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On the Serre Functor in the Category of Strict Polynomial Functors

Marcin Chałupnik[1]

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Abstract
We study a Serre functor in functor categories related to the category $\mathcal{P}_d$ of strict polynomial functors over a field of positive characteristic. Our main result shows that the derived category of the category of affine strict polynomial functors in some cases carries the structure of Calabi–Yau category. We also re-obtain the Poincaré duality formulas for Ext groups in $\mathcal{P}_d$ and construct a certain recollement diagram relating the derived categories of affine and ordinary strict polynomial functors.

Keywords Strict polynomial functor · Serre functor · Calabi–Yau category

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1 Introduction

In the present article we study the Serre functor in the derived category of the category $\mathcal{P}_d$ of strict polynomial functors of degree d over a field of positive characteristic. Although the existence of Serre functor in our context follows from general theory, and it was constructed in [15], we investigate its interplay with various structures existing on $\mathcal{P}_d$ (Frobenius twist, affine subcategories, blocks), which has some interesting consequences.

We start by studying the Serre functor $\mathbf{S}$ in the bounded derived category of $\mathcal{P}_d$. The main goal of Section 2 is to show that the Poincaré type formulas for Ext groups in $\mathcal{P}_d$ obtained in [6] and [17] are a consequence of the interplay between $\mathbf{S}$ and the Frobenius twist, which motivates the further part of the paper. Namely we obtain

Corollary 2.4 Let λ be a Young diagram of weight d which is single in its block (we call such a diagram and its block *basic*), let μ be any Young diagram of weight

✉ Marcin Chałupnik
mchal@mimuw.edu.pl

1 Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02–097 Warsaw, Poland

dp^i. Let F_λ, F_μ be the corresponding simple objects. Then there is an isomorphism of linear spaces:

$$\mathrm{Ext}^s_{\mathcal{P}_{dp^i}}(F_\lambda^{(i)}, F_\mu) \simeq \mathrm{Ext}^{2d(p^i-1)-s}_{\mathcal{P}_{dp^i}}(F_\lambda^{(i)}, F_\mu)^*,$$

as a formal consequence of basic properties of $\mathbf{S}$.

The main part of the paper consists of Sections 3–5. Our goal is to put the Poincaré duality formula from Corollary 2.4 into a wider categorical context. To this end we turn attention to the category $\mathcal{P}_d^{af_i}$ of i-affine strict polynomial functors of degree d, and we introduce a (somewhat weaker version of) Serre functor on its derived category $\mathcal{DP}_d^{af_i}$. In general, a Serre functor produces Poincaré duality in Ext groups when it acts on some object as the shift functor. Indeed, we see in our Proposition 2.3 that this exactly happens for some Frobenius twisted strict polynomial functors (in fact, Corollaries 2.4 and 2.5 are formal consequences of Proposition 2.3). We provide a categorical interpretation of this phenomenon by investigating a Serre functor on the derived category of the category $\mathcal{DP}_d^{af_i}$ of i-affine strict polynomial functors of degree d (c.f. [7]), which is a full triangulated subcategory of $\mathcal{DP}_{dp^i}$ generated the i-times Frobenius twisted functors (c.f. [7]). Namely, we find certain subcategories, called "basic affine semiblocks" of $\mathcal{DP}_{dp^i}$, which correspond to "basic blocks" appearing in Cor. 2.5, on which the Serre functor is isomorphic to the shift functor (such categories are called Calabi–Yau). Thus we have succeeded in providing a categorical interpretation of both assumptions in Corollary 2.4: we specialize to $\mathcal{DP}_d^{af_i}$ because $F_\lambda^{(i)}$ is twisted and we restrict to the image of the block containing F_λ to take advantage of the fact that λ is basic.

Below we describe the contents of the article in more detail.

Section 2 studies basic properties of the Serre functor in $\mathcal{P}_d$ (Th. 2.2, Corollary 2.3) and shows how they lead to the Poincaré duality formulas in $\mathcal{P}_d$ (Cor. 2.4). We also point out that in Cor. 2.5 we obtain a new example of Poincareé duality, which suggests that the approach to the Poincaré duality presented here may be more flexible than that from [6].

Then in Section 3 we study a Serre functor in $\mathcal{DP}_d^{af_i}$. We start with reviewing basic properties of the categories $\mathcal{P}_d^{af_i}$ and $\mathcal{DP}_d^{af_i}$. This is mainly a recollection of some facts from [7] where these concepts were introduced and adapting them to a slightly more general setting of "multiple twists" in which we work in the present article. A new ingredient is the "affine Kuhn duality" which is useful in all kinds of duality issues. In order to have this piece of structure we were forced (in contrast to [7]) to allow unbounded complexes in our derived categories. In the next subsection we use the affine Kuhn duality to obtain the "affine recollement diagram" which is not used in the rest of the paper but may be of independent interest. Then we proceed to define the Serre functor $\mathbf{S}^{af_i}$ in $\mathcal{DP}_d^{af_i}$. However, we need to adapt this notion to the fact that $\mathcal{DP}_d^{af_i}$ has infinite dimensional Hom-spaces. Hence, technically, we define "a weak Serre functor " (Definition 3.14, 3.15) on $\mathcal{DP}_d^{af_i}$.

In Section 4 we introduce "the semiblock decomposition" of $\mathcal{DP}_d^{af_i}$. This is a collection of reflective subcategories of $\mathcal{DP}_d^{af_i}$ indexed by the set of blocks in $\mathcal{P}_d$.

They generate $\mathcal{DP}_d^{af_i}$ but in contrast to genuine blocks they are not orthogonal. We believe that this structure deserves a further investigation, however in the present article we content ourselves to introducing the affine derived Kan extension and Serre functor on the semiblocks. This is a non-trivial task due to the non-orthogonality of semiblocks.

In Section 5 we focus on the basic semiblocks, i.e., the subcategories of $\mathcal{DP}_d^{af_i}$ which correspond to the blocks containing a single simple object. We establish the main result of the article

Theorem 5.1 For any basic Young diagram λ, the category $\mathcal{DP}_\lambda^{af_i,b}$ is Calabi–Yau of dimension $2d(p^i-1)$, the category $\mathcal{DP}_\lambda^{af_i}$ is weak Calabi–Yau of dimension $2d(p^i-1)$.

thus providing the promised categorical interpretation of our Poincaré duality formulas. We finish our article by giving various explicit descriptions of basic semiblocks as categories of DG modules over certain graded algebras (Proposition 5.5, Corollary 5.6) which should make them easier to handle.

2 Serre Functor in $\mathcal{DP}_d$

Let $\mathcal{P}_d$ be the category of strict polynomial functors of degree d over a fixed field $\mathbf{k}$ of characteristic $p>0$ as defined in [11]. Since $\mathcal{P}_d$ is an artinian category of finite homological dimension, it follows from [3] that its bounded derived category possesses a Serre functor. The Serre functor on $\mathcal{DP}_d$ was studied by Krause in [15] who described it explicitly and established its basic properties. In the present section we recall Krause's approach, and show how by using the interplay between the Serre functor and the Frobenius twist to re-obtain (and slightly extend) the Poincaré dualities from [6].

We start with recalling standard notations concerning strict polynomial functors. Let $\mathcal{V}$ stand for the category of finite dimensional vectors spaces over $\mathbf{k}$, and let $\Gamma^d\mathcal{V}$ stand for the category of dth divided powers over $\mathcal{V}$. By this we mean that the objects of $\Gamma^d\mathcal{V}$ are those of $\mathcal{V}$ but

$$\mathrm{Hom}_{\Gamma^d\mathcal{V}}(V,W) := \Gamma^d(\mathrm{Hom}(V,W)),$$

where for a vector space X, $\Gamma^d(X)$ stands for the space of symmetric d-tensors on X. Then the category $\mathcal{P}_d$ of strict polynomial functors of degree d is the category $\mathbf{k}$-linear functors from $\Gamma^d\mathcal{V}$ to $\mathcal{V}$ (c.f. [12, Sect. 3]). For a finite dimensional $\mathbf{k}$-vector space U we define the strict polynomial functor $S^d_{U^*}\in\mathcal{P}_d$ by the formula

$$V\mapsto S^d(U^*\otimes V).$$

Then by the Yoneda lemma (c.f. [11, Th. 2.10]) we have the natural in U and $F\in\mathcal{P}_d$ isomorphism

$$\mathrm{Hom}_{\mathcal{P}_d}(F,S^d_{U^*})\simeq F(U)^*.$$

It immediately follows from this formula that $S^d_{U^*}$ is injective and it was shown in [11, Th. 2.10] that if $\dim(U)\geq d$ then $S^d_{U^*}$ is a cogenerator of $\mathcal{P}_d$. Dually, we have a family of projective objects $\Gamma^d_{U^*}$ for which the Yoneda lemma gives the isomorphism

$$\mathrm{Hom}_{\mathcal{P}_d}(\Gamma^d_{U^*}, F) \simeq F(U).$$

Let $\mathcal{DP}_d$ denote the bounded derived category of $\mathcal{P}_d$. In order to describe explicitly the Serre functor on $\mathcal{DP}_d$, we shall regard the assignment $(V, W) \mapsto S^d(V^* \otimes W) =: S^d(I^* \otimes I)$ as a strict polynomial bifunctor of degree (d, d) in the sense of [10]. It was shown in [7, Prop. 4.1], [5, pp. 10020–10021] that the category $\mathcal{P}^d_d$ of strict polynomial bifunctors of degree (d, d) has a finite homological dimension and that taking Hom with respect to the covariant variable produces the left balanced functor:

$$\mathcal{H}om_{\mathcal{P}_d} : (\mathcal{P}^d_d)^{op} \times \mathcal{P}_d \longrightarrow \mathcal{P}_d,$$

which has the total derived functor:

$$\mathcal{RH}om_{\mathcal{P}_d} : (\mathcal{DP}^d_d)^{op} \times \mathcal{DP}_d \longrightarrow \mathcal{DP}_d.$$

Definition 2.1 We define a functor $\mathbf{S} : \mathcal{DP}_d \longrightarrow \mathcal{DP}_d$ by the formula:

$$\mathbf{S}(F) := \mathcal{RH}om_{\mathcal{P}_d}(S^d(I^* \otimes I), F).$$

Practically, for $F \in \mathcal{P}_d$, we have:

$$H^*(\mathbf{S}(F))(V) \simeq \mathrm{Ext}^*_{\mathcal{P}_d}(S^d_{V^*}, F).$$

One can see that when we explicitly write down the definition of Serre functor given in [15] in terms of the monoidal structure on $\mathcal{P}_d$, we get $(-)^\# \circ \mathbf{S} \circ (-)^\#$, where $(-)^\#$ stands for the Kuhn duality. Conjugating by the Kuhn duality is a consequence of the fact that, as we will see in a moment, we obtain the left Serre functor (which is easier to describe in terms of Hom-functors), while Krause considers the right one.

Now we gather the basic properties of $\mathbf{S}$ (c.f. [15, Cor. 5.5]):

Theorem 2.2 *The functor $\mathbf{S}$ satisfies the following properties:*

1. *There is a natural in U isomorphism in $\mathcal{DP}_d$:*

$$\mathbf{S}(S^d_{U^*}) \simeq \Gamma^d_{U^*}.$$

2. *There is an isomorphism of functors:*

$$\mathbf{S} \simeq \Theta \circ \Theta$$

where Θ is the "Koszul duality" functor from [4] *given by the formula:*

$$\Theta(F) := \mathcal{RH}om_{\mathcal{P}_d}(\Lambda^d(I^* \otimes I), F),$$

where $\Lambda^d(I^* \otimes I)(V, W) := \Lambda^d(V^* \otimes W)$.

3. *For any $F \in \mathcal{DP}_d$, $G \in \mathcal{DP}_{d'}$ there are isomorphisms in respectively $\mathcal{DP}_{dp^i}$, $\mathcal{DP}_{d+d'}$:*
   - $\mathbf{S}(F^{(i)}) \simeq \mathbf{S}(F)^{(i)}[-2d(p^i - 1)]$
   - $\mathbf{S}(F \otimes G) \simeq \mathbf{S}(F) \otimes \mathbf{S}(G)$.

4. *$\mathbf{S}$ is a self-equivalence of $\mathcal{DP}_d$.*

5. *There is a natural in $F, G \in \mathcal{DP}_d$ isomorphism*

$$Hom_{\mathcal{DP}_d}(F, G) \simeq Hom_{\mathcal{DP}_d}(\mathbf{S}(G), F)^*,$$

that is, $\mathbf{S}$ is a left Serre functor in the sense of [2].

