K-theory of finite groups. *J. Noncommut. Geom.* **9**(3), 877–937 (2015)
10. Masuoka, A.: Calculations of some groups of Hopf algebra extensions. *J. Algebra* **191**(2), 568–588 (1997)
11. Masuoka, A.: Hopf algebra extensions and cohomology. In: *New directions in Hopf algebras*, volume 43 of *Math. Sci. Res. Inst. Publ.*, pages 167–209. Cambridge Univ. Press, Cambridge, (2002)
12. Nikshych, D.: Classifying braidings on fusion categories. In *Tensor categories and Hopf algebras*, volume 728 of *Contemp. Math.*, pages 155–167. Amer. Math. Soc., [Providence], RI, [2019] (2019)
13. Naidu, D., Nikshych, D., Witherspoon, S.: Fusion subcategories of representation categories of twisted quantum doubles of finite groups. *Int. Math. Res. Not. IMRN* **22**, 4183–4219 (2009)
14. Nichols, W.D., Richmond, M.B.: The Grothendieck group of a Hopf algebra. *J. Pure Appl. Algebra* **106**(3), 297–306 (1996)
15. Sekine, Y.: An example of finite-dimensional Kac algebras of Kac-Paljutkin type. *Proc. Amer. Math. Soc.* **124**(4), 1139–1147 (1996)

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