Proof To see the first part we recall that since $S^d_{U^*}$ is injective, we have a chain of quasi-isomorphisms:

$$\mathbf{S}(S^d_{U^*})(V) = \mathrm{Hom}_{\mathcal{DP}_d}(S^d_{V^*}, S^d_{U^*}) \simeq \mathrm{Hom}_{\mathcal{P}_d}(S^d_{V^*}, S^d_{U^*}) \simeq S^d_{V^*}(U^*)^* \simeq \Gamma^d_{U^*}(V)$$

by the Yoneda lemma.

In fact, [4, Prop. 2.2] can be easily extended to the "parameterized version":

$$\Theta((S_\lambda)_{U^*}) \simeq (W_{\tilde{\lambda}})_{U^*},$$

where S_λ stands for the Schur functor associated to the Young diagram λ and $W_{\tilde{\lambda}}$ stands for the Weyl functor associated to the transposed Young diagram $\tilde{\lambda}$. From this we obtain the isomorphisms: $\Theta(S^d_{U^*}) \simeq \Lambda^d_{U^*}$ and $\Theta(\Lambda^d_{U^*}) \simeq \Gamma^d_{U^*}$ which give the second part.

The formulas from part 3 follow from the analogous facts holding for Θ [4, Prop. 2.6].

The fact that $\mathbf{S}$ is an equivalence follows from a general argument (we will resort to it in a more general setting in Section 3), but it will be useful to explicitly describe the inverse of $\mathbf{S}$. Namely, it follows from [4, Def. 2.3, Cor. 2.4] that the"right Serre functor" $\mathbf{S}_r := (-)^\# \circ \mathbf{S} \circ (-)^\#$ where $(-)^\#$ is the Kuhn duality, is the inverse of $\mathbf{S}$.

In order to obtain the last part, it suffices to establish a natural in U isomorphism

$$\mathrm{Hom}_{\mathcal{DP}_d}(F, S^d_{U^*}) \simeq \mathrm{Hom}_{\mathcal{DP}_d}(\mathbf{S}(S^d_{U^*}), F)^*.$$

By the first part and the injectivity of $S^d_{U^*}$ and projectivity of $\Gamma^d_{U^*}$ it reduces to

$$\mathrm{Hom}_{\mathcal{P}_d}(F, S^d_{U^*}) \simeq \mathrm{Hom}_{\mathcal{P}_d}(\Gamma^d_{U^*}, F)^*,$$

which follows from the Yoneda lemma. □

The fact that $\mathbf{S}$ is a Serre functor can be used to obtain the Poincaré like formulas for the Ext groups, provided that we are able to compute $\mathbf{S}(F)$ in some interesting cases. We shall illustrate this idea by re-obtaining the most important example of the Poincaré duality formula for Ext groups in $\mathcal{P}_d$ established in [6].

Let λ be a Young diagram of weight (=size) d which is a p-core. We recall that the blocks in $\mathcal{P}_d$ are indexed by the p-core Young diagrams of weight $d - jp$ for $j \geq 0$ (c.f. [19, Section 5], it was overlooked in [6] that the description of blocks for $\mathcal{P}_d$ is simpler than that for the Schur algebra in general). Thus the block labeled by λ contains only one simple object F_λ. We call such a Young diagram λ and the corresponding block *basic*.

Proposition 2.3 *Let λ be a basic Young diagram of weight d. Then*

$$\mathbf{S}(F_\lambda^{(i)}) \simeq F_\lambda^{(i)}[-2d(p^i - 1)].$$

Proof Since F_λ is single in its block, we have the isomorphisms: $F_\lambda \simeq S_\lambda \simeq W_\lambda$. Therefore:

$$\Theta(F_\lambda) \simeq \Theta(S_\lambda) = W_{\widetilde{\lambda}}.$$

Now, since $\widetilde{\lambda}$ is a p-core, also $F_{\widetilde{\lambda}}$ is single in its block. Hence we obtain:

$$\Theta(W_{\widetilde{\lambda}}) \simeq \Theta(S_{\widetilde{\lambda}}) = W_\lambda \simeq F_\lambda.$$

Thus we see that $\mathbf{S}(F_\lambda) \simeq \Theta^2(F_\lambda) \simeq F_\lambda$ and our formula follows from Theorem 2.2(3). □

The Poincaré duality formula [6, Example 3.3] is a formal consequence of Proposition 2.3.

Corollary 2.4 *Let λ be a basic Young diagram of weight d, μ be any Young diagram of weight dp^i, and F_λ, F_μ be the corresponding simple objects. Then*

$$Ext^s_{\mathcal{P}_{dp^i}}(F_\lambda^{(i)}, F_\mu) \simeq Ext^{2d(p^i-1)-s}_{\mathcal{P}_{p^i d}}(F_\lambda^{(i)}, F_\mu)^*.$$

Proof By applying first the Kuhn duality $(-)^\#$ (we recall that simple objects are self-dual with respect to $(-)^\#$), then the Serre functor and finally using Proposition 2.3 we obtain:

$$\mathrm{Ext}^s_{\mathcal{P}_{p^i d}}(F_\lambda^{(i)}, F_\mu) \simeq \mathrm{Ext}^s_{\mathcal{P}_{p^i d}}(F_\mu, F_\lambda^{(i)}) = \mathrm{Hom}_{\mathcal{DP}_{dp^i}}(F_\mu, F_\lambda^{(i)}[s]) \simeq$$

$$\mathrm{Hom}_{\mathcal{DP}_{dp^i}}(\mathbf{S}(F_\lambda^{(i)}[s]), F_\mu)^* \simeq \mathrm{Hom}_{\mathcal{DP}_{dp^i}}(F_\lambda^{(i)}[s-2d(p^i-1)], F_\mu)^* \simeq$$

$$\mathrm{Ext}^{2d(p^i-1)-s}_{\mathcal{P}_{dp^i}}(F_\lambda^{(i)}, F_\mu)^*.$$

□

The next example shows that our approach to the Poincaré duality is more flexible than that used in [6].

Corollary 2.5 *There is a natural in $U \in \mathcal{V}$ and $F \in \mathcal{P}_{dp^i}$ isomorphism*

$$Ext^s_{\mathcal{P}_{p^i d}}(\Gamma^{d(i)}_{U^*}, F) \simeq Ext^{2d(p^i-1)-s}_{\mathcal{P}_{p^i d}}(\Gamma^{d(i)}_{U^*}, F^\#)^*.$$

Proof By applying the right Serre functor and the Kuhn duality we obtain

$$\mathrm{Ext}^s_{\mathcal{P}_{p^i d}}(\Gamma^{d(i)}_{U^*}, F) \simeq \mathrm{Ext}^s_{\mathcal{P}_{p^i d}}(F, \mathbf{S}_r(\Gamma^{d(i)}_{U^*}))^* \simeq \mathrm{Ext}^s_{\mathcal{P}_{p^i d}}(F, S^{d(i)}_{U^*}[-2d(p^i-1)])^* \simeq$$

$$\simeq \mathrm{Ext}^s_{\mathcal{P}_{p^i d}}(\Gamma^{d(i)}_{U^*}[2d(p^i-1)], F^\#)^* \simeq \mathrm{Ext}^{2d(p^i-1)-s}_{\mathcal{P}_{p^i d}}(\Gamma^{d(i)}_{U^*}, F^\#)^*.$$

□

Let us observe that our Corollary 2.5 does not follow from [6, Th. 3.2], since $\Gamma^d_{U^*}$ does not satisfy the assumption that its endomorphism algebra (which is the Schur algebra) is a symmetric Frobenius algebra. This shows that our current approach to the Poincaré duality is more direct than that of [6] and may have a wider range of applications.

In the further part of the paper we will try to put the mechanism which turns the Serre duality into the Poincaré duality into a wider categorical context.

3 Serre Functor in Affine Categories

The aim of this section is to introduce a suitable version of Serre functor on the category of affine strict polynomial functors. The notion of affine strict polynomial functor was introduced in [7]. In the present paper we generalize it in two directions: firstly we consider the case of multiple twists (i.e., our category $\mathcal{P}_d^{af_i}$ for $i = 1$ corresponds to the original category $\mathcal{P}_d^{af}$ from [7]), and secondly we allow unbounded graded objects. Both changes are rather innocuous but for the reader's convenience we briefly explain in Section 3.1 how to adapt the ideas of [7] to this more general context. Then in Section 3.2 we introduce the affine Kuhn duality. This structure was not investigated in [7] since it requires unbounded complexes (the Kuhn dual of a complex bounded below is bounded above). However, in order to produce the Poincaré duality at the level of affine categories we need this structure. This is the reason for which we switched in the present article to unbounded graded objects and complexes. In Section 3.3 we derive from the results of Section 3.2 the existence of a certain recollement diagram. This fact is interesting for its own but will not be used in the further part of the article. Finally, in Section 3.4 we endow the derived category of $\mathcal{P}_d^{af_i}$ with a Serre functor.

3.1 Review of *i*-Affine Functors

In this subsection we briefly describe the theory of i-affine strict polynomial functors, which for $i = 1$ specializes to the theory of affine strict polynomial functors developed in [7]. In fact the proofs of all the results of [7, Sections 2–3] carry over to the current situation, hence here we just set up the framework and terminology and formulate relevant facts. However handling the formality phenomena, which corresponds to [7, Section 4] requires a substantial extension of the tools used there, hence we discuss the relevant material in greater detail.

We start with a graded algebra $A_i := \mathbf{k}[x_1, x_2, \ldots, x_i]/(x_1^p, x_2^p, \ldots x_i^p)$ for $|x_j| = 2p^j$ (the reader of [7] sees that for $i = 1$ we get the graded algebra A studied there) and set $\Gamma^d\mathcal{V}_{A_i}$ to be the following graded $\mathbf{k}$-linear category. The objects of $\Gamma^d\mathcal{V}_{A_i}$ are finite dimensional vector spaces, though we follow the convention taken in [7, Section 2] and label them as $V \otimes A_i$ where V is a finite dimensional vector space. The morphisms are given as

$$\mathrm{Hom}_{\Gamma^d\mathcal{V}_{A_i}}(V \otimes A_i, W \otimes A_i) := \Gamma^d(\mathrm{Hom}(V, W) \otimes A_i).$$

Let $\mathcal{V}^f$ stand for the category of graded vector spaces over $\mathbf{k}$, finite dimensional in each degree. An i-affine strict polynomial functor of degree d is a graded functor from $\Gamma^d\mathcal{V}_{A_i}$ to $\mathcal{V}^f$. We point out for the difference with [7] here: we do not assume that our functors are bounded below (i.e., we replace the category $\mathcal{V}^{f+}$ from [7] with $\mathcal{V}^f$). The i-affine strict polynomial functors of degree d form the $\mathbf{k}$-linear graded abelian category $\mathcal{P}_d^{af_i}$ with morphisms being the natural transformations. For any finite dimensional vector space U we have the representable i-affine strict polynomial functors of degree d, $h^{U\otimes A_i}$ given by the formula

$$V \otimes A_i \mapsto \mathrm{Hom}_{\Gamma^d \mathcal{V}_{A_i}}(U \otimes A_i, V \otimes A_i) = \Gamma^d(\mathrm{Hom}(U, V) \otimes A_i),$$

and by the Yoneda lemma [7, Prop. 2.2] we have

$$\mathrm{Hom}_{\mathcal{P}_d^{af_i}}(h^{U\otimes A_i}, F) \simeq F(U \otimes A_i).$$

Similarly, we have the corepresentable functor $c^*_{U\otimes A_i}$ given by

$$V \otimes A_i \mapsto \mathrm{Hom}_{\Gamma^d \mathcal{V}_{A_i}}(V \otimes A_i, U \otimes A_i)^*$$

where $(-)^*$ stands for the graded $\mathbf{k}$-linear dual. This time the Yoneda lemma gives

$$\mathrm{Hom}_{\mathcal{P}_d^{af_i}}(F, c^*_{U\otimes A_i}) \simeq F(U \otimes A_i)^*.$$

Analogously to the non-affine case, $\mathcal{P}_d^{af_i}$ is equivalent to a certain module category. Namely, we define the i-affine Schur algebra $S_{d,n}^{af_i} := \Gamma^d(\mathrm{End}(\mathbf{k}^n) \otimes A_i)$. Then $F(\mathbf{k}^n \otimes A_i)$ is naturally a graded $S_{d,n}^{af_i}$-module and we have (c.f. [7, Prop. 2.5])

Proposition 3.1 *If $n \geq d$ then*

$$ev_n : \mathcal{P}_d^{af_i} \longrightarrow S_{d,n}^{af_i}\text{-}mod^f,$$

where $S_{d,n}^{af}$-mod^f is the category of finite dimensional in each degree $\mathbf{Z}$-graded $S_{d,n}^{af}$ modules, is an equivalence of graded abelian categories.

The forgetful functor $z : \Gamma^d\mathcal{V}_{A_i} \longrightarrow \Gamma^d\mathcal{V}$ induces an exact functor $z^* : \mathcal{P}_d^f \longrightarrow \mathcal{P}_d^{af_i}$ where $\mathcal{P}_d^f$ stands for the category of graded functors from $\Gamma^d\mathcal{V}$ regarded as the graded category concentrated in degree 0 to $\mathcal{V}^f$. This fact is not entirely trivial, since it relies on the possibility of canonical extension of the domain of a strict polynomial functor to the graded spaces (see the discussion in [7, p. 657]). The functor z^* has a right adjoint $t^* : \mathcal{P}_d^{af_i} \longrightarrow \mathcal{P}_d^f$ which is explicitly given as $t^*(F)(V) := F(V \otimes A_i)$.

The relation between $\mathcal{P}_d^{af_i}$ and $\mathcal{P}_{dp^i}$ is much deeper and it emerges only at the level of derived categories. In order to develop homological algebra in $\mathcal{P}_d^{af_i}$ we regard it as a DG category, with the trivial differentials, and we apply the machinery of homological algebra for DG categories as developed in [13] (see also [14], [7, Section 3]). Namely we consider the DG category $\mathcal{KP}_d^{af_i}$ of graded functors from $\Gamma^d\mathcal{V}_{A_i}$ to the category of complexes of finite dimensional in each degree vector spaces. The derived category $\mathcal{DP}_d^{af_i}$ is obtained from $\mathcal{KP}_d^{af_i}$ by inverting the class of quasi-isomorphisms. It is very convenient to perform this localization process by applying the formalism of Quillen model categories. We recall that the category of all graded functors from $\Gamma^d\mathcal{V}_{A_i}$ to the category of complexes of vector spaces can be equipped with either of two model structures: the projective one in which every object is fibrant and $h^{U\otimes A_i}$ are cofibrant (among others) and the injective one in which every object is cofibrant and $c^*_{U\otimes A_i}$ are fibrant. In both cases the weak equivalences are quasi-isomorphisms [14, Section 3]. In order to apply this machinery to the subcategory $\mathcal{KP}_d^{af_i}$ we need a fact analogous to [7, Th. 3.4].

Proposition 3.2 *Any $F \in \mathcal{KP}_d^{af_i}$ has a cofibrant resolution $p(F)$ and fibrant resolution $i(F)$ inside $\mathcal{KP}_d^{af_i}$.*

Proof This fact follows from the "local finiteness" of $\mathcal{KP}_d^{af_i}$, by which we mean: □

Lemma 3.3 *Any $F \in \mathcal{KP}_d^{af_i}$ has a filtration by finite dimensional sub-objects which stabilizes in each degree.*

Proof of the lemma It will be more convenient to work with the complex of $S_{d,d}^{af_i}$ modules $M := F(\mathbf{k}^d \otimes A_i)$ (c.f. Proposition 3.1). We take a set $\{x_s\}_{s=1}^{\infty}$ which generates M as a $\mathbf{k}$ module and there is only a finite number of x_s' in each degree. Let M_j be the smallest subcomplex of $S_{d,d}^{af_i}$ modules in M containing $\{x_1, \dots, x_j\}$. Then

$$M_j = A_i\langle x_1, \dots, x_j\rangle + A_i\langle d(x_1), \dots, d(x_j)\rangle,$$

hence it is finite dimensional and we obtain the required filtration. □

We will find a cofibrant resolution, the proof in the fibrant case is similar (it could also be deduced from the cofibrant case by means of the affine Kuhn duality discussed in the next subsection). We will use a construction analogous to that appearing in the last part of the proof of [7, Th. 3.4]. For $F \in \mathcal{KP}_d^{af_i}$ let F_j be a filtration of F by finite dimensional subcomplexes. By the argument used in the proof of [7, Th. 3.4] we can construct for each $j \geq 0$ a cofibrant replacement $P_j \simeq F_j$ satisfying the following properties:

1. P_j^k is finite dimensional for all j, k.
2. $\inf\{k : P_j^k \neq 0\} \geq \inf\{k : F_j^k \neq 0\} - d_0$, where d_0 is homological dimension of $\mathcal{P}_d$.
3. P_j embeds into P_{j+1},
4. The quotient P_{j+1}/P_j is cofibrant.

Then it is easy to see that $P := \bigcup_j F_j$ is a cofibrant replacement of F and that $P \in \mathcal{KP}_d^{af_i}$.

Now one can proceed along the lines of [7, Sections 4–5] to establish the adjunction between the triangulated categories $\mathcal{DP}_d^{af_i}$ and $\mathcal{DP}_{dp^i}$ (from now on also $\mathcal{DP}_{dp^i}$ stands for the unbounded derived category). In fact the only place when the arguments of [7] require a substantial modification is the proof of the formality result analogous to [7, Th. 4.3].

Namely, let $\Gamma^d(I^* \otimes I^{(i)}) \in \mathcal{P}_{dp^i}^d$ be the bifunctor given by the formula $(V, W) \mapsto \Gamma^d(\mathrm{Hom}(V, W^{(i)})$ and let X be a projective resolution of $\Gamma^d(I^* \otimes I^{(i)})$ in $\mathcal{P}_{dp^i}^d$. We introduce the DG category $\Gamma^d\mathcal{V}_X^{op}$ whose objects are finite dimensional vector spaces over $\mathbf{k}$ and

$$\mathrm{Hom}^n_{\Gamma^d\mathcal{V}_X^{op}}(V, V') := \mathrm{Hom}_{\mathcal{P}_{dp^i}}(X(V', -), X(V, -)[n]),$$

where $\mathrm{Hom}_{\mathcal{P}_{dp^i}}$ stands for the Hom complex (i.e., we do not require that the maps preserve differentials). Then we have:

Theorem 3.4 *The assignment $V \otimes A_i \mapsto V$ extends to a quasi-isomorphism of DG categories $\phi : \Gamma^d\mathcal{V}_{A_i} \simeq \Gamma^d\mathcal{V}_X^{op}$.*

Proof The proof is analogous to that of [7, Th. 4.3]. The only difference is that we need more "Touzé classes" than those constructed in [5, Prop.3.2], since we need to have "natural enough" classes in $\mathrm{Ext}^*_{\mathcal{P}_{dp^i}}(\Gamma^d(V'^*\otimes(-)^{(i)}, \Gamma^d(V^*\otimes(-)^{(i)})$ instead of $\mathrm{Ext}^*_{\mathcal{P}_{dp^i}}(\Gamma^d(V'^*\otimes(-)^{(1)}, \Gamma^d(V^*\otimes(-)^{(1)})$ used in the proof of [7, Th. 4.3]. However, we constructed the required classes in [8], where we studied formality phenomena for affine functor categories in an even more general context. Namely, we have □

Lemma 3.5 [8, Lemma 3.5] *There exist classes*

$$c[d]^{(i)} \in Ext^{2dp^{i-1}}_{\mathcal{P}^{dp^i}_{dp^i}}(\Gamma^{dp^i}(I^* \otimes I), \Gamma^d(I^{*(i)} \otimes I^{(i)}))$$

such that $c[1]^{(i)} \neq 0$ for all $i \geq 1$, and are compatible with cup product, i.e.,

$$\Delta_*(c[d]^{(i)}) = (c[1]^{(i)})^{\cup d}$$

where $\Delta : \Gamma^d \longrightarrow I^d$ is the standard embedding.

Having at our disposal the classes $c[d]^{(i)}$ we can perform the proof exactly along the lines of the proof of [7, Th. 4.3].

Thanks to Theorem 3.4 we can factorize the Frobenius twist functor at the level of derived categories

$$\mathbf{C}_{I^{(i)}} : \mathcal{DP}_d \longrightarrow \mathcal{DP}_{dp^i}$$

through the full embedding

$$\mathbf{C}^{af_i} : \mathcal{DP}_d^{af_i} \longrightarrow \mathcal{DP}_{dp^i},$$

and analogous factorization holds for the functor $\mathbf{K}_{I(i)}$ right adjoint to $\mathbf{C}_{I(i)}$. More precisely, we have the following analog of [7, Th. 5.1]:

Theorem 3.6 *There exist exact functors:* $\boldsymbol{C}^{af_i} : \mathcal{DP}_d^{af_i} \longrightarrow \mathcal{DP}_{dp^i}$, $\boldsymbol{K}^{af_i} : \mathcal{DP}_{dp^i} \longrightarrow \mathcal{DP}_d^{af_i}$ *satisfying the following properties:*

1. $\boldsymbol{C}^{af_i} \circ z^* \simeq \boldsymbol{C}$, $t^* \circ \boldsymbol{K}^{af_i} \simeq \boldsymbol{K}^r$.
2. $\boldsymbol{K}^{af_i}$ *is right adjoint to* $\boldsymbol{C}^{af_i}$.
3. $\boldsymbol{K}^{af_i} \circ \boldsymbol{C}^{af_i} \simeq Id_{\mathcal{DP}_d^{af}}$.
4. $\boldsymbol{C}^{af_i}$ *is fully faithful.*
5. *The triangulated quotient category* $\mathcal{DP}_{dp^i}/\mathcal{DP}_d^{af_i}$ *is equivalent to the Verdier localization of* $\mathcal{DP}_{dp^i}$ *with respect to the essential image of* $\boldsymbol{C}^{af_1}$ *(see, e.g., [16]).*

We refer the reader for the construction of $\mathbf{C}^{af_i}$ and $\mathbf{K}^{af_i}$ (or rather their specializations for $i = 1$) to [7, Section 5]. What will be important in the present article are their properties listed in Theorem 3.6, especially part (4) which allows one to regard $\mathcal{DP}_d^{af_i}$ as a full subcategory of $\mathcal{DP}_{dp^i}$.

At last, we shall also use occasionally use the category $\mathcal{KP}_d^{af_i,b}$ which is the full subcategory of $\mathcal{KP}_d^{af_i}$ consisting of the complexes chain homotopic to bounded cofibrant objects. Clearly, all objects $X \in \mathcal{KP}_d^{af_i,b}$ are still compact when regarded as objects in $\mathcal{DP}_d^{af_i}$, by which we mean that the functor $\mathrm{Hom}_{\mathcal{DP}_d^{af_i}}(X, -)$ commutes with infinite coproducts in $\mathcal{DP}_d^{af_i}$ whenever they exist. When we localize $\mathcal{KP}_d^{af_i,b}$ with respect to the class of quasi-isomorphisms, we get the bounded derived category $\mathcal{DP}_d^{af_i,b}$ which is a full subcategory of $\mathcal{DP}_d^{af_i}$.

3.2 Affine Kuhn Duality

We recall that the Kuhn duality $(-)^\#$ is the contravariant self-equivalence of $\mathcal{P}_d$ given by the formula $F^\#(V) := F(V^*)^*$. Since $(-)^\#$ is exact, it extends to both bounded and unbounded derived categories of $\mathcal{P}_d$. Now we would like to have an analogous self-equivalence on $\mathcal{P}_d^{af_i}$ and its derived category. However, we face the problem that, technically, the formula $F^\#(V \otimes A_i) = (F((V \otimes A_i)^*))^*$ does not make sense since $(V \otimes A_i)^*$ is not an object of $\mathcal{V}_{A_i}$. To cope with this problem we recall that by [20, Sect. 2.5] any strict polynomial functor can be extended to the functor on the category of graded spaces and the same construction works for affine strict polynomial functors. Then we observe that since $A_i^* \simeq A_i[2(p^i - 1)]$ as graded A_i modules, we just have $(V \otimes A_i)^* \simeq V^* \otimes A_i^* \simeq V^* \otimes A_i[2(p^i - 1)]$. Thus we formally define

$$F^\#(V \otimes A_i) := F^{gr}(V^* \otimes A_i[2d(p^i - 1)])^*$$

where F^{gr} is the extension of F to the graded spaces we have just discussed. Now, since all graded spaces considered are finite dimensional in each degree, the operation $(-)^\#$ is involutive and exact. The latter property allows one to extend it to $\mathcal{DP}_d^{af_i}$ in the obvious way. As one can expect, the affine Kuhn duality takes representable functors to corepresentable ones. However, also here, some shifting phenomena emerge. The best way to capture them is to allow representable and corepresentable functors to be labeled by graded spaces, which again is justified by using graded extensions of functors. Now we gather the basic properties of the Kuhn duality

Proposition 3.7

1. $(h^{U\otimes A_i})^{\#} \simeq c^*_{U^*\otimes A_i^*} \simeq c^*_{U^*\otimes A_i}[-2d(p^i-1)]$.
2. *Let* $\chi_{s,j} \in \mathcal{P}_1^{af_i}$ *for* $1 \leq s \leq i$, $0 \leq j \leq p-1$ *be defined as*
$$\chi_{s,j}(V\otimes A_i) := (x_s)^j \cdot (V\otimes A_i)/(x_s)^{j+1}\cdot (V\otimes A_i).$$
Then $(\chi_{s,j})^{\#} = \chi_{s,p-1-j}$.
3. *We have an isomorphism of functors on* $\mathcal{DP}_d$: $z^* \circ (-)^{\#} \simeq (-)^{\#} \circ z^*$.
4. *The functor* $h^* := (-)^{\#} \circ t^* \circ (-)^{\#}$ *is left adjoint to* z^*.
5. *Explicitly:* $h^*(F)(V) \simeq F(V\otimes A_i^*)$, *hence* $h^* \simeq t^*[2d(p^i-1)]$.

Proof For the first part we compute

$$(h^{U\otimes A_i})^{\#}(V\otimes A_i) = (\Gamma^d(\mathrm{Hom}_{A_i}(U\otimes A_i, (V\otimes A_i)^*)))^* \simeq (\Gamma^d(\mathrm{Hom}_{A_i}(V\otimes A_i, (U\otimes A_i)^*))) =$$

$$c_{U^*\otimes A_i^*}(V\otimes A_i).$$

The isomorphism $c^*_{U^*\otimes A_i^*} \simeq c^*_{U^*\otimes A_i}[-2d(p^i-1)]$ follows from the elementary properties of graded extension of a functor.

The second part follows from the fact that

$$\chi_{s,j}(V\otimes A_i) \simeq V\otimes A\otimes \ldots \otimes A^{(s-1)} \otimes \mathbf{k}[-2p^{s-1}j] \otimes A^{(s+1)} \otimes \ldots A^{(i)}.$$

The third part is obvious, the fourth part follows formally from the third part and $\{z^*, t^*\}$ adjunction. The fifth part is obvious. □

Now we would like to show that the affine Kuhn duality preserves the bounded derived category $\mathcal{DP}_d^{af_i,b}$. For this we need the following important technical fact.

Proposition 3.8 *For any* $U \in \mathcal{V}$, $c^*_{U\otimes A_i}$ *has a cofibrant resolution in* $\mathcal{KP}_d^{af_i,b}$. *Therefore* $c^*_{U\otimes A_i}$ *is a compact object of* $\mathcal{DP}_d^{af_i}$.

Proof Let $P_U^{\bullet}$ be a finite projective resolution of $S^d_{U^*}$ in $\mathcal{P}_d$. Then

$$z^*(P_U^{\bullet}) \simeq z^*(S^d_{U^*}) = c^*_{U\otimes A_i}[-2d(p^i-1)].$$

Since z^* preserves cofibrant objects, $z^*(P_U^{\bullet})[2d(p^i-1)]$ is a cofibrant replacement of $c^*_{U\otimes A_i}$. To conclude the proof we observe that since $P_U^{\bullet}$ is finite, $z^*(P_U^{\bullet})[2d(p^i-1)]$ is bounded. □

This immediately gives

Corollary 3.9 *If $F \in \mathcal{DP}_d^{af_i,b}$ then $F^\# \in \mathcal{DP}_d^{af_i,b}$.*

Proof Any $F \in \mathcal{DP}_d^{af_i,b}$ has a finite filtration with relatively projective (i.e., summands in the sums of shifts of $h^{U\otimes A_i}$'s) subquotients. Therefore it suffices to show that the Kuhn dual of a relatively projective object belongs to $\mathcal{DP}_d^{af_i,b}$. This follows from the fact that any relatively projective object is a summand in the finite sum of shifted representable objects and Propositions 3.7(1) and 3.8. □

We finish our discussion of the affine Kuhn duality by investigating its compatibility with $\mathbf{C}^{af_i}$ and $\mathbf{K}^{af_i}$.

Proposition 3.10 *We have the following isomorphisms of functors:*

1.
$$(-)^\# \circ \boldsymbol{K}^{af_i} \circ (-)^\# \simeq \boldsymbol{K}^{af_i}$$
as functors between $\mathcal{DP}_{dp^i}$ and $\mathcal{DP}_d^{af_i}$,
2.
$$(-)^\# \circ \boldsymbol{C}^{af_i} \circ (-)^\# \simeq \boldsymbol{C}^{af_i}$$
as functors between $\mathcal{DP}_d^{af_i,b}$ and $\mathcal{DP}_{dp^i}^b$.

Proof We shall compare contravariant functors $(-)^\# \circ \mathbf{K}^{af_i}$ and $\mathbf{K}^{af_i} \circ (-)^\#$. Let X be a projective resolution of the bifunctor $\Gamma^d(I^* \otimes I^{(i)})$. Then we have

$$(-)^\# \circ \mathbf{K}^{af_i}(F)(V) = \mathrm{Hom}_{\mathcal{P}_{dp^i}}(X(V,-), F)^\# = \mathrm{Hom}_{\mathcal{P}_{dp^i}}(X(V^*,-), F[2d(p^i-1)])^*.$$

Then by applying the right Serre functor we get

$$\mathrm{Hom}_{\mathcal{P}_{dp^i}}(X(V^*,-), F[2d(p^i-1)])^* \simeq \mathrm{RHom}_{\mathcal{P}_{dp^i}}(F, \mathbf{S}_r(X(V^*,-))[-2d(p^i-1)]).$$

Let $Y \in \mathcal{DP}_{dp^i}^d$ be an injective resolution of the complex of bifunctors $(V, W) \mapsto \mathbf{S}_r(X(V^*, W))$. Then we have an isomorphism of functors

$$(-)^\# \circ \mathbf{K}^{af_i} \simeq \mathrm{Hom}_{\mathcal{P}_{dp^i}}((-), Y[-2d(p^i-1)]).$$

Now let us look at $\mathbf{K}^{af_i} \circ (-)^\#$. This time we obtain

$$\mathbf{K}^{af_i} \circ (-)^\#(F)(V) = \mathrm{Hom}_{\mathcal{P}_{dp^i}}(X(V,-), F^\#) \simeq \mathrm{Hom}_{\mathcal{P}_{dp^i}}(F, X(V,-)^\#).$$

Now, let $Z \in \mathcal{DP}_{dp^i}^d$ be an injective resolution of the complex of bifunctors $(V, W) \mapsto X(V,-)^\#$ (in fact it may be shown that $P(V,-)^\#$ is already injective). Then we have an isomorphism of functors

$$\mathbf{K}^{af_i} \circ (-)^\# \simeq \mathrm{Hom}_{\mathcal{P}_{dp^i}}((-), Z).$$

Now, we recall that Y is isomorphic in $\mathcal{DP}_{dp^i}^d$ to the complex of bifunctors $(V, W) \mapsto \mathbf{S}_r(\Gamma_{V^*}^{d(i)})(W)$ and Z is isomorphic in $\mathcal{DP}_{dp^i}^d$ to the complex of bifunctors $(V, W) \mapsto$

$S^{d(i)}_{V^*}(W)$. Thus, by Theorem 2.2(1),(3) and the injectivity of Y and Z there exists a quasi-isomorphism $\alpha : Y \longrightarrow Z$ and also its quasi-inverse can be realized as a genuine map of complexes. Then the postcomposing with α gives an isomorphism of functors

$$\alpha_* : (-)^\# \circ \mathbf{K}^{af_i} \longrightarrow \mathbf{K}^{af_i} \circ (-)^\#,$$

which finishes the proof of the first part of the proposition.

For the second part we recall that $\mathbf{C}^{af} = \mathbf{L}T_X \circ \phi^{*-1}$ where $\phi^{*-1} : \mathcal{DP}^{af}_d \longrightarrow \mathcal{DP}_X$ is an equivalence between $\mathcal{DP}^{af}_d$ and the derived category of certain intermediate DG category $\mathcal{P}_X$ and T_X is a tensor functor in a sense of [13, Section 6]. Moreover, $\mathcal{DP}_X$ can be equipped with the Kuhn duality by the construction analogous to that applied to $\mathcal{DP}^{af}_d$ and ϕ^{*-1} and T_X clearly commutes with the dualities. Then our assertion essentially follows from the acyclicity of $c^*_{U\otimes A_i}$ with respect to T_X. Let us make this statement precise.

Let $Q : \mathcal{C} \longrightarrow \mathcal{D}$ be a left Quillen functor between Quillen model categories. We call $A \in \mathcal{C}$ *Q-acyclic* if the augmentation map $\mathbf{L}Q(A) \longrightarrow Q(A)$ is a weak equivalence. □

Lemma 3.11 *Let $F \in \mathcal{KP}^{af_i,b}_d$ be bounded and fibrant. Then F is $T_X \circ \phi^{*-1}$-acyclic.*

Proof Let us first assume that $F = c^*_{U\otimes A_i}$. Then we have

$$\mathbf{L}(T_X \circ \phi^{*-1})(c_{U\otimes A_i}) = \mathbf{C}^{af_i}(c_{U\otimes A_i}) = \mathbf{C}^{af_i}(z^*(S^d_{U^*})[2d(p^i-1)]) \simeq$$
$$\mathbf{C}_{I^{(i)}}(S^d_{U^*}[2d(p^i-1)]) = S^{d(i)}_{U^*}[2d(p^i-1)].$$

On the other hand, since T_X commutes with $(-)^\#$, we have

$$T_X(c^*_{U\otimes A_i}) = X(U,-)^\# \simeq S^{d(i)}_{U^*}[-2d(p^i-1)].$$

Moreover, since also $T_X(c^*_{U\otimes A_i})$ was described in terms of the complex X, we see that the quasi-isomorphism

$$\mathbf{L}(T_X \circ \phi^{*-1})(c_{U\otimes A_i}) \simeq (T_X \circ \phi^{*-1})(c_{U\otimes A_i})$$

is realized by the augmentation map. This shows that $c^*_{U\otimes A_i}$, and hence also its direct summands, are $T_X \circ \phi^{*-1}$-acyclic. Now, let F be an arbitrary bounded fibrant object. Then it has a finite filtration with subquotients being finite sums of direct summands of $c^*_{U\otimes A_i}$. Our lemma follows from a general fact that if in a short exact sequence of objects of $\mathcal{KP}^{af_i,f}_d$, the outer terms are acyclic, so is the middle term. □

In order to apply Lemma 3.11 we consider two skeletons of $\mathcal{DP}^{af_i,b}_d$: $\mathcal{C}^b$ consisting of bounded cofibrant objects and $\mathcal{F}^b$ consisting of bounded fibrant objects. Then $(\mathcal{C}^b)^\# = \mathcal{F}^b$ and when we restrict our functors to $\mathcal{C}^b$, we obtain

$$\mathbf{C}^{af}_i \circ (-)^\# \simeq T_X \circ \phi^{*-1} \circ (-)^\# \simeq (-)^\# \circ T_X \circ \phi^{*-1} \simeq (-)^\# \circ \mathbf{C}^{af_i}.$$

We finish this subsection by providing an example showing that $\mathbf{C}^{af_i}$ commutes with $(-)^\#$ only for compact objects. For this we need a few basic computations in $\mathcal{DP}^{af_1}_1$.

Proposition 3.12 *We have the following isomorphisms:*

1. *Let* $|y| = -2(p-1)$, $|z| = -1$. *There is an isomorphism of graded rings:*

$$Hom^*_{\mathcal{DP}_1^{af_1}}(\chi_{1,0}, \chi_{1,0}) = \begin{cases} \boldsymbol{k}[y] \otimes \Lambda(z) & \text{if } p > 2 \\ \boldsymbol{k}[z] & \text{if } p = 2. \end{cases}$$

2. *There is an isomorphism in* $\mathcal{DP}_1^{af_1}$: $\boldsymbol{K}^{af_1}(S^p) \simeq \chi_{1,0}$.

Proof It will be easier to write down explicit formulas when interpreting $\mathcal{P}_1^{af_1}$ as A_1-mod^f via Proposition 3.1. Under this identification $\chi_{1,0}$ just corresponds to the trivial A_1 module $\mathbf{k}$. Thus as its cofibrant replacement we can take the suitably shifted periodic resolution:

$$\ldots \xrightarrow{\cdot x_1} A_1[-2p] \xrightarrow{\cdot x_1^{p-1}} A_1[-2] \xrightarrow{\cdot x_1} A_1.$$

Therefore we derive the first part of Prop. 3.12 from the classical computation of the cohomology ring of the cyclic group $\mathbf{Z}/p$ with the only difference following from our graded setting being the negative degrees of the multiplicative generators. In order to get the second part we observe that in the special case of $d = i = 1$ as a resolution X used to construct $\mathbf{K}^{af_i}$ we can just take a complex of bifunctors of the form:

$$(V, W) \mapsto W^* \otimes C^\bullet(V)$$

where $C^\bullet$ is any projective resolution of $I^{(1)}$. Hence using the graded module description of $\mathcal{P}_1^{af_1}$ again, we obtain

$$H^*(\mathbf{K}^{af_1}(F)) = \mathrm{Ext}^*_{\mathcal{P}_p}(I^{(1)}, F)$$

for any $F \in \mathcal{P}_p$. Applying this formula to $F = S^p$ we get

$$H^*(\mathbf{K}^{af_1}(S^p)) = H^0(\mathbf{K}^{af_1}(S^p)) = \mathbf{k}.$$

Now we observe that since $H^*(\mathbf{K}^{af_1}(S^p))$ is concentrated in a single degree, $\mathbf{K}^{af_1}(S^p)$ is formal. This finishes the proof. □

Now we are ready to provide the promised example. By Proposition 3.7(2) we have:

$$\chi_{1,0}^\# \simeq \chi_{1,p-1} \simeq \chi_{1,0}[-2(p-1)].$$

Therefore if we had

$$\mathbf{C}^{af_1}(\chi_{1,0}^\#) \simeq (\mathbf{C}^{af_1}(\chi_{1,0}))^\#,$$

it would imply that

$$(\mathbf{C}^{af_1}(\chi_{1,0}))^\# \simeq \mathbf{C}^{af_1}(\chi_{1,0})[-2(p-1)].$$

But this is impossible, because, as we will see, $H^*(\mathbf{C}^{af_1}(\chi_{1,0}))$ evaluated on a one-dimensional space is bounded below but not bounded above. To this end we observe that by the Yoneda lemma and Proposition 3.12(2):

$$H^n(\mathbf{C}^{af_1}(\chi_{1,0}))(\mathbf{k})^* \simeq \mathrm{Hom}^{-n}_{\mathcal{DP}_p}(\mathbf{C}^{af_1}(\chi_{1,0}), S^p) \simeq \mathrm{Hom}^{-n}_{\mathcal{DP}_1^{af_1}}(\chi_{1,0}, \mathbf{K}^{af_1}(S^p)) \simeq$$

$$\mathrm{Hom}^{-n}_{\mathcal{DP}_p}(\chi_{1,0}, \chi_{1,0}),$$

and the latter group is non-trivial when $n = 2j(p-1)$ or $n = 2j(p-1) - 1$ for $j \geq 0$, by Proposition 3.12(1).

3.3 Affine Recollement Diagram

In this short subsection we show how to extend the adjunction $\{\mathbf{C}^{af_i}, \mathbf{K}^{af_i}\}$ to a recollement of triangulated categories, thus answering question posed in [5] and partially addressed in [7]. We do not use the results of this subsection elsewhere in the article but we decided to discuss them here, since they are formal consequences of Proposition 3.7.

Let us define the functor $\mathbf{C}^{af_i,r} : \mathcal{DP}_d^{af_i} \longrightarrow \mathcal{DP}_{dp^i}$ as the composite

$$\mathbf{C}^{af_i,r} := (-)^{\#} \circ \mathbf{C}^{af_i} \circ (-)^{\#}.$$

Then we have

Theorem 3.13 *The functor $\mathbf{C}^{af_i,r}$ is right adjoint to $\mathbf{K}^{af_i}$.*

Moreover, the triple $\{\mathbf{K}^{af_i}, \mathbf{C}^{af_i}, \mathbf{C}^{af_i,r}\}$ is a part of recollement diagram of triangulated categories:

$$\ker(\mathbf{K}^{af_i}) \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathcal{DP}_{dp^i} \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathcal{DP}_d^{af_i}$$

Proof The adjunction is a formal consequence of commuting $\mathbf{K}^{af_i}$ with $(-)^{\#}$. Namely, for any $X \in \mathcal{DP}_{dp^i}$ and $Y \in \mathcal{DP}_d^{af_i}$ we have

$$\mathrm{Hom}_{\mathcal{DP}_{dp^i}}(X, \mathbf{C}^{af_i,r}(Y)) = \mathrm{Hom}_{\mathcal{DP}_{dp^i}}(X, (\mathbf{C}^{af_i}(Y^{\#}))^{\#}) \simeq \mathrm{Hom}_{\mathcal{DP}_{dp^i}}(\mathbf{C}^{af_i}(Y^{\#}), X^{\#}) \simeq$$

$$\mathrm{Hom}_{\mathcal{DP}_d}(Y^{\#}, \mathbf{K}^{af_i}(X^{\#})) \simeq \mathrm{Hom}_{\mathcal{DP}_d}((\mathbf{K}^{af_i}(X^{\#}))^{\#}, Y) \simeq \mathrm{Hom}_{\mathcal{DP}_d}((\mathbf{K}^{af_i}(X), Y).$$

The fact that our two-sided adjunction can be extended to a recollement diagram follows from [9, Prop. 2.1] for $j^* = \mathbf{K}^{af_i}$, $j_* = \mathbf{C}^{af_i}$, $j_! = \mathbf{C}^{af_i,r}$ and the fact that $\mathbf{K}^{af_i} \circ \mathbf{C}^{af_i} \simeq Id$. □

The idea of finding a recollement of triangulated categories analogous to the recollement of abelian categories $\mathcal{P}_d \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathcal{P}_{dp^1}$ considered by Kuhn [18] was the main motivation for introducing the category $\mathcal{DP}_d^{af}$ in [7]. Unfortunately, infinite homological dimension of $\mathcal{DP}_d^{af_i}$ generates serious problems. In particular, it implies that $\mathbf{C}^{af_i}$ does not preserve infinite products, hence it cannot have a left adjoint. More concretely, if we try to define the left adjoint by conjugating $\mathbf{K}^{af_i}$ by the Kuhn duality, the construction fails because $\mathbf{C}^{af_i}$ does not commute with $(-)^{\#}$ on the whole $\mathcal{DP}_d^{af_i}$ but merely on $\mathcal{DP}_d^{af_i,b}$. Thus, as we remarked at the end of [7], we could only obtain recollement between bounded derived categories, which is less useful, because (co)fibrant replacements are usually not bounded. However, this obstacle disappears when we change our setting into the dual one. In other words, we should rather think of $\mathcal{DP}_d^{af_i}$ as a quotient category of $\mathcal{DP}_{dp^i}$ instead of a subcategory. This change of perspective is quite surprising when we think of abelian counterparts which inspired our work. On the other hand, it is coherent with [6, Section 3] where we observed that the approach to the Poincaré duality based on the "left–right homological shift" is formally analogous to the "dual Verdier duality".

At last, let us take a closer look at the recollement diagram in the bounded case. Strictly speaking, we have a recollement between $\mathcal{DP}_d^{af_i,b}$ and

$$\mathcal{X} := (\mathbf{C}^{af_i})^{-1}(\mathcal{DP}_d^{af_i,b}).$$

However, the situation simplifies drastically, since here $\mathbf{C}^{af_i}$ and $\mathbf{K}^{af_i}$ are mutually two-sided adjoint. Therefore our recollement splits into the orthogonal decomposition

$$\mathcal{X} \simeq \mathrm{im}(\mathbf{C}^{af_i}) \times \ker(\mathbf{K}^{af_i}).$$

Thus we see that the affine-bounded and affine-unbounded cases are fundamentally different, since we have no analogous orthogonal decomposition of $\mathcal{DP}_{dp^i}$.

At last, we warn the reader that there is no orthogonal decomposition of $\mathcal{DP}_{dp^i}^b$, since it is strictly larger than $\mathcal{X}$.

3.4 Serre Functor in $\mathcal{DP}_d^{af_i}$

In this subsection we introduce (a suitably modified version of) Serre functor on $\mathcal{DP}_d^{af_i}$. In fact a genuine Serre functor exists only on $\mathcal{DP}_d^{af_i,b}$ but this is not very useful for us since the objects of $\mathcal{DP}_d^{af_i}$ rarely have cofibrant replacements in $\mathcal{DP}_d^{af_i,b}$. On the other hand one cannot hope for a Serre functor in $\mathcal{DP}_d^{af_i}$, since it is not even a Hom-finite category. What we really have is the following weaker version of Serre functor:

Definition 3.14 Let $\mathcal{C}$ be a $\mathbf{k}$-linear category. A $\mathbf{k}$-linear functor $\mathbf{S} : \mathcal{C} \longrightarrow \mathcal{C}$ is a weak (left) Serre functor if:

1. There is a natural in X, Y isomorphism
$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{S}(X), Y) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, X)^*$$
whenever X or Y is compact.
2. $\mathbf{S}$ is an auto-equivalence.

For example, if $\mathcal{C}$ is the bounded derived category of category of finitely generated modules over a finite dimensional algebra of finite homological dimension then by [3] $\mathcal{C}$ possesses a Serre functor. In that case, it is easy to see that it extends to a weak Serre functor on the unbounded derived category. However, our situation is quite different, since $\mathcal{P}_d^{af_i}$ is of infinite homological dimension. By this we mean that a cofibrant replacements of objects of $\mathcal{P}_d^{af_i}$ does not necessarily belong to the smallest triangular subcategory of $\mathcal{DP}_d^{af_i}$ contain relatively projective objects, which is in our case equivalent to $\mathcal{DP}_d^{af_i,b}$. The crucial property of $\mathcal{DP}_d^{af_i}$ which makes a construction analogous to that used in Section 2 work is the fact that $c^*_{U\otimes A_i}$ is compact (Proposition 3.5).

Technically, in order to ensure that our functors are well defined we have to consider affine bifunctors. Namely we introduce the category $\mathcal{P}_d^{e,af_i}$ of i-affine bifunctors of bi-degree (d,e) as the category of $\mathbf{k}$-linear graded functors from $\Gamma^e(\mathcal{V}_{A_i})^{op}\otimes\Gamma^d(\mathcal{V}_{A_i})$ to $\mathcal{V}$. The homological algebra in $\mathcal{P}_d^{e,af_i}$ can be developed analogously to that in $\mathcal{P}_d^{af_i}$. Let us take, like in Section 2, $P^\bullet$, the projective resolution of the bifunctor $(V,W)\mapsto S^d(V^*\otimes W)$. Now similarly to the case of functors in one variable, we have the exact functor $z^*:\mathcal{P}_d^e\longrightarrow\mathcal{P}_d^{e,af_i}$ which extends to the category of complexes and preserves cofibrant objects. Then it is easy to see that

$$C:=z^*(P^\bullet)[2d(p^i-1)]$$

is a cofibrant replacement of the affine bifunctor $c^*(I^*\otimes I)$ defined by the formula:

$$(V\otimes A_i, W\otimes A_i)\mapsto c^*_{V\otimes A_i}(W\otimes A_i),$$

(c.f. Proposition 3.8). For the future use we remark that C is bounded. Then, analogously to Section 2 we put:

Definition 3.15 We define a functor $\mathbf{S}^{af_i}:\mathcal{DP}_d^{af_i}\longrightarrow\mathcal{DP}_d^{af_i}$ by the formula

$$\mathbf{S}^{af_i}(F):=\mathcal{RH}om_{\mathcal{P}_d^{af_i}}(c^*(I^*\otimes I),F).$$

We collect the basic properties of $\mathbf{S}^{af_i}$ which will be needed for the applications described in Section 5.

Theorem 3.16 *The functor $\mathbf{S}^{af_i}$ satisfies the following properties:*

1. *There is a natural in $U\otimes A_i$ isomorphism in $\mathcal{DP}_d^{af_i}$*

$$\mathbf{S}^{af_i}(c^*_{U\otimes A_i})\simeq h^{U\otimes A_i}.$$

2. *$\mathbf{S}^{af_i}$ is an auto-equivalence of $\mathcal{DP}_d^{af_i}$.*
3. *$\mathbf{S}^{af_i}$ restricted to $\mathcal{DP}_d^{af_i,b}$ is a left Serre functor and it is a weak left Serre functor on the whole $\mathcal{DP}_d^{af_i}$.*
4. *There are isomorphisms of functors*

$$\mathbf{S}^{af_i}\circ z^*[2d(p^i-1)]\simeq z^*\circ\mathbf{S},\quad \mathbf{S}\circ t^*\simeq t^*\circ\mathbf{S}^{af_i}[2d(p^i-1)],$$
$$\mathbf{S}^{af_i}\circ\mathbf{K}^{af_i}\simeq\mathbf{K}^{af_i}\circ\mathbf{S},\quad \mathbf{S}\circ\mathbf{C}^{af_i}\simeq\mathbf{C}^{af_i}\circ\mathbf{S}^{af_i}.$$

Proof Since $c^*_{U\otimes A_i}$ is fibrant in the injective Quillen structure we get

$$\mathbf{S}^{af}(c^*_{U\otimes A_i})(V\otimes A_i)\simeq \mathrm{Hom}_{\mathcal{P}_d^{af_i}}(c^*_{V\otimes A_i},c^*_{U\otimes A_i})\simeq(c^*_{V\otimes A_i}(U\otimes A_i))^*=$$

$$\Gamma^d(\mathrm{Hom}(U,V)\otimes A_i)=h^{U\otimes A_i}(V\otimes A_i).$$

In order to show that $\mathbf{S}^{af_i}$ is an equivalence we observe that by the first part it is faithfully full on the subcategory consisting of corepresentable objects. Then, by

Lemma 3.3 and the fact that $\mathbf{S}^{af_i}$ preserves infinite coproducts, whenever they exist, our assertion follows from [13, Lemma 4.2].

In order to get the third part, we establish a natural in $F \in \mathcal{DP}_d^{af_i}$ and $U \otimes A_i$ isomorphism

$$\mathrm{Hom}_{\mathcal{DP}_d^{af_i}}(F, c^*_{U\otimes A_i}) \simeq \mathrm{Hom}_{\mathcal{DP}_d^{af_i}}(\mathbf{S}^{af_i}(c^*_{U\otimes A_i}), F)^*,$$

which by the first part and the fact that $h^{U\otimes A_i}$ is cofibrant and $c^*_{U\otimes A_i}$ is fibrant reduces to the isomorphism

$$\mathrm{Hom}_{\mathcal{P}_d^{af_i}}(F, c^*_{U\otimes A_i}) \simeq \mathrm{Hom}_{\mathcal{P}_d^{af_i}}(h^{U\otimes A_i}, F)^*,$$

which follows from the Yoneda lemma. This shows that $\mathbf{S}^{af_i}$ restricted to $\mathcal{DP}_d^{af_i,b}$ is a left Serre functor. In the unbounded case we observe that if $Z \in \mathcal{DP}_d^{af_i,b}$ then for any $W \in \mathcal{DP}_d^{af_i}$, the spaces $\mathrm{Hom}_{\mathcal{DP}_d^{af_i}}(Z, W)$ and $\mathrm{Hom}_{\mathcal{DP}_d^{af_i}}(W, Z)$ are finite dimensional. Thus we see that if X or Y is compact, we still have the required isomorphism.

In order to obtain the first isomorphism in part 4, we recall that $z^*(\Gamma^d_{U^*}) = h^{U\otimes A_i}$ and $z^*(S^d_{U^*}) = c^*_{U\otimes A_i}[-2d(p^i-1)]$. Hence we get natural in U isomorphisms:

$$\mathbf{S}^{af_i} \circ z^*(S^d_{U^*}) = \mathbf{S}^{af_i}(c^*_{U\otimes A_i}[-2d(p^i-1)]) = h^{U\otimes A_i}[-2d(p^i-1)]$$

and

$$z^* \circ \mathbf{S}(S^d_{U^*}) = z^*(\Gamma^d_{U^*}) = h^{U\otimes A_i}.$$

The second isomorphism follows from the facts that $t^*(c^*_{U\otimes A_i}) = S^d_{(U\otimes A_i)^*}$ and $t^*(h^{U\otimes A_i}) = \Gamma^d_{(U\otimes A_i)^*}[-2d(p^i-1)]$.

The proof of the last two isomorphisms is analogous to that of Proposition 3.7. The last formula holds on the whole $\mathcal{DP}_d^{af_i}$ because $\mathbf{C}^{af_i}$ commutes with direct colimits. □

Remark When we compose the isomorphisms from part 4, we obtain the formulas:

$$\mathbf{S} \circ \mathbf{K}^r \simeq \mathbf{K}^r \circ \mathbf{S}[2d(p^i-1)], \quad \mathbf{S} \circ \mathbf{C} \simeq \mathbf{C} \circ \mathbf{S}[-2d(p^i-1)],$$

from which, in particular, Theorem 2.2.(3.1) follows. Thus we see that the shift phenomenon which produces the Poincaré duality is related to the scalar extension from $\mathcal{P}_d$ to $\mathcal{P}_d^{af_i}$.

4 Affine Semiblocks

In this section we introduce certain subcategories of $\mathcal{P}_d^{af_i}$ we call semiblocks, which correspond to the blocks in $\mathcal{P}_d$. This structure may be interesting for its own but in the present article we are mainly interested in the Serre functor restricted to the semiblocks, since, as it will be shown in the next section, in certain cases it enjoys very special properties.

We recall that the category $\mathcal{P}_d$ admits decomposition into the blocks:

$$\mathcal{P}_d \simeq \mathcal{P}_{\lambda^1} \times \ldots \times \mathcal{P}_{\lambda^s}$$

and the set of blocks is indexed by the family $\lambda^1, \ldots, \lambda^s$ of p-core Young diagrams of weight $d - pj$ for some $j \geq 0$ (see, e.g., [19, Section 5]). By the Yoneda lemma, there is the corresponding decomposition of the bifunctor $(V, W) \mapsto \Gamma^d(\mathrm{Hom}(V, W))$ into the "block bifunctors":

$$\Gamma^d(\mathrm{Hom}(V, W)) \simeq B_{\lambda^1}(V, W) \oplus \ldots \oplus B_{\lambda^s}(V, W).$$

The Cauchy decomposition [1, Th. III.1.4] provides the filtration of bifunctor $\Gamma^d(\mathrm{Hom}(V, W))$ with the associated object

$$\bigoplus_{\mu \in Y_d} W_\mu(V^*) \otimes W_\mu(W)$$

where Y_d stands for the set of Young diagrams of weight d. Hence each $B_\lambda(V, W)$ has the filtration with the associated object

$$\bigoplus_{\mu \in Y_{\lambda^j}} W_\mu(V^*) \otimes W_\mu(W)$$

where Y_{λ^j} is the set of Young diagrams of degree d belonging to the block labeled by λ^j.

Moreover, the bifunctor B_{λ^j} can be used to form the category $B_{\lambda^j}\mathcal{V}$ whose objects are finite vector spaces and

$$\mathrm{Hom}_{B_{\lambda^j}\mathcal{V}}(V, W) := B_{\lambda^j}(V, W).$$

Then the category $\mathcal{P}_{\lambda^j}$ can be identified with the category of $\mathbf{k}$-linear functors from $B_{\lambda^j}\mathcal{V}$ to the category of finite dimensional vector spaces over $\mathbf{k}$. The main objective of the present section is to define the affine counterpart of $\mathcal{P}_\lambda$, relate it to $\mathcal{P}_\lambda$ and $\mathcal{P}_{dp^i}$, and equip it with a Serre functor.

Let us fix a p-core Young diagram λ of weight $|\lambda| = d - pj$ and let $B_\lambda^{(i)}$ denote the bifunctor $(V, W) \mapsto B_\lambda(V, W^{(i)})$. We introduce the graded category $B_\lambda\mathcal{V}_{A_i}$ with the objects being finite dimensional vector spaces and the morphisms given by the formula

$$\mathrm{Hom}_{B_\lambda\mathcal{V}_{A_i}}(V \otimes A_i, V' \otimes A_i) := B_\lambda(V, V' \otimes A_i),$$

where we choose to label the objects by $V \otimes A_i$ in order to make our terminology coherent with that used in Section 3 and [7]. Thanks to the Collapsing Conjecture [6, Cor. 3.7], the Hom spaces in $B_\lambda\mathcal{V}_{A_i}$ admit descriptions as appropriate Ext groups. Namely, we have natural in V, V' isomorphisms

$$\mathrm{Ext}^*_{\mathcal{P}_{dp^i}}(B_\lambda^{(i)}(V', -), B_\lambda^{(i)}(V, -)) \simeq \mathrm{Ext}^*_{\mathcal{P}_d}(B_\lambda(V', -), B_\lambda(V, - \otimes A_i)) \simeq$$

$$B_\lambda(V, V' \otimes A_i).$$

We define $\mathcal{P}_\lambda^{af_i}$ as the category of graded $\mathbf{k}$-linear functors from $B_\lambda\mathcal{V}_{A_i}$ to the category of $\mathbf{Z}$-graded finite dimensional in each degree vector spaces.

The category $\mathcal{P}_\lambda^{af_i}$ shares with $\mathcal{P}_d^{af_i}$ its basic properties. In particular, we have the representable functor $h_\lambda^{U\otimes A_i}$ in $\mathcal{DP}_\lambda^{af_i}$ given explicitly by the formula

$$h_\lambda^{U\otimes A_i}(V\otimes A_i) := \mathrm{Hom}_{B_\lambda\mathcal{V}_{A_i}}(U\otimes A_i, V\otimes A_i),$$

the corepresentable functor $c^*_{\lambda,U\otimes A_i}$, and the block affine Kuhn duality. Also the analog of Proposition 3.1 holds. Let us call *the block affine Schur algebra* the graded algebra

$$S_{\lambda,n}^{af_i} := \mathrm{Hom}_{B_\lambda\mathcal{V}_{A_i}}(\mathbf{k}^n\otimes A_i, \mathbf{k}^n\otimes A_i) \simeq B_\lambda(\mathbf{k}^n, \mathbf{k}^n\otimes A_i).$$

Then

Proposition 4.1 *For any $n\geq d$, the evaluation functor $F\mapsto F(\boldsymbol{k}^n\otimes A_i)$ gives an equivalence of graded abelian categories:*

$$\mathcal{P}_\lambda^{af_i}\simeq S_{\lambda,n}^{af_i}-\mathrm{mod}^f,$$

where $S_{\lambda,n}^{af_i}$-mod^f stands for the category of $\mathbf{Z}$-graded $S_{\lambda,n}^{af_i}$-modules finite dimensional in each degree.

The adjunction $\{z^*, t^*\}$ between $\mathcal{P}_d$ and $\mathcal{P}_d^{af_i}$ clearly extends to the adjunction $\{z_\lambda^*, t_\lambda^*\}$ between $\mathcal{P}_\lambda$ and $\mathcal{P}_\lambda^{af_i}$.

Now, let us observe an important yet unfortunate phenomenon: when we try to decompose $\mathcal{P}_d^{af_i}$, into the product of $\mathcal{P}_{\lambda^j}^{af_i}$ we face a problem that for $\lambda\neq\lambda'$, $\mathcal{P}_\lambda^{af_i}$ and $\mathcal{P}_{\lambda'}^{af_i}$ are not orthogonal as subcategories of $\mathcal{P}_d^{af_i}$. We will come back to this observation later, since it is best understood at the level of derived categories.

Now we turn to describing relation between $\mathcal{P}_\lambda^{af_i}$ and $\mathcal{P}_d^{af_i}$ more precisely. Let

$$i_\lambda : B_\lambda(V, W\otimes A_i)\longrightarrow \Gamma^d(\mathrm{Hom}(V, W\otimes A_i))$$

be the natural embedding and

$$\pi_\lambda : \Gamma^d(\mathrm{Hom}(V, W\otimes A_i))\longrightarrow B_\lambda(V, W\otimes A_i)$$

be the natural projection. Then the composite $\epsilon_\lambda := i_\lambda\circ\pi_\lambda$ can be thought of as an idempotent endofunctor on $\Gamma^d\mathcal{V}_{A_i}$ (being the identity on the objects). Thus the category $B_\lambda\mathcal{V}_{A_i}$ can be identified with the category $\epsilon_\lambda(\Gamma^d\mathcal{V}_{A_i})\epsilon_\lambda$ whose objects are those of $\Gamma^d\mathcal{V}_{A_i}$ but

$$\mathrm{Hom}_{\epsilon_\lambda(\Gamma^d\mathcal{V}_{A_i})\epsilon_\lambda}(V\otimes A_i, V'\otimes A_i) := \epsilon_\lambda(\mathrm{Hom}_{\Gamma^d\mathcal{V}_{A_i}}(V\otimes A_i, V'\otimes A_i))\epsilon_\lambda.$$

Then the assignment

$$(V, V')\mapsto \epsilon_\lambda(\mathrm{Hom}_{\Gamma^d\mathcal{V}_{A_i}}(V\otimes A_i, V'\otimes A_i))$$

defines a $\Gamma^d\mathcal{V}_{A_i}$–$B_\lambda\mathcal{V}_{A_i}$ bimodule in the terminology of [13, Sect. 6]. Hence we get a pair of functors $j_{\lambda!}$, j_λ^* which satisfy the following properties.

Proposition 4.2 1. *The functor $j_{\lambda!}:\mathcal{P}_\lambda^{af_i}\longrightarrow\mathcal{P}_d^{af_i}$ is a full embedding.*
2. *The functor j_λ^* is right adjoint to $j_{\lambda!}$.*

Proof The adjunction follows from the machinery of standard functors developed in [13, Sect. 6]. The full embedding is a formal consequence of the fact that $j_\lambda^* \circ j_{\lambda !} \simeq id_{\mathcal{P}_\lambda^{af_i}}$ which follows from the fact that e_λ is an idempotent. □

Remark Proposition 4.2 may be thought of as a categorification of [9, Prop. 2.1]. This explains our choice of notations with $j_!, j^*$ instead of T_X, H_X used in [13]. In particular, the functor j_λ^* has a right adjoint $j_{\lambda *}$ and the triple $\{j_{\lambda !}, j_{\lambda *}, j_\lambda^*\}$ form a part of recollement diagram of graded abelian categories. In fact we could derive Proposition 4.2 directly from [9, Prop. 2.1] by invoking our Proposition 4.1 but we prefer to consistently work in functor categories.

We also mention that Proposition 4.2 carries over to the level of derived categories which was the main objective of [13] and [9] and which will be discussed in the next paragraph.

Namely, we define $\mathcal{DP}_\lambda^{af_i}$ as the derived category of DG category $\mathcal{KP}_\lambda^{af_i}$ in the manner analogous to that in Section 3. The adjunctions $\{z_\lambda^*, t_\lambda^*\}$ and $\{j_{\lambda !}, j_{\lambda *}\}$ carry over to the derived categories and, as we have already mentioned, the analog of Proposition 4.2 holds. In particular we still have a full embedding which will be denoted by the same symbol as its graded abelian counterpart:

$$j_{\lambda !} : \mathcal{DP}_\lambda^{af_i} \longrightarrow \mathcal{DP}_d^{af_i},$$

which allows us to regard $\mathcal{DP}_\lambda^{af_i}$ as a full subcategory of $\mathcal{DP}_d^{af_i}$. Then it is clear that our construction is compatible with the scalar extension from $\mathcal{DP}_d$ to $\mathcal{DP}_d^{af_i}$:

Proposition 4.3 *There are isomorphisms of functors between $\mathcal{DP}_\lambda$ and $\mathcal{DP}_d^{af_i}$:*

$$z^* \circ b_{\lambda !} \simeq j_{\lambda !} \circ z_\lambda^*, \quad t_\lambda^* \circ j_\lambda^* \simeq b_\lambda^* \circ t^*,$$

where $b_{\lambda !}$ and $b_{\lambda *}$ are induced respectively by the embedding of and the projection onto the block.

Now we would like to construct a block version of the affine derived Kan extension in order to relate $\mathcal{DP}_\lambda^{af_i}$ to $\mathcal{DP}_{dp^i}$. For this we need an analog of the formality result [7, Th. 4.2]. Let $B_\lambda^{(i)}$ denote the bifunctor $(V, W) \mapsto B_\lambda(V, W^{(i)})$ and let X_λ be a projective resolution of $B_\lambda^{(i)}$ in $\mathcal{P}_{dp^i}^d$. We introduce a DG category $\Gamma^d \mathcal{V}_{X_\lambda}$ with the objects being finite dimensional vector spaces and

$$\mathrm{Hom}_{\Gamma^d \mathcal{V}_{X_\lambda}}(V, V') := \mathrm{Hom}_{\mathcal{P}_{dp^i}}(X_\lambda(V, -), X_\lambda(V', -)).$$

Then $B_\lambda \mathcal{V}_{A_i}$ is clearly the cohomology category of $\Gamma^d \mathcal{V}_{X_\lambda}^{op}$ but we have a much stronger result (c.f. [7, Th. 4.3]):

Proposition 4.4 *The identity on the objects extends to a quasi-isomorphism of DG categories $\phi_\lambda : B_\lambda \mathcal{V}_{A_i} \simeq \Gamma^d \mathcal{V}_{X_\lambda}^{op}$.*

Proof Since $\Gamma^d(\mathrm{Hom}(V, W^{(i)})) \simeq B_\lambda^{(i)}(V, W) \oplus B'(V, W)$, we can obtain X, the projective resolution of $\Gamma^d(\mathrm{Hom}(V, W^{(i)}))$, as the direct sum $X = X_\lambda \oplus X'$ of projective resolutions of $B_\lambda^{(i)}$ and B'. Let

$$i_\lambda : \mathrm{Ext}^*_{\mathcal{P}_{dp^i}}(B_\lambda^{(i)}(V', -), B_\lambda^{(i)}(V, -)) \longrightarrow \mathrm{Ext}^*_{\mathcal{P}_{dp^i}}(\Gamma^d(V', (-)^{(i)}), \Gamma^d(V, (-)^{(i)}))$$

be the embedding induced by the decomposition $\Gamma^d(I^{(i)} \otimes I^*) \simeq B_\lambda^{(i)} \oplus B'$ (we have already encountered this embedding when constructing the idempotent functor ϵ_λ). We similarly define the projection

$$\widetilde{\pi}_\lambda : \mathrm{Hom}_{\mathcal{P}_{dp}}(X(V', -), X(V, -)) \longrightarrow \mathrm{Hom}_{\mathcal{P}_{dp^i}}(X_\lambda(V', -), X_\lambda(V, -)).$$

We define $\phi_\lambda : B_\lambda \mathcal{V}_{A_i} \longrightarrow \Gamma^d \mathcal{V}_{X_\lambda}$ as the composite $\phi_\lambda := \widetilde{\pi}_\lambda \circ \phi \circ i_\lambda$ where $\phi : \Gamma^d \mathcal{V}_{A_i} \longrightarrow \Gamma^d \mathcal{V}_X$ is the transformation from [7, Theorem 4.2] or rather its multitwist analog (as we mentioned in Section 3, this generalization is not entirely trivial, we again refer the reader to [8, Theorem 3.1] where an analogous construction is conducted in even greater generality). Then the fact that ϕ_λ is a quasi-isomorphism follows from the fact that ϕ is a quasi-isomorphism and that it commutes with the idempotent $\epsilon_\lambda := i_\lambda \circ \pi_\lambda$ and its $\Gamma^d \mathcal{V}_{X_\lambda}$-analog $\widetilde{\epsilon}_\lambda := \widetilde{i}_\lambda \circ \widetilde{\pi}_\lambda$. □

Thanks to Proposition 4.4 we are able to construct the block affine derived Kan extension. We summarize its basic properties below

Proposition 4.5 *There exist functors* $\boldsymbol{C}_\lambda^{af_i} : \mathcal{DP}_\lambda^{af_i} \longrightarrow \mathcal{DP}_{dp^i}$ *and* $\boldsymbol{K}_\lambda^{af_i} : \mathcal{DP}_{dp^i} \longrightarrow \mathcal{DP}_\lambda^{af_i}$ *satisfying the following properties:*

1. $\boldsymbol{K}_\lambda^{af_i}$ *is right adjoint to* $\boldsymbol{C}_\lambda^{af_i}$.
2. $\boldsymbol{C}_\lambda^{af_i}$ *is a full embedding.*
3. *The functors* $\boldsymbol{C}_\lambda^{af_i}$ *(restricted to* $\mathcal{DP}_\lambda^{af_i,b}$*) and* $\boldsymbol{K}_\lambda^{af_i}$ *commute with the Kuhn duality.*
4. *There are isomorphisms of functors between* $\mathcal{DP}_\lambda$ *and* $\mathcal{DP}_{dp^i}$*:*

$$\boldsymbol{C}^{af_i} \circ j_{\lambda!} \simeq \boldsymbol{C}_\lambda^{af_i}, \quad j_\lambda^* \circ \boldsymbol{K}^{af_i} \simeq \boldsymbol{K}_\lambda^{af_i}.$$

Proof The proofs of parts 1, 2, 3 are analogous to those of [7, Th. 5.1] and our Theorem 3.6, Proposition 3.10. The compatibility formula in part 4 follows immediately from the construction of the considered functors. □

Having at our disposal the block affine derived Kan extension we can offer a better explanation of the phenomenon of non-orthogonality of semiblocks. Namely let us take $F \in \mathcal{P}_\lambda$, $G \in \mathcal{P}_{\lambda'}$ for $\lambda \neq \lambda'$. Then

$$\mathrm{Hom}^*_{\mathcal{DP}_d^{af_i}}(j_{\lambda!}(z_\lambda^*(F)), j_{\lambda'!}(z_{\lambda'}^*(G))) \simeq \mathrm{Hom}^*_{\mathcal{DP}_{dp^i}}(\mathbf{C}^{af_i}(j_\lambda^*(z_\lambda^*(F))), \mathbf{C}^{af_i}(j_{\lambda'}^*(z_{\lambda'}^*(G)))) \simeq$$

$$\simeq \mathrm{Ext}^*_{\mathcal{P}_{dp^i}}(F^{(i)}, G^{(i)})$$

and the latter Ext groups may well be non-trivial. Thus we see that the reason for the non-orthogonality of semiblocks is simply that the Frobenius twist transfers all the blocks from $\mathcal{P}_d$ into the single (principal) block in $\mathcal{P}_{dp^i}$.

We finish this section by endowing the category $\mathcal{DP}_\lambda^{af_i}$ with a Serre functor. This may be achieved by a construction analogous to that given in the global (affine) case. We consider the affine corepresentable bifunctor $c_\lambda^*(I^* \otimes I)$ given by the formula

$$(V \otimes A_i, W \otimes A_i) \mapsto c^*_{\lambda, V\otimes A_i}(W \otimes A_i).$$

Definition 4.6 We define the block affine Serre functor $\mathbf{S}_\lambda^{af_i} : \mathcal{DP}_\lambda^{af_i} \longrightarrow \mathcal{DP}_\lambda^{af_i}$ by the formula

$$\mathbf{S}_\lambda^{af_i}(F) := \mathcal{RH}om_{\mathcal{P}_\lambda^{af_i}}(c_\lambda^*(I^* \otimes I), F).$$

Then we have

Theorem 4.7 *The functor $\boldsymbol{S}_\lambda^{af_i}$ satisfies the following properties:*

1. *There is a natural in $U \otimes A_i$ isomorphism in $\mathcal{DP}_\lambda^{af_i}$*
$$\boldsymbol{S}_\lambda^{af_i}(c^*_{\lambda, U\otimes A_i}) \simeq h_\lambda^{U\otimes A_i}.$$
2. *$\boldsymbol{S}_\lambda^{af_i}$ is an auto-equivalence of $\mathcal{DP}_\lambda^{af_i}$.*
3. *$\boldsymbol{S}_\lambda^{af_i}$ restricted to $\mathcal{DP}_\lambda^{af_i,b}$ is a left Serre functor and it is a weak left Serre functor on the whole $\mathcal{DP}_\lambda^{af_i}$.*
4. *There are isomorphisms of functors*
$$\boldsymbol{S}_\lambda^{af_i} \circ z_\lambda^*[2d(p^i-1)] \simeq z_\lambda^* \circ \boldsymbol{S}_\lambda, \quad \boldsymbol{S}_\lambda \circ t_\lambda^* \simeq t_\lambda^* \circ \boldsymbol{S}_\lambda^{af_i}[2d(p^i-1)],$$
$$\boldsymbol{S}_\lambda^{af_i} \circ \boldsymbol{K}_\lambda^{af_i} \simeq \boldsymbol{K}_\lambda^{af_i} \circ \boldsymbol{S}, \quad \boldsymbol{S}_\lambda \circ \boldsymbol{C}_\lambda^{af_i} \simeq \boldsymbol{C}_\lambda^{af_i} \circ \boldsymbol{S}_\lambda^{af_i},$$
where $\boldsymbol{S}_\lambda$ is $\boldsymbol{S}$ restricted to the block $\mathcal{DP}_\lambda$.

The proof of Theorem 3.16 carries over to the current situation.

5 Basic Affine Semiblocks and Calabi–Yau Categories

In this section we show that the affine Serre functor when restricted to certain semiblocks in $\mathcal{DP}_d^{af_i}$ is isomorphic to the shift functor. We also show that in that case our functor category is equivalent to the category of finite dimensional graded modules over a certain explicitly described graded algebra, which refines Proposition 4.1.

5.1 The Calabi–Yau Structure on Basic Affine Semiblocks

We recall that a block in $\mathcal{P}_d$ is called basic if it contains a single simple object. Hence the basic blocks are indexed by p-core Young diagrams of weight d and we also call such Young diagrams basic. So, let us fix a basic Young diagram λ. Then

$S_\lambda \simeq W_\lambda \simeq F_\lambda$. Moreover, S_λ is injective and projective and every object of P_λ is a direct sum of S_λ, therefore the category $\mathcal{P}_\lambda$ is semisimple.

We recall that a triangulated category $\mathcal{T}$ with a Serre functor $\mathbf{S}_\mathcal{T}$ is called *Calabi–Yau of dimension n* if there is an isomorphism of functors $\mathbf{S}_\mathcal{T} \simeq \mathrm{id}[n]$. Then we call a triangulated category $\mathcal{T}$ *weak Calabi–Yau of dimension n* if it has a weak Serre functor $\mathbf{S}_\mathcal{T}$ such that $\mathbf{S}_\mathcal{T} \simeq \mathrm{id}[n]$.

Theorem 5.1 *For any basic Young diagram λ, the category $\mathcal{DP}_\lambda^{af_i,b}$ is Calabi–Yau of dimension $2d(p^i-1)$, the category $\mathcal{DP}_\lambda^{af_i}$ is weak Calabi–Yau of dimension $2d(p^i-1)$.*

Proof The theorem is a formal consequence of the following properties of the bifunctor B_λ (the crucial second property is specific to basic blocks). □

Lemma 5.2 *There are the following isomorphisms of bifunctors:*

1. $B_\lambda(V, W\otimes A_i) \simeq B_\lambda(V\otimes A_i^*, W)$ *for any* λ.
2. $B_\lambda(V, W) \simeq S_\lambda(V^*)\otimes S_\lambda(W)$ *for basic* λ.

Proof of the Lemma We recall that by the Yoneda lemma

$$B_\lambda(V, W) = \mathrm{Hom}_{\mathcal{P}_d}(B_\lambda(W,-), B_\lambda(V,-))$$

and a general fact that

$$\mathrm{Hom}_{\mathcal{P}_d}(F(-\otimes X), G) \simeq \mathrm{Hom}_{\mathcal{P}_d}(F, G(-\otimes X^*))$$

for any graded space X and $F, G \in \mathcal{P}_d$. This gives the first isomorphism.

The second isomorphism immediately follows from the Cauchy decomposition and the fact that $S_\lambda \simeq W_\lambda$ for basic λ. □

We recall that we deal with left Serre functors, hence we should show that $\mathbf{S}_\lambda^{af_i} \simeq id[-2d(p^i-1)]$.

Since

$$\mathbf{S}_\lambda^{af_i}(F)(V\otimes A_i) \simeq \mathrm{RHom}_{\mathcal{P}_\lambda^{af_i}}(c^*_{\lambda, V\otimes A_i}, F)$$

and by the Yoneda lemma

$$F(V\otimes A_i) = \mathrm{RHom}_{\mathcal{P}_\lambda^{af_i}}(h_\lambda^{V\otimes A_i}, F),$$

it suffices to find a natural in V isomorphism

$$c^*_{\lambda, V\otimes A_i} \simeq h_\lambda^{V\otimes A_i}[2d(p^i-1)].$$

On the one hand we have:

$$h_\lambda^{V\otimes A_i}(W) = \mathrm{Hom}_{B_\lambda \mathcal{V}_{A_i}}(V, W) = (B_\lambda(V, W\otimes A_i) \simeq S_\lambda(V^*)\otimes S_\lambda(W\otimes A_i).$$

On the other hand, by using the both parts of Lemma 5.2, we obtain:

$$c^*_{\lambda, V\otimes A_i}(W) = (\mathrm{Hom}_{B_\lambda \mathcal{V}_{A_i}}(W, V))^* = (B_\lambda(W, V\otimes A_i))^* \simeq (B_\lambda(W\otimes A_i^*, V))^* \simeq$$

$$(S_\lambda(W^* \otimes A_i) \otimes S_\lambda(V))^* = S_\lambda(V^*) \otimes S_\lambda(W \otimes A_i^*).$$

Since $A_i^* \simeq A_i[2(p^i - 1)]$ we have an isomorphism of functors

$$S_\lambda(- \otimes A_i^*) \simeq S_\lambda(- \otimes A_i)[2d(p^i - 1)]$$

which completes the proof.

Corollary 5.3 *The category $\mathcal{DP}_1^{af_i,b}$ is Calabi–Yau of dimension $2(p^i - 1)$ and $\mathcal{DP}_1^{af_i}$ is weak Calabi–Yau of dimension $2(p^i - 1)$.*

Proof The corollary follows from Theorem 5.1 and the fact that $\mathcal{P}_1$ consists of a single block which is obviously basic. □

This fact has the following global generalization.

Proposition 5.4 *For any $d < p$, the category $\mathcal{DP}_d^{af_i,b}$ is Calabi–Yau of dimension $2d(p^i - 1)$ and $\mathcal{DP}_d^{af_i}$ is weak Calabi–Yau of dimension $2d(p^i - 1)$.*

Proof In fact for $d < p$ all the blocks in $\mathcal{P}_d$ are basic but since $\mathcal{P}_d^{af_i}$ is not a product of its affine semiblocks, our statement cannot be directly deduced from Theorem 5.1. Instead one can repeat the proof of Theorem 5.1 in the present context. The crucial fact is that $\Gamma^d \simeq S^d$ if $d < p$. We leave the straightforward details to the reader. □

As we have said in the Introduction, the Calabi–Yau structure on $\mathcal{DP}_\lambda^{af_i}$ provides sort of categorical interpretation of the Poincaré duality. Hence it is not surprising that one can deduce Corollary 2.4 from Theorem 5.1 (and the compatibility of the (block) affine derived Kan extension with the Kuhn duality).

Namely, by the block affine derived Kan extension we obtain

$$\mathrm{Ext}^s_{\mathcal{P}_{p^i d}}(F_\lambda^{(i)}, F_\mu) \simeq \mathrm{Hom}_{\mathcal{DP}_\lambda^{af_i}}(z_\lambda^*(F_\lambda), \mathbf{K}_\lambda^{af_i}(F_\mu)[s]).$$

Then we apply the Calabi–Yau isomorphism (we emphasize the fact that we need “the weak Calabi–Yau structure” here, since $\mathbf{K}^{af_i}$ does not preserve compact objects)

$$\mathrm{Hom}_{\mathcal{DP}_\lambda^{af_i}}(z_\lambda^*(F_\lambda), \mathbf{K}_\lambda^{af_i}(F_\mu)[s]) \simeq \mathrm{Hom}_{\mathcal{DP}_\lambda^{af_i}}(\mathbf{K}_\lambda^{af_i}(F_\mu)[s - 2d(p^i - 1)], z_\lambda^*(F_\lambda))^*.$$

Next we apply the Kuhn duality and use the fact that it commutes with z^* (Proposition 3.7(3)) and with $\mathbf{K}_\lambda^{af_i}$ (Proposition 3.10(1))

$$\mathrm{Hom}_{\mathcal{DP}_\lambda^{af_i}}(\mathbf{K}_\lambda^{af_i}(F_\mu)[s-2d(p^i-1)], z_\lambda^*(F_\lambda))^* \simeq \mathrm{Hom}_{\mathcal{DP}_\lambda^{af_i}}(z_\lambda^*(F_\lambda^\#)[s-2d(p^i-1)], \mathbf{K}_\lambda^{af_i}(F_\mu^\#))^*.$$

At last we come back to $\mathcal{DP}_{dp^i}$:

$$\mathrm{Hom}_{\mathcal{DP}_\lambda^{af_i}}(z_\lambda^*(F_\lambda^\#)[s - 2d(p^i - 1)], \mathbf{K}_\lambda^{af_i}(F_\mu^\#))^* \simeq \mathrm{Hom}_{\mathcal{DP}_{dp^i}}(F_\lambda^{(i)\#}[s - 2d(p^i - 1)], F_\mu^\#))^*$$

and by using the self-duality of simples we finally obtain our formula

$$\mathrm{Hom}_{\mathcal{DP}_{dp^i}}(F_\lambda^{(i)\#}[s-2d(p^i-1)], F_\mu^{\#}))^* \simeq \mathrm{Ext}^{2d(p^i-1)-s}_{\mathcal{P}_{dp^i}}(F_\lambda^{(i)}, F_\mu)^*.$$

Of course this approach is technically much more involved than that taken in Section 2, but it shows how classical and affine phenomena are related and also explains why we insist on considering weak Serre functors.

5.2 $\mathcal{P}_\lambda^{af_i}$ as a Module Category

In this subsection we provide various descriptions of $\mathcal{P}_\lambda^{af_i}$ as a category of graded modules over a certain explicitly described graded algebra. We recall that for a finite dimensional graded algebra S, we denote by $S\text{-mod}^f$ the graded abelian category of $\mathbf{Z}$-graded S-modules finite dimensional in each degree. Then as we remember from Proposition 4.1, for any λ, $i \geq 1$, $n \geq d$, the category $\mathcal{P}_\lambda^{af_i}$ is equivalent to $S_{d,n}^{af_i}\text{-mod}^f$, where $S_{d,n}^{af_i}$ is the block affine Schur algebra. However, this fact is not very useful in practice, since this graded algebra is quite complicated. Luckily, in the case of basic block the situation massively simplifies. First of all, as we observed in Lemma 5.2 we have an isomorphism of graded vector spaces

$$B_\lambda(\mathbf{k}^d, \mathbf{k}^d \otimes A_i) \simeq S_\lambda(\mathbf{k}^{d*}) \otimes S_\lambda(\mathbf{k}^d \otimes A_i).$$

However, in order to understand the multiplicative structure it is better to take a different point of view. Namely, by Lemma 5.2 we have a decomposition

$$B_\lambda(\mathbf{k}^d, -) \simeq \bigoplus_{j=1}^{s_{\lambda,d}} S_\lambda$$

where $s_{\lambda,d} = \dim(S_\lambda(\mathbf{k}^d))$. Let us define a graded algebra

$$A_{i,\lambda} := \mathrm{Ext}^*_{\mathcal{P}_{dp^i}}(S_\lambda^{(i)}, S_\lambda^{(i)}).$$

Then we have isomorphisms of graded algebras

$$B_\lambda(\mathbf{k}^d, \mathbf{k}^d \otimes A_i) \simeq \mathrm{Ext}^*_{\mathcal{P}_{dp^i}}(B_\lambda^{(i)}(-, \mathbf{k}^d), B_\lambda^{(i)}(-, \mathbf{k}^d)) \simeq \mathrm{Ext}^*_{\mathcal{P}_{dp^i}}(\bigoplus_{j=1}^{s_{\lambda,d}} S_\lambda^{(i)}, \bigoplus_{j=1}^{s_{\lambda,d}} S_\lambda^{(i)}) \simeq$$

$$M_{s_{\lambda,d}}(A_{i,\lambda}).$$

Since any matrix algebra is Morita equivalent to the ground algebra, we obtain

Proposition 5.5 *For any basic Young diagram λ, the categories $\mathcal{P}_\lambda^{af_i}$ and $A_{i,\lambda}-\mathrm{mod}^f$ are equivalent as graded abelian categories.*

Finally, let us take a closer a look at the graded algebra $A_{i,\lambda}$. Firstly, by the Collapsing Conjecture [5, Cor. 3.7]:

$$A_{i,\lambda} \simeq \mathrm{Hom}_{\mathcal{P}_d}(S_\lambda, S_{\lambda,A_i}).$$

The dimension of the latter algebra can be explicitly expressed in terms of the Littlewood-Richardson numbers. This point of view also allows one to describe the multiplication: it comes as the composite of scalar extension, Hom-multiplication and the multiplication in A_i:

$$\mathrm{Hom}_{\mathcal{P}_d}(S_\lambda, S_{\lambda,A_i}) \otimes \mathrm{Hom}_{\mathcal{P}_d}(S_\lambda, S_{\lambda,A_i}) \longrightarrow$$
$$\mathrm{Hom}_{\mathcal{P}_d}(S_\lambda, S_{\lambda,A_i}) \otimes \mathrm{Hom}_{\mathcal{P}_d}(S_{\lambda\otimes A_i}, S_{\lambda,A_i\otimes A_i}) \longrightarrow \mathrm{Hom}_{\mathcal{P}_d}(S_\lambda, S_{\lambda,A_i\otimes A_i}) \longrightarrow$$
$$\mathrm{Hom}_{\mathcal{P}_d}(S_\lambda, S_{\lambda,A_i}).$$

A bit different description of $A_{i,\lambda}$ is perhaps even more down to earth. It follows from the fact that since λ is basic, S_λ is a direct summand in I^d. Hence there exists an idempotent $e_\lambda \in \mathbf{k}[\Sigma_d]$ such that $S_\lambda = e_\lambda I^d$. Therefore we get

$$A_{i,\lambda} \simeq e_\lambda(\mathrm{Ext}^*_{\mathcal{P}_{dp^i}}(I^{d(i)}, I^{d(i)}))e_\lambda \simeq e_\lambda(A_i^{\otimes d} \otimes \mathbf{k}[\Sigma_d])e_\lambda.$$

A subtle point here is that even if we would take the whole S_λ-isotypical summand in I^d and the corresponding central idempotent e'_λ, this e'_λ is not central in the algebra $A_i^{\otimes d} \otimes \mathbf{k}[\Sigma_d]$. Hence $A_{i,\lambda}$ is not Morita equivalent to a direct factor in $A_i^{\otimes d} \otimes \mathbf{k}[\Sigma_d]$. This is another manifestation of the fact that affine semiblocks are not genuine blocks.

All these descriptions drastically simplify for $d = 1$. In this case we just obtain

Corollary 5.6 *The categories $\mathcal{P}_1^{af_i}$ and $A_i-\mathrm{mod}^f$ are equivalent as graded abelian categories.*

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Declarations

Conflict of Interest The author declares no competing interests.

